The classifying space of the 1+1 dimensional G-cobordism category

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Abstract

The 1+1 G-cobordism category, with G a finite group, is important in the construction of G-topological field theories which are completely determined by a G-Frobenius algebra, see [MS06, Tur10, Kau03]. We give a description of the classifying space of this category generalizing the work presented by Ulrike Tillmann in [Til96]. Moreover, we compute the connected components and the fundamental group of this classifying space and we study the classifying spaces of some important subcategories. Finally, we present some relations between the rank of the fundamental group of the G-cobordism category, with some related data as the number of (abelian) subgroups of the group G.

1 Introduction

Nowadays the study of the classifying space of cobordism categories is rapidly gaining importance due to the proof by Madsen and Weiss in [MW07], of Mumford's conjeture about the stable cohomology of the moduli space of Riemann surfaces. Initially, the cobordism category was introduced by Segal in [Seg74], for the study of conformal field theories in dimension 1+1 (where 1 represents the dimension of the objects and 1+1=2 the dimension of the morphisms of the cobordism category). Tillmann in [Til96], was the first who provided a study of its classifying space and in collaboration with Galatius, Madsen and Weiss culminated in the calculation of the homotopy type of the cobordism category in any dimension, see [GMTW]. They showed that in every dimension the classifying space of the cobordism category has the homotopy type of the infinite loop space of a certain Thom-spectra. For G a finite group, the G-cobordism category was introduced by Turaev in [Tur10], with a homotopical version given by a background space with base point, that in our case is the classifying space BG. The definition we use in this article for the G-cobordism category was given by Segal and Moore in [MS06] and by Kaufmann in [Kau03]. Let $\mathscr{S}^G$ be the G-cobordism category in dimension $1+1$, the principal result of this article is the calculation of the homotopy type of the classifying space

$$B\mathscr{S}^G \simeq \frac{G}{[G,G]} \times X_G \times T^{r(G)}\,, \tag{1}$$

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where X_G is the homotopy fiber of the classifying map of a certain functor and $T^{r(G)}$ is the direct product of $r(G)$-circles, with $r(G)$ a positive integer which depends on G. In addition, we prove that the connected components are parameterized by the abelianization of G, thus $\pi_0(\mathscr{S}^G) \cong G/[G,G]$, and the fundamental group satisfies the isomorphism $\pi_1(\mathscr{S}^G) \cong \mathbb{Z}^{r(G)}$. Calculations of the number $r(G)$ for some groups G, are presented in table (17) which provide unknown relations with geometric group theory through an approximation of the number of (abelian) subgroups of the group G. Some future extensions of our work consist to substitute the group G by a groupoid $\mathcal{G}$, see [Pha10], and to consider an arbitrary space as the background space in the homotopical version, see [Tur10, BT99]. An interesting application is the utility of these techniques to classify G-equivariant invertible topological field theories in dimension two, see [JT13].

This article is organized as follows. In **section 2** we define the G-cobordism category in terms of the elementary components given by the G-principal bundles over the circle, the pair of pants, the cylinder and the disc. In **section 3** we prove for the G-cobordism category, that the connected components of the classifying space are parameterized by the abelianization of the group G and, in addition, we study the classifying space of some important subcategories. In **section 4** we prove that the category of fractions of the G-cobordism category, can be obtained with a quotient category followed by inverting the G-closed surfaces; this fact permit us to reinterpret its fundamental group and allow us to prove that $\pi_1(\mathscr{S}^G) \cong \mathbb{Z}^{r(G)}$. Finally, in **section 6** we report some results which include a set of equations which determine completely the morphisms of the G-cobordism category (this equations were proved in [Seg11]). These equations depend only on the axioms of a G-Frobenius algebra [MS06]. We implement these equations in MATLAB [MAT] to obtain the data of table (17); we use the web page http://oeis.org/ see [oei], to discover some related relations.

2 Definition of the G-cobordism category

For G a finite group, the G-cobordism category in dimension 1+1, denoted by $\mathscr{S}^G$, has principal G-bundles over the circle as objects and principal G-bundles over surfaces (up to homeomorphisms) as morphisms. We can take the principal G-bundles over the circle as based maps $P \xrightarrow{\pi} S^1$, with P and S^1 based spaces and π a based map. They are in correspondence with the elements of the group G by the lifting of the base space starting in the base point of the total space (this elements are called G-circles). We denote by P_g the total space associated to the principal G-bundle for $g \in G$. Therefore the objects of the category $\mathscr{S}^G$ are described by sequences $(g_1, g_2, ..., g_n)$ of elements in G except for some of them that are the empty set $\emptyset$. The order of the sequences reflects downwards in our pictures. For the morphisms of $\mathscr{S}^G$ consider $\overline{g} := (g_1, g_2, ..., g_n)$ and $\overline{h} := (h_1, h_2, ..., h_m)$ sequences as before, denote $\bigsqcup_i P_{g_i}$ and $\bigsqcup_j P_{h_j}$ their corresponding total spaces; a morphism from $\overline{g}$ to $\overline{h}$ is composed by a cobordism M of the based spaces and a principal G-bundle $P \xrightarrow{\pi} M$ with $P|_{\partial_{in}} = \bigsqcup_i P_{g_i}$ and $P|_{\partial_{out}} = \bigsqcup_j P_{h_j}$. Two cobordisms are identified if there is a homeomorphism $M \longrightarrow M'$ of principal G-bundles fixing the boundary.

For the morphisms of $\mathscr{S}^G$ we do not have an explicit expression, but we can study them

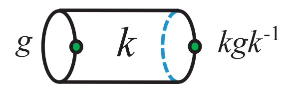

Figure 1: A G-cylinder from g to kgk^{-1}.

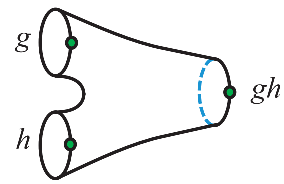

Figure 2: A G-pair of pants from $g \sqcup h$ to gh.

through their decomposition in elementary parts. The elementary parts are the principal G-bundles over the cylinder, the pair of pants and the disk; the description is as follows:

A. For $g, h \in G$, the morphisms from g to h (with base space the cylinder) are in one-to-one correspondence with the elements of the set $\{k : h = kgk^{-1}\}$ up to the identification[1] $k \sim h^n k g^m$, where $n, m \in \mathbb{Z}$. A typical element is given in Figure 1. This correspondence is given by means of the homotopy lifting property applied to the base space, a cylinder, with starting point P_g.

B. For the principal G-bundles over the pair of pants we describe the *thin* ones. We consider the set of principal G-bundles over the wedge of two circles $S^1 \vee S^1$, since G is finite this is in bijection with the group homomorphisms from the fundamental group of $S^1 \vee S^1$ to G, i.e. with $G \times G$. A basic element is given in Figure 2. and for a generic principal G-bundle over the pair of pants, we make compositions with principal G-bundles over the cylinder.

C. The disk is contractile, therefore it has only one principal G-bundle over it which is trivial.

This elements are called G-cylinders, G-pair of pants and the disc respectively. Every G-cobordism is constructed by composition of these elements, together with the corresponding elements with the reverse orientation. These are restricted up to some constrains which are given in section 6. Finally, the composition of two morphisms is done in such a way that the base points match.

3 Analysis of subcategories

We start this section with the proof that the connected components of the classifying space of $\mathscr{S}^G$ is the abelianization $G/[G,G]$. This reduces the calculation of the classifying space of

[1]This identification is given by the action of the Dehn twists.

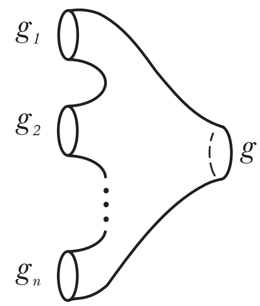

Figure 3: A G-pair of pants with multiple legs.

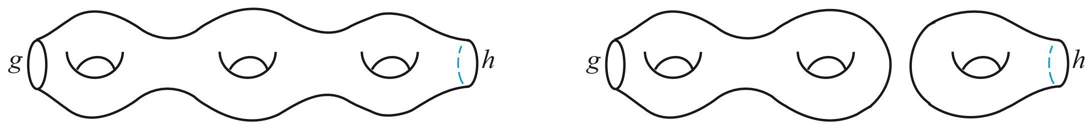

Figure 4: G-cobordisms from g to h.

$\mathscr{S}^G$ to the connected component of the empty 1-manifold $\emptyset$. In addition we study below the classifying space of some smaller subcategories of $\mathscr{S}^G$.

