

Exclusivity structures and graph representatives of local complementation orbits

Adán Cabello ^{*} Matthew G. Parker [†] Giannicola Scarpa [‡] Simone Severini [§]

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Abstract

We describe a construction that maps any connected graph G on three or more vertices into a larger graph, $H(G)$, whose independence number is strictly smaller than its Lovász number which is equal to its fractional packing number. The vertices of $H(G)$ represent all possible events consistent with the stabilizer group of the graph state associated to G , and exclusive events are adjacent. Mathematically, the graph $H(G)$ corresponds to the orbit of G under local complementation. Physically, the construction translates into graph-theoretic terms the connection between a graph state and a Bell inequality maximally violated by quantum mechanics. In the context of zero-error information theory, the construction suggests a protocol achieving the maximum rate of entanglement-assisted capacity, a quantum mechanical analogue of the Shannon capacity, for each $H(G)$. The violation of the Bell inequality is expressed by the one-shot version of this capacity being strictly larger than the independence number. Finally, given the correspondence between graphs and exclusivity structures, we are able to compute the independence number for certain infinite families of graphs with the use of quantum non-locality, therefore highlighting an application of quantum theory in the proof of a purely combinatorial statement.

1 Introduction

Partitioning a phase space into orbits is a central step in the study of any physical or formal dynamics. Immediately after the introduction of graph (/stabilizer) states in quantum coding theory [8, 14, 32] and in the context of measurement-based quantum computation [27], it was evident that the related task, when considering the dynamics at the subsystems level, requires approaches of combinatorial flavour. While substantial attention has been given to orbits obtained by the application of local unitaries (with a clear motivation coming from the classification of multipartite entanglement [16]), the question to decide whether two graph states are equivalent under the action of the local Clifford group has been settled, by showing [24] that the equivalence classes are in one-to-one correspondence with local complementation orbits (also called *Kotzig orbits*

^{*}Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain. adan@us.es

[†]Department of Informatics, University of Bergen, P.O. Box 7803, Bergen N-5020, Norway. Matthew.Parker@ii.uib.no

[‡]CWI, Science Park 123, 1098 XG Amsterdam, the Netherlands. g.scarpa@cwi.nl

[§]Department of Computer Science, and Department of Physics & Astronomy, University College London, WC1E 6BT London, United Kingdom. simoseve@gmail.com

[18]). In other words, the existence of a sequence of local complementations relating the associated graphs guarantees equivalence under local Clifford operations and *viz.*

Even though this link does not embrace full local unitary equivalence, having now counterexamples to the LU-LC conjecture [19, 28], it unveils a rich interface between the structure of useful multi-qubit systems and a number of mathematical ideas. Indeed, local complementation (or, equivalently, κ -transformation) is a fundamental operation for studying circle graphs [18]. This notion has been instrumental for unifying certain properties of Eulerian tours and matroids via isotropic systems [3], constructs associated to vector spaces over $GF(2)$; and it appears in string reconstruction problems (related to DNA sequencing) and graph polynomials [2].

Given an equivalence class induced by local complementation, in the present work we shall describe a method for constructing a larger graph associated to the equivalence class. The method makes use of the stabilizer group of an arbitrary graph state from the class. Each of these graphs is identified with an exclusivity structure and a related *non-contextuality inequality* (for short, NC inequality). Such an inequality is an upper bound on the sum of probabilities of a set of events, with some exclusivity constraints (a technical discussion about events and exclusivity will be given in Section 3). NC inequalities are satisfied by any non-contextual hidden variable theory, *i.e.*, any physical theory for which the probability of seeing an event is independent of the choice of measurements. Quantum mechanics or more general theories can violate such inequalities. For more details, see [7]. In this reference, the graph (and more generally an hypergraph), whose vertices are events, is employed to characterize the correlations for classical and general probabilistic theories satisfying that the sum of probabilities of pairwise exclusive events cannot be larger than 1. The maximum values for the three physical theories: classical, general, and quantum, were computed through the three well-known combinatorial parameters: the independence number, the fractional packing number, and the Lovász number, respectively. As a consequence, it becomes evident that quantum and general probabilistic correlations satisfying that the sum of probabilities of pairwise exclusive events cannot be larger than 1 have semidefinite and linear characterizations, respectively. Quantum mechanics is *sandwiched* between the other two theories.

The framework introduced in [7] permits to quantitatively discuss classical, quantum, and more general theories through the analysis of a single mathematical object and to have a general technique to single out quantum correlations with *ad hoc* degree of contextuality. For example, a generic graph with independence number strictly smaller than the Lovász number is associated to a NC inequality violated by quantum mechanics. If, in addition, the graph has equal Lovász and fractional packing number, then it can be associated to a NC inequality that is “maximally violated” by quantum mechanics, meaning that no general probabilistic theory satisfying that the sum of probabilities of pairwise exclusive events cannot be larger than 1 can achieve a larger value. (Of course, there are graphs for which all theories coincide, as, for example, perfect graphs.)

In the present paper, we propose a construction that translates into the combinatorial language developed in [7] the connection between every graph state of three or more qubits and a Bell inequality maximally violated by quantum mechanics found in [15]. Namely, we describe a construction that maps any graph on three or more vertices G into a larger graph, $H(G)$, such that its independence number is strictly smaller than its Lovász number which is equal to its fractional packing number. The vertices of $H(G)$ represent all possible events consistent with the stabilizer group of the graph state associated to G and exclusive events are adjacent.

The construction has also applications in zero-error information theory. It leads to a straightforward protocol achieving the maximum rate of zero-error entanglement-assisted capacity [9, 12]. We

conjecture that this quantity for a graph $H(G)$ is always strictly larger than its Shannon capacity. A proof of this statement would possibly require a rank bound *a la Haemers*. While it is difficult to compute this bound in general, it may be easier in our case, since $H(G)$ has a very particular structure because of the connection with the stabilizer group. The violation of the Bell inequality is here expressed by the one-shot version of this capacity being strictly larger than the independence number. The correspondence between graphs and exclusivity structures allows us compute the independence number of the graphs $H(K_n)$, where K_n is the complete graph, by taking advantage of well-known techniques used in quantum non-locality.

Since two graphs yield the same (up to isomorphism) graph in our construction if and only if they are equivalent under local complementation, the construction can be interpreted as a method to represent local complementation orbits. Somehow this is in analogy with the notion of a two-graph, a well-studied mathematical object which represents equivalence classes under the operation of switching (see Ch. 11 of [13]).

Our work is innovative with respect to [7] and [15] in many ways. First, we present a characterization of local complementation orbits, a result of pure combinatorial nature. Our representative graphs are obtained via a construction inspired by quantum mechanics. We find tools to analyze the properties of such graphs in [15]. Second, by using the results in [7], we discover that local complementation orbits are naturally associated to Bell inequalities. We improve the mathematical representation of such Bell inequalities by writing down an explicit operational form, namely, a *pseudo-telepathy game*. The form that we have introduced is often easier to use in both theoretical purposes and the design of laboratory experiments. Third, we introduce a novel connection between the results presented in [7] and [15], and zero-error information theory. Such a connection is also interesting on its own, since it provides a way to construct families of channels with a separation between classical and quantum capacities starting from *any* graph.

The remainder of the work is organized as follows. The next section introduces the required terminology and notions: the language of graph theory, non-locality, and channel capacities. The construction is described in Section 3. Section 4 discusses the relevant graph-theoretic parameters. Section 5 contains examples. We highlight that physical arguments can be useful to consider difficult tasks such as computing the independence number. Section 6 is devoted to zero-error capacities. We show that our construction produces infinite families of graphs for which the use of entanglement gives the maximum possible zero-error capacity. Section 7 classifies graphs (or local complementation orbits) according to the objects obtained via the construction. We point out a link with Boolean functions and propose a conjecture about connectedness of the graph representatives.

2 Preliminaries

2.1 Graphs, graph parameters and graph states

A (simple) *graph* $G = (V, E)$ is an ordered pair: $V(G)$ is a set whose elements are called *vertices*; $E(G) \subset V(G) \times V(G)$ is a set whose elements are called *edges*. The set $E(G)$ does not contain an edge of the form $\{i, i\}$, for every $i \in V(G)$. The vertices forming an edge are said to be *adjacent*. We denote by $\mathcal{N}(i)$ the *neighbourhood* of the vertex i , *i.e.*, $\mathcal{N}(i) = \{j \in V \mid (i, j) \in E\}$. An *independent set* in a graph G is a set of mutually non-adjacent vertices. The *independence number* of a graph G , denoted by $\alpha(G)$, is the size of the largest independent set of G . A *subgraph* $H = (V, E)$ of a graph $G = (V, E)$, is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph*

$H = (V, E)$ of a graph $G = (V, E)$ with respect to $V(H) \subseteq V(G)$ is a graph with vertex set $V(H)$ and edge set $E(H) = \{\{i, j\} \mid i, j \in V(H) \text{ and } \{i, j\} \in E(G)\}$. A *clique* in a graph G is a subgraph whose vertices are all adjacent to each other.

