The maximum number of faces of the Minkowski sum of three convex polytopes

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Abstract

We derive tight expressions for the maximum number of k-faces, $0 \leq k \leq d-1$, of the Minkowski sum, $P_1+P_2+P_3$, of three d-dimensional convex polytopes P_1, P_2 and P_3, as a function of the number of vertices of the polytopes, for any $d \geq 2$. Expressing the Minkowski sum of the three polytopes as a section of their Cayley polytope $\mathcal{C}$, the problem of counting the number of k-faces of $P_1+P_2+P_3$, reduces to counting the number of $(k+2)$-faces of the subset of $\mathcal{C}$ comprising of the faces that contain at least one vertex from each P_i. In two dimensions our expressions reduce to known results, while in three dimensions, the tightness of our bounds follows by exploiting known tight bounds for the number of faces of r d-polytopes, where $r \geq d$. For $d \geq 4$, the maximum values are attained when P_1, P_2 and P_3 are d-polytopes, whose vertex sets are chosen appropriately from three distinct d-dimensional moment-like curves.

Key words: high-dimensional geometry, discrete geometry, combinatorial geometry, combinatorial complexity, Cayley trick, tight bounds, Minkowski sum, convex polytopes

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1 Introduction

We study the Minkowski sum of three d-dimensional convex polytopes, or, simply d-polytopes, and derive tight upper bounds for the number of its k-faces, for $0 \leq k \leq d-1$, with respect to the number of vertices of the summands. Given two convex polytopes P_1 and P_2, their *Minkowski sum* $P_1 + P_2$ is the set $\{p_1 + p_2 \mid p_1 \in P_1, p_2 \in P_2\}$. This definition extends to any number of summands and also, to non-convex sets of points. The Minkowski sum of convex polytopes is itself a convex polytope, namely, the convex hull of the Minkowski sum of the vertices of its summands.

Minkowski sums are widespread operations in Computational Geometry and find applications in a wide range of areas such as robot motion planning [Lat91], pattern recognition [TRH00], collision detection [LM04], Computer-Aided Design, and, very recently, Game Theory. They reflect geometrically some algebraic operations, and capture important properties of algebraic objects, such as polynomial systems. This makes them especially useful in Computational Algebra, see e.g., [GS93, Stu96, CLO05].

The geometry of the Minkowski sum can be derived from that of its summands: its *normal fan* is the *common refinement* of the normal fans of the summands (see [Zie95] for definitions and details). However, its combinatorial structure is not fully understood, partially due to the fact that most algorithms for computing Minkowski sums have focused on low dimensions (see, e.g., [Fog08] for algorithms computing Minkowski sums in three dimensions). The recent development of algorithms that target high dimensions [Fuk04], has led to a more extensive study of their properties (see, e.g., [Wei07]).

A natural and fundamental question regarding the combinatorial properties of Minkowski sums, concerns their complexity measured as a function of the vertices, or the facets of the summands. A complete answer, in terms of the number of vertices or facets of the summands, does not yet exist although for certain classes of polytopes the question has been resolved (see Section 1). Most of the known results offer tight bounds with respect to the number of vertices of the summands; deriving tight upper bounds with respect to the number of facets seems much harder. Knowing the complexity of Minkowski sums is crucial in developing algorithms for their computation, since it allows to quantify their efficiency.

Preliminaries. Let P be a d-polytope; its dimension is the dimension of its affine span. The faces of P are $\emptyset, P$, and the intersections of P with its supporting hyperplanes. The former faces are called improper while the latter faces are called proper. Each face of P is itself a polytope, and a face of dimension k is called a k-face. Faces of P of dimension $0, 1, d-2$ and $d-1$ are called vertices, edges, ridges, and facets, respectively.

A d-dimensional *polytopal complex, or simply d-complex*, $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{R}^d$ such that (i) $\emptyset \in \mathcal{C}$, (ii) if $P \in \mathcal{C}$ then all the faces of P are also in $\mathcal{C}$ and (iii) the intersection $P \cap Q$ for two polytopes P and Q in $\mathcal{C}$ is a face of both. The dimension $\dim(\mathcal{C})$ of $\mathcal{C}$ is the largest dimension of a polytope in $\mathcal{C}$. A polytopal complex is called *pure* if all its maximal (with respect to inclusion) faces have the same dimension. In this case the maximal faces are called the *facets* of $\mathcal{C}$. A polytopal complex is simplicial if all its faces are simplices. Finally, a polytopal complex $\mathcal{C}'$ is called a *subcomplex* of a polytopal complex $\mathcal{C}$ if all faces of $\mathcal{C}'$ are also faces of $\mathcal{C}$. For a polytopal complex $\mathcal{C}$, the *star* of v in $\mathcal{C}$, denoted $\mathrm{star}(v, \mathcal{C})$, is the subcomplex of $\mathcal{C}$ consisting of all faces that contain v, and their faces. The *link* of v, denoted by $\mathcal{C}/v$, is the subcomplex of $\mathrm{star}(v, \mathcal{C})$ consisting of all the faces of $\mathrm{star}(v, \mathcal{C})$ that do not contain v.

One important class of polytopal complexes arises from polytopes. More precisely, a d-polytope P, together with all its faces and the empty set, form a d-complex, denoted by $\mathcal{C}(P)$. The only maximal face of $\mathcal{C}(P)$, which is clearly the only facet of $\mathcal{C}(P)$, is the polytope P itself. Moreover, all proper faces of P form a pure $(d-1)$-complex, called the *boundary complex* $\mathcal{C}(\partial P)$, or simply ∂P, of P. The facets of ∂P are just the facets of P.

For a d-polytope P, or its boundary complex ∂P, we can define its f-vector as $\boldsymbol{f}(P) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$, where $f_k = f_k(P)$ denotes the number of k-faces of P and $f_{-1}(P) := 1$ corresponds to the empty face of P. From the f-vector of P we define its h-vector as the vector $\boldsymbol{h}(P) = (h_0, h_1, \ldots, h_d)$, where $h_k = h_k(P) = \sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{d-k} f_{i-1}(P)$, $0 \leq k \leq d$.

Let $\mathcal{C}$ be a pure simplicial polytopal d-complex. A shelling $\mathsf{S}(\mathcal{C})$ of $\mathcal{C}$ is a linear ordering $F_1, F_2, \ldots, F_s$ of the facets of $\mathcal{C}$ such that for all $1 < j \leq s$ the intersection, $F_j \cap \left(\bigcup_{i=1}^{j-1} F_i\right)$, of the facet F_j with the previous facets is non-empty and pure $(d-1)$-dimensional. In other words, for every $i < j$ there exists some $\ell < j$ such that the intersection $F_i \cap F_j$ is contained in $F_\ell \cap F_j$, and such that $F_\ell \cap F_j$ is a facet of F_j.

Every pure polytopal complex that has a shelling is called *shellable*. In particular, the boundary complex of a polytope is always shellable (cf. [BM71]). Consider a pure shellable simplicial polytopal complex $\mathcal{C}$ and let $\mathsf{S}(\mathcal{C}) = \{F_1, \ldots, F_s\}$ be a shelling order of its facets. The *restriction* $\mathsf{R}(F_j)$ of a facet F_j is the set of all vertices $v \in F_j$ such that $F_j \setminus \{v\}$ is contained in one of the earlier facets.[1] The main observation here is that when we construct $\mathcal{C}$ according to the shelling $\mathsf{S}(\mathcal{C})$, the new faces at the j-th step of the shelling are exactly the vertex sets G with $\mathsf{R}(F_j) \subseteq G \subseteq F_j$ (cf. [Zie95, Section 8.3]). Moreover, notice that $\mathsf{R}(F_1) = \emptyset$ and $\mathsf{R}(F_i) \neq \mathsf{R}(F_j)$ for all $i \neq j$.

Previous work. The complexity of Minkowski sums depends on the geometry of their summands. Worst-case tight upper bounds offer the best possible alternative when the geometric characteristics of a specific instance of the problem are not accounted for. Gritzman and Sturmfels [GS93] have been the first to derive tight upper bounds for the number of k-faces of $P_1 + \cdots + P_r$, namely:

$$f_k(P_1 + \cdots + P_r) \leq 2\binom{m}{k} \sum_{j=0}^{d-k-1} \binom{m-k-1}{j}, \qquad 0 \leq k \leq d-1, \qquad d, r \geq 2,$$

where m denotes the number of non-parallel edges of $P_1, \ldots, P_r$. Equality occurs when P_i are generic zonotopes, i.e., when each P_i is a Minkowski sum of edges, and the generating edges of all polytopes are in general position.

Our knowledge of tight upper bounds for $f_k(P_1 + \cdots + P_r)$ as a function of the number of vertices or facets of the summands is much more limited, while the problem of finding such tight bounds is far from being fully understood and resolved. Given two polygons P_1, P_2 in two dimensions, with n_1, n_2 vertices (or edges) respectively, their Minkowski sum can have at most $n_1 + n_2$ vertices; clearly, this bound holds also for the number of edges of $P_1 + P_2$, and generalizes in the obvious way for any number of summands (cf. [dBvKOS00]).

In three or more dimensions, Fukuda and Weibel [FW07] have shown what they call the *trivial upper bound*: given r d-polytopes $P_1, P_2, \ldots, P_r$ in $\mathbb{R}^d$, where $d \geq 3$ and $r \geq 2$, we have

$$f_k(P_1 + P_2 + \cdots + P_r) \leq \Phi_{k+r}(n_1, n_2, \ldots, n_r), \tag{1}$$

where n_i is the number of vertices of P_i, $1 \leq i \leq r$, and

$$\Phi_\ell(n_1, n_2, \ldots, n_r) = \sum_{\substack{1 \leq s_i \leq n_i \\ s_1 + \ldots + s_r = \ell}} \prod_{i=1}^{r} \binom{n_i}{s_i}, \qquad \ell \geq r, \qquad s_i \in \mathbb{N}.$$

In the same paper, Fukuda and Weibel have shown that the trivial upper bound is tight for: (i) $d \geq 4$, $2 \leq r \leq \lfloor \frac{d}{2} \rfloor$ and for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - r$, and (ii) for the number of vertices, $f_0(P_1 + P_2 + \cdots + P_r)$, of $P_1 + P_2 + \cdots + P_r$, when $d \geq 3$ and $2 \leq r \leq d-1$. For $r \geq d$, Sanyal

[1] For simplicial faces, we identify the face with its defining vertex set.