Proposition 1. *The connected components of the classifying space $B\mathscr{S}^G$ are in bijection with the elements of the abelianization of the group, i.e. with the quotient $G/[G,G]$.*

Proof. We have the following three facts: first, the empty set and the trivial G-circle are connected through the disc; second, we can connect every sequence $(g_1, g_2, ..., g_n)$ of elements in G with the product $g = \prod_i^n g_i$ by the G-pair of pants with multiple legs depicted in Figure 3; and finally, the G-cobordisms with empty boundary are superfluous to determine the connected components. Therefore, it is enough to consider the two different morphisms given in Figure 4. Since every handle has as boundary a commutator, see Figure 5. Moreover, the morphisms exemplified in Figure 4, are compositions of handles with G-pair of pants with multiple legs. Consequently, the G-circles associated to $g, h \in G$ are connected in $\mathscr{S}^G$ if and only if they differ by an element of the commutator group. □

It is well known that the classifying space of a symmetric monoidal category is an H-space. Moreover, if it has a group structure in the connected components, compatible with the product of the H-space, then every pair of connected components are homotopy equivalent by multiplication by an element (an H-space with this property is called a *grouplike*, see [May74]). Consequently, we conclude the following result.

Corollary 2. *The classifying space $B\mathscr{S}^G$ is of the homotopy type of the product of $G/[G,G]$ with the connected component of the empty 1-manifold $\emptyset$.*

Denote $\mathscr{S}_0^G$ the full subcategory of $\mathscr{S}^G$ with only one object given by the empty set.

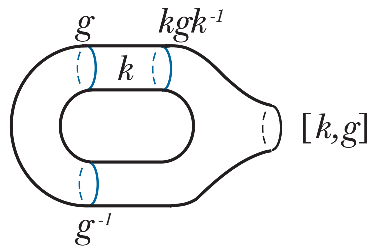

Figure 5: A G-cobordisms with boundary the commutator $[k, g]$.

Proposition 3. *The classifying space $B\mathscr{S}_0^G$ is homotopic to the infinite dimensional torus T^∞.*

Proof. This category is endowed with the structure of an abelian monoid, infinitely generated, without torsion and hence of the form $\mathbb{N}^\infty$. Since the classifying map of the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ is a homotopy equivalence and since the classifying space of $\mathbb{Z}$ is the circle, then $B\mathscr{S}_0^G \simeq T^\infty$ where the infinite torus has the direct limit topology induced from the finite dimensional subtori. $\square$

Now, we denote by $\mathscr{S}_{>0}^G$ the subcategory of $\mathscr{S}^G$ with the same objects of $\mathscr{S}^G$ except for the empty set and where each connected component of every morphism has non empty initial boundary and non empty final boundary. We can simplify the classifying space of this category in terms of a smaller full subcategory, denoted by $\mathscr{S}_1^G$, which each object is a G-circle (i.e. one principal G-bundle over the circle) and the morphisms are connected G-cobordisms.

Theorem 4. *The inclusion functor $\mathscr{S}_1^G \longrightarrow \mathscr{S}_{>0}^G$ has a left adjoint Φ.*

Proof. We proceed to define the adjoint $\Phi : \mathscr{S}_{>0}^G \longrightarrow \mathscr{S}_1^G$. On objects this is defined by multiplication $(g_1, g_2, ..., g_n) \longmapsto g = \prod_i g_i$ and for morphisms we take the following construction: take Σ a G-cobordism in $\mathscr{S}_{>0}^G$ from (g_i) to (h_j), $g = \prod_i g_i$ and $h = \prod_j h_j$; then we compose Σ with a G-pair of pants with multiple legs with one exit and the same entries that the exits of Σ (as in Figure 3); subsequently, for the resulting connected cobordism by Cerf theory [Cer70], we can find a Morse function for a representant of Σ, such that there exists $t \in [0, 1]$ with the property that the inverse image of $[0, t]$ is a pair of pants with multiple legs, and the inverse image of t is a circle; finally, we take $\Phi(\Sigma)$ as the class in $\mathscr{S}_1^G$ of the pre-image of $[t, 1]$ by the last Morse function, see the Figure 6. The functorial property of Φ is illustrated in Figure 7. Eventually, in Figure 6, we express the adjointness of Φ by the commutativity of the diagram

$$\begin{tikzcd} (g_i) \arrow[r, "p_{(g_i)}"] \arrow[d, "\Sigma"'] & g \arrow[d, "\Phi(\Sigma)"] \\ (h_j) \arrow[r, "p_{(h_j)}"'] & h \end{tikzcd},$$

where $p_{(g_i)}$ and $p_{(h_j)}$ are G-pair of pants with multiple legs. $\square$

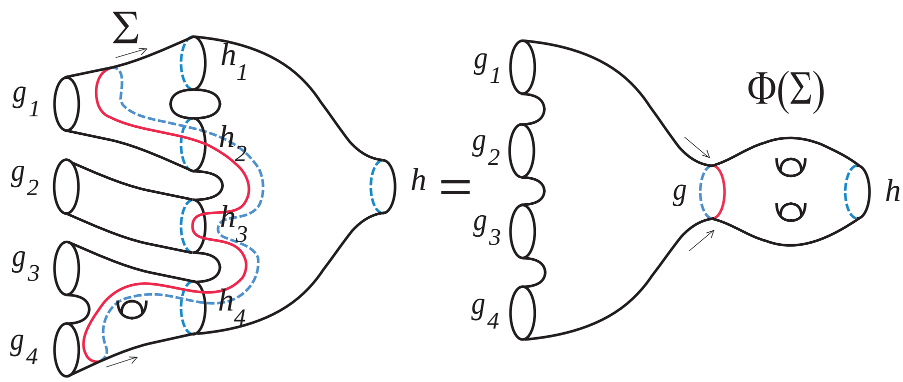

Figure 6: The adjointness between the inclusion $\mathscr{S}^G_1 \longrightarrow \mathscr{S}^G_{>0}$ and Φ.

Let $\mathcal{M}$ the monoid conformed by element in $\mathscr{S}^G_1$ exemplified in Figure 8. Thus they are represented by connected G-cobordisms which start and end in the trivial G-bundle and which only (possible) non-trivial G-cylinders are positioned at the top. The monoid $\mathcal{M}$ is relative easy since it is abelian, finitely generated and does not have torsion. We prove the first two properties in the Figures 9 and 10 respectively, This monoid does not have torsion because of the existence of the genus map, which associates to any G-cobordism the genus of the base space. Consequently, this monoid is the direct sum $\mathbb{N}^{r(G)}$ for some positive integer $r(G)$. We will prove latter that the group completion of $\mathcal{M}$ is isomorphic to the fundamental group of the whole category $\mathscr{S}^G$. Some calculations of the number $r(G)$ are presented in table (17).

There is a subcategory $\mathscr{S}^G_b$ of $\mathscr{S}^G$ similar to $\mathscr{S}^G_{>0}$. This category has the same objects as $\mathscr{S}^G$ and each connected component of every morphism has non empty final boundary[2]. Similarly, we can define a functor $\Phi : \mathscr{S}^G_b \longrightarrow \mathscr{S}^G_1$ which is adjoint to the inclusion $\mathscr{S}^G_1 \hookrightarrow \mathscr{S}^G_b$. We recall from [Seg68] that a natural transformation between two functors translate into a homotopy equivalence between the classifying maps of the functors. Thus the two inclusions $\mathscr{S}^G_1 \hookrightarrow \mathscr{S}^G_{>0}$ and $\mathscr{S}^G_1 \hookrightarrow \mathscr{S}^G_b$ are homotopy equivalences in classifying spaces.

4 The fundamental group

Now we make a further analysis of the fundamental group of the whole category $\mathscr{S}^G$. The category $\mathscr{S}^G$ is essentially the union of the categories $\mathscr{S}^G_0$ and $\mathscr{S}^G_{>0}$ with the disc from the empty 1-manifold to the trivial G-circle and vice versa as additional generators. We prove in the last section that the connected components satisfy the identity $\pi_0(\mathscr{S}^G) = G/[G,G]$. In this section we prove that the fundamental group of the classifying space of $\mathscr{S}^G$ satisfies the isomorphism $\pi_1(\mathscr{S}^G) \cong \mathbb{N}^{r(G)}$.

Let $\widehat{\mathscr{S}^G}$ be the quotient category $\mathscr{S}^G$ with the equivalent relation generated by the identification given in Figure 11, where the components are described as follows: take M a G-cobordism

[2] Note that by vacuity the empty manifold satisfies to be the identity of the empty set.