An *orthogonal representation* of G is a map from $V(G)$ to \mathbb{C}^k for some k , such that adjacent vertices are mapped to orthogonal vectors. An orthogonal representation is *faithful* when vertices u and v are mapped to orthogonal vectors if and only if $\{u, v\} \in E(G)$. The *Lovász number* [20] is defined as follows:

$$\vartheta(G) = \max \sum_{i=1}^n |\langle \psi | v_i \rangle|^2, \quad (1)$$

where the maximum is taken over all unit vectors ψ and all orthogonal representations, $\{v_i\}$, of G . The *fractional packing number* is defined by the following linear program:

$$\alpha^*(G, \Gamma) = \max \sum_{i \in V} w_i, \quad (2)$$

where the maximum is taken over all $0 \leq w_i \leq 1$ under the restriction $\sum_{i \in C_j} w_i \leq 1$ [30], and for all cliques $C_j \in \Gamma$, where Γ is the set of all cliques of G . In this paper, by fractional packing number, we mean $\alpha^*(G, \Gamma)$ and denote it as $\alpha^*(G)$.

Given a graph $G = (V, E)$, the *graph state* $|G\rangle$ (see, for example, [16, 32]) associated to G is the unique n -qubit state such that

$$g_i |G\rangle = |G\rangle \text{ for } i = 1, \dots, n, \quad (3)$$

where g_i is the generator labeled by a vertex $i \in V$ of the stabilizer group of $|G\rangle$. A generator for $i \in V(G)$ is defined as

$$g_i = X^{(i)} \bigotimes_{j \in \mathcal{N}(i)} Z^{(j)}, \quad (4)$$

where $X^{(i)}$, $Y^{(i)}$, and $Z^{(i)}$ denote the Pauli matrices (sometimes denoted as σ_x , σ_y , and σ_z) acting on the i -th qubit. Therefore, g_i can be obtained directly and univocally from G . The stabilizer group of the state $|G\rangle$ is the set S of the stabilizing operators s_j of $|G\rangle$ defined by the product of any number of generators g_i . Note that, for convenience, we shall remove the identity element from S . Therefore, the set S contains $2^n - 1$ elements.

Given a graph $G = (V, E)$, the operation of *local complementation* (LC) on $i \in V$ transforms G into a graph G^i on the same set of vertices. To obtain G^i , we replace the induced subgraph of G on $\mathcal{N}(i)$ by its complement. It is easy to verify that $(G^i)^i = G$. The set of graphs is partitioned into *LC orbits* (also known as *Kotzig orbits*) by the repeated action of local complementation on each graph [3]. The LC orbits are then equivalence classes.

2.2 Non-locality

We assume familiarity with the basics of quantum information theory. The reader can find a good introduction in [25, Chapter 2]. A *non-local game* is an experimental setup between a referee and two players, Alice and Bob. (It can also be defined with more players, but we do not consider this case here.) The game is not adversarial, but the players collaborate with each other. They are allowed to arrange a strategy beforehand, but they are not allowed to communicate during the game. The referee sends Alice an input $x \in X$ and sends Bob an input $y \in Y$, according to a fixed

and known probability distribution π on $X \times Y$. Alice and Bob answer with $a \in A$ and $b \in B$ respectively, and the referee declares the outcome of the game according to a verification function $V : A \times B \times X \times Y \rightarrow \{\text{win} = 1, \text{lose} = 0\}$. So, the non-local game is completely specified by the sets X, Y, A, B , the distribution π , and the verification function V .

A *classical strategy* is *w.l.o.g.* a pair of functions $s_A : X \rightarrow A$ and $s_B : Y \rightarrow B$ for Alice and Bob, respectively. A *quantum strategy* consists of a shared bipartite entangled state $|\psi\rangle$ and POVMs $\{P_a^x\}_{a \in A}$, for every $x \in X$ for Alice and $\{P_b^y\}_{b \in B}$, for every $y \in Y$ for Bob. On input x , Alice uses the positive operator valued measurement (POVM) $\{P_a^x\}_{a \in A}$ to measure her part of the entangled state and Bob does similarly on his input y . Alice (resp. Bob) answers with a (resp. b) corresponding to the obtained measurement outcome. Therefore, the probability to output a, b given x, y is $\Pr(a, b|x, y) = \langle \psi | P_a^x \otimes P_b^y | \psi \rangle$. The classical and quantum values (or winning probabilities) for the game are:

$$\omega_c = \max_{s_A, s_B} \sum_{x, y, a, b} \pi(x, y) V(s_A(x), s_B(y), x, y), \quad (5a)$$

$$\omega_q = \max_{|\psi\rangle, \{P_a^x\}, \{P_b^y\}} \sum_{x, y, a, b} \pi(x, y) V(a, b, x, y) \langle \psi | P_a^x \otimes P_b^y | \psi \rangle. \quad (5b)$$

A *Bell inequality* for a non-local game is a statement of the form $\omega_c \leq t$ for $t \in [0, 1]$. It is violated by quantum mechanics if $\omega_q > t$. A non-local game is called a *pseudo-telepathy* game if $\omega_c < \omega_q = 1$, *i.e.* quantum players win with certainty, while classical players have nonzero probability to lose.

Non-local games are a special form of Bell experiment. In general, a *Bell operator* \mathcal{B} is a linear combination of observables and a Bell inequality is a statement of the form

$$\max |\langle \mathcal{B} \rangle| \leq t, \quad (6)$$

where the maximum runs over classical states. A quantum state is said to *violate* the Bell inequality if $|\langle \mathcal{B} \rangle| > t$.

2.3 Channel capacity

Zero-error information theory was initiated in [29]; a review is [17]. A *classical channel* \mathcal{C} with input set X and output set Y is specified by a conditional probability distribution $\mathcal{C}(y|x)$, the probability to produce output y upon input x . (Precisely this is a discrete, memoryless, stationary channel.) Two inputs $x, x' \in X$ are *confusable* if there exists $y \in Y$ such that $\mathcal{C}(y|x) > 0$ and $\mathcal{C}(y|x') > 0$. We then define the *confusability graph of channel* \mathcal{C} , $G(\mathcal{C})$, as the graph with vertex set X and edge set $\{(x, x') : x, x' \text{ are distinct and confusable}\}$.

The *one-shot zero-error capacity* of \mathcal{C} , $c_0(\mathcal{C})$, is the size of a largest set of non-confusable inputs. This is just the independence number $\alpha(G(\mathcal{C}))$ of the confusability graph. In the entanglement-assisted setting, the sender (Alice) and receiver (Bob) share an entangled state ρ and can perform local quantum measurements on their part of ρ .

The general form of an entanglement-assisted protocol used by Alice to send one out of q messages to Bob with a single use of the classical channel \mathcal{C} can be described as follows (also see [9]). For each message $m \in [q]$, Alice has a POVM $\mathcal{E}^m = \{E_1^m, \dots, E_{|X|}^m\}$ with $|X|$ outputs. To send message m , she measures her subsystem using \mathcal{E}^m and sends through the channel the observed

$x \in X$. Bob receives some $y \in Y$ with $\mathcal{C}(y|x) > 0$. If the right condition holds (as we will explain below), Bob can recover m with certainty using a projective measurement on his subsystem.

It is not hard to state a necessary and sufficient condition for the success of the protocol. If Alice gets outcome $x \in X$ upon measuring \mathcal{E}^m , Bob's part of the entangled state collapses to $\beta_x^m = \text{tr}_A((E_x^m \otimes I)\rho)$. Given the channel's output y , Bob can recover m if and only if

$$\forall m \neq m', \forall \text{ confusable } x, x' \text{ tr}(\beta_x^m \beta_{x'}^{m'}) = 0. \quad (7)$$

Bob can recover the message with a projective measurement on the mutually orthogonal supports of

$$\sum_{x : \mathcal{C}(y|x) > 0} \beta_x^m, \quad (8)$$

for all messages m . In such a case we say that, assisted by the entangled state ρ , Alice can use the POVMs $\mathcal{E}^1, \dots, \mathcal{E}^q$ as her strategy for sending one out of q messages with a single use of \mathcal{C} .

The *entanglement-assisted one-shot zero-error channel capacity* of \mathcal{C} , $c_0^*(\mathcal{C})$, is the maximum integer q such that there exists a protocol for which condition (7) holds.