[San09] has shown that the trivial bound for $f_0(P_1+P_2+\cdots+P_r)$ cannot be attained, since in this case:

$$f_0(P_1+P_2+\cdots+P_r) \leq \left(1-\frac{1}{(d+1)^d}\right)\prod_{i=1}^{r} n_i < \prod_{i=1}^{r} n_i.$$

Karavelas and Tzanaki [KT11] recently extended the range of of d, r and k for which the trivial upper bound (1) is attained. More precisely, they showed that for any $d \geq 3$, $2 \leq r \leq d-1$ and for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor - r$, there exist r neighborly d-polytopes $P_1, P_2, \ldots, P_r$ in $\mathbb{R}^d$, for which the number of k-faces of their Minkowski sum attains the trivial upper bound. Recall that a d-polytope P is neighborly if any subset of $\lfloor \frac{d}{2} \rfloor$ or less vertices is the vertex set of a face of P. Tight bounds for $f_0(P_1+P_2+\cdots+P_r)$, where $r \geq d$, have very recently been shown by Weibel [Wei12], namely:

$$f_0(P_1+P_2+\cdots+P_r) \leq \alpha + \sum_{j=1}^{d-1}(-1)^{d-1-j}\binom{r-1-j}{d-1-j}\sum_{S\in\mathfrak{S}_j^r}\left(\prod_{i\in S} f_0(P_i)-\alpha\right),$$

where $\mathfrak{S}_j^r$ is the family of subsets of $\{1,2,\ldots,r\}$ of cardinality j, and $\alpha = 2(d-2\lfloor \frac{d}{2} \rfloor)$.

Tight bounds for *all* face numbers, i.e., for all $0 \leq k \leq d-1$, expressed as a function of the number of vertices or facets of the summands, are known only for *two* d-polytopes when $d \geq 3$. Fukuda and Weibel [FW07] have shown that, given two 3-polytopes P_1 and P_2 in $\mathbb{R}^3$, the number of k-faces of P_1+P_2, $0 \leq k \leq 2$, is bounded from above as follows:

$$\begin{aligned} f_0(P_1+P_2) &\leq n_1 n_2, \\ f_1(P_1+P_2) &\leq 2n_1 n_2 + n_1 + n_2 - 8, \\ f_2(P_1+P_2) &\leq n_1 n_2 + n_1 + n_2 - 6, \end{aligned} \tag{2}$$

where n_i is the number of vertices of P_i, $i=1,2$. These bounds are tight. Weibel [Wei07] has derived analogous tight expressions in terms of the number of facets m_i of P_i, $i=1,2$:

$$\begin{aligned} f_0(P_1+P_2) &\leq 4m_1 m_2 - 8m_1 - 8m_2 + 16, \\ f_1(P_1+P_2) &\leq 8m_1 m_2 - 17m_1 - 17m_2 + 40, \\ f_2(P_1+P_2) &\leq 4m_1 m_2 - 9m_1 - 9m_2 + 26. \end{aligned} \tag{3}$$

Weibel's expression for $f_2(P_1+P_2)$ (cf. rel. (3)) has been generalized to the number of facets of the Minkowski sum of any number of 3-polytopes by Fogel, Halperin and Weibel [FHW09]; they have shown that, for $r \geq 2$, the following tight bound holds:

$$f_2(P_1+P_2+\cdots+P_r) \leq \sum_{1\leq i<j\leq r}(2m_i-5)(2m_j-5) + \sum_{i=1}^{r} m_i + \binom{r}{2},$$

where $m_i = f_2(P_i)$, $1 \leq i \leq r$. Finally, Karavelas and Tzanaki [KT12] have shown that for any two d-polytopes P_1 and P_2 in $\mathbb{R}^d$, where $d \geq 4$, and for all $1 \leq k \leq d$, we have:

$$f_{k-1}(P_1+P_2) \leq f_k(C_{d+1}(n_1+n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor}\tbinom{d+1-i}{k+1-i}\left(\tbinom{n_1-d-2+i}{i}+\tbinom{n_2-d-2+i}{i}\right), \tag{4}$$

where $n_i = f_0(P_i)$, $i=1,2$, and $C_d(n)$ stands for the cyclic d-polytope with n vertices. The bounds in (4) have been shown to be tight, and match the corresponding, previously known, bounds for 2- and 3-polytopes (cf. rel. (2)).

Overview. In this work we continue the line of research in [KT12], extending the methods to the case of three d-polytopes in $\mathbb{R}^d$. This turns out to be far from trivial. Allowing just one more summand significantly raises the problem's intricacy. On the other hand, the case of three d-polytopes provides a valuable insight towards our ultimate goal, the general case of r d-polytopes, for any $d, r \geq 2$. Using the tools and methodology applied in this paper, some of the results obtained here can be generalized to the case $d, r \geq 2$ (see Section 7), while others still remain elusive.

We state our main result, also presented in Theorem 14. Let P_1, P_2 and P_3 be three d-polytopes in $\mathbb{R}^d$, $d \geq 2$, with $n_i \geq d+1$ vertices, $1 \leq i \leq 3$. Then, for all $1 \leq k \leq d$, we have:

$$f_{k-1}(P_1+P_2+P_3) \leq f_{k+1}(C_{d+2}(n_{[3]})) - \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k+2-i} \sum_{\emptyset \subset S \subset [3]} (-1)^{|S|} \binom{n_S - d - 3 + i}{i}$$
$$- \delta \binom{\lfloor \frac{d}{2} \rfloor + 1}{k - \lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{3} \binom{n_i - \lfloor \frac{d}{2} \rfloor - 2}{\lfloor \frac{d}{2} \rfloor + 1},$$

where $[3] = \{1,2,3\}$, $\delta = d - 2\lfloor \frac{d}{2} \rfloor$, and $n_S = \sum_{i \in S} n_i$, $\emptyset \subset S \subseteq [3]$. Moreover, for any $d \geq 2$, there exist three d-polytopes in $\mathbb{R}^d$ for which the bounds above are attained.

To establish the upper bounds we first lift the three d-polytopes in $\mathbb{R}^{d+2}$ using an affine basis of $\mathbb{R}^2$, and form the convex hull $\mathcal{C}$ of the embedded polytopes in $\mathbb{R}^{d+2}$. $\mathcal{C}$ is known as the *Cayley polytope* of the P_i's (see Section 2). Exploiting the bijection between the set $\mathcal{F}_{[3]}$, consisting of the k-faces of $\mathcal{C}$ that contain vertices from each P_i, and the $(k-2)$-faces of $P_1+P_2+P_3$, we reduce the derivation of upper bounds for $f_{k-2}(P_1+P_2+P_3)$ to deriving upper bounds for $f_k(\mathcal{F}_{[3]})$, $2 \leq k \leq d+1$.

The rest of our proof follows the main steps of McMullen's proof of the Upper bound Theorem for polytopes [McM70]. In Section 3 we add auxiliary vertices to appropriate faces of the Cayley polytope $\mathcal{C}$, resulting in a simplicial polytope $\mathcal{Q}$ whose face set contains $\mathcal{F}_{[3]}$. We then consider the f-vector $\boldsymbol{f}(\partial\mathcal{Q})$ and the h-vector $\boldsymbol{h}(\partial\mathcal{Q})$ of $\partial\mathcal{Q}$ and derive expressions for their entries via the corresponding vectors for $\mathcal{F}_{[3]}$. Using these expressions, we continue by deriving Dehn-Sommerville-like equations for $\mathcal{F}_{[3]}$. As an intermediate step we define the subcomplex $\mathcal{K}_{[3]}$ of $\mathcal{C}$ as the closure under subface inclusion of $\mathcal{F}_{[3]}$, and derive expressions for its f- and h-vectors (cf. relations (5) and (12) with $R = [3]$). This allows us to write the Dehn-Sommerville-like equations for $\mathcal{F}_{[3]}$ in the very concise form:

$$h_{d+2-k}(\mathcal{F}_{[3]}) = h_k(\mathcal{K}_{[3]}), \qquad 0 \leq k \leq d+2.$$

In Section 4 we establish a recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F}_{[3]})$ (see Lemma 7). Our starting point is a well known relation by McMullen (cf. rel. (17)), and the expressions for the h-vector of $\partial\mathcal{Q}$ already established in the previous section. The recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F}_{[3]})$ is then used in Section 5 to prove upper bounds on the elements of $\boldsymbol{h}(\mathcal{F}_{[3]})$ and $\boldsymbol{h}(\mathcal{K}_{[3]})$. These upper bounds combined with the Dehn-Sommerville-like equations for $\mathcal{F}_{[3]}$, yield refined upper bounds for the values $h_k(\mathcal{F}_{[3]})$ when $k > \lfloor \frac{d+2}{2} \rfloor$. We end by establishing our upper bounds on the number of k-faces, $0 \leq k \leq d-1$, of the Minkowski sum of three d-polytopes by computing $\boldsymbol{f}(\mathcal{F}_{[3]})$ from $\boldsymbol{h}(\mathcal{F}_{[3]})$. At the same time we establish conditions on a subset of the elements of the vectors $\boldsymbol{f}(\mathcal{F}_R)$, $\emptyset \subset R \subseteq [3]$, that are sufficient and necessary in order for the upper bounds in the number of k-faces of $P_1+P_2+P_3$ to be tight for all k ($\mathcal{F}_R$ stands for the set of faces of $\mathcal{C}$ that have at least one vertex from each P_i for all $i \in R$).

In Section 6 we describe the constructions that establish the tightness of our upper bounds. For $d = 2$ and $d = 3$ we rely on previous results. For $d \geq 4$ we define three convex d-polytopes, whose vertices lie on three distinct moment-like d-curves, and show that the sets $\mathcal{F}_R$, $\emptyset \subset R \subseteq [3]$, associated with them satisfy the sufficient and necessary conditions mentioned above. We conclude with Section 7, where we discuss the case of four or more summands and directions for future work.

2 The Cayley trick

Recall that [3] stands for the set $\{1, 2, 3\}$, and denote by $\mathfrak{S}_j^3 := \{R \subseteq [3] \mid |R| = j\}$, the set of all subsets of [3] of cardinality j, for $1 \leq j \leq 3$. To keep the notation lean, in the rest of this paper we shall denote $\mathfrak{S}_j^3$ as $\mathfrak{S}_j$. Consider three d-polytopes P_1, P_2 and P_3 in $\mathbb{R}^d$, and choose the basis $\boldsymbol{e}_{2,1} = (0, 0)$, $\boldsymbol{e}_{2,2} = (1, 0)$, $\boldsymbol{e}_{2,3} = (0, 1)$, as the preferred affine basis of $\mathbb{R}^2$. The *Cayley embedding* of the P_i's is defined via the maps $\mu_i(\boldsymbol{x}) = (\boldsymbol{e}_{2,i}, \boldsymbol{x})$, and we denote by $\mathcal{C}$ the $(d + 2)$-polytope we get by taking the convex hull of the sets $\mathcal{V}_i = \{\mu_i(\boldsymbol{v}) \mid v \in V_i\}$, where V_i is the vertex set of P_i. This is known as the *Cayley polytope* of the P_i. Similarly, by taking appropriate affine bases we define the Cayley polytope $\mathcal{C}_R$ of all polytopes P_i, $i \in R$, where $R \in \mathfrak{S}_j$, $j = 1, 2$. These are the Cayley polytopes of all pairs of P_i's and, trivially, the P_i's themselves. Clearly, $\mathcal{C}_R \equiv P_i$, for $R \in \mathfrak{S}_1$. Moreover, $\mathcal{C} \equiv \mathcal{C}_{[3]}$.