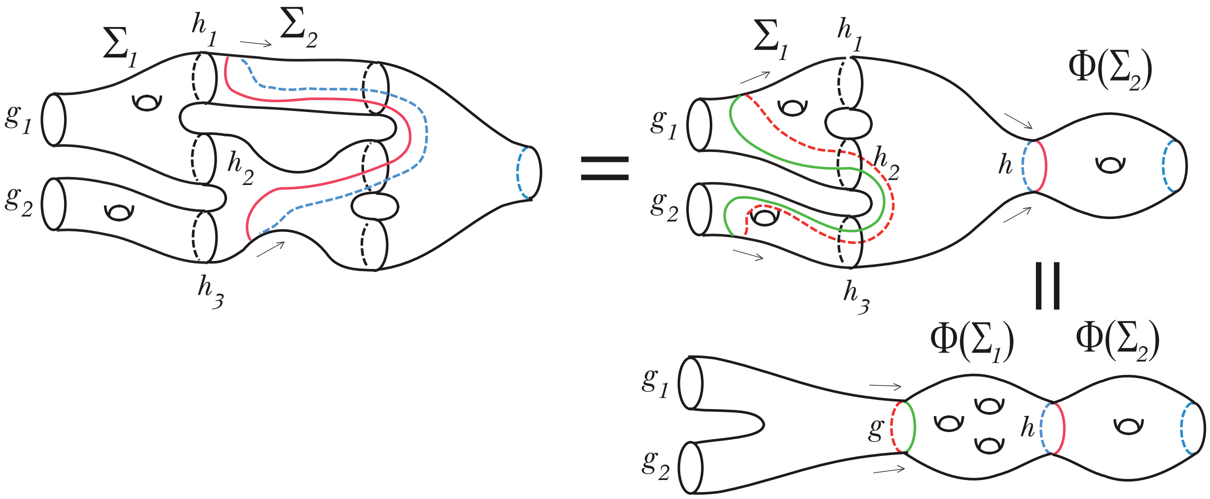

Figure 7: Functorial property of the functor Φ.

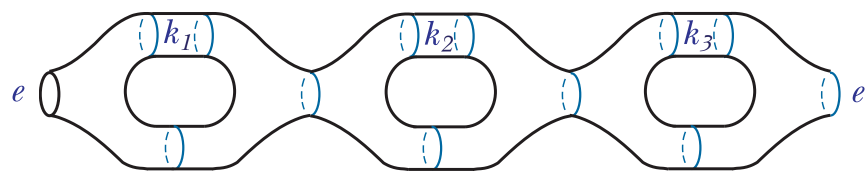

Figure 8: An element of the monoid $\mathcal{M}$.

from $\overline{g} := (g_1, ..., g_n)$ to $\overline{h} := (h_1, ..., h_m)$ and let N be any (connected) G-cobordism from $\overline{h}$ to the empty set, N exists because we can restrict the whole category $\mathscr{S}^G$ to the full subcategory where every object is connected to the empty set; in addition we denote by M' the composition of M with N and $\overline{M}'$ is M' with the reverse orientation. By the equivalence relation generated we refer that two G-cobordisms are identified if we can get one from the other through a finite number of steps, given by the last identification. Denote by $\mathscr{G}^G = \mathscr{S}^G[\mathscr{S}^{G^{-1}}]$ the category of fractions obtained by inverting all the morphisms of $\mathscr{S}^G$, see [GZ67, GM03]. We denote by $\mathcal{I}$ the subset of morphisms of $\mathscr{S}^G$ given by disjoint union of G-surfaces (G-cobordisms with empty boundary) with trivial G-cylinders. This set is a localizing set in the sense of [GM03] or check the appendix 7.1 for a definition. The importance of a localizing set is that it lets us to give an explicit description of the category of fractions associated. We prove in the following proposition that indeed making a quotient category and localization in the set $\mathcal{I}$, it is enough to give a description of the category of fractions $\mathscr{G}^G = \mathscr{S}^G[\mathscr{S}^{G^{-1}}]$. Thus the following result.

Proposition 5. *There is an isomorphism between* $\mathscr{G}^G = \mathscr{S}^G[\mathscr{S}^{G^{-1}}]$ *and* $\widehat{\mathscr{S}^G}[\mathcal{I}^{-1}]$.

Proof. In Figure 12 we see that the set $\mathcal{I}$ is a localizing set inside $\widehat{\mathscr{S}^G}$. The universal properties of a quotient category and of a category of fractions, assure the existence of unique functors

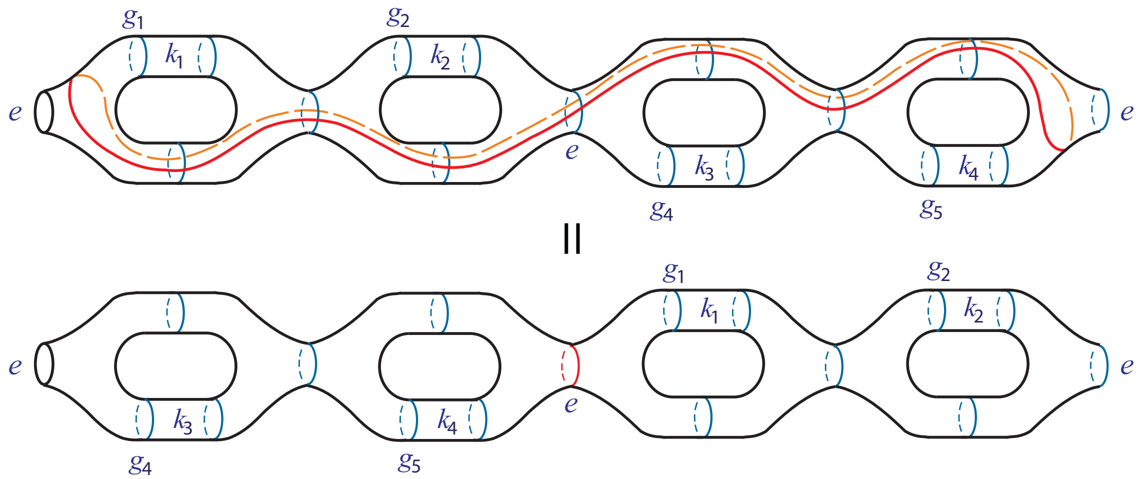

Figure 9: The monoid $\mathcal{M}$ is abelian.

$F : \widehat{\mathscr{S}^G} \longrightarrow \mathscr{G}^G$ and $E : \widehat{\mathscr{S}^G}[\mathcal{I}^{-1}] \longrightarrow \mathscr{G}^G$ such that the following diagrams commute

$$\begin{tikzcd} \mathscr{S}^G \arrow[r, "P"] \arrow[dr, "J"'] & \widehat{\mathscr{S}^G} \arrow[d, "F"] \\ & \mathscr{G}^G \end{tikzcd} , \qquad \begin{tikzcd} \widehat{\mathscr{S}^G} \arrow[r, "J'"] \arrow[dr, "F"'] & \widehat{\mathscr{S}^G}[\mathcal{I}^{-1}] \arrow[d, "E"] \\ & \mathscr{G}^G \end{tikzcd} .$$

By the representation of "roofs" of the category of fractions, we can see that the composition $L := J' \circ P$ inverts any morphism of $\mathscr{S}^G$. For this, take M a G-cobordism from $\overline{g} := (g_1, ..., g_n)$ to $\overline{h} := (h_1, ..., h_m)$ and let N be a (connected) G-cobordism from $\overline{h}$ to the empty set, denote by M' the composition of M with N and $\overline{M}'$ denotes M' with the reverse orientation. Thus the composition in Figure 13 is equivalent to the identity map if and only if we have the identification of Figure 11. As a consequence, there is a unique functor $D : \mathscr{G}^G \longrightarrow \widehat{\mathscr{S}^G}[\mathcal{I}^{-1}]$ such that the diagram

$$\begin{tikzcd} \mathscr{S}^G \arrow[r, "J"] \arrow[dr, "L"'] & \mathscr{G}^G \arrow[d, "D"] \\ & \widehat{\mathscr{S}^G}[\mathcal{I}^{-1}] \end{tikzcd}$$

is commutative. Therefore we have the sequence of identities

$$E \circ D \circ J = E \circ L = E \circ J' \circ P = F \circ P = J ,$$

and by the universal property of the category of fractions we conclude that $E \circ D = 1$; reciprocally, since the functor P is an epimorphism in the category of small categories[3], then the identity

[3] An epimorphism in the category of small categories is a full and an essentially surjective functor.