We are now ready to outline the setting where Alice and Bob share a maximally entangled state in the above protocol. We will refer to this particular case later in section 6. Let the (canonical) maximally entangled state of local dimension n be defined as follows:

$$|\Psi\rangle := \frac{1}{\sqrt{n}} \sum_{i \in [n]} |i\rangle|i\rangle, \quad (9)$$

where $\{|i\rangle\}_{i \in [n]}$ is the standard basis of \mathbb{C}^n . When Alice and Bob share a maximally entangled state and Alice performs a projective measurement observing P_a , Bob's part of the state collapses to $\text{tr}_A((P_a \otimes I)|\Psi\rangle\langle\Psi|) = P_a^\top/n$. This implies that Bob can distinguish between P_a and P_b perfectly if and only if $\text{tr}(P_a P_b) = 0$. Therefore, if Alice uses projective measurement $\{P_x^m\}_{x \in X}$ for message m and players share a maximally entangled state, then Condition (7) is true if and only if

$$\forall m \neq m', \forall \text{ confusable } x, x' \text{ tr}(P_x^m P_{x'}^{m'}) = 0. \quad (10)$$

Considering more than a single use of the channel, one can define the asymptotic zero-error channel capacity $\Theta(\mathcal{C})$ and the asymptotic entanglement-assisted zero-error channel capacity $\Theta^*(\mathcal{C})$ by $\Theta(\mathcal{C}) := \lim_{k \rightarrow \infty} (c_0(\mathcal{C}^{\otimes k}))^{1/k}$ and $\Theta^*(\mathcal{C}) := \lim_{k \rightarrow \infty} (c_0^*(\mathcal{C}^{\otimes k}))^{1/k}$.

Since $c_0^*(\mathcal{C})$ depends solely on the confusability graph $G(\mathcal{C})$ [9], we can talk about $c_0^*(G)$ for a graph G , meaning the entanglement-assisted one-shot zero-error capacity of a channel with confusability graph G . Similarly we can talk about quantities $c_0(G)$, $\Theta(G)$, and $\Theta^*(G)$.

3 Construction

Let G be a graph on n vertices and consider the n -qubit graph state $|G\rangle$. Let S be the stabilizer group of G . For each $s_j \in S$, with $s_j = \bigotimes_{k=1}^n O^{(k)}$, let $w_j = |\{O^{(k)} : O^{(k)} \neq I\}|$ be the *weight* of s_j . Let $S_j = \{S_{(i,j)} : i = 1, 2, \dots, 2^{w_j-1}\}$ be the set of the *events* of s_j , *i.e.* the measurement outcomes that occur with non-zero probability when the system is in state $|G\rangle$ and the stabilizing operators s_j are measured with single-qubit measurements. The set of all events is $\mathcal{S} = \bigcup_{j=1,2,\dots,2^n-1} S_j$. Two events are *exclusive* if there exists a $k \in \{1, \dots, n\}$ for which the same single-qubit measurement gives a different outcome.

A graph representing a Kotzig orbit can be naturally defined as follows:

Definition 1 Let G be a graph. Let S be the stabilizer group of the graph state of G . We denote by $H(G)$ the graph whose vertices are the events in S and the edges are all the pairs of exclusive events.

We give an example for events and exclusiveness. Let $n = 3$ and $s_2 = ZXZ$ (we omit the superscripts for simplicity). This means that $ZXZ|G\rangle = |G\rangle$, *i.e.*, if the system is prepared in $|G\rangle$ and s_2 is measured by measuring Z on the first qubit (with possible results -1 or 1), X on the second qubit, and Z on the third qubit, then the product of the three results must be 1 . Therefore, $S_2 = \{zxz, \underline{zxz}, \underline{zxz}, \underline{zxz}\}$, where hereafter \underline{zxz} denotes the event “the result 1 is obtained when Z is measured on qubit 1, the result -1 is obtained when X is measured on qubit 2, and the result -1 is obtained when Z is measured on qubit 3”. As another example: if $n = 2$ and $s_1 = XZ$, then $S_1 = \{zx, \underline{zx}\}$.

We now give an example of a graph representing a Kotzig orbit. Let us consider $P_3 = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$, the path on three vertices. We construct $H(P_3)$. The stabilizer group S (minus the identity) has the following elements: $s_1 = g_1 = XZI$, $s_2 = g_2 = ZXZ$, $s_3 = g_3 = IZX$, $s_4 = g_1g_2 = YYZ$, $s_5 = g_1g_3 = XIX$, $s_6 = g_2g_3 = ZYY$, and $s_7 = g_1g_2g_3 = -YXY$. For all $j = 1, \dots, 2^3 - 1$, obtain all possible events (*i.e.*, those which can happen with non-zero probability) when three qubits are prepared in the state $|G\rangle$ and three parties measure the observables corresponding to s_j . For instance, when $j = 1$, Alice measures $X^{(1)}$, Bob measures $Z^{(2)}$, and Charlie does not perform any measurement. Since the three qubits are in state $|G\rangle$, there are only two possible outcomes: Alice obtains $X^{(1)} = +1$ and Bob obtains $Z^{(2)} = +1$, denoted as xzI ; or Alice obtains $X^{(1)} = -1$ and Bob obtains $Z^{(2)} = -1$, denoted as \underline{xzI} . For $j = 2$, the only events that can occur are zxz , \underline{zxz} , \underline{zxz} , and \underline{zxz} . The other events for the remaining j 's are obtained in a similar way. Now, let us construct the graph $H(P_3)$: the vertices represent possible events; two vertices are adjacent if and only events are exclusive (*e.g.*, xzI and \underline{xzI}). Notice that each s_j of weight w_j generates 2^{w_j-1} vertices. A drawing of $H(P_3)$ is in Fig. 1.

Each $H(G)$ can be interpreted as in [7]. Every graph is in fact associated to an NC inequality and is constructed by expressing the linear combination of joint probabilities of events in the NC inequality as a *sum* S . For a graph in [7], an event in S is represented by a vertex and exclusive events are represented by edges. Constructing such a graph from the inequality is straightforward, when the absolute values of the coefficients in the linear combination are natural numbers (which, to our knowledge, is always the case for all relevant NC inequalities). As already mentioned in the introduction, this graph-theoretic framework can be used to single out games with *ad hoc* quantum advantage and quantum correlations with *ad hoc* degree of contextuality (see [23, 1], respectively).

3.1 Local complementation orbits

If we apply the method to graphs G and G' in the same orbit under local complementation [16, 6] then *we obtain the same graph* H . The reason is that the graph states $|G\rangle$ and $|G'\rangle$ share the same set of perfect correlations (up to relabeling), so also share the same graph in which all possible exclusive events are adjacent.

This paper constructs $H(G)$ from G , as described earlier, where each of the $2^n - 1$ operators, s_j , generated by G , in turn generates a clique of size 2^{w_j-1} in H , where w_j is the weight of operator s_j . In this section, we present a classification of all $H(G)$ from all graphs G for $n < 7$. This classification is greatly simplified by the fact that if two graphs, G and G' , are in the same local complementation (LC) orbit, then $H(G) = H(G')$. So we need only classify for one representative

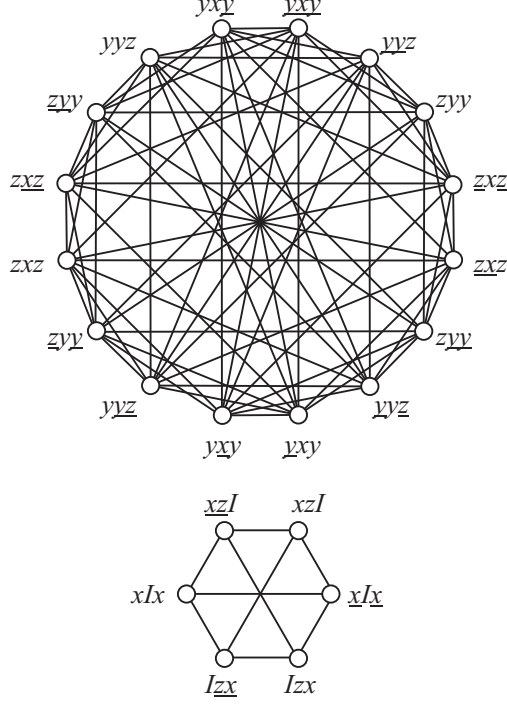


Figure 1: The graph $H(P_3)$ associated to the path on three vertices, P_3 , consists of two connected components: the upper component in the drawing is CS , the complement of the Shrikhande graph [31]; the lower component is $Ci_8(1,3)$, the 6-vertex (1,3)-circulant graph. We have $\alpha(CS) = 3$, $\vartheta(CS) = \alpha^*(CS) = 4$, $\alpha[Ci_8(1,3)] = \vartheta[Ci_8(1,3)] = \alpha^*[Ci_8(1,3)] = 3$. Therefore, $\alpha(H(P_3)) = 6$, while $\vartheta(H(P_3)) = \alpha^*(P_3) = 7$.

from each orbit. A choice of representatives for $n = 2, \dots, 6$, for connected graphs only, are listed in the second column of table 1.

The action of local complementation on vertex v of graph G to yield graph G' can be realised, in the context of graph states, by a specific local unitary action:

$$\begin{aligned}
 G' &= G^v \\
 \Downarrow & \quad \Downarrow \\
 |G'\rangle &= \omega^7 T^{(v)} N^{(v)} \prod_{i \in \mathcal{N}_v} T^{(i)} |G\rangle,
 \end{aligned} \tag{11}$$

where $N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, and $\omega = e^{\frac{7\pi i}{4}}$.