For any $\emptyset \subset R \subseteq [3]$, let $\mathcal{V}_R$ denote the union of the sets $\mathcal{V}_i$, $i \in R$. In the sequel we shall identify $\mathcal{C}_R \subset \mathbb{R}^{d+|R|-1}$, for all $R \in \mathfrak{S}_j$, $j = 1, 2$, with the affinely isomorphic and combinatorially equivalent polytope $\mathrm{conv}(\mathcal{V}_R) \subset \mathcal{C} \subset \mathbb{R}^{d+2}$. This will allow us to study properties of these subsets of $\mathcal{C}$ by examining the corresponding Cayley polytopes which lie in lower dimensional spaces.

We shall denote by $\mathcal{F}_R$, $\emptyset \subset R \subseteq [3]$, the set of proper faces of $\mathcal{C}$, with the property that $F \in \mathcal{F}_R$ if $F \cap \mathcal{V}_i \neq \emptyset$, for all $i \in R$. In other words, $\mathcal{F}_R$ consists of all the faces of $\mathcal{C}$ that have at least one vertex from each $\mathcal{V}_i$, for all $i \in R$. Clearly, if $|R| \geq 2$, then $f_0(\mathcal{F}_R) = 0$. Moreover, if $R \in \mathfrak{S}_1$ then $\mathcal{F}_R \equiv \partial P_i$. The dimension of $\mathcal{F}_R$ is the maximum dimension of the faces in $\mathcal{F}_R$, i.e., $\dim(\mathcal{F}_R) = \max_{F \in \mathcal{F}_R} \dim(F) = d + |R| - 2$.

We call $\overline{W}$ the d-flat of $\mathbb{R}^{d+2}$:

$$\overline{W} = \{\tfrac{1}{3}\boldsymbol{e}_{2,1} + \tfrac{1}{3}\boldsymbol{e}_{2,2} + \tfrac{1}{3}\boldsymbol{e}_{2,3}\} \times \mathbb{R}^d,$$

and consider the *weighted* Minkowski sum $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$. Note that this is nothing more than $P_1 + P_2 + P_3$, scaled down by $\frac{1}{3}$, hence these two sums are combinatorially equivalent. The *Cayley trick* [HRS00] says that the intersection of $\overline{W}$ with $\mathcal{C}$ is combinatorially equivalent (isomorphic) to the weighted Minkowski sum $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$, hence also to the unweighted Minkowski sum $P_1 + P_2 + P_3$ (see also Fig. 1). Moreover, every face of $P_1 + P_2 + P_3$ is the intersection of a face of $\mathcal{F}_{[3]}$ with $\overline{W}$. This implies that:

$$f_{k-1}(P_1 + P_2 + P_3) = f_{k+1}(\mathcal{F}_{[3]}), \qquad 1 \leq k \leq d.$$

To compute the upper bounds for the number of k-faces of $P_1 + P_2 + P_3$, in the rest of the paper we assume that $\mathcal{C}$ is *"as simplicial as possible"*, i.e., all faces of $\mathcal{C}$ are simplicial except for the trivial faces of $\mathcal{C}_R$, for all $\emptyset \subset R \subseteq [3]$. Otherwise, we can employ the so called *bottom-vertex triangulation* [Mat02], where we triangulate every face of $\mathcal{C}$ except the trivial faces of $\mathcal{C}_R$ for all $\emptyset \subset R \subseteq [3]$. The resulting complex is polytopal and all of its faces are simplicial, except from the seven trivial faces above. Moreover, it has the same number of vertices as $\mathcal{C}$, while the number of its k-faces is never less than the number of k-faces of $\mathcal{C}$.

Under the *"as simplicial as possible"* assumption above, the faces in $\mathcal{F}_R$ are simplicial. We shall denote by $\mathcal{K}_R$ the *closure*, under subface inclusion, of $\mathcal{F}_R$, i.e., $\mathcal{K}_R$ contains all the faces in $\mathcal{F}_R$ and all the faces that are subfaces of faces in $\mathcal{F}_R$. It is easy to see that $\mathcal{K}_R$ does not contain any of the trivial faces of $\mathcal{C}_S$, $S \subseteq R$, and, thus, $\mathcal{K}_R$ is a pure simplicial $(d + |R| - 2)$-complex, whose facets are precisely the facets in $\mathcal{F}_R$. It is also clear that $\mathcal{F}_R \equiv \mathcal{K}_R \equiv \partial P_R$, for $R \in \mathfrak{S}_1$. Moreover, $\mathcal{K}_{[3]}$ is the boundary complex $\partial\mathcal{C}$ of the Cayley polytope $\mathcal{C}$, except for its three facets (i.e., $(d + 1)$-faces) $\mathcal{C}_R$, $R \in \mathfrak{S}_2$, and its three ridges (i.e., d-faces) P_i, $1 \leq i \leq 3$.

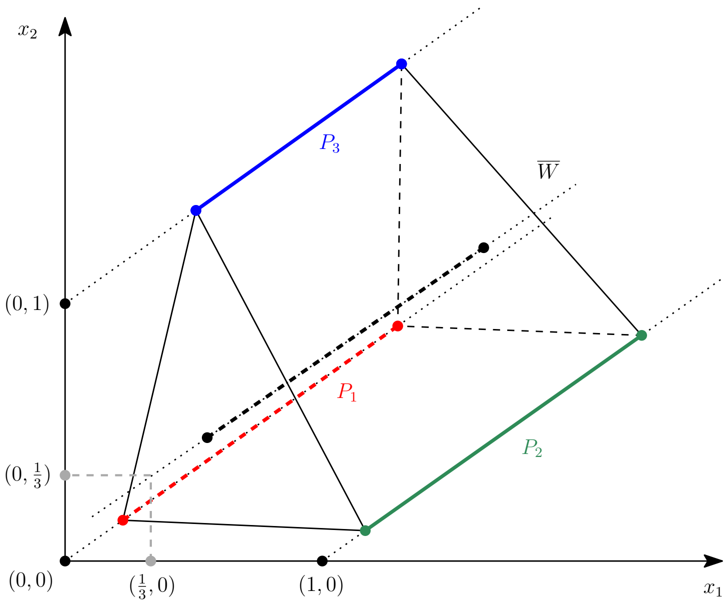

Figure 1: Schematic of the Cayley trick for three polytopes. The three polytopes P_1, P_2 and P_3 are shown in red, green and blue, respectively. The polytope $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$ is shown in black.

Consider a k-face F of $\mathcal{K}_R$, $\emptyset \subset R \subseteq [3]$. By the definition of $\mathcal{K}_R$, F is either a k-face of $\mathcal{F}_R$, or a k-face of $\mathcal{F}_S$ for some nonempty subset S of R. Hence

$$f_k(\mathcal{K}_R) = \sum_{\emptyset \subset S \subseteq R} f_k(\mathcal{F}_S), \qquad -1 \leq k \leq d + |R| - 2, \tag{5}$$

where, in order for the above equation to hold for $k = -1$, we set $f_{-1}(\mathcal{F}_R) = (-1)^{|R|-1}$. In what follows we use the convention that $f_k(\mathcal{F}_R) = 0$, for any $k < -1$ or $k > d + |R| - 2$.

3 f-vectors, h-vectors and Dehn-Sommerville-like equations

We are going to define auxiliary vertices in $\mathbb{R}^{d+2}$ not contained in $\mathcal{V}_i, i = 1, 2, 3$. For every $\emptyset \subset R \subset [3]$ we add a vertex y_R in the relative interior of $\mathcal{C}_R$ and, following [ES74], we consider the complex arising by taking successive stellar subdivisions of $\partial\mathcal{C}$ as follows:

(i) we form the complex arising from $\partial\mathcal{C}$ by taking the stellar subdivisions $\mathrm{st}(y_{\{i\}}, \mathcal{C}_{\{i\}})$ for all $1 \leq i \leq 3$, then

(ii) we form the complex arising from the one constructed in the previous step by taking the stellar subdivisions $\mathrm{st}(y_R, \mathcal{C}'_R)$ for every $R \in \mathfrak{S}_2$, where $\mathcal{C}'_R$ is the complex obtained by taking, for every $S \subset R$, the stellar subdivision of y_S over the boundary complex of $\mathcal{C}_S$.

This complex is polytopal and isomorphic to the boundary complex of a $(d+2)$-polytope which we shall denote as $\mathcal{Q}$ (see also Fig. 2). The boundary complex $\partial\mathcal{Q}$ is a simplicial $(d+1)$-sphere. The simpliciality of $\partial\mathcal{Q}$ will allow us to utilize its Denh-Sommerville equations in order to prove Dehn-Sommerville-like equations for $\mathcal{F}_{[3]}$ in the upcoming Lemma 3. We denote by $\mathcal{V} := \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \{y_R \mid \emptyset \subset R \subset [3]\}$ the vertex set of $\mathcal{Q}$.

Let us count the k-faces of $\partial\mathcal{Q}$. Suppose that F is a k-face of $\partial\mathcal{Q}$. We distinguish between the following cases depending on the number of auxiliary vertices, y_R, that F contains:

(i) F does not contain any additional auxiliary vertices. Then, it can be a k-face of any $\mathcal{F}_R, R \in \mathfrak{S}_1$, or it can be a k-face of any of the $\mathcal{F}_R, R \in \mathfrak{S}_2$, or it can be a k-face of $\mathcal{F}_{[3]}$. This gives a total of $f_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_1} f_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_2} f_k(\mathcal{F}_R)$ k-faces of $\partial\mathcal{Q}$.