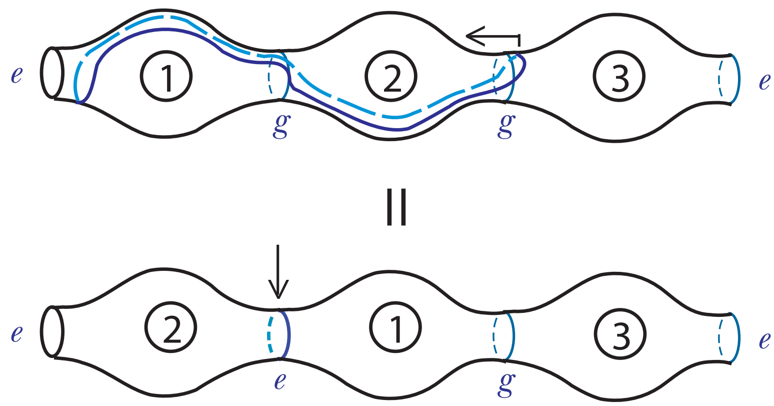

Figure 10: The monoid $\mathcal{M}$ is finitely generated.

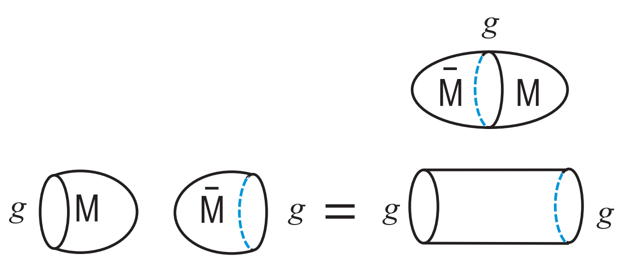

Figure 11: Identification for the quotient category $\widehat{\mathscr{S}^G}$.

$J' \circ P = L = D \circ J = D \circ F \circ P$ implies that $J' = D \circ F$, and consequently $D \circ E \circ J' = D \circ F = J'$ and by the universal property of the category of fractions we have $D \circ E = 1$.

□

Theorem 6. *The fundamental group $\pi_1(B\mathscr{S}^G)$ is isomorphic to the direct product $\mathbb{Z}^{r(G)}$, with $r(G)$ a positive integer which depends on the group G.*

Proof. We know by [Qui73] that the fundamental group of the classifying space of a small category, based on an object of the category, is isomorphic with the restriction of the category of fractions to this object, see [GZ67]. Let $\mathscr{G}^G_e$ be the group defined as the full subcategory of $\mathscr{G}^G = \mathscr{S}^G[\mathscr{S}^{G^{-1}}]$ with only one object given by the trivial G-bundle. By Proposition 5 this group is composed by roofs of the form

$$\begin{tikzcd} & e \arrow[dl, "z_-"'] \arrow[dr, "z_+"] & \\ e & & e \end{tikzcd} \tag{2}$$

where z_+ and z_- are the positive and negative part, given by G-surfaces (with empty boundary),

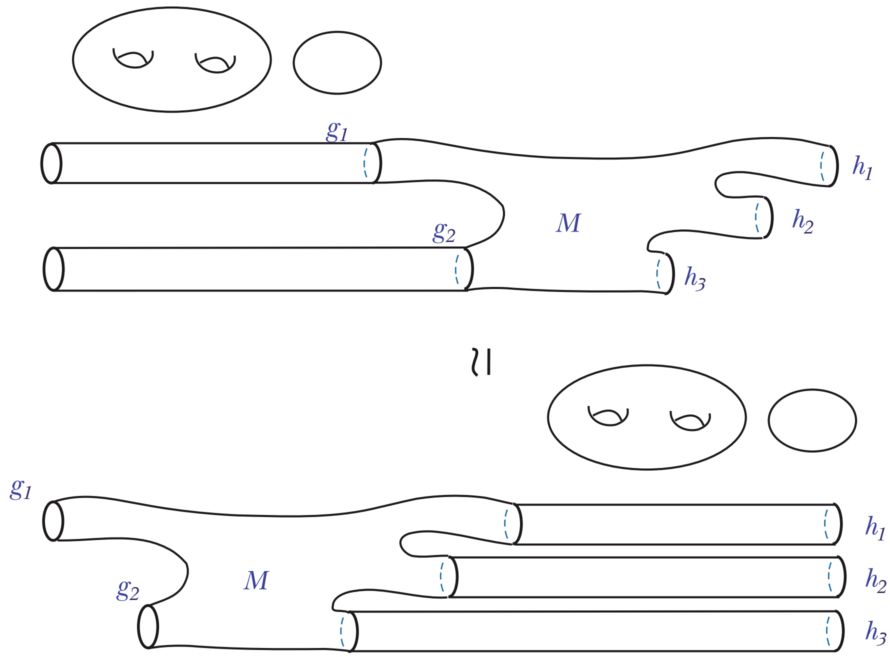

Figure 12: The set $\mathcal{I}$ is a localizing set.

inside $\widehat{\mathscr{S}^G}$. We recall that in the monoid $\mathcal{M}$ every element is represented by a connected G-cobordism which starts and ends in the trivial G-bundle and which only (possible) non-trivial G-cylinders are positioned at the top (we give an example in Figure 8). We proved in the last section that this monoid is isomorphic to the direct product $\mathbb{N}^{r(G)}$ for some positive integer $r(G)$. There exists a morphism of monoids $j : \mathbb{N}^{r(G)} \longrightarrow \mathscr{G}_e^G$, defined for a connected G-cobordism Σ which starts and ends in the trivial G-bundle, by

$$\begin{tikzcd} & e \arrow[dl, "1"'] \arrow[dr, "\Sigma"] & \\ e & & e\,. \end{tikzcd}$$

We can associate to j, the comma category $e\diagdown j$ with objects roofs of the form (2) and morphisms given by elements Σ inside $\mathcal{M} = \mathbb{N}^{r(G)}$. Now we are going to prove that $e\diagdown j$ is a filtrated category. This is just to check two properties that are given in the appendix 7.2. Since the classifying space of a filtrated category is contractible, then we can use a theorem proved by Daniel Quillen, in [Qui73] or check the appendix 7.4, in order to prove that j is a homotopy equivalence in classifying spaces.

For the first axiom of a filtrated category we take two objects of $e\diagdown j$ as follows

$$\begin{tikzcd} & e \arrow[dl, "z_-"'] \arrow[dr, "z_+"] & \\ e & & e \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & e \arrow[dl, "z'_-"'] \arrow[dr, "z'_+"] & \\ e & & e\,. \end{tikzcd} \tag{3}$$

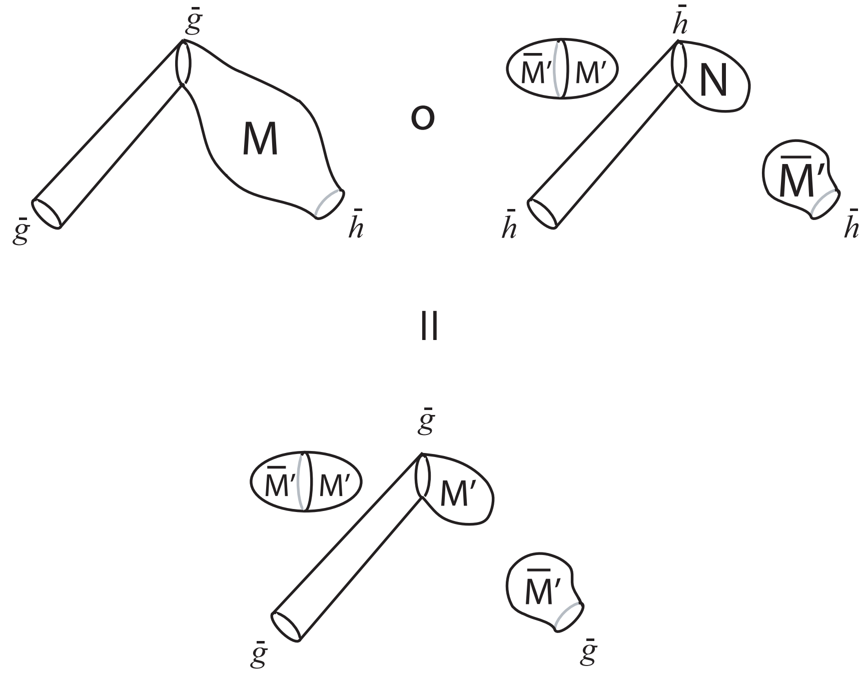

Figure 13: Composition of two roofs.

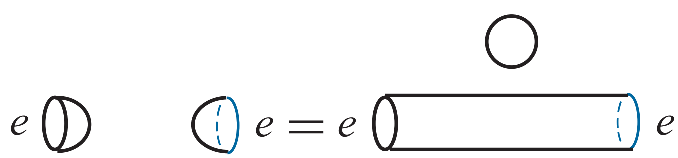

Figure 14: Identification inside the category of fraction.