Here, $S = \{s_j, j = 1 \dots 2^n - 1\}$ is the stabilizer group associated with the graph G , where we omit s_0 for convenience. Similarly, let $S' = \{s'_j, j = 1 \dots 2^n - 1\}$ be the stabilizer group associated with $G' = G^v$. We show how to obtain S' from S . Let $s_j = (-1)^{c_j} \prod_{k=1}^n O_j^{(k)}$, where $O_j \in \{I, X, Z, Y\}$, and $c_j \in \{0, 1\}$.

Define the mapping $\mathcal{L}_v : \{I, X, Z, Y\} \rightarrow \{I, X, Z, Y\}$ as follows:

$$\begin{aligned} \mathcal{L}_v : \quad & X^{(k)} \rightarrow X^{(k)}, Z^{(k)} \rightarrow Y^{(k)}, Y^{(k)} \rightarrow Z^{(k)}, & k = v \\ & X^{(k)} \rightarrow Y^{(k)}, Z^{(k)} \rightarrow Z^{(k)}, Y^{(k)} \rightarrow X^{(k)}, & k \in \mathcal{N}_v \\ & I \rightarrow I, X \rightarrow X, Z \rightarrow Z, Y \rightarrow Y, & \text{otherwise.} \end{aligned} \quad (12)$$

Moreover, define $y_v(s_j) = |\{k \mid O_j^{(k)} = Y^{(k)} \text{ and } k \in v \cup \mathcal{N}_v\}|$. In words, $y_v(s_j)$ is the total number of Y matrices at tensor positions $v \cup \mathcal{N}_v$ of s_j .

Lemma 2 *The action of LC at vertex v of G maps G to G' and S to S' , where*

$$s'_j = (-1)^{c_j + y_v(s_j)} \prod_{k=1}^n \mathcal{L}_v(O_j^{(k)}), \quad j = 0 \dots 2^n - 1. \quad (13)$$

This action is a permutation, $(I)(X)(ZY)$, of the Pauli matrices at tensor position v of each s_j and a permutation, $(I)(Z)(XY)$, of the Pauli matrices at tensor positions in \mathcal{N}_v of each s_j , followed by a global multiplication by $(-1)^{y_v(s_j)}$.

For example, consider the graph G with two edges $\{1, 2\}$ and $\{1, 3\}$. Then $G' = G^1$ is the graph with edges $\{1, 2\}, \{1, 3\}, \{2, 3\}$ (so the star ST_3 and the complete graph K_3 are in the same Kotzig orbit). We have that $S(G) = \{XZZ, ZXI, YYZ, ZIX, YZY, IXX, -XYY\}$ and that $S(G^1) = S' = \{XZZ, ZXZ, YYI, ZZX, YIY, IYY, -XXX\}$. For example $s_j = YYZ$ is mapped to $s'_j = (-1)^{y_1(s_j)} ZXZ = ZXZ$, where $v = 1$, $\mathcal{N}_v = \{2, 3\}$, and $y_1(s_j) = 2$ as Y occurs at tensor positions 1 and 2 of s_j , where $1, 2 \in v \cup \mathcal{N}_v$.

Proof. We use (11). For vertex v we replace U with $\omega^7 TNUN^\dagger T^\dagger \omega$, for each of $U \in \{X, Z, Y\}$ to obtain $\{X, Y, -Z\}$. Similarly, for vertices in \mathcal{N}_v we replace U with $\omega^7 TUT^\dagger \omega$, for each of $U \in \{X, Z, Y\}$ to obtain $\{Y, Z, -X\}$. ■

Theorem 3 *Let G^L be the Kotzig orbit of graphs generated by the action of successive local complementation on G . Then*

$$H(G') = H(G''), \quad \forall G', G'' \in G^L. \quad (14)$$

Proof. Every vertex in $H(G)$ represents a measurement, s_j , of $|G\rangle$, combined with a certain measurement result, as specified by the bars under x, y , and z , as appropriate. This measurement is equivalent to a measurement, s'_j of $|G'\rangle$, where $s'_j, |G'\rangle$, and the new measurement results are obtained from s_j and $|G\rangle$ by the same local unitary transform, namely the transform in (11). Since the two measurement scenarios are equivalent, then the edge relationship between vertices in $H(G)$ is preserved in $H(G')$, *i.e.*, $H(G) = H(G')$. The theorem is then extended to any two $G', G'' \in G^L$ as G'' can be obtained from G' by a series of local complementations. ■

Proposition 3 implies that we only have to classify for one representative member, G , (arbitrarily chosen) of each Kotzig orbit, G^L , of n -vertex graphs. Table 1 classifies, computationally, graphs $H(G) = (V^H, E^H)$ for $n = 2, \dots, 6$ from $G = (V, E)$, where $|G^L|$ is the size of the Kotzig orbit of G up to re-labeling (graph isomorphism).

4 Parameters

Theorem 4 *Let H be a graph representing a Kotzig orbit. Then, for $n > 2$,*

$$\alpha(H) < \vartheta(H) = \alpha^*(H) = 2^n - 1. \quad (15)$$

We firstly give an intuition of the statement, explaining how the Theorem can be seen as a consequence of the results in [7, 15]. A formal and stand-alone proof will follow later in this section. Let $|G\rangle$ be the graph state with corresponding graph G . It was shown in [15] that the sum of the elements of the stabilizer group of $|G\rangle$, $\sum_{j=1}^{2^n-1} s_j$ is a Bell operator such that $\sum_{j=1}^{2^n-1} s_j |G\rangle = (2^n - 1)|G\rangle$ and $\max |\langle S \rangle| < 2^n - 1$ when restricting to classical states (where S is the stabilizer group defined earlier). In words, the graph state $|G\rangle$ violates the corresponding Bell inequality up to its algebraic maximum. This fact together with [7, Equation 6] enforces that $\alpha(H) < \vartheta(H)$ and $\vartheta(H) = \alpha^*(H) = 2^n - 1$. The construction in Definition 1 simply transforms the Bell operator, originally written as a sum of mean values, into a sum of probabilities of events, in order to construct the graph associated with the exclusivity structure.

The statement, therefore, combines known facts from quantum information in a novel way in order to prove a purely graph theoretical result.

The proof of Theorem 4 requires the following definition:

Definition 5 (Canonical orthogonal representation) *Let $H = (V, E)$ be a graph representing a Kotzig orbit. Let $S_{(i,j)} = (s_{(i,j)}^{(1)}, s_{(i,j)}^{(2)}, \dots, s_{(i,j)}^{(n)})$ be the event at vertex $(i, j) \in V(H)$, where $i = 1 \dots 2^n - 1$, $j = 1 \dots 2^{w_i-1}$, and $s_{(i,j)}^{(k)} \in \{I, x, \underline{x}, y, \underline{y}, z, \underline{z}\}$, for each $k = 1, 2, \dots, n$. Let $|s_{(i,j)}^{(k)}\rangle$ be defined as follows:*

$$\begin{array}{c|cccccc} s_{(i,j)}^{(k)} & x & \underline{x} & y & \underline{y} & z & \underline{z} & I \\ \hline |s_{(i,j)}^{(k)}\rangle & |+\rangle & |-\rangle & |y_+\rangle & |y_-\rangle & |0\rangle & |1\rangle & |\psi\rangle \end{array}. \quad (16)$$

Here, $|\psi\rangle$ is an arbitrary ray in \mathbb{C}^2 and $|y_+\rangle, |y_-\rangle$ are the eigenvectors of the Pauli matrix Y with eigenvalue $+1$ and -1 , respectively. The canonical orthogonal representation of H is the set of vectors $\{|s_{(i,j)}\rangle := |s_{(i,j)}^{(1)}\rangle \otimes |s_{(i,j)}^{(2)}\rangle \otimes \dots \otimes |s_{(i,j)}^{(n)}\rangle : (i, j) \in V(H)\}$.

For example, in $H(P_3)$ (see Fig. 1), the element of the canonical orthogonal representation of the vertex labeled by $\underline{xI\underline{x}}$ is $|-\rangle \otimes |\psi\rangle \otimes |-\rangle$. Notice that if $|\psi\rangle$ is chosen to be non-orthogonal to any of the vectors $|+\rangle, |-\rangle, \dots, |1\rangle$ then the representation is faithful.