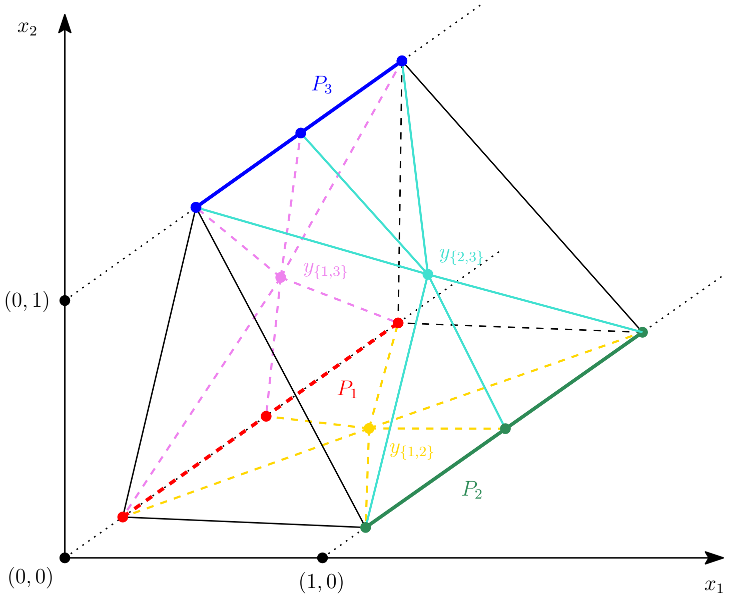

Figure 2: The $(d+2)$-polytope $\mathcal{Q}$.

(ii) F contains one auxiliary vertex. Then, it can consist of a $(k-1)$-face of:

(a) $\mathcal{F}_R, R \in \mathfrak{S}_1$ and vertex y_R, (e.g., a $(k-1)$-face of ∂P_1 and vertex $y_{\{1\}}$), or

(b) $\mathcal{F}_R, R \in \mathfrak{S}_2$ and vertex y_R, (e.g., a $(k-1)$-face of $\mathcal{F}_{\{1,2\}}$ and vertex $y_{\{1,2\}}$), or

(c) $\mathcal{F}_S, S \in \mathfrak{S}_1$ and vertex y_R, where $S \subset R \in \mathfrak{S}_2$, (e.g., a $(k-1)$-face of ∂P_1 and vertex $y_{\{1,2\}}$ or vertex $y_{\{1,3\}}$),

for a total of faces equal to:

$$\overbrace{\sum_{R\in\mathfrak{S}_1} f_{k-1}(\mathcal{F}_R)}^{\text{case (a)}} + \overbrace{\sum_{R\in\mathfrak{S}_2} f_{k-1}(\mathcal{F}_R)}^{\text{case (b)}} + \overbrace{\sum_{R\in\mathfrak{S}_2}\sum_{\emptyset\subset S\subset R} f_{k-1}(\mathcal{F}_S)}^{\text{case (c)}} = \sum_{R\in\mathfrak{S}_2} f_{k-1}(\mathcal{F}_R) + 3\sum_{R\in\mathfrak{S}_1} f_{k-1}(\mathcal{F}_R).$$

(iii) F contains two auxiliary vertices. Then, it can consist of a $(k-2)$-face of $\mathcal{F}_R, R \in \mathfrak{S}_1$ and vertices y_R and y_S, where $S \in \mathfrak{S}_2$ such that $R \subset S$, (e.g., a $(k-2)$-face of ∂P_1 and vertices $y_{\{1\}}$ and either $y_{\{1,2\}}$ or $y_{\{1,3\}}$), for a total of $2\sum_{R\in\mathfrak{S}_1} f_{k-2}(\mathcal{F}_R)$ faces.

Summing over all previous cases we obtain the following relation, for all $0 \le k \le d+1$:

$$f_k(\partial\mathcal{Q}) = f_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2}[f_k(\mathcal{F}_R) + f_{k-1}(\mathcal{F}_R)] + \sum_{R\in\mathfrak{S}_1}[f_k(\mathcal{F}_R) + 3f_{k-1}(\mathcal{F}_R) + 2f_{k-2}(\mathcal{F}_R)]. \quad (6)$$

Relation (6) also holds for $k \in \{-1, 0\}$, since, by convention, we have set $f_l(\mathcal{F}_S) = 0$ for all $l < -1$ and $\emptyset \subset S \subseteq [3]$.

Denote by $\mathcal{Y}$ a generic subset of faces of $\mathcal{C}$. $\mathcal{Y}$ will either be a subcomplex of the boundary complex $\partial\mathcal{C}$ of $\mathcal{C}$, or one of the $\mathcal{F}_R$'s. Let δ be the dimension of $\mathcal{Y}$. Then we can define the h-vector of $\mathcal{Y}$ as

$$h_k(\mathcal{Y}) = \sum_{i=0}^{\delta+1}(-1)^{k-i}\binom{\delta+1-i}{\delta+1-k} f_{i-1}(\mathcal{Y}). \quad (7)$$

Another quantity that will be heavily used in the rest of the paper is that we call the m-order g-vector of $\mathcal{Y}$, the k-th element of which is given by the following recursive formula:

$$g_k^{(m)}(\mathcal{Y}) = \begin{cases} h_k(\mathcal{Y}), & m = 0, \\ g_k^{(m-1)}(\mathcal{Y}) - g_{k-1}^{(m-1)}(\mathcal{Y}), & m > 0. \end{cases} \quad (8)$$

Observe that for $m = 0$ we get the h-vector of $\mathcal{Y}$, while for $m = 1$ we get what is typically known as the g-vector of $\mathcal{Y}$. Clearly, $\boldsymbol{g}^{(m)}(\mathcal{Y})$ is the m-order backward finite difference of $\boldsymbol{h}(\mathcal{Y})$, which suggests the following lemma (see Section A.1 of Appendix A for the proof):

Lemma 1. *For any $k, m \geq 0$, we have:*

$$g_k^{(m)}(\mathcal{Y}) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} h_{k-i}(\mathcal{Y}). \tag{9}$$

We next define the summation operator $\mathcal{S}_k(\cdot; D, \nu)$ whose action on $\mathcal{Y}$ is as follows:

$$\mathcal{S}_k(\mathcal{Y}; D, \nu) = \sum_{i=0}^{D+1} (-1)^{k-i} \binom{D+1-i}{D+1-k} f_{i-1-\nu}(\mathcal{Y}). \tag{10}$$

Regarding the action of $\mathcal{S}_k(\cdot; D, \nu)$ on $\mathcal{Y}$, it is easy to verify the following (see Section A.1 of Appendix A for the proof):

Lemma 2. *Let δ be the dimension of $\mathcal{Y}$, $\nu \geq 0$, $\delta \leq D$, and $D - \delta - \nu \geq 0$. Then for any $k \geq 0$ we have:*

$$\mathcal{S}_k(\mathcal{Y}; D, \nu) = g_{k-\nu}^{(D-\delta-\nu)}(\mathcal{Y}). \tag{11}$$

In the following lemma we relate the h-vectors of $\mathcal{F}_R$ and $\mathcal{K}_R$ with each other, and with the h-vector of $\partial\mathcal{Q}$. The last among the relations proved in the following lemma can be thought of as the analogue of the Dehn-Sommerville equations for $\mathcal{F}_{[3]}$ and $\mathcal{K}_{[3]}$.

Lemma 3. *The following relations hold:*

$$h_k(\mathcal{K}_R) = \sum_{\emptyset \subset S \subseteq R} g_k^{(|R|-|S|)}(\mathcal{F}_S), \qquad 0 \leq k \leq d + |R| - 1, \qquad \emptyset \subset R \subseteq [3]. \tag{12}$$

$$h_k(\partial\mathcal{Q}) = h_k(\mathcal{F}_{[3]}) + \sum_{R \in \mathfrak{S}_2} h_k(\mathcal{F}_R) + \sum_{R \in \mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)], \qquad 0 \leq k \leq d+2. \tag{13}$$

$$h_{d+2-k}(\mathcal{F}_{[3]}) = h_k(\mathcal{K}_{[3]}), \qquad 0 \leq k \leq d+2. \tag{14}$$

Proof. Relation (12) follows directly from the application of the summation operator $\mathcal{S}_k(\cdot; d + |R| - 2, 0)$ to relation (5). More precisely, from (5) we get, for all $0 \leq k \leq d + |R| - 1$,

$$\mathcal{S}_k(\mathcal{K}_R; d + |R| - 2, 0) = \sum_{\emptyset \subset S \subseteq R} \mathcal{S}_k(\mathcal{F}_S; d + |R| - 2, 0). \tag{15}$$

Relation (12) now immediately follows by noticing that:

- By applying Lemma 2 on the right-hand-side of (15), with $\delta \leftarrow d+|R|-2$, $D \leftarrow d+|R|-2$ and $\nu \leftarrow 0$, we get

$$\mathcal{S}_k(\mathcal{K}_R; d + |R| - 2, 0) = g_{k-0}^{((d+|R|-2)-(d+|R|-2)-0)}(\mathcal{K}_R) = h_k(\mathcal{K}_R).$$

- Similarly, by applying Lemma 2 on the left hand side of (15), with $\delta \leftarrow d + |S| - 2$, $D \leftarrow d + |R| - 2$, $\nu \leftarrow 0$, we get:

$$\mathcal{S}_k(\mathcal{F}_S; d + |R| - 2, 0) = g_{k-0}^{((d+|R|-2)-(d+|S|-2)-0)}(\mathcal{F}_S) = g_k^{(|R|-|S|)}(\mathcal{F}_S).$$

To prove (13), we apply the summation operator $\mathcal{S}_k(\cdot; d+1; 0)$ to the $(d+1)$-complex $\partial\mathcal{Q}$. Using relation (6), we get, for all $0 \leq k \leq d+2$:

$$\begin{aligned}\mathcal{S}_k(\partial\mathcal{Q}; d+1; 0) = \mathcal{S}_k(\mathcal{F}_{[3]}; d+1; 0) &+ \sum_{R\in\mathfrak{S}_2} [\mathcal{S}_k(\mathcal{F}_R; d+1; 0) + \mathcal{S}_k(\mathcal{F}_R; d+1; 1)] \\ &+ \sum_{R\in\mathfrak{S}_1} [\mathcal{S}_k(\mathcal{F}_R; d+1; 0) + 3\mathcal{S}_k(\mathcal{F}_R; d+1; 1) + 2\mathcal{S}_k(\mathcal{F}_R; d+1; 2)],\end{aligned}$$

which, using Lemma 2, gives, for all $0 \leq k \leq d+2$:

$$g_k^{(0)}(\partial\mathcal{Q}) = g_k^{(0)}(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} [g_k^{(1)}(\mathcal{F}_R) + g_{k-1}^{(0)}(\mathcal{F}_R)] + \sum_{R\in\mathfrak{S}_1} [g_k^{(2)}(\mathcal{F}_R) + 3g_{k-1}^{(1)}(\mathcal{F}_R) + 2g_{k-2}^{(0)}(\mathcal{F}_R)].$$

Relation (13) follows by expanding $\boldsymbol{g}^{(m)}(\cdot)$, $1 \leq m \leq 2$, according to Lemma 1, and gathering common terms.