There is a relation inside the category of fractions generated by the identification given in Figure 14. Indeed, this identification is implied by the property that for a morphism in a groupoid the inverse is unique (we apply this property to the composition given in Figure 15). Moreover, we have also the property given in Figure 16, which is read sphere kills handle with trivial G-bundle. Thus we can connect different surfaces adding a disjoint sphere, which can disappear if we can compose a handle with trivial G-bundle. We take the morphisms in $\mathcal{M}$ associated to

$$e \xrightarrow{z_- + z'_+} e \quad \text{and} \quad e \xrightarrow{z'_- + z_+} e \ ,$$

and we have the compositions

$$\begin{tikzcd}[column sep=small] & e \arrow[dl, "z_-"'] \arrow[dr, "z_+"] & & e \arrow[dl, "1"'] \arrow[dr, "z_- + z'_+"] & \\ e & & e & & e \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & e \arrow[dl, "z'_-"'] \arrow[dr, "z'_+"] & & e \arrow[dl, "1"'] \arrow[dr, "z'_- + z_+"] & \\ e & & e & & e\,. \end{tikzcd} \tag{4}$$

Figure 15: Composition of a sphere with a negative sphere.

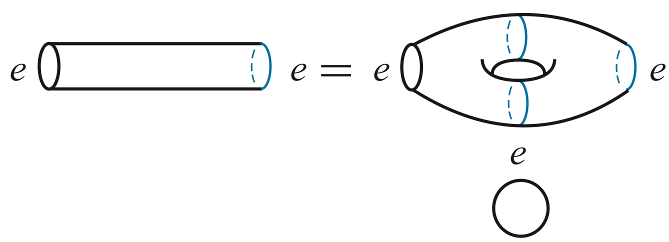

Figure 16: Sphere kills handle with trivial G-bundle.

We can use the following property of a localizing set

$$\begin{tikzcd}
& e \arrow[dl, "1"'] \arrow[dr, "z"] & \\
e \arrow[d, "z"'] \arrow[drr, "z"] & & e \arrow[dll, "1"'] \arrow[d, "1"] \\
e & & e\,.
\end{tikzcd} \tag{5}$$

Therefore, both composition in (4) are equal to

$$\begin{tikzcd}
& e \arrow[dl, "1"'] \arrow[dr, "z_+ + z'_+"] & \\
e & & e\,.
\end{tikzcd}$$

For the second axiom of a filtrated category we take the commutative diagram

$$\begin{tikzcd}
& e \arrow[dl, "w"'] \arrow[dr, "w'"] & & \\
e \arrow[d, "z_-"'] \arrow[drr, "z_+"] & & e \arrow[dll, "z_-"'] \arrow[d, "z_+"] & \\
e & & e \arrow[dr, shift left, "\Gamma"] \arrow[dr, shift right, "\Gamma'"'] & \\
& & & e\,.
\end{tikzcd}$$

So we have $w + z_- = w' + z_-$ and $(w + z_+) \cdot \Gamma' = (w' + z_+) \cdot \Gamma$. Set $N = z_- + w + z_+ = z_- + w' + z_+$. Similar as before N can be modified in order to have an element inside $\mathcal{M}$. Since we prove that

$\mathcal{M}$ is abelian, then we can modify the identity $N \cdot \Gamma' = N \cdot \Gamma$ to have $\Gamma \cdot \widetilde{N} = \Gamma' \cdot \widetilde{N}$ where $\widetilde{N}$ is again inside $\mathcal{M}$. Therefore, the category $e \diagdown j$ is filtrated with contractible classifying space and hence j is a homotopy equivalence in classifying spaces. $\square$

5 Homotopy type of the G-cobordism category

In this section we give an analog of the functor Φ constructed in theorem 4. This functor is used below in theorem 8, to split the classifying space of the whole category $\mathscr{S}^G$ in terms of the abelianization of G, an homotopy fiber and a $r(G)$-torus.

We have denote by $\mathscr{G}^G$ the category of fractions of $\mathscr{S}^G$, thus $\mathscr{G}^G = \mathscr{S}^G[\mathscr{S}^{G^{-1}}]$. This category has already been described in proposition 5. We define the functor $\Phi : \mathscr{S}^G \longrightarrow \mathscr{G}^G$ as follows: for a morphism $\Sigma = \Sigma_1 \sqcup \Sigma_2$, with Σ_1 a G-cobordism in $\mathscr{S}_b^G$ [4] and Σ_2 with empty outgoing boundary. We compose Σ_1 with a G-pair of pants with multiple legs as in Figure 3, with one exit and the same entries that the exits of Σ_1. Then we glue each connected component of Σ_2 with the last G-pair of pants by a connected sum, where this connected sum is independent of the way we do it because on a surface any two simple contractible closed curves are related by an orientation-preserving homeomorphism[5]. Subsequently, for the resulting connected cobordism, by Cerf theory [Cer70], we can find a Morse function with the property that there exists $t \in [0, 1]$ such that the inverse image of $[0, t]$ is a G-pair of pants with multiple legs as in Figure 3, and with a circle as the inverse image of t. Finally, we suppose that Σ_2 has c_2 connected components, and we take $\Phi(\Sigma)$ as the class, inside the category of fractions $\mathscr{G}^G$, of the disjoint union of the pre-image of $[t, 1]$ by the last Morse function, with a disjoint union of c_2 spheres. We illustrate this construction in the Figure 17. For the minimal surfaces the functor Φ is defined in Figure 18. The construction of Φ is functorial since we can eliminate the cases where we have adjacent critical points of index 0 and 2 [6]. We illustrate some of them in the Figure 19. Therefore, every connected component with empty outgoing boundary can be identified with a unique critical point of index 2.

Remark: 7. It is important to mention that this functor is not an extension of the functor constructed in theorem 4. For example it sends all the G-cylinders, which start and end in the trivial G-bundle, to the identity.

Let X_G denote the homotopy fiber of Φ. Consider the sequence of functors

$$\mathbb{N}^{r(G)} = \mathcal{M} \hookrightarrow \mathscr{S}^G \longrightarrow \mathscr{G}^G \,, \tag{6}$$

with image the fundamental group based at the trivial G-bundle. The composition restricts to the inclusion $\mathbb{N}^{r(G)} \hookrightarrow \pi_1(\mathscr{S}^G) \cong \mathbb{Z}^{r(G)}$, and the induced map of classifying spaces is a homotopy equivalence. A splitting principle for infinite loop spaces, see [**?**], allows us to conclude the following result.

[4] Every connected component of a morphism in $\mathscr{S}_b^G$ has non-empty final boundary.

[5] Recall that every principal G-bundle over the disk is trivial, consequently, every connected sum extends to the total space of the principal G-bundles.

[6] They locally are of the form $\pm(x^2 + y^2)$.

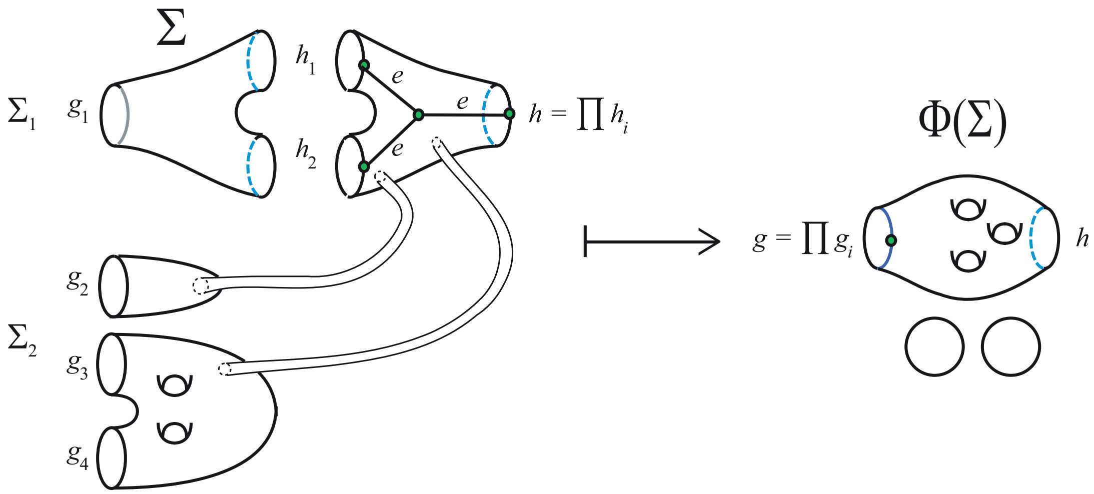

Figure 17: The functor $\Phi : \mathscr{S}^G \longrightarrow \mathscr{G}^G$.