Proof of Theorem 4. Let $H = H(G)$ be a graph representing a Kotzig orbit of a graph G . The proof is structured in three parts: (1) we prove that $\vartheta(H) \geq 2^n - 1$; (2) we prove that $\alpha^*(H) \leq 2^n - 1$; (3) finally, we prove that $\alpha(H) < 2^n - 1$. The first two parts together prove that $\vartheta(H) = \alpha^*(H) = 2^n - 1$, since $\vartheta(G) \leq \alpha^*(G)$, for any graph G (see, e.g., [7]). We begin with the first part:

(1) It follows directly from Eq. (3) that $\sum_{i=1}^{2^n-1} \langle G | s_i | G \rangle = 2^n - 1$. We know that the eigenvectors with eigenvalue $+1$ of each operator s_i are in one-to-one correspondence with the vertices of a clique in H : $|s_{(i,1)}\rangle, |s_{(i,2)}\rangle, \dots, |s_{(i,2^{w_i-1})}\rangle$. These are elements of the canonical orthogonal representation of H . From the definition of the stabilizer group, for all $s_i \in S$ and for all eigenvectors $|s_{(i,j)}\rangle$

($j = 1, 2, \dots, 2^{w_i-1}$) with eigenvalue -1 , we have $\langle \underline{s}^{(i,j)} | G \rangle = 0$, because $|G\rangle$ is in the $+1$ eigenspace. Now, let $s_i = \sum_j \lambda_{ij} |s_{(i,j)}\rangle \langle s_{(i,j)}|$ be an Hermitian eigendecomposition of s_i . Thus,

$$2^n - 1 = \sum_{i=1}^{2^n-1} \langle G | s_i | G \rangle \quad (17)$$

$$= \sum_{i=1}^{2^n-1} \sum_j \lambda_{ij} \langle G | s_{(i,j)} \rangle \langle s_{(i,j)} | G \rangle \quad (18)$$

$$= \sum_{i=1}^{2^n-1} \sum_{j:\lambda_{ij}=1} \langle G | s_{(i,j)} \rangle \langle s_{(i,j)} | G \rangle \quad (19)$$

$$= \sum_{i=1}^{2^n-1} \sum_{j:\lambda_{ij}=1} |\langle G | s_{(i,j)} \rangle|^2 \quad (20)$$

$$\leq \vartheta(H), \quad (21)$$

where the inequality in the last line follows because a canonical orthogonal representation of H together with the state $|G\rangle$ represents a feasible solution for the semidefinite formulation of the Lovász number in Eq. (1).

(2) From the linear programming formulation of the fractional packing number (see Eq. (2)), it is easy to see that a partition of the set of vertices into k cliques gives an upper bound to $\alpha^*(H)$. To see this choose one vertex, say i , per clique and set its weight $w_i = 1$. We get a partition of H into $2^n - 1$ cliques if we consider the events associated with each s_i .

(3) We use an argument very similar to [15, Lemma 1 and Theorem 1]. If the number of vertices of G is two then the result does not hold as $\alpha(H) = 2^2 - 1$ (by direct calculation). Each connected graph with more than two vertices has a subgraph with three vertices. For each of those we can see (also by direct calculation; see Table 1 of section 3.1) that $\alpha(H) < 7$. Therefore, we just need to show that if G' is a subgraph of G with n' vertices and $\alpha(H') < 2^{n'} - 1$, where H' is the representative of the Kotzig orbit of G' , then $\alpha(H) < 2^n - 1$ for $n > 2$. Notice that S' , the stabilizer group of G' , is a subset of S . Therefore, in the graph H we find cliques associated with S' , but containing slightly different events. For each $s'_i \in S'$, the corresponding $s_i \in S$ has the same structure, with eventually some additional Z operators. Let \tilde{H} be the subgraph of H induced by the vertices in cliques associated with the elements of S' . We need to show that if in H' there is no vertex per clique to form a maximal independent set then neither are there in \tilde{H} . Therefore, $\alpha(H) < 2^n - 1$. Towards a contradiction, suppose there is an independent set L of \tilde{H} such that $|L| = 2^{n'} - 1$. We distinguish two cases:

- If the events at the vertices in L do not have any \underline{z} element then we can map them to an independent set in H' of size $2^{n'} - 1$, just by ignoring the additional Z operators. This contradicts the hypothesis that $\alpha(H') < 2^{n'} - 1$.
- If the events at the vertices in L do have \underline{z} elements then we can find another independent set J with the same cardinality such that the events at its vertices do not have any \underline{z} element. We can find J as follows. It is easy to check that an operator s_i has the form $O^{(1)} \dots Z^{(\ell)} \dots O^{(n)}$ if and only if it has an odd number of $X^{(k)}$ and $Y^{(k)}$, with $\{\ell, k\} \in E(H)$. Therefore, complementing $\underline{z}^{(\ell)}$ and all occurrences of $X^{(k)}$ and $Y^{(k)}$ in the events at the vertices of the

independent set L , we obtain the events in J with the desired properties, and so we are back to the previous case.

■

The Bell inequalities described by the graph H are exactly the same as in [15], but in the form of a pseudo-telepathy game.

Definition 6 For any graph G on n vertices, let us define an n -player game for G as follows. The input set for each player is $\{X, Y, Z, I\}$ and the output set is $\{+1, -1\}$. The set of valid inputs is the set of elements of the stabilizer group of $|G\rangle$. The players win on input $s_i \in S$ if and only if the sign of the product of their outputs equals the sign of s_i .

Corollary 7 The graph game for G is a pseudo-telepathy game.

Proof. It is easy to see that if the players share the graph state $|G\rangle$ and each player performs the measurement corresponding to her input, then they always win. On the other hand, we show that a classical strategy for the game can be used to construct an independent set of $H = H(G)$ and *viz.*

We now consider the first direction. If there exists a strategy which answers correctly to k questions then there exists an independent set with k elements. A classical strategy is *w.l.o.g.* a set of functions for each player from the input set to the output set. Therefore, for all the winning inputs s_i , there will be a single output (a_1, \dots, a_n) , corresponding to a vertex of H . It is easy to verify that there cannot be an edge between any pair of these vertices. Since the strategy wins on k input pairs, the independent set has k elements.

For the other direction, we show that if there exists an independent set L of H having size k , then there exists a strategy for the game on G that answers correctly to at least k of the $2^n - 1$ questions. By the structure of H , the independent set L cannot contain vertices i, j such that, for the same input x , $a_\ell^{(i)} \neq a_\ell^{(j)}$ for some $\ell \in \{1, \dots, n\}$. Hence, we have the following strategy: on input x , each player outputs the unique a determined by the vertices in the independent set. The size k of the independent set implies that the players answer correctly to at least k input pairs. ■

5 Examples

Proposition 8 Let K_n be the complete graph on n vertices. Then,

$$\alpha(H(K_n)) = \begin{cases} 2^{\frac{n-3}{2}} + 3 \cdot 2^{n-2} - 1, & \text{for } n \text{ odd;} \\ 2^{\frac{n}{2}-1} - 2^{n-2} + 2^n - 1, & \text{for } n \text{ even.} \end{cases} \quad (22)$$

Proof. As said before, $H(G)$ can be associated to the Bell inequality in which the Bell operator is the sum of all stabilizer operators of $|G\rangle$ [15]. The first observation is that the graph state associated to $G = K_n$ is the n -qubit Greenberger-Horne-Zeilinger (GHZ) state [16]. The second observation is that, in that case, the Bell inequality corresponding to $H(G)$ is the sum of a well-known Bell inequality maximally violated by the n -qubit GHZ state [22] plus a trivial Bell inequality not

violated by quantum mechanics [5]. For example, for $n = 3$, the Bell inequality corresponding to $H(K_3)$ is $\beta_3 = \mu_3 + \tau_3 \leq 2 + 3$, where

$$\mu_3 = \langle XZZ \rangle + \langle ZXZ \rangle + \langle ZZX \rangle - \langle XXX \rangle, \quad (23a)$$

$$\tau_3 = \langle Y Y I \rangle + \langle Y I Y \rangle + \langle I Y Y \rangle; \quad (23b)$$

recall that $\langle XZZ \rangle$ denotes the mean value of the product of the outcomes of the measurement of X on qubit 1, Z on qubit 2, and Z on qubit 3. The inequality $\mu_3 \leq 2$ is the well-known Bell inequality introduced in [22], while $\tau_3 \leq 3$ is a trivial inequality not violated by quantum mechanics [15, 5]. The sum β_3 has 7 terms, with four terms generating cliques of size $2^3 - 1$ and the other three terms generating cliques of size $2^2 - 1$. Then, we can see that $\alpha(H(K_3))$ is equal to the maximum number of quantum predictions that a deterministic local theory can simultaneously satisfy. By quantum predictions we mean: $XZZ = 1$, $ZXZ = 1$, $ZZX = 1$, $XXX = -1$, $IYY = 1$, $YIY = 1$, and $IYY = 1$. In this case, the maximum number of quantum predictions that a deterministic local theory can simultaneously satisfy is 6: 3 out of $XZZ = 1$, $ZXZ = 1$, $ZZX = 1$, $XXX = -1$, plus the other 3 ($IYY = 1$, $YIY = 1$, and $IYY = 1$). Equivalently, $\alpha(H(K_3))$ is the maximum quantum violation, denoted by β_{QM} , minus the minimum number of quantum predictions which *cannot* be satisfied by a deterministic local theory. Since the minimum number of quantum predictions which cannot be satisfied by a deterministic local hidden variable theory is β_{QM} minus the maximum value of the Bell operator for a deterministic local theory, denoted by β_{LHV} , and all of them divided by two, then

$$\alpha(H(K_n)) = \frac{\beta_{\text{QM}}(n) + \beta_{\text{LHV}}(n)}{2}. \quad (24)$$