To prove what we named the Dehn-Sommerville-like equations for $\mathcal{F}_{[3]}$ (cf. (14)), we replace k by $d+2-k$ in (13), to get, for all $0 \leq k \leq d+2$:

$$h_{d+2-k}(\partial\mathcal{Q}) = h_{d+2-k}(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} h_{d+2-k}(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_1} [h_{d+2-k}(\mathcal{F}_R) + h_{d+1-k}(\mathcal{F}_R)]. \tag{16}$$

Using the above relation, in conjunction with (13), the Dehn-Sommerville equations for $\partial\mathcal{Q}$ become:

$$\begin{aligned}h_{d+2-k}(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} h_{d+2-k}(\mathcal{F}_R) &+ \sum_{R\in\mathfrak{S}_1} [h_{d+2-k}(\mathcal{F}_R) + h_{d+1-k}(\mathcal{F}_R)] \\ &= h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} h_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)].\end{aligned}$$

Using the Dehn-Sommerville equations for $\mathcal{F}_R$, $R \in \mathfrak{S}_1$, as well as the Dehn-Sommerville-like equations for $\mathcal{F}_R$, $R \in \mathfrak{S}_2$ (cf. [KT12, rel. (3.10)]), we get:

$$\begin{aligned}h_{d+2-k}(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} [h_{k-1}(\mathcal{F}_R) + \sum_{\emptyset\subset S\subset R} g_{k-1}(\mathcal{F}_S)] &+ \sum_{R\in\mathfrak{S}_1} [h_{k-2}(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)] \\ = h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} h_k(\mathcal{F}_R) &+ \sum_{R\in\mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)].\end{aligned}$$

Finally, solving in terms of $h_{d+2-k}(\mathcal{F}_{[3]})$, we arrive at the following:

$$\begin{aligned}h_{d+2-k}(\mathcal{F}_{[3]}) &= h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} h_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)] \\ &\quad - \sum_{R\in\mathfrak{S}_2} [h_{k-1}(\mathcal{F}_R) + \sum_{\emptyset\subset S\subset R} g_{k-1}(\mathcal{F}_S)] - \sum_{R\in\mathfrak{S}_1} [h_{k-2}(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)] \\ &= h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} h_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)] \\ &\quad - \sum_{R\in\mathfrak{S}_2} h_{k-1}(\mathcal{F}_R) - 2\sum_{R\in\mathfrak{S}_1} g_{k-1}(\mathcal{F}_S) - \sum_{R\in\mathfrak{S}_1} [h_{k-2}(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)] \\ &= h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} [h_k(\mathcal{F}_R) - h_{k-1}(\mathcal{F}_R)] \\ &\quad + \sum_{R\in\mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R) - 2g_{k-1}(\mathcal{F}_S) - h_{k-2}(\mathcal{F}_R) - h_{k-1}(\mathcal{F}_R)]\end{aligned}$$

$$
\begin{aligned}
&= h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} g_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_1} [h_k(\mathcal{F}_R) - 2h_{k-1}(\mathcal{F}_R) + h_{k-2}(\mathcal{F}_R)] \\
&= h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} g_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_1} g_k^{(2)}(\mathcal{F}_R) \\
&= h_k(\mathcal{K}_{[3]}),
\end{aligned}
$$

where for the last equality we used relation (12) for $R \equiv [3]$. □

4 Recurrence relations

Recall that we denote by $\mathcal{V}$ the vertex set of $\partial\mathcal{Q}$ and by $\mathcal{V}_i$ the (Cayley embedding of the) vertex set of ∂P_i, $1 \le i \le 3$. Let $\mathcal{Y}/v$ denote the link of vertex v of $\mathcal{Y}$ in the simplicial complex $\mathcal{Y}$. McMullen [McM70] showed that for any d-dimensional polytope P the following relation holds:

$$
(k+1)h_{k+1}(\partial P) + (d-k)h_k(\partial P) = \sum_{v\in\mathrm{vert}(\partial P)} h_k(\partial P/v), \qquad 0 \le k \le d-1. \tag{17}
$$

Applying relation (17) to the $(d+2)$-dimensional polytope $\mathcal{Q}$, we have, for all $0 \le k \le d+1$:

$$
(k+1)h_{k+1}(\partial\mathcal{Q}) + (d+2-k)h_k(\partial\mathcal{Q}) = \sum_{v\in\mathcal{V}} h_k(\partial\mathcal{Q}/v) = \sum_{v\in\mathcal{V}_{[3]}} h_k(\partial\mathcal{Q}/v) + \sum_{\emptyset\subset R\subset[3]} h_k(\partial\mathcal{Q}/y_R), \tag{18}
$$

where we used the fact that $\mathcal{V}$ is the disjoint union of the vertex sets $\mathcal{V}_{[3]} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ and $\{y_R \mid \emptyset \subset R \subset [3]\}$.

Lemma 4. *The h-vectors of the complexes $\partial\mathcal{Q}/v$, $v \in \mathcal{V}_i$, $i = 1, 2, 3$, $\partial\mathcal{Q}/y_R$, $R \in \mathfrak{S}_1$, and $\partial\mathcal{Q}/y_R$, $R \in \mathfrak{S}_2$ are given by the following relations:*

$$
h_k(\partial Q/v) = h_k(\mathcal{K}_{[3]}/v) + \sum_{\{i\}\subseteq R\subset[3]} h_{k-1}(\mathcal{K}_R/v) + h_{k-2}(\mathcal{K}_{\{i\}}/v), \quad v \in \mathcal{V}_i, \quad i \in [3], \tag{19}
$$

$$
h_k(\partial\mathcal{Q}/y_R) = h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R), \qquad R \in \mathfrak{S}_1, \tag{20}
$$

$$
h_k(\partial\mathcal{Q}/y_R) = \sum_{\emptyset\subset S\subseteq R} h_k(\mathcal{F}_S), \qquad R \in \mathfrak{S}_2. \tag{21}
$$

Proof. We start by proving relation (19). Without loss of generality we assume that $v \in \mathcal{V}_1$; the cases $v \in \mathcal{V}_2$ and $v \in \mathcal{V}_3$ are entirely analogous.

Let F be a k-face of $\partial Q/v$. We have the following cases depending on the number of additional points y_R, $\emptyset \subset R \subset [3]$, that F contains:

(i) F does contain any additional points. Then, it is a k-face of $\mathcal{K}_{[3]}$.

(ii) F contains one additional point. Then, it can consist of a $(k-1)$-face of:

  (a) $\mathcal{K}_{\{1\}} (\equiv \partial P_1)$ and point $y_{\{1\}}$, or

  (b) $\mathcal{K}_{\{1,2\}}$, and point $y_{\{1,2\}}$, or

  (c) $\mathcal{K}_{\{1,3\}}$, and point $y_{\{1,3\}}$.

(iii) F contains two additional points. Then, it can consist of a $(k-2)$-face of $\mathcal{K}_{\{1\}}$ and points $y_{\{1\}}$ and $y_{\{1,2\}}$, or points $y_{\{1\}}$ and $y_{\{1,3\}}$.

Summing over all previous cases we obtain the following relation:

$$f_k(\partial Q/v) = \overbrace{f_k(\mathcal{K}_{[3]}/v)}^{\text{case (i)}} + \overbrace{\sum_{\{1\}\subseteq R\subset[3]} f_{k-1}(\mathcal{K}_R/v)}^{\text{case (ii)}} + \overbrace{2f_{k-2}(\mathcal{K}_{\{1\}}/v)}^{\text{case (iii)}}, \quad v \in \mathcal{V}_1. \tag{22}$$

We apply the summation operator $\mathcal{S}_k(\cdot; d, 0)$ to the d-complex $\partial Q/v$ and obtain:

$$g_k^{(0)}(\partial Q/v) = g_k^{(0)}(\mathcal{K}_{[3]}/v) + \sum_{\{1\}\subseteq R\subset[3]} g_{k-1}^{(2-|R|)}(\mathcal{K}_R/v) + 2g_{k-2}^{(0)}(\mathcal{K}_{\{1\}}/v),$$

which finally gives, for any $v \in \mathcal{V}_1$:

$$\begin{aligned}
h_k(\partial Q/v) &= h_k(\mathcal{K}_{[3]}/v) + \Big(g_{k-1}(\mathcal{K}_{\{1\}}/v) + \sum_{\{1\}\subset R\subset[3]} h_{k-1}(\mathcal{K}_R/v)\Big) + 2h_{k-2}(\mathcal{K}_{\{1\}}/v) \\
&= h_k(\mathcal{K}_{[3]}/v) + h_{k-1}(\mathcal{K}_{\{1\}}/v) - h_{k-2}(\mathcal{K}_{\{1\}}/v) + \sum_{\{1\}\subset R\subset[3]} h_{k-1}(\mathcal{K}_R/v) + 2h_{k-2}(\mathcal{K}_{\{1\}}/v) \\
&= h_k(\mathcal{K}_{[3]}/v) + \sum_{\{1\}\subseteq R\subset[3]} h_{k-1}(\mathcal{K}_R/v) + h_{k-2}(\mathcal{K}_{\{1\}}/v).
\end{aligned}$$

To prove (20) consider a k-face of $\partial Q/y_R$, $R \in \mathfrak{S}_1$. Such a face is either a k-face of $\mathcal{F}_R$, or consists of a $(k-1)$-face of $\mathcal{F}_R$ and point y_S for any $S \in \mathfrak{S}_2$ such that $S \supset R$. Note that there exactly two such points y_S. Hence:

$$f_k(\partial \mathcal{Q}/y_R) = f_k(\mathcal{F}_R) + 2f_{k-1}(\mathcal{F}_R), \qquad R \in \mathfrak{S}_1. \tag{23}$$

Applying the summation operator $\mathcal{S}_k(\cdot; d, 0)$ to the simplicial d-complex $\partial \mathcal{Q}/y_R$, $R \in \mathfrak{S}_1$, and using relation (23) and Lemma 2, we get, for any $R \in \mathfrak{S}_1$:

$$\begin{array}{rclcl}
h_k(\partial \mathcal{Q}/y_R) &=& g_k^{(0)}(\partial \mathcal{Q}/y_R) &=& \mathcal{S}_k(\partial \mathcal{Q}/y_R; d, 0) \\
&=& \mathcal{S}_k(\mathcal{F}_R; d, 0) + 2\mathcal{S}_k(\mathcal{F}_R; d, 1) &=& g_k^{(1)}(\mathcal{F}_R) + 2g_{k-1}^{(0)}(\mathcal{F}_R) \\
&=& h_k(\mathcal{F}_R) - h_{k-1}(\mathcal{F}_R) + 2h_{k-1}(\mathcal{F}_R) &=& h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R).
\end{array}$$