Figure 18: The functor Φ for the minimal surfaces.

Theorem 8. *There is a simply connected infinite loop space X_G such that $B\mathscr{S}^G$ is homotopic to the product space $\frac{G}{[G,G]} \times X_G \times T^{r(G)}$.*

The space X_G has a categorical description since the functor $\Phi : \mathscr{S}^G \longrightarrow \mathscr{G}^G$ has a groupoid as target category. This implies the assumptions of the theorem of Quillen, see [Qui73] or the appendix theorem 14. As a consequence, the space X_G is of the homotopy type of the classifying space of the comma category e/Φ, i.e. $B(e/\Phi)$ (see the appendix 7.3 for the definition of the comma category).

6 An approximation to the number of subgroups of a finite group

An unsolved problem in geometric group theory is to give an explicit formula for the number of subgroups of a (finite) non-abelian group. Even for abelian groups, this is a complicated task, see [Tǐ0, Cǒ4]. There are families of groups which admit a nice description of this number. For example the number of divisors of the integer n, denoted by $\tau(n)$, is the number of subgroups of the cyclic group $\mathbb{Z}_n$. For the Dihedral group D_{2n}, Stephan A. Cavior in [Cav75], proved that the

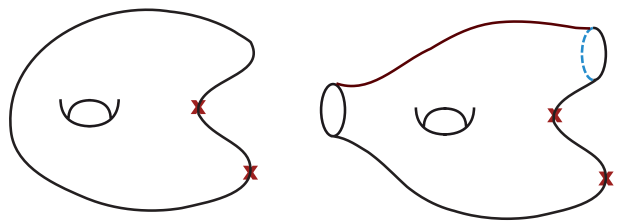

Figure 19: Elimination of adjacent critical points.

number of subgroups is given by $\tau(n)+\sigma(n)$, where $\sigma(n)$ is the sum of the divisors. Similarly, for the dicyclic groups Dic_n, the number of subgroups of Dic_n coincides with $\tau(2n)+\sigma(n)$. For all the groups of order less than thirty, a list of the groups and the number of their subgroups is presented by G. A. Miller in [Mil40].

We prove below that for the cyclic group $G=\mathbb{Z}_n$, with n a positive integer, the number $r(\mathbb{Z}_n)$ coincides with the number of subgroups of $\mathbb{Z}_n$, or $\tau(n)$. Thus at first sight we might think that this could be the case for any arbitrary finite group. But by a computational implementation with MATLAB, see [MAT], we find that the first counterexample is $\mathbb{Z}_2^3$ for the abelian groups, and the dihedral group D_{12} for the nonabelian groups, see table (17). In addition, we observe that the more the group G splits, then the number $r(G)$ is far to the number of subgroups of G. Thus it would be interesting to think if the number $r(G)$ associated to a simple group coincides with its number of subgroups. Moreover, we found that the number $r(\mathbb{Z}_2^n)$ follows the sequence $2, 5, 15, 51, 187, 715, \ldots$ which writes as $(2^n+1)(2^{n-1}+1)/3$. To my knowledge, see [oei], this number represents the dimension of the universal embedding of the symplectic dual polar space, see [BB03], or is the number of isomorphism classes of regular four folding coverings of a graph with respect to the identity automorphism, see [HK93], or the density of a language L_c with $c=4$, see [MR05]. We prove another variant for this number with the identity

$$r(\mathbb{Z}_2^n)=\frac{(2^n+1)(2^n-1)}{3}\,. \tag{7}$$

For any prime number p we generalize the formula (7) as follows

$$r(\mathbb{Z}_p^n)=\frac{p^{2n-1}+p^{n+1}-p^{n-1}+p^2-p-1}{p^2-1}\,.$$

It could be interesting to find a formula for the sequences $r(\mathbb{Z}_k^n)$ for a general positive integer k. Finally, the number $r(G)$ decomposes as a finite sum $r_1(G)+r_2(G)+\ldots$, where each subindex corresponds to the genus of the generator. We prove below that, for the dihedral and dicyclic groups, the number $r_1(G)$, associated to these groups, coincides with the number of abelian subgroups.

The definition of the positive integer $r(G)$ includes a quotient monoid composed by n-tuples of pairs

$$(g_1,k_1)(g_2,k_2)\cdots(g_n,k_n) \tag{8}$$

with $g_i, k_i \in G$, $1 \leq i \leq n$, such that

$$[k_n, g_n][k_{n-1}, g_{n-1}] \cdots [k_1, g_1] = e \,,$$

where $e \in G$ is the identity of the group. We say that the element (8) has genus n. These elements make up a monoid with the concatenation as product. The equivalence relation, by which we are going to make the quotient, is generated by some relations which are motivated by the definition of a G-Frobenius algebra, see [MS06, Tur10, Kau03]. These relations are proved in [Seg11] and are the following:

A. For generators of genus 1, with the form (g, k), the application of a Dehn twist, see [FM12], over a trivial G-cylinder gives the following equation

$$(g, k) \sim (g, h^n k g^m) \,,$$

where $h = kgk^{-1}$ and $n, m \in \mathbb{Z}$; since $[k, g] = e$, then this equation simplifies to the following

$$(g, k) \sim (g, kg^m) \,. \tag{9}$$

B. We consider generators of genus 2, with the form $(g_1, k)(g_2, k^{-1})$; there are some identifications in the G-pair of pants, which imply the equation

$$(g_1, k)(g_2, k^{-1}) \sim (g_1^{-1}, k^{-1})((k^{-1})^2 g_2^{-1} k^2, k) \,.$$

C. The interchange of critical points of index one for adjacent genus, clockwise and counterclockwise, induces two equations for generators of genus 2,

$$(g, k)(g', k') \sim ([g, k]g', k')k'(g, k)k'^{-1} \;,\; (g, k)(g', k') \sim k^{-1}(g', k')k(k^{-1}[k, g]kg, k) \,.$$

D. The interchange of the two generators of the fundamental group (non-homotopic to a boundary) of a torus with one boundary circle, corresponds to the following identification

$$(g, k) \sim (gkg^{-1}, g^{-1}) \,,$$

this equation simplifies, for generators of genus 1, as follows

$$(g, k) \sim (k, g^{-1}) \,. \tag{10}$$

The explicit equations for generators of genus bigger than 2 are combinations of the last equations.

Proposition 9. *For a finite group G we have an action of the special linear group $\mathrm{SL}(2, \mathbb{Z}_2)$ into the set of commuting elements $\{(k, g) : [k, g] = e\}$ whose number of orbits is the number of generators of genus 1, i.e. $r_1(G)$.*

Proof. First let G be an abelian group, the special linear group $\mathrm{SL}(2,\mathbb{Z}_2)$ is generated by two matrices,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the action of the first matrix, from left to right, gives the equation (9), while the action of the second matrix gives the equation (10). For a general group we only need to prove that the action is well defined, but this follows since the action in any of the two coordinates produces products $g^m k^n$ with m, n integers. The proposition follows since g and k commute. $\square$

Theorem 10. *For the cyclic groups $\mathbb{Z}_n$, the number $r(\mathbb{Z}_n)$ coincides with the number of subgroups of $\mathbb{Z}_n$.*

Proof. For the cyclic group $G = \mathbb{Z}_n$ and $(g,k) \in \mathbb{Z}_n \times \mathbb{Z}_n$ we have the division algorithm

$$\begin{aligned} k &= q_1 g + r_1 \\ -g &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &\vdots \\ \pm r_{m-1} &= q_{m+1} r_m \end{aligned}$$

with $0 \leq |r_{m-1}| < ... < |r_1| < |k|$. Thus by the equations (9) and (10) we have the following sequence of identifications

$$(g,k) \sim (g,r_1) \sim (r_1,-g) \sim (r_1,r_2) \sim (r_2,-r_1) \sim (r_2,-r_3) \sim \cdots \sim (\pm r_m, 0)\,.$$

Since $\sim$ is an equivalence relation and $(p,0) \sim (np, mp)$ for some $n, m \in \mathbb{Z}$ (and similar for q), then $(p,0) \sim (q,0)$ if and only if $\langle p \rangle = \langle q \rangle$. Therefore, $r(\mathbb{Z}_n)$ is the number of subgroups of $\mathbb{Z}_n$. $\square$

Theorem 11. *The numbers $r_1(D_{2n})$ and $r_1(Dic_n)$ coincide with the number of abelian subgroups of D_{2n} and Dic_n respectively.*

Proof. A presentation for the dihedral group is as follows

$$D_{2n} = \langle r, s | r^n = s^2 = 1, sr = r^{-1}s \rangle\,.$$

We know that any subgroup of D_{2n} is a subgroup of $\langle r \rangle$ or is of the form $\langle sr^i, r^m \rangle$ for $m = n/d$ with d a divisor of n. Thus any subgroup of D_{2n} is cyclic or is generated by two elements. We can find a correspondence between the abelian subgroups of D_{2n} and the generators of genus 1 as follows. If the subgroup is cyclic with generator a, then we assign the class of the pair $(a,0)$. If the subgroup is generated by two elements, say a and b, then we assign the class of the pair (a,b). This assignment is clearly surjective and, by the application of the equations (9) and (10), it is injective.