This expression is very useful since, for the Bell inequalities for the n -qubit GHZ states [15],

$$\beta_{\text{QM}}(n) = \mu_{\text{QM}}(n) + \tau_{\text{QM}}(n), \quad (25)$$

$$\beta_{\text{LHV}}(n) = \mu_{\text{LHV}}(n) + \tau_{\text{LHV}}(n).$$

The interesting point is that the values of $\mu_{\text{QM}}(n)$ and $\mu_{\text{LHV}}(n)$ are well-known [22], and

$$\tau_{\text{QM}}(n) = \tau_{\text{LHV}}(n) = \beta_{\text{QM}}(n) - \mu_{\text{QM}}(n) \quad (26)$$

(Recall that $\beta_{\text{QM}}(n) = 2^n - 1$.) For n odd, $\mu_{\text{QM}}(n) = 2^{n-1}$, and $\mu_{\text{LHV}}(n) = 2^{(n-1)/2}$; for n even, $\mu_{\text{QM}}(n) = 2^{n-1}$ and $\mu_{\text{LHV}}(n) = 2^{n/2}$. Inserting these numbers in Eq. (25) and then in Eq. (24), we obtain the statement. ■

6 Zero-error capacity

In this section, we show that for every graph G on n vertices the graph $H(G)$ has zero-error entanglement-assisted capacity $2^n - 1$. Theorem 4 states that $\alpha(H(G)) < 2^n - 1$. The result gives a separation between $c_0^*(H(G))$ and $c_0(H(G))$. It is known that for all graphs, the Lovász number upper bounds the entanglement-assisted Shannon capacity [12]. Therefore, $c_0^*(H(G))$ saturates its upper bound.

There are few (and very recently discovered) classes of graphs for which this separation is known. For example, one is based on the Kochen-Specker theorem [9] and other ones are based on variations of orthogonality graphs [4, 21]. Here, we present a new family of graphs and a construction method,

which can also be interpreted as a graph theoretic technique of independent interest. The most important point is that every graph gives rise to a member of the family through our construction. This property opens directions for future studies, for example, identifying subclasses or hierarchies where the separation is large or is easy to quantify.

Theorem 9 *Let H be a graph from a Kotzig orbit. Then $c_0^*(H) = 2^n - 1$.*

Proof. From Theorem 4 and [12, Corollary 14] we obtain the upper bound

$$c_0^*(H(G)) \leq \vartheta(H(G)) = 2^n - 1.$$

We need to show a matching lower bound on $c_0^*(H(G))$. We do this by exhibiting a strategy for entangled parties to send one out of $2^n - 1$ messages in the zero-error setting through a channel with confusability graph H . The strategy is as follows. Alice and Bob share a maximally entangled state of local dimension $2^n(n - 1)$. Observe that H can be partitioned into $2^n - 1$ cliques, one for each element of the stabilizer group. The clique corresponding to $s_i \in S$ consists of the vertices associated with the mutually exclusive events in the set S_i ; we denote by S_i the set of events related to s_i as in Section 3. For each $i \in \{1, 2, \dots, 2^n - 1\}$, Alice performs a projective measurement on her part of the shared state. The outcomes of the measurement are the elements of S_i . Since the parties share a maximally entangled state, Alice's strategy has to satisfy two properties to be correct:

1. For each $i \in \{1, 2, \dots, 2^n - 1\}$, the projectors associated to elements of S_i form a projective measurement (because Alice needs to perform a projective measurement for each message i to be sent).
2. For each edge $\{u, v\} \in E(H(G))$, projectors associated with u and v must be orthogonal (to satisfy the zero-error constraint).

The next step is to exhibit projectors in Alice's strategy and show that both properties are satisfied. In what follows we use the notation in Definition 5.

We begin by examining the case where s_i does not contain any identity operator. In this case, each projective measurement will consist of projectors of rank 1 acting on $\mathbb{C}^{2^n(n-1)}$. Order the elements of S_i arbitrarily. Let s_i be of the form $O^{(1)} \dots O^{(n)}$, where $O^{(k)} \in \{X, Y, Z\}$. Define for each $s_{(i,j)}^{(k)}$ the *occurrence number* $\nu(i, j, k)$ based on a chosen ordering: if the same eigenvector of $O^{(k)}$ occurs in $s_{(i,j)}^{(k)}$ for the ℓ -th time in the chosen ordering then $\nu(i, j, k) = \ell$. Construct projectors starting from the canonical orthogonal representation and an ancillary space of dimension $n - 1$. For $s_{(i,j)}^{(k)}$, let

$$P_{(i,j)} = \bigotimes_{k=1}^n |s_{(i,j)}^{(k)}\rangle\langle s_{(i,j)}^{(k)}| \otimes |\nu(i, j, k)\rangle\langle \nu(i, j, k)|. \quad (27)$$

We show that Property 1 is satisfied. These projectors are mutually orthogonal for all vertices (i, j) . We need to prove that their sum is the identity. From the structure of the events in S_i we observe that, for each O_k , the eigenvectors with eigenvalue $+1$ (and -1) occur in half of the elements of S_i . Therefore, in the construction of the projectors, a pair of ± 1 eigenvectors for each O_k is summed for each ancillary subspace. The sum of each subspace is the identity. Hence, the total sum is the

identity for the whole space. We show now that also Property 2 is satisfied. If two projectors are in the same clique, orthogonality follows from the discussion above. Consider now two projectors of adjacent vertices from two different cliques that project to the same ancillary subspace. Since we started from an orthogonal representation, those projectors are orthogonal.

Now, consider the more general case where s_i can contain identity operators. Let s_i be of the form $O^{(1)} \dots O^{(n)}$, where $O^{(k)} \in \{I, X, Y, Z\}$. We assume that s_i has weight w . First consider the case where the first w operators are different from identity, $O^{(1)}, O^{(2)}, \dots, O^{(w)} \neq I$. To construct the projective measurement for S_i , we initially construct the projectors for the first w operators as in the previous case. We obtain rank-1 projectors acting on $\mathbb{C}^{2^w(w-1)}$. Choose a basis for $\mathbb{C}^{2^n(n-1)-2^w(w-1)}$ and let the projectors be

$$Q_{(i,j)} = \sum_{\ell=1}^{2^n(n-1)-2^w(w-1)} P_{(i,j)} \otimes |\ell\rangle\langle\ell|. \quad (28)$$

This ensures that the dimensions match and that Properties 1 and 2 hold. To finish the proof, we need to prove the general case where identity operators are in arbitrary positions and not all at the end. In this case, split the construction into subspaces so that each subspace has all the identities at the end. Obtain the projectors for the subspaces as described above and then obtain the final projectors by making tensor products of the projectors for the subspaces. ■

We immediately have the following corollary from Theorems 4 and 9 and the Lovász number upper bound on Θ^* .

Corollary 10 *Let H be a graph from a Kotzig orbit. Then, $c_0(H) < c_0^*(H) = \Theta^*(H)$.*

7 Classification

Let λ_H be the number of connected components of H - we follow λ_H by the number of vertices in each connected component, *e.g.*, $\lambda_H = 2[6, 16]$ means that there are two connected components, one with 6 vertices and one with 16 vertices. The degree sequence of H , D_H , is written as $a, b/c, d/e, f/\dots$, meaning that there are b vertices of degree a , d vertices of degree c , f vertices of degree e , etc.

For each Kotzig orbit, we wish to compute the independence number, $\alpha(H)$. This is also listed in Table 1. Each stabilizing operator of S is multiplied by a global coefficient $+1$ or -1 . For instance, for the operators of $H(P_3)$ (see Fig 1 and associated discussion), there are six $+1$ coefficients and one -1 coefficient. If one selects one vertex from each of the 6 cliques in H generated by the 6 operators of this example which have a $+1$ coefficient, then one can be sure that they are mutually unconnected in H . (More specifically, one can select for each operator the event where all the measurements gave outcome $+1$. This event is always present if the operator has coefficient $+1$.) So $\alpha(H) \geq 6$, *i.e.* the number of $+1$ coefficients in S for a given G yields a lower bound on $\alpha(H)$, for each G . This idea leads to the following lemma. Let $\beta(G)$ be the number of operators in S with a -1 coefficient where, in general, $\beta(G)$ is not an invariant of the Kotzig orbit of G . Let $\beta_{\min}(G^L) = \min\{\beta(G), G \in G^L\}$ and $\beta_{\max}(G^L) = \max\{\beta(G), G \in G^L\}$, where G^L is the set of graphs in the Kotzig orbit of G .