To prove (21) consider a k-face of $\partial \mathcal{Q}/y_R$, $R \in \mathfrak{S}_2$. This is either a k-face of $\mathcal{F}_S$, for any $\emptyset \subset S \subseteq R$, or consists of a $(k-1)$-face of $\mathcal{F}_S$ and point y_S for any $\emptyset \subset S \subset R$. Hence, for any $R \in \mathfrak{S}_2$, we have:

$$f_k(\mathcal{F}_R/y_R) = \sum_{\emptyset\subset S\subseteq R} f_k(\mathcal{F}_S) + \sum_{\emptyset\subset S\subset R} f_{k-1}(\mathcal{F}_S) = f_k(\mathcal{F}_R) + \sum_{\emptyset\subset S\subset R} [f_k(\mathcal{F}_S) + f_{k-1}(\mathcal{F}_S)]. \tag{24}$$

Applying the summation operator $\mathcal{S}_k(\cdot; d, 0)$ to the d-dimensional complex $\partial \mathcal{Q}/y_R$, $R \in \mathfrak{S}_2$, and using relation (24), along with Lemma 2, we get, for any $R \in \mathfrak{S}_2$:

$$\begin{aligned}
h_k(\partial \mathcal{Q}/y_R) = \mathcal{S}_k(\partial \mathcal{Q}/y_R; d, 0) &= \mathcal{S}_k(\mathcal{F}_R; d, 0) + \sum_{\emptyset\subset S\subset R} [\mathcal{S}_k(\mathcal{F}_S; d, 0) + \mathcal{S}_k(\mathcal{F}_S; d, 1)] \\
&= g_k^{(0)}(\mathcal{F}_R) + \sum_{\emptyset\subset S\subset R} [g_k^{(1)}(\mathcal{F}_S) + g_{k-1}^{(0)}(\mathcal{F}_S)] \\
&= h_k(\mathcal{F}_R) + \sum_{\emptyset\subset S\subset R} h_k(\mathcal{F}_S) \\
&= \sum_{\emptyset\subset S\subseteq R} h_k(\mathcal{F}_S).
\end{aligned}$$

□

The following two lemmas are essential in the proof of the upcoming recurrence relation in Lemma 7.

Lemma 5. *The following relation holds, for all* $0 \le k \le d+1$*:*

$$(k+1)h_{k+1}(\mathcal{F}_{[3]}) + (d+2-k)h_k(\mathcal{F}_{[3]}) = \sum_{\emptyset \subset R \subseteq [3]} (-1)^{3-|R|} \sum_{v \in \mathcal{V}_R} g_k^{(3-|R|)}(\mathcal{K}_R/v). \tag{25}$$

Sketch of proof. The complete proof may be found in Section A.2 of Appendix A. Our starting point is relation (18). We first substitute $h_k(\partial Q)$ and $h_{k+1}(\partial Q)$ on the left-hand side of (18) with their relevant expressions from (13). We then group the terms so that we get a sum of:

(i) the left-hand side of (25),

(ii) $(k+1)h_{k+1}(\mathcal{F}_R) + (d+1-k)h_k(\mathcal{F}_R)$, $R \in \mathfrak{S}_2$

(iii) $(k+1)h_{k+1}(\mathcal{F}_R) + (d-k)h_k(\mathcal{F}_R)$ and $kh_k(\mathcal{F}_R) + (d-k-1)h_{k-1}(\mathcal{F}_R)$ with $R \in \mathfrak{S}_1$

(iv) additional terms.

As will be described below, the intuition behind this grouping is to substitute the terms in (ii) and (iii) by sums involving quantities of the form $g_k^{(m)}(\mathcal{K}_S/v)$. These quantities will be grouped with the terms obtained from a similar expansion of the term $h_k(\partial\mathcal{Q}/v)$ appearing in the right-hand side of (18), yielding the right-hand side of (25).

In the proof of [KT12, Lemma 3.2], the sum in item (ii) above is shown to be equal[2] to $\sum_{i\in R}\sum_{v\in\mathcal{V}_i}[h_k(\mathcal{K}_R/v) - g_k(\mathcal{K}_{\{i\}}/v)]$. For (iii) we use (17) combined with the fact that for any $R \in \mathfrak{S}_1$, $\mathcal{F}_R \equiv \partial P_R$. On the right-hand side of (18) we substitute $h_k(\partial Q/v)$ and $h_k(\partial Q/y_R)$ using the relations in Lemma 4. Finally, we equate our expansions of the left- and right-hand side of (18) and notice that the terms in (iv) and the expressions for $h_k(\partial Q/y_R)$ cancel-out. Recalling that $g_k = h_k - h_{k-1}$ and $g_k^{(2)} = h_k - 2h_{k-1} + h_{k-2}$, we appropriately regroup the remaining terms to obtain the desired expression. ☐

Lemma 6. *The following relation holds, for all* $0 \le k \le d+1$*:*

$$\sum_{\emptyset \subset R \subseteq [3]} (-1)^{3-|R|} \sum_{v \in \mathcal{V}_R} g_k^{(3-|R|)}(\mathcal{K}_R/v) \le \sum_{\emptyset \subset R \subseteq [3]} (-1)^{3-|R|} \sum_{v \in \mathcal{V}_R} g_k^{(3-|R|)}(\mathcal{K}_R). \tag{26}$$

Proof. Let us first observe that, by rearranging terms, we can rewrite relation (26) as follows:

$$\sum_{i=1}^{3}\sum_{v\in\mathcal{V}_i}\sum_{\{i\}\subseteq R\subseteq[3]} (-1)^{3-|R|} g_k^{(3-|R|)}(\mathcal{K}_R/v) \le \sum_{i=1}^{3}\sum_{v\in\mathcal{V}_i}\sum_{\{i\}\subseteq R\subseteq[3]} (-1)^{3-|R|} g_k^{(3-|R|)}(\mathcal{K}_R). \tag{27}$$

Clearly, to show that relation (27) holds, it suffices to prove that:

$$\sum_{\{i\}\subseteq R\subseteq[3]} (-1)^{3-|R|} g_k^{(3-|R|)}(\mathcal{K}_R/v) \le \sum_{\{i\}\subseteq R\subseteq[3]} (-1)^{3-|R|} g_k^{(3-|R|)}(\mathcal{K}_R), \quad v \in \mathcal{V}_i, \quad i \in [3]. \tag{28}$$

In the rest of the proof we shall prove relation (28) for $i = 1$ and for any $v \in \mathcal{V}_1$. The cases $i = 2$ and $i = 3$ are entirely similar.

Fix a vertex $v \in \mathcal{V}_1$. Let $\partial\mathcal{Q}'$ be the polytopal $(d+1)$-complex that we get by removing from $\partial\mathcal{Q}$ the faces that are incident to $y_{\{2,3\}}$ (see Fig. 3(left)). It is straightforward to see that: (1) the stars of v in $\mathcal{Q}$ and $\partial\mathcal{Q}'$ coincide (the faces incident to $y_{\{2,3\}}$ contain vertices from $\mathcal{V}_{\{2,3\}} \cup \{y_{\{2\}}, y_{\{3\}}\}$ only), and (2) $\partial\mathcal{Q}'$ is shellable. To verify the latter consider a shelling $\mathsf{S}(\partial\mathcal{Q})$ of $\partial\mathcal{Q}$ that shells the star of $y_{\{2,3\}}$ in $\partial\mathcal{Q}$ last; the shelling order that we get by removing

[2]The expression in [KT12] is written differently; it is equivalent, however, to the expression stated here.

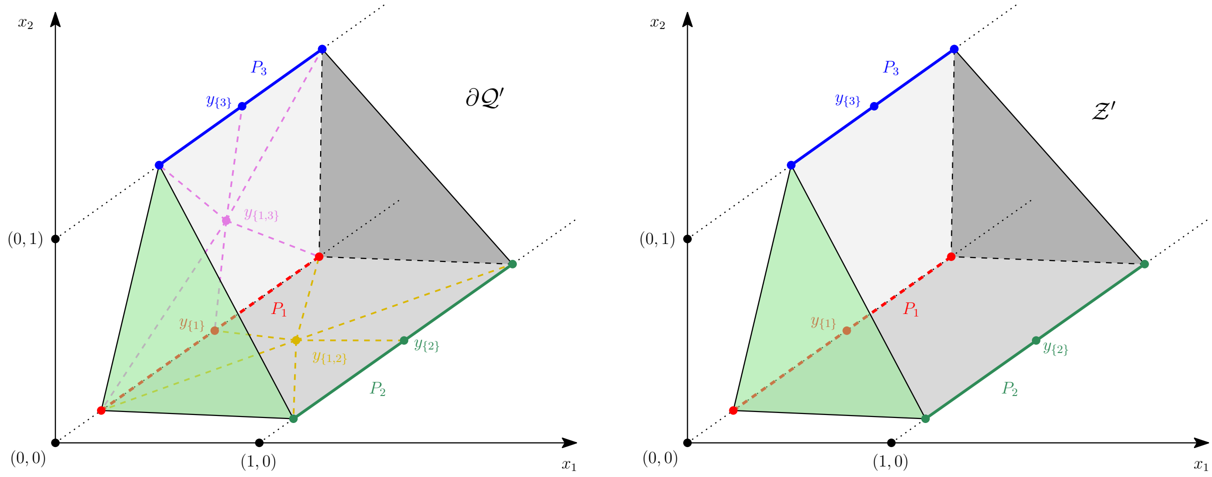

Figure 3: Left: the $(d+1)$-complex $\partial\mathcal{Q}'$ that we get from $\partial\mathcal{Q}$ be removing all faces incident to $y_{\{2,3\}}$. Right: the $(d+1)$-complex $\mathcal{Z}'$ that we get from the Cayley polytope $\mathcal{C}_{[3]}$ of P_1, P_2 and P_3, after we: (i) have performed stellar subdivisions using the vertices $y_{\{1\}}, y_{\{2\}}$ and $y_{\{3\}}$ (which yields the $(d+1)$-polytope $\mathcal{Z}$), and (ii) have removed the facet $\mathcal{Q}_{\{2,3\}}$ from $\mathcal{Z}$.

from $\mathsf{S}(\partial\mathcal{Q})$ the facets that are incident to $y_{\{2,3\}}$ is clearly a shelling order for $\partial\mathcal{Q}'$. Let $\mathcal{S}_R$, $R \in \{\{1,2\},\{1,3\}\}$, be the star of y_R in $\partial\mathcal{Q}'$ (which actually coincides with the star of y_R in $\partial\mathcal{Q}$). Let $\mathcal{X}$ denote the set of faces of $\partial\mathcal{Q}'$ that are either faces in $\mathcal{S}_{\{1,2\}}$ or faces in $\mathcal{S}_{\{1,3\}}$, and let $\mathcal{G}$ denote the set of faces of $\partial\mathcal{Q}'$ that are either faces in $\mathcal{F}_{[3]}$ or faces in $\mathcal{F}_{\{2,3\}}$. Notice that the sets $\mathcal{X}$ and $\mathcal{G}$ form a disjoint union of the faces in $\partial\mathcal{Q}'$, which implies that:

$$f_k(\partial\mathcal{Q}') = f_k(\mathcal{X}) + f_k(\mathcal{G}), \qquad -1 \le k \le d+1. \tag{29}$$