A presentation for the dicyclic group is as follows

$$Dic_n = \langle r, s | r^{2n} = 1, r^n = s^2, sr = r^{-1}s \rangle \,.$$

The arguments of the last paragraph work, where we consider the pair (r^m, sr^i) as a representant for the group $\langle sr^i, r^m, r^n \rangle$. □

Theorem 12. *For the group $\mathbb{Z}_p^n$, with p a prime number, we have the identity*

$$r(\mathbb{Z}_p^n) = \frac{p^{2n-1} + p^{n+1} - p^{n-1} + p^2 - p - 1}{p^2 - 1} \,.$$

Proof. For $r_p^n := r(\mathbb{Z}_p^n)$, let $F(n)$ be the number $r_p^{n+1} - r_p^n$. We will prove that

$$F(n) = p^{n-1}(p^n + p - 1) \,. \tag{11}$$

Since $r_p^n = (r_p^n - r_p^{n-1}) + (r_p^{n-1} - r_p^{n-2}) + ... + (r_p^3 - r_p^2) + (r_p^2 - r_p^1) + r_p^1$, where $r_p^1 = 2$ by Theorem 10. Thus $r_p^n = p^{n-2}(p^{n-1} + p - 1) + p^{n-3}(p^{n-2} + p - 1) + ... + p(p^2 + p - 1) + (p + p - 1) + 2$ and as a consequence we have the following equations

$$\begin{aligned}
r_p^n &= \sum_{i=0}^{n-2} p^{2i+1} + (p-1) \sum_{i=0}^{n-2} p^i + 2 \\
&= p\frac{(p^2)^{n-1} - 1}{p^2 - 1} + (p-1)\frac{p^{n-1} - 1}{p - 1} + 2 \\
&= \frac{p^{2n-1} - p + (p^{n-1} - 1)(p^2 - 1)}{p^2 - 1} \\
&= \frac{p^{2n-1} + p^{n+1} - p^{n-1} + p^2 - p - 1}{p^2 - 1} \,.
\end{aligned}$$

The formula (11) follows by induction, applying the following identity

$$F(n) = pF(n-1) + p^{2n-2}(p-1) \,. \tag{12}$$

where $F(0) = r_p^1 - r_p^0 = 2 - 1 = 1$. It rests to prove the identity (12). Let $\mathrm{Mat}(n \times 2, \mathbb{Z}_p)$ be the matrices with coefficients in $\mathbb{Z}_p$, then the equations (9) and (10) are given by the generators of the special linear group $\mathrm{SL}(2, \mathbb{Z}_p)$ as follows

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} .$$

Thus the number r_p^n is the same as the number of orbits of the quotient of $\mathrm{Mat}(n \times 2, \mathbb{Z}_p)$ by the group $\mathrm{SL}(2, \mathbb{Z}_p)$. By definition, $F(n)$ consists of elements in $\mathrm{Mat}(n \times 2, \mathbb{Z}_p)$ such that the last column is different from zero. There are three cases to consider:

A. The representatives of the classes have zeros in the coordinate n, i.e. the matrix has the form
$$\begin{pmatrix} \cdots & 0 & i \\ \cdots & 0 & j \end{pmatrix}$$
with $i, j \neq 0$, at least one, and for this case the number of classes is $F(n-1)$.

B. The representatives of the classes have zeros in the second row for the last two columns, i.e. the matrix has the form
$$\begin{pmatrix} \cdots & i & j \\ \cdots & 0 & 0 \end{pmatrix}$$
with $i \neq 0$ and $j \neq 0$. For these elements the stabilizer group is the same, before and after erasing the last column, so we have $(p-1)F(n-1)$ classes, where we multiply by $p-1$ since we can not take the zero value for j.

C. The last case is composed by classes with representative of the form
$$\begin{pmatrix} \cdots & i & 0 \\ \cdots & 0 & j \end{pmatrix} \tag{13}$$
with $i \neq 0$ and $j \neq 0$. Every stabilizer of an element of the form (13), has to be the identity, then the classes have cardinality the order of $\mathrm{SL}(2, \mathbb{Z}_p)$ which is $p(p^2-1)$. For the calculation of the number of classes we consider all the matrix of the form (13), inside $\mathrm{Mat}(n \times 2, \mathbb{Z}_p)$; for this we multiply $p^{n-1}p^{n-1}$, given by the first $n-1$ columns, with the index $|\mathrm{GL}(2, \mathbb{Z}_p)/\,\mathrm{SL}(2, \mathbb{Z}_p)| = p-1$ since the last columns represents an element in $\mathrm{Gl}(2, \mathbb{Z}_p)$. Therefore, the classes are $p^{2n-2}(p-1)$. The sum of the numbers associated to these three cases finalize the proof of the theorem.

□

7 Appendix

7.1 Category of Fractions

For $\mathcal{C}$ a category and J a subset of morphisms of $\mathcal{C}$ we say that J is a localizing set if we have the following properties:

A. The set J contains the identities, i.e. $1_x \in J$ for any object of $\mathcal{C}$, the set J is closed under composition, i.e. $s \circ t \in J$ for any $s, t \in J$ whenever the composition is defined.

B. For any morphism f of $\mathcal{C}$, $s \in J$ with common end, there exist morphisms g in $\mathcal{C}$ and $t \in J$ such that the following square
$$\begin{tikzcd} w \arrow[r, "g"] \arrow[d, "t"'] & z \arrow[d, "s"] \\ x \arrow[r, "f"'] & y \end{tikzcd} \tag{14}$$
is commutative.

C. Let f, g be two morphisms from x to y; the existence of $s \in J$ with $s \circ f = s \circ g$ is equivalent to the existence of $t \in J$ with $f \circ t = g \circ t$.

An important result of having a localizing set is the simplicity under which one can write the category of fractions, see [GM03]. Let J be a localizing subset of the morphisms in a category $\mathcal{C}$. The category of fractions $\mathcal{C}[J^{-1}]$ can be described as follows: the objects of $\mathcal{C}[J^{-1}]$ are the same as in $\mathcal{C}$ and one morphism $x \longrightarrow y$ in $\mathcal{C}[J^{-1}]$ is a class of "roofs", i.e. of diagrams (s, f), in $\mathcal{C}$ of the form

$$\begin{tikzcd}
 & x' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
x & & y\,,
\end{tikzcd}$$

where $s \in J$ and f is a morphism in $\mathcal{C}$, and two roofs are equivalent $(s, f) \sim (t, g)$ if and only if there exists a third roof (r, h) forming a commutative diagram of the form

$$\begin{tikzcd}
 & & x''' \arrow[dl, "r"'] \arrow[dr, "h"] & & \\
 & x' \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & x'' \arrow[dlll, "t" near end] \arrow[dr, "g"] & \\
x & & & & y\,.
\end{tikzcd} \tag{15}$$

Moreover, the identity morphism $1 : x \longrightarrow x$ is the class of the roof $(1_x, 1_x)$ and the composition of two roofs (s, f) and (t, g), is the class of the roof $(s \circ t', g \circ f')$ obtained by using the first square in (14) as follows,

$$\begin{tikzcd}
 & & x'' \arrow[dl, "t'"'] \arrow[dr, "f'"] & & \\
 & x' \arrow[dl, "s"'] \arrow[dr, "f"] & & y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
x & & y & & z
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & x'' \arrow[dl, "st'"'] \arrow[dr, "gf'"] & \\
x & & y\,.
\end{tikzcd}$$

7.2 Filtered categories

The basic example of a filtered category is a *directed set*. A directed set is a nonempty set, together with a reflexive and transitive binary relation, with the additional property that every pair of elements has an upper bound. A category J is filtered if it is not empty and it has the next two properties:

A. For every two objects j and j' in J there exist an object k and two morphisms $f : j \longrightarrow k$ and $f' : j' \longrightarrow k$ in J.