Lemma 11 *Given a graph G , we have*

$$\alpha(H) \geq 2^n - 1 - \beta_{\min}(G^L) \geq 2^n - 1 - \beta(G). \quad (29)$$

Table 1 lists both the computed independence number, $\alpha(H)$, for each H , and the range of lower bounds, $\beta_{\min}(G^L) - \beta_{\max}(G^L)$, on $\alpha(H)$. One observes that the lower bound is often tight but not always. For example, for the graph $G = 15, 25, 34, 45$, computations show that $\beta_{\min}(G^L) = 6$ and $\beta_{\max}(G^L) = 12$. So $\alpha(H) \geq 2^5 - 1 - 6 = 25$. In this case the bound is tight as $\alpha(H)$ is computed to be 25.

The *symmetrical form* of K_n makes it relatively easy to prove that, for $n > 2$, $H(ST_n) = H(K_n)$ is comprised of two disjoint subgraphs, and the results of section 5 allow us to identify these two disjoint subgraphs, *i.e.* $H(K_n) = H(\mu_n) + H(\tau_n)$, where $H(\mu_n)$ has 2^{2n-2} vertices and $H(\tau_n)$ has $\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2^{2i-1} \binom{n}{2i}$ vertices - (see the equations in (24) for the case of $n = 3$). It is likewise easy to show that $\beta(K_n) = \sum_{i=1}^{\lfloor \frac{n+1}{4} \rfloor} \binom{n}{4k-1}$, and therefore we know that $\alpha(H(K_n)) \geq 2^n - 1 - \sum_{i=1}^{\lfloor \frac{n+1}{4} \rfloor} \binom{n}{4k-1}$.

The evaluation of $\beta(G)$ can be translated to the following problem.

Lemma 12 *Define the Boolean function, $f_G(z_1, z_2, \dots, z_n) : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ such that*

$$f_G(z_1, z_2, \dots, z_n) = \sum_{\{i,j\}, \{j,k\} \in E(G), i < j < k} z_i z_j z_k. \quad (30)$$

Let $wt(f)$ be the weight of f , defined to be the number of ones in its truth-table). Then,

$$\beta(G) = wt(f_G). \quad (31)$$

Proof. (Sketch) Consider the subgraph of G at vertices i, j, k , where we assume $\{i, j\}, \{j, k\} \in E(G)$. One can confirm that $s = g_i g_j g_k$ is an operator with a global coefficient of -1 and we can represent this in f_G by the cubic term $z_i z_j z_k$. Moreover this must be true for each such pair of edges in G . Likewise consider the five vertices i, j, k, l, m in G where we assume that $\{i, j\}, \{j, k\}, \{k, l\}, \{l, m\} \in E(G)$. One can confirm that $s = g_i g_j g_k g_l g_m$ is an operator with a global coefficient of -1 , and the operators $s' = g_i g_j g_k$, $s'' = g_j g_k g_l$, and $s''' = g_k g_l g_m$ also have global coefficients equal to -1 . We represent this situation in f_G by the sum of the cubic terms $z_i z_j z_k + z_j z_k z_l + z_k z_l z_m$. The lemma follows from an elaboration of this argument. Specifically the global coefficient of $s = \prod_{j=1}^n g_j^{z_j}$ is $(-1)^{f_G(z_1, z_2, \dots, z_n)}$. ■

Lemma 12 allows us to obtain an equation for $\beta(C_n)$. We evaluate $\beta(C_n)$ by computer for $n = 3, 4, 5, 6, 7, 8, \dots$ to be $1, 4, 6, 18, 36, 80, \dots$, respectively and use [26] to find the sequence A051253 and the reference [10] where a recurrence formula is provided for $\beta(C_n) = wt(f_G) = wt(z_1 z_2 z_3 + z_2 z_3 z_4 + \dots + z_{n-2} z_{n-1} z_n + z_{n-1} z_n z_1 + z_n z_1 z_2)$, namely

$$\beta(C_{n+3}) = 2(\beta(C_{n+1}) + \beta(C_n) + 2^{n-1}). \quad (32)$$

We also offer the following conjecture, based on the results of Table 1.

Conjecture 13 *The graph $H(G)$ is always connected except when G is in the Kotzig orbit of the star graph, ST_n , in which case $H(G)$ splits into 3 disjoint components for $n = 2$, and 2 disjoint components for $n > 2$.*

Table 1: $H(G)$ for $n = 2, \dots, 6$

n	G	$ G^L $	$ V^H $	λ_H	$\alpha(H)$	$\beta_{\min}(G^L) - \beta_{\max}(G^L)$	D_H
2	12	1	6	3[2, 2, 2]	3	0	1, 6/
3	12, 13	2	22	2[6, 16]	6	1 - 1	3, 6/9, 16
4	14, 24, 34	2	84	2[20, 64]	13	4 - 4	11, 12/13, 2/17, 6/43, 64
	14, 23, 24, 34	4	76	1	13	2 - 4	15, 4/25, 4/27, 4/29, 32/35, 32
5	15, 25, 35, 45	2	316	2[60, 256]	25	10 - 10	33, 20/45, 10/49, 30/195, 256/
	15, 25, 34, 45	6	280	1	25	6 - 12	49, 6/51, 2/89, 12/91, 12/107, 24/129, 72/133, 24/159, 128/
	15, 24, 25, 34, 35	10	268	1	25	6 - 12	55, 4/99, 16/101, 4/103, 4/107, 16/117, 64/129, 32/137, 32/139, 32/147, 64/
	12, 13, 24, 35, 45	3	256	1	25	6 - 10	99, 40/117, 120/135, 96
6	16, 26, 36, 46, 56	2	1206	2[182, 1024]	51	20 - 20	101, 30/143, 30/147, 90/151, 2/167, 30/841, 1024/
	16, 26, 36, 45, 56	6	1030	1	51	16 - 28	145, 12/189, 2/207, 2/211, 6/299, 24/301, 24/343, 4/351, 28/
							383, 32/455, 32/459, 96/587, 256/665, 512/
	16, 26, 35, 45, 46	16	976	1	49	14 - 30	169, 6/207, 2/349, 12/351, 12/357, 24/383, 8/411, 48/449, 8/
							453, 24/471, 32/473, 16/477, 48/481, 48/483, 48/533, 128/557, 128/
							565, 64/567, 64/611, 256/
	15, 26, 36, 45, 56	4	1044	1	49	14 - 30	173, 12/321, 36/323, 36/479, 192/541, 64/545, 192/679, 512/
	15, 26, 35, 36, 45, 46	5	958	1	51	20 - 28	213, 6/355, 32/395, 12/397, 12/441, 96/515, 384/547, 192/
							549, 192/559, 4/567, 28/
	16, 24, 35, 46, 56	10	958	1	51	12 - 28	181, 4/213, 2/323, 4/325, 4/347, 8/349, 8/379, 32/401, 64/455, 8/
							459, 24/477, 32/487, 64/489, 64/515, 256/593, 256/599, 32/603, 96/
	15, 26, 34, 35, 46, 56	25	940	1	49	14 - 30	187, 4/337, 4/339, 4/357, 16/379, 16/383, 32/431, 64/435, 32/
						469, 32/471, 32/493, 32/495, 32/497, 128/521, 128/529, 64/531, 64/	
						575, 128/601, 64/603, 64/	
16, 24, 26, 35, 36, 45	21	922	1	47	16 - 28	193, 2/355, 16/357, 16/405, 16/413, 64/425, 64/441, 8/475, 32/	
						477, 32/479, 64/503, 256/511, 32/513, 32/557, 256/563, 8/567, 24/	
14, 25, 36, 45, 46, 56	5	990	1	47	20 - 28	197, 6/355, 12/357, 12/433, 192/473, 96/487, 8/491, 24/579, 192/	
						581, 192/625, 256/	
12, 13, 25, 36, 45, 46	16	904	1	45	18 - 30	355, 32/395, 24/407, 96/431, 48/485, 384/539, 192/565, 64/567, 64/	
12, 13, 14, 23, 25, 36, 45, 46, 56	2	936	1	45	22 - 26	427, 360/571, 576/	

The conjecture is verified, computationally, for $n = 2, 3, 4, 5$. A potential way to prove it is to try to construct two connected components by adding codewords and show that this forces one to be in the ST_n orbit.