Notice that $\mathcal{X}$ is a $(d+1)$-complex, whereas $\mathcal{G}$ is a set of faces with maximal dimension $d+1$. By applying the summation operator $\mathbb{S}_k(\cdot; d+1, 0)$ to (29), we immediately get the corresponding h-vector relation:

$$h_k(\partial\mathcal{Q}') = h_k(\mathcal{X}) + h_k(\mathcal{G}), \qquad 0 \le k \le d+2. \tag{30}$$

We claim that there exists a specific shelling $\mathsf{S}(\partial\mathcal{Q}')$ of $\partial\mathcal{Q}'$, which actually is an initial segment of a shelling of $\partial\mathcal{Q}$ that shells the star of $y_{\{2,3\}}$ last, with the property that the corresponding shelling order has the facets in $\mathcal{X}$ before the facets in $\mathcal{G}$. We will postpone the proof of this claim, and we will assume for now that the claim holds true. Consider the specific shelling of $\partial\mathcal{Q}'$ just mentioned, and notice that the facets in $\mathcal{G}$ are actually the facets in $\mathcal{F}_{[3]}$. The existence of this particular shelling $\mathsf{S}(\partial\mathcal{Q}')$ also implies that $\mathcal{X}$ is shellable, and the shelling $\mathsf{S}(\mathcal{X})$ of $\mathcal{X}$ induced by $\mathsf{S}(\partial\mathcal{Q}')$ coincides with $\mathsf{S}(\partial\mathcal{Q}')$ as long as it visits the facets of $\mathcal{X}$. As a result of this, and as long as we shell $\mathcal{X}$, we get a contribution of $+1$ to both $h_k(\partial\mathcal{Q}')$ and $h_k(\mathcal{X})$ for every restriction of $\mathsf{S}(\partial\mathcal{Q}')$ of size k. After the shelling $\mathsf{S}(\partial\mathcal{Q}')$ has left $\mathcal{X}$, a restriction of size k for $\mathsf{S}(\partial\mathcal{Q}')$ contributes $+1$ to $h_k(\partial\mathcal{Q}')$, does not contribute to $h_k(\mathcal{X})$ ($\mathcal{X}$ has already been fully constructed), and, thus, by relation (30), contributes $+1$ to $h_k(\mathcal{G})$. In other words, for this particular shelling $\mathsf{S}(\partial\mathcal{Q}')$ of $\partial\mathcal{Q}'$, $h_k(\mathcal{G})$ counts the number of restrictions of size k that correspond to the facets of $\partial\mathcal{Q}'$ that are also facets of $\mathcal{G}$ (and, of course, of $\mathcal{F}_{[3]}$).

The same argumentation can be applied to the links of vertices $v \in \mathcal{V}_1$: $\partial\mathcal{Q}'/v$ can be seen as the disjoint union of the sets $\mathcal{X}/v$ and $\mathcal{G}/v$, while the particular shelling $\mathsf{S}(\partial\mathcal{Q}')$ of $\partial\mathcal{Q}'$ that shells $\mathcal{X}$ first, induces a particular shelling $\mathsf{S}(\partial\mathcal{Q}'/v)$ for $\partial\mathcal{Q}'/v$ that shells the facets of $\partial\mathcal{Q}'/v$ in $\mathcal{X}/v$ first. From these observations we immediately arrive at the following h-vector relation for $\partial\mathcal{Q}'/v$, $\mathcal{X}/v$ and $\mathcal{G}/v$:

$$h_k(\partial\mathcal{Q}'/v) = h_k(\mathcal{X}/v) + h_k(\mathcal{G}/v), \qquad 0 \le k \le d+1, \tag{31}$$

from which we argue, as above, that $h_k(\mathcal{G}/v)$ counts the number of restrictions of size k for $\mathsf{S}(\partial\mathcal{Q}'/v)$ that correspond to the facets of $\partial\mathcal{Q}'/v$ that are also facets of $\mathcal{G}/v$ (or $\mathcal{F}_{[3]}/v$).

Let us now consider the dual graph $G^\Delta(\partial\mathcal{Q})$ of $\partial\mathcal{Q}$, oriented according to the shelling $\mathsf{S}(\partial\mathcal{Q})$, as well as the dual graph $G^\Delta(\partial\mathcal{Q}/v)$ of $\partial\mathcal{Q}/v$, also oriented according to the shelling $\mathsf{S}(\partial\mathcal{Q}/v)$. We will denote by $\mathcal{V}^\Delta(\mathcal{Y})$ the subset of vertices of $G^\Delta(\partial\mathcal{Q})$ that are the duals of the facets in $\partial\mathcal{Q}$ that belong to $\mathcal{Y}$, where $\mathcal{Y}$ stands for a subset of the set of faces of $\partial\mathcal{Q}$.

Since $\mathsf{S}(\partial\mathcal{Q}/v)$ is induced from $\mathsf{S}(\partial\mathcal{Q})$, $G^\Delta(\partial\mathcal{Q}/v)$ is isomorphic to the subgraph of $G^\Delta(\partial\mathcal{Q})$ defined over $\mathcal{V}^\Delta(\mathrm{star}(v,\partial\mathcal{Q}))$. Moreover, $h_k(\partial\mathcal{Q})$ counts the number of vertices of $\mathcal{V}^\Delta(\partial\mathcal{Q})$ with in-degree equal to k, while $h_k(\mathcal{G})$ counts the number of vertices of $\mathcal{V}^\Delta(\mathcal{G})$ of in-degree k in $G^\Delta(\partial\mathcal{Q})$ (for the particular shelling $\mathsf{S}(\partial\mathcal{Q})$ of $\partial\mathcal{Q}$ that we have chosen). Consequently, $h_k(\mathcal{G})$ counts the number of vertices of $\mathcal{V}^\Delta(\mathcal{G})$ of in-degree k in $G^\Delta(\partial\mathcal{Q})$; in an analogous manner, we can conclude that $h_k(\mathcal{G}/v)$ counts the number of vertices of $\mathcal{V}^\Delta(\mathrm{star}(v,\mathcal{G}))$ with in-degree k in $G^\Delta(\partial\mathcal{Q}/v)$. Since, however, $G^\Delta(\partial\mathcal{Q}/v)$ is the subgraph of $G^\Delta(\partial\mathcal{Q})$ that corresponds to the face v^Δ of $G^\Delta(\partial\mathcal{Q})$, the number of vertices of $\mathcal{V}^\Delta(\mathrm{star}(v,\mathcal{G}))$ with in-degree k cannot exceed the number of vertices of $\mathcal{V}^\Delta(\mathcal{G})$ with in-degree k. Hence,

$$h_k(\mathcal{G}/v) \leq h_k(\mathcal{G}), \qquad 0 \leq k \leq d+2. \tag{32}$$

On the other hand, recall that $\mathcal{G}$ is the disjoint union of $\mathcal{F}_{[3]}$ and $\mathcal{F}_{\{2,3\}}$. Using expressions (5), in conjunction with the fact that $\mathcal{F}_S \equiv \mathcal{K}_S$ for $S \in \mathfrak{S}_1$, we have, for all $-1 \leq k \leq d+1$:

$$\begin{aligned}
f_k(\mathcal{G}) &= f_k(\mathcal{F}_{[3]}) + f_k(\mathcal{F}_{\{2,3\}}) \\
&= \overbrace{f_k(\mathcal{K}_{[3]}) - \sum_{R\in\mathfrak{S}_2} f_k(\mathcal{F}_R) - \sum_{R\in\mathfrak{S}_1} f_k(\mathcal{K}_R)}^{f_k(\mathcal{F}_{[3]})} + \overbrace{f_k(\mathcal{K}_{\{2,3\}}) - \sum_{i\in\{2,3\}} f_k(\mathcal{K}_{\{i\}})}^{f_k(\mathcal{F}_{\{2,3\}}} \\
&= f_k(\mathcal{K}_{[3]}) - \sum_{R\in\mathfrak{S}_2}\Big[f_k(\mathcal{K}_R) - \sum_{i\in R} f_k(\mathcal{K}_{\{i\}})\Big] - \sum_{R\in\mathfrak{S}_1} f_k(\mathcal{K}_R) + f_k(\mathcal{K}_{\{2,3\}}) - \sum_{i\in\{2,3\}} f_k(\mathcal{K}_{\{i\}}) \\
&= f_k(\mathcal{K}_{[3]}) - \sum_{R\in\mathfrak{S}_2} f_k(\mathcal{K}_R) + 2\sum_{R\in\mathfrak{S}_1} f_k(\mathcal{K}_R) - \sum_{R\in\mathfrak{S}_1} f_k(\mathcal{K}_R) + f_k(\mathcal{K}_{\{2,3\}}) - \sum_{i\in\{2,3\}} f_k(\mathcal{K}_{\{i\}}) \\
&= f_k(\mathcal{K}_{[3]}) - \sum_{\{1\}\subset R\subset[3]} f_k(\mathcal{K}_R) + f_k(\mathcal{K}_{\{1\}}) \\
&= \sum_{\{1\}\subseteq R\subseteq[3]} (-1)^{3-|R|} f_k(\mathcal{K}_R).
\end{aligned} \tag{33}$$

By a similar argument, we can arrive that the following expression for $f_k(\mathcal{G}/v)$:

$$f_k(\mathcal{G}/v) = \sum_{\{1\}\subseteq R\subseteq[3]} (-1)^{3-|R|} f_k(\mathcal{K}_R/v), \qquad -1 \leq k \leq d. \tag{34}$$

By applying the summation operators $\mathcal{S}_k(\cdot; d+1, 0)$ and $\mathcal{S}_k(\cdot; d, 0)$ to relations (33) and (34), respectively, we get the corresponding h-vector relations:

$$\begin{aligned}
h_k(\mathcal{G}) &= \sum_{\{1\}\subseteq R\subseteq[3]} (-1)^{3-|R|} g_k^{(3-|R|)}(\mathcal{K}_R), \qquad 0 \leq k \leq d+2, \\
h_k(\mathcal{G}/v) &= \sum_{\{1\}\subseteq R\subseteq[3]} (-1)^{3-|R|} g_k^{(3-|R|)}(\mathcal{K}_R/v), \qquad 0 \leq k \leq d+1.
\end{aligned} \tag{35}$$

Relation (28) (for $i = 1$) follows by substituting the expressions for $h_k(\mathcal{G})$ and $h_k(\mathcal{G}/v)$ from (35) in (32).