B. For every two parallel arrows $i \underset{v}{\overset{u}{\rightrightarrows}} j$ in J, there exist an object k and an arrow $w : j \longrightarrow k$ such that $w \circ u = w \circ v$.

7.3 Comma categories

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and y an object of $\mathcal{D}$. We define the *coma* category $y \diagdown F$ by the category with objects pairs (v, x) with x an object in $\mathcal{C}$ and $v : y \longrightarrow F(x)$ a morphism in $\mathcal{D}$; a morphism $f : (v, x) \longrightarrow (v', x')$ is given by a morphism $f : x \longrightarrow x'$ in $\mathcal{C}$ with $F(f) \circ v = v'$. The category $y \diagdown F$ is filtrated if we have the following two conditions: every two objects (x, v) and (x', v') can be equalized by the existence of $u : x \longrightarrow x''$, $u' : x' \longrightarrow x''$ with the commutative of the following diagram,

$$\begin{tikzcd} y \arrow[r, "v"] \arrow[d, "v'"'] & F(x) \arrow[d, "F(u)"] \\ F(x') \arrow[r, "F(u')"'] & F(x'')\, ; \end{tikzcd}$$

and the second consists that for parallel morphisms $u, u' : x \longrightarrow x''$ with $F(u)v = F(u')v$ there exists $w : x' \longrightarrow x'$ with $v'' = F(w)v'$ and $wu = wu'$ as in the following diagram,

$$\begin{tikzcd} y \arrow[r, "v"] \arrow[dr, "v'"] \arrow[ddr, "v''"'] & F(x) \arrow[d, shift right, "F(u)"'] \arrow[d, shift left, "F(u')"] \\ & F(x') \arrow[d, "F(w)"] \\ & F(x'')\, . \end{tikzcd}$$

A filtrated category has contractible classifying space, since it can be constructed by a inductive limit of categories, each of them, with final object.

7.4 Theorem A and B of Quillen

Now we describe two notable theorems given by Daniel Quillen in [Qui73] known as theorem A and theorem B. For a functor between small categories $F : \mathcal{C} \longrightarrow \mathcal{C}'$ we have the following theorems.

Theorem 13 (Quillen, theorem A in [Qui73]). *If the category $y \setminus F$ is contractible for every object y of $\mathcal{C}'$, then the functor F is a homotopy equivalence (in classifying spaces).*

Theorem 14 (Quillen, theorem B in [Qui73]). *If for every arrow $y \longrightarrow y'$ in $\mathcal{C}'$, the induced functor $y' \setminus F \longrightarrow y \setminus F$ is a homotopy equivalence, then for any object y of $\mathcal{C}'$ the cartesian square of categories*

$$
\begin{tikzcd}
y \setminus F \arrow[r, "j"] \arrow[d, "f'"'] & \mathcal{C} \arrow[d, "f"] \\
y \setminus \mathcal{C}' \arrow[r, "j'"'] & \mathcal{C}'
\end{tikzcd}
\qquad
\begin{aligned}
j(x,v) &= x \\
f'(x,v) &= (fx, v) \\
j'(y',v) &= y'
\end{aligned}
$$

is homotopy-cartesian[7].

Consequently, we have an exact homotopy sequence

$$\cdots \longrightarrow \pi_{i+1}(\mathcal{C}', y) \longrightarrow \pi_i(y \setminus F, x) \xrightarrow{j_*} \pi_i(\mathcal{C}, x) \xrightarrow{f_*} \pi_i(\mathcal{C}', y) \longrightarrow \cdots \tag{16}$$

7.5 Some calculations with Matlab

With the help of [MAT] and [GAP] we complete the following table which was motivated by [Mil40]. This table contains for each group G the number of subgroups, the number of abelian subgroups and the number $r(G)$ associated. We denote by a sum $r_1(G) + r_2(G) + \cdots$ the decomposition of the number $r(G)$ in the components of different genus. Note that in the following table we color the abelian groups with orange and the nonabelian groups with blue. Also note that we color with red the cases when we check that the number $r(G)$ does not satisfies to be the number of subgroups of G and we color with green the cases when the generator $r_1(G)$ does not satisfies to be the number of abelian subgroups.

LIST OF THE GROUPS AND THE NUMBER OF THEIR SUBGROUPS (17)

ORDERS	DESCRIPTION OF THE GROUPS	SUBGROUPS	ABE-SUBGROUPS	GENERATOR'S
4	Cyclic($\mathbb{Z}_4$), $\mathbb{Z}_2^2$	3,5	3,5	3,5
6	$\mathbb{Z}_6$, symmetric(Σ_3)	4,6	4,5	4,5+1
8	$\mathbb{Z}_8$, octic(D_8), quaternion(Q_8)	4,10,6	4,9,5	4,9+1,5+1
	$\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2^3$	8,16	8,16	8,15
9	$\mathbb{Z}_9$, $\mathbb{Z}_3^2$	3,6	3,6	3,7
10	$\mathbb{Z}_{10}$, dihedral(D_{10})	4,8	4,7	4,7+1
12	$\mathbb{Z}_{12}$, tetrahedral(A_4), D_{12}	6,10,16	6,9,13	6,9+1,13+14+_
	Dicyclic(Dic_3), $\mathbb{Z}_2^2 \times \mathbb{Z}_3$	8,10	7,10	7+1,10
14	$\mathbb{Z}_{14}$, D_14	4,10	4,9	4,9+1
15	$\mathbb{Z}_{15}$	4	4	4
16	$\mathbb{Z}_{16}$, Dic_4, D_{16}, $(Q_8 \times \mathbb{Z}_2)$	5,11,19,19	5,8,16,14	5,8+4+_ ,16+4+_,14+10+_
	$\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_4^2$, $\mathbb{Z}_4 \times \mathbb{Z}_2^2$, $\mathbb{Z}_2^4$	11,15,27,67	11,15,27,67	11,16,25,51
	Modular group of order 16	11	10	10+1
	Quasihedral of order 16	15	12	12+4+_
	$D_8 \rtimes \mathbb{Z}_2$	35	30	28+_
	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	23	22	21+_
	$G_{4,4}$	15	14	10+7+_

[7] A square is homotopy-cartesian if the canonical map to the homotopy-fiber-product is a homotopy equivalence.

	$Q_8 \rtimes \mathbb{Z}_2$	23	18	$20+_-$
18	$\mathbb{Z}_{18}$, $\mathbb{Z}_3 \times \mathbb{Z}_6$, D_{18}	6,12,16	6,12,12	$6,14,12+_-$
	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $\Sigma_3 \times \mathbb{Z}_3$	28,14	15,12	$16+_-,13+13$
20	$\mathbb{Z}_{20}$, $\mathbb{Z}_{10} \times \mathbb{Z}_2$, D_{20}	6,10,22	6,10,19	$6,10\ ,19+_-$
	Dic_5, metacyclic	10,14	9,12	$9+1,\ 12+3+_-$
21	$\mathbb{Z}_{21}$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	4,10	4,9	4,9+1
22	$\mathbb{Z}_{22}$, D_{22}	4,14	4,13	$4,13+4+_-$
24	$\mathbb{Z}_{24}$,$\mathbb{Z}_2 \times \mathbb{Z}_{12}$,$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$	8,16,32	8,16,32	8,16,30
	$D_8 \times \mathbb{Z}_3$,$Q_8 \times \mathbb{Z}_3$	20,12	18,10	$18+_-,10+_-$
	$Sl(2,3)$, $A_4 \times \mathbb{Z}_2$	15,26	13,24	$13+_-,23+_-$
	Σ_4, D_{24}, Dic_6	30,34,18	21,24,12	$21+3+_-\ ,24+_-\ ,12+$
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$, $\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	54,22	43,19	$40+_-,19+_-$
	$\mathbb{Z}_4 \times \Sigma_3$,$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	26,10	21,9	$21+_-,9+_-$
	$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	30	22	22+
25	$\mathbb{Z}_{25}$, $\mathbb{Z}_5^2$	3,8	3,8	3,11
26	$\mathbb{Z}_{26}$, D_{26}	4,16	4,15	$4,15+_-$
27	$\mathbb{Z}_{27}$, $\mathbb{Z}_9 \times \mathbb{Z}_3$, $\mathbb{Z}_3^3$	4,10,28	4,10,28	4,12,40
	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$	19,10	18,9	$22+_-,10+_-$
28	$\mathbb{Z}_{28}$, $\mathbb{Z}_{14} \times \mathbb{Z}_2$, D_{28}, Dic_7	6,10,28,12	6,10,25,11	$6,10,25+_-\ ,11+_-$

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