By using code-theoretic techniques, we can lower and upper bound the size of cliques in $H(G)$. Remember that S is the stabilizer group generated by G , comprising $2^n - 1$ operators, s_j , and that $w_j = w(s_j)$ is the *weight* of operator s_j . Let $w_{\max}(S) = \max_{s_j \in S}(w(s_j))$ and $w_{\min}(S) = \min_{s_j \in S}(w(s_j))$. We have that $w_{\max}(S) = n$, for any S , because, for any graph G , then $\prod_{i=0}^{n-1} g_i$ is always an operator of weight n . It is also well-known that the stabilizer group for a graph state characterizes a self-dual additive code over \mathbb{F}_4 of length n whose minimum distance, d_H , is given by $w_{\min}(S)$ [11]. So, in subsequent discussions, we refer to $w_{\min}(S)$ and $w_{\max}(S)$ by d_H and n , respectively. Let $\mathcal{C}(H)$ be the set of maximal cliques in H , where each $c \in \mathcal{C}(H)$ is a subset of $V(H)$, the set of vertices of H , over which there is a clique in H . Let $\tilde{\omega}(H)$ and $\omega(H)$ be the minimum and maximum size of a maximal clique, respectively, in graph H . So $\tilde{\omega}(H) = \min\{|c| : c \in \mathcal{C}(H)\}$ and $\omega(H) = \max\{|c| : c \in \mathcal{C}(H)\}$. Recall that $\omega(H)$ denotes the clique number of H .

Theorem 14 *For $H(G)$, the graph generated from the stabilizer set S ,*

$$\begin{aligned} \tilde{\omega}(H) \leq |c| \leq \omega(H), \quad \forall c \in \mathcal{C}(H), \\ \text{where } \tilde{\omega}(H) = 2^{d_H-1}, \omega(H) = 2^{n-1}, \quad (33) \\ \text{and both upper and lower bounds are tight.} \end{aligned}$$

Proof. Consider an arbitrary set of two operators, $R = \{IXZYZ, -YXZYY\}$. Then both operators have XZY at their second, third, and fourth tensor positions. We say that the two operators have a set overlap of XZY and this overlap is of size $\mu_R = 3$. Let $R \subset S$ and consider an arbitrary splitting of the assignments to XZY of $xzy, \underline{xzy}, \underline{xzy}, xz\underline{y}, \underline{xzy}$ for operator $IXZYZ$ and $xzy, \underline{xzy}, xzy$ for operator $-YXZYY$. Then $H(G)$ contains a size-11 clique over the vertices $I\underline{xzyz}, I\underline{xzyz}, I\underline{xzyz}, I\underline{xzyz}, I\underline{xzyz}$ and $y\underline{xzyy}, \underline{y\underline{xzyy}}, y\underline{xzyy}, \underline{y\underline{xzyy}}, \underline{y\underline{xzyy}}, \underline{y\underline{xzyy}}$. It is straightforward to verify that this clique cannot be extended by adding another vertex from the subset of vertices in H originating from the operators in R . More generally, any splitting of the 2^3 assignments to XZY is possible, each giving a different clique. For our example, the operator $IXZYZ$ contributes $5 \cdot 2^{w(IXZYZ) - \mu_R - 1}$ vertices to the clique and operator $-YXZYY$ contributes $3 \cdot 2^{w(-YXZYY) - \mu_R - 1}$ vertices to the clique. So this clique is of size $2^{-\mu_R - 1}(5 \cdot 2^{w(IXZYZ)} + 3 \cdot 2^{w(-YXZYY)}) = 11$. It is a maximal clique if and only if there is no $R' \subset S$ such that $R \subset R'$ and $\mu_{R'} = \mu_R$.

If, instead, $R = \{IXZYZ, -YXZYY, XXZ XI\}$, then the set overlap is reduced to XZ and of size $\mu_R = 2$. As an example, consider just the partition where \underline{xz} is assigned to operator $IXZYZ$, xz, \underline{xz} to operator $-YXZYY$, and xz to operator $XXZ XI$. Then this assignment identifies the vertices $I\underline{xzyz}, I\underline{xzyz}$, and $\underline{y\underline{xzyy}}, y\underline{xzyy}, y\underline{xzyy}, \underline{y\underline{xzyy}}, \underline{y\underline{xzyy}}, \underline{y\underline{xzyy}}$, and $\underline{xxzxI}, \underline{xxzxI}$. The operator $IXZYZ$ contributes $1 \cdot 2^{w(IXZYZ) - \mu_R - 1}$ vertices to the clique, operator $-YXZYY$ contributes $2 \cdot 2^{w(-YXZYY) - \mu_R - 1}$ vertices to the clique, and operator $XXZ XI$ contributes $1 \cdot 2^{w(XXZ XI) - \mu_R - 1}$ vertices to the clique. So this clique is of size $2^{-\mu_R - 1}(1 \cdot 2^{w(IXZYZ)} + 2 \cdot 2^{w(-YXZYY)} + 1 \cdot 2^{w(XXZ XI)}) = 12$. It is a maximal clique if and only if there is no $R' \subset S$ such that $R \subset R'$ and $\mu_{R'} = \mu_R$.

Therefore, a strategy to find all maximal cliques in $H(G)$ is to find all subsets $R \subset S$ where $\mu_R > 0$, and such that there is no $R' \subset S$ where $R \subset R'$ and $\mu_{R'} = \mu_R$. Then for each of these subsets, R , and for each $|R|$ -wise ordered partition, $p = \{a_{s_j} \mid s_j \in R\}$ of 2^{μ_R} , where

$\sum_{s_j \in R} a_{s_j} = 2^{\mu_R}$, and for each assignment of the integers in set $\{0, 1, \dots, 2^{\mu_R} - 1\}$ according to partition p , there is associated a maximal clique, $K_{R,p}$, of size,

$$|K_{R,p}| = 2^{-\mu_R-1} \sum_{s_j \in R} a_{s_j} 2^{w(s_j)}. \quad (34)$$

Let $w^-(s_j) = \min\{w(s_j) \mid s_j \in R\}$. Then, for the specific $s_j \in R$ where $w(s_j) = w^-(s_j)$ we can minimise the clique size by assigning $a_j = 2^{\mu_R}$. So a special case of the above equation identifies a maximal clique of minimum size $2^{w^-(s_j)-1}$. Similarly let $w^+(s_j) = \max\{w(s_j) \mid s_j \in R\}$. Then, for the specific $s_j \in R$ where $w(s_j) = w^+(s_j)$ we can maximise the clique size by assigning $a_j = 2^{\mu_R}$. A special case of the above equation identifies a maximal clique of maximum size $2^{w^+(s_j)-1}$. We arrive at both $\tilde{\omega}(H) = 2^{d_H-1}$ and $\omega(H) = 2^{n-1}$ by observing that all members of S occur in at least one R identified. ■

In a step towards enumerating the maximal cliques of $H(G)$, we first enumerate the number of maximal cliques of size given by (34), as generated by a fixed R and $|R|$ -wise ordered partition, p . Let $R_k \subset R$ satisfy $R_k = \{s_j \in R \mid j < k\}$, and let $b_k = \sum_{s_j \in R_k} a_j$.

Lemma 15 *For fixed R and p , the graph $H(G)$, contains $\#K_{R,p}$ maximal cliques of size $|K_{R,p}|$, where*

$$\#K_{R,p} = \prod_{s_j \in R} \binom{2^{\mu_R - b_j}}{a_j}. \quad (35)$$

Proof. The lemma follows immediately by counting the number of ways that one can assign to the integers an $|R|$ -way partition of 2^{μ_R} . For instance, for $\mu_R = 2$ and partition $\{1, 2, 1\}$, one can assign the integers as $\{\{0\}, \{1, 2\}, \{3\}\}$, $\{\{0\}, \{1, 3\}, \{2\}\}$, $\{\{0\}, \{2, 3\}, \{1\}\}$, $\{\{1\}, \{0, 2\}, \{3\}\}$, $\{\{1\}, \{0, 3\}, \{2\}\}$, $\{\{1\}, \{2, 3\}, \{0\}\}$, $\{\{2\}, \{0, 1\}, \{3\}\}$, $\{\{2\}, \{0, 3\}, \{1\}\}$, $\{\{2\}, \{1, 3\}, \{0\}\}$, $\{\{3\}, \{0, 1\}, \{2\}\}$, $\{\{3\}, \{0, 2\}, \{1\}\}$, $\{\{3\}, \{1, 2\}, \{0\}\}$ - in total $\binom{2^2}{1} \times \binom{2^2-1}{2} \times \binom{2^2-3}{1} = 4 \times 3 \times 1 = 12$ ways. ■

Although, in Theorem 14, we have obtained tight lower and upper bounds for the size of the maximal cliques of $H(G)$ in the general case, in terms of d_H , and n , respectively, it remains open to obtain tight lower and upper bounds for the size of the maximal independent sets of $H(G)$ in the general case. However we have obtained a lower bound, $\beta(G)$, on $\alpha(H(G))$, although this bound is not tight in general. It also remains open to provide equations for $|V^H|$ in the general case. Given that this paper highlights the gap between $\alpha(H(G))$ and $2^n - 1$, it is particularly desirable to develop equations for $\alpha(H(G))$ in the general case. Of course, it would be nice to find a formula linking $\alpha(H(G))$ and $\alpha(G)$ - however it should be noted that $\alpha(G)$ is not an invariant of the Kotzig orbit of G .

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