To finish our proof, it remains to establish our claim that there exists a specific shelling $\mathsf{S}(\partial\mathcal{Q}')$ of $\partial\mathcal{Q}'$ with the property that the facets of $\mathcal{X}$ appear in the shelling before the facets of

$\mathcal{G}$. Let us start with some definitions: we denote by $\mathcal{Z}$ the $(d+1)$-complex we get by performing the stellar subdivisions on $\mathcal{C}_{[3]}$ using the vertices y_R, $R \in \mathfrak{S}_1$ (see also Fig. 3(right)), and by $\mathcal{Q}_R$, $R \in \mathfrak{S}_2$ the $(d+1)$-complex that we get by performing stellar subdivisions on the non-simplicial proper faces of $\mathcal{C}_R$, namely the faces $\mathcal{C}_S$, $\emptyset \subset S \subset R$. Notice that $\mathcal{Q}_R$, $R \in \mathfrak{S}_2$, is nothing but a facet of $\mathcal{Z}$, while $\partial\mathcal{Q}_R$ is actually the link of y_R in $\partial\mathcal{Q}$. In fact, we can separate the facets of $\mathcal{Z}$ in two categories; they are either (1) facets of the form $\mathcal{Q}_R$, $R \in \mathfrak{S}_2$, which are non-simplicial, or (2) facets in $\mathcal{G}$ (or $\mathcal{F}_{[3]}$), which are simplicial. Moreover, notice that $\text{star}(y_R, \mathcal{Z})$, $R \in \mathfrak{S}_1$, consists of the faces belonging to the two facets $\mathcal{Q}_S$, $R \subset S \subset [3]$ of $\mathcal{Z}$. Since stellar subdivisions produce polytopal complexes [ES74], $\mathcal{Z}$ is polytopal and, thus, shellable. In fact, there exists a particular (line) shelling $\mathsf{S}(\mathcal{Z})$ of $\mathcal{Z}$ in which the facets of $\text{star}(y_{\{1\}}, \mathcal{Z})$ appear first, while $\mathcal{Q}_{\{2,3\}}$ is the last facet in $\mathsf{S}(\mathcal{Z})$. More precisely, for this particular shelling of $\mathcal{Z}$, the two facets $\mathcal{Q}_{\{1,2\}}$ and $\mathcal{Q}_{\{1,3\}}$ appear first, followed by the facets in $\mathcal{G}$, which, in turn, are followed by the facet $\mathcal{Q}_{\{2,3\}}$.

Let us call $\mathcal{Z}'$ the $(d+1)$-complex we get by removing $\mathcal{Q}_{\{2,3\}}$ from $\mathcal{Z}$. The complex $\mathcal{Z}'$ is shellable (it follows from the fact that $\mathsf{S}(\mathcal{Z})$ has $\mathcal{Q}_{\{2,3\}}$ as its last facet), while the particular line shelling $\mathsf{S}(\mathcal{Z})$ of $\mathcal{Z}$ described above, yields a shelling $\mathsf{S}(\mathcal{Z}')$ for $\mathcal{Z}'$ in which the facets $\mathcal{Q}_{\{1,2\}}$ and $\mathcal{Q}_{\{1,3\}}$ appear first, followed by the facets in $\mathcal{G}$. Notice that if we perform stellar subdivisions on the two non-simplicial facets $\mathcal{Q}_{\{1,2\}}$ and $\mathcal{Q}_{\{1,3\}}$ of $\mathcal{Z}'$ (using the vertices $y_{\{1,2\}}$ and $y_{\{1,3\}}$), we arrive at the simplicial $(d+1)$-complex $\partial\mathcal{Q}'$ described earlier. Furthermore, from the particular shelling $\mathsf{S}(\mathcal{Z}')$ of $\mathcal{Z}'$ described above, we may obtain the sought-for shelling for $\partial\mathcal{Q}'$ that shells $\mathcal{X}$ first and $\mathcal{G}$ last. To see this, notice that given any shelling order for ∂P_i, $i = 1, 2, 3$, we may construct a shelling for $\mathcal{Q}_R$, $R \in \{\{1,2\}, \{1,3\}\}$, that: (1) shells $\text{st}(y_{\{1\}}, \mathcal{Q}_R)$ first, (2) shells $\text{st}(y_{R\setminus\{i\}}, \mathcal{Q}_R)$ last, and (3) the shelling order of the facets in both stars is the order implied by the shellings of the boundary complexes ∂P_i and $\partial P_{R\setminus\{i\}}$. This implies that if we choose shelling orders for $\partial\mathcal{Q}_{\{1,2\}}$ and $\partial\mathcal{Q}_{\{1,3\}}$ that respect a common shelling order for ∂P_1, we can replace the facets $\mathcal{Q}_{\{1,2\}}$ and $\mathcal{Q}_{\{1,3\}}$ in $\mathsf{S}(\mathcal{Z}')$ by the facets in $\text{star}(y_{\{1,2\}}, \partial\mathcal{Q}')$ and $\text{star}(y_{\{1,3\}}, \partial\mathcal{Q}')$, respectively, (the shelling orders of $\partial\mathcal{Q}_{\{1,2\}}$ and $\partial\mathcal{Q}_{\{1,3\}}$ are "inherited" in the shelling orders for $\text{star}(y_{\{1,2\}}, \partial\mathcal{Q}')$ and $\text{star}(y_{\{1,3\}}, \partial\mathcal{Q}')$) and arrive at a shelling order for $\partial\mathcal{Q}'$ with the desired property. □

Using inequality (26) in Lemma 6, we arrive at the following recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F}_{[3]})$; its proof may be found in Section A.2 in Appendix A.

Lemma 7. *For all $0 \leq k \leq d+1$, we have:*

$$h_{k+1}(\mathcal{F}_{[3]}) \leq \frac{n_{[3]} - d - 2 + k}{k+1} h_k(\mathcal{F}_{[3]}) + \sum_{i=1}^{3} \frac{n_i}{k+1} g_k(\mathcal{F}_{[3]\setminus\{i\}}). \tag{36}$$

Sketch of proof. Using Lemma 6, we can bound the left hand side of relation (25) by the right hand side of relation (26), which involves g-vectors, or various orders, of the complexes $\mathcal{K}_R$, where $\emptyset \subset R \subseteq [3]$. These can be substituted by their equal values from relation (12) with $R = [3]$ and for all $R \in \mathfrak{S}_2$. This gives an inequality involving h-vectors and g-vectors of $\mathcal{F}_{[3]}$ and $\mathcal{F}_R$, $R \in \mathfrak{S}_2$, which simplifies to relation (36). □

5 Upper bounds

In this section we establish upper bounds for the number of $(k+2)$-faces of $\mathcal{F}_{[3]}$, $0 \leq k \leq d-1$, which immediately yield upper bounds for the number of k-faces of $P_1 + P_2 + P_3$. Our starting point is the recurrence relation (36). We shall first prove a few lemmas that establish bounds for the g-vector of $\mathcal{F}_R$, $R \in \mathfrak{S}_2$, and the h-vectors of $\mathcal{F}_{[3]}$ and $\mathcal{K}_{[3]}$.

Lemma 8. *Let R be a non-empty subset of $[3]$ of cardinality 2. Then, for all $0 \leq k \leq d+2$, we have:*
$$g_k(\mathcal{F}_R) \leq \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\binom{n_S - d - 3 + k}{k}. \tag{37}$$
Equality holds for some k, where $0 \leq k \leq \lfloor\frac{d+1}{2}\rfloor$, if and only if $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\binom{n_S}{l}$, for all $0 \leq l \leq k$.

Proof. The bound clearly holds, as equality, for $k = 0$. For $k \geq 1$, from [KT12, Lemma 3.2] we have:
$$h_k(\mathcal{F}_R) \leq \tfrac{n_R-d-2+k}{k} h_{k-1}(\mathcal{F}_R) + \sum_{\emptyset\subset S\subset R} \tfrac{n_{R\setminus S}}{k} g_{k-1}(\mathcal{F}_S). \tag{38}$$
Subtracting $h_{k-1}(\mathcal{F}_R)$ from both sides of (38) we get:
$$g_k(\mathcal{F}_R) \leq \tfrac{n_R-d-2}{k} h_{k-1}(\mathcal{F}_R) + \sum_{\emptyset\subset S\subset R} \tfrac{n_{R\setminus S}}{k} g_{k-1}(\mathcal{F}_S). \tag{39}$$
Using now the upper bounds for $h_{k-1}(\mathcal{F}_R)$, $g_{k-1}(\mathcal{F}_S)$, $\emptyset \subset S \subset R$, and noting that $n_R - d - 2 \geq 2(d+1) - d - 2 = d > 0$, we deduce, for any $k \geq 1$:
$$\begin{aligned}
g_k(\mathcal{F}_R) &\leq \tfrac{n_R-d-2}{k} \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\tbinom{n_S-d-3+k}{k-1} + \sum_{\emptyset\subset S\subset R} \tfrac{n_{R\setminus S}}{k}\tbinom{n_S-d-3+k}{k-1} \\
&= \tfrac{n_R-d-2}{k}\tbinom{n_R-d-3+k}{k-1} - \sum_{\emptyset\subset S\subset R} \tfrac{n_R-d-2}{k}\tbinom{n_S-d-3+k}{k-1} + \sum_{\emptyset\subset S\subset R} \tfrac{n_{R\setminus S}}{k}\tbinom{n_S-d-3+k}{k-1} \\
&= \tfrac{n_R-d-2+k}{k}\tbinom{n_R-d-3+k}{k-1} - \tbinom{n_R-d-3+k}{k-1} - \sum_{\emptyset\subset S\subset R} \tfrac{n_R-d-2-n_{R\setminus S}}{k}\tbinom{n_S-d-3+k}{k-1} \\
&= \tbinom{n_R-d-2+k}{k} - \tbinom{n_R-d-3+k}{k-1} - \sum_{\emptyset\subset S\subset R} \tfrac{n_S-d-2}{k}\tbinom{n_S-d-3+k}{k-1} \\
&= \tbinom{n_R-d-3+k}{k} - \sum_{\emptyset\subset S\subset R} \left[\tfrac{n_S-d-2+k}{k}\tbinom{n_S-d-3+k}{k-1} - \tbinom{n_S-d-3+k}{k-1}\right] \\
&= \tbinom{n_R-d-3+k}{k} - \sum_{\emptyset\subset S\subset R} \left[\tbinom{n_S-d-2+k}{k} - \tbinom{n_S-d-3+k}{k-1}\right] \\
&= \tbinom{n_R-d-3+k}{k} - \sum_{\emptyset\subset S\subset R} \tbinom{n_S-d-3+k}{k} \\
&= \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\tbinom{n_S-d-3+k}{k}.
\end{aligned}$$

We focus now on the equality claim. Suppose first that $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\binom{n_S}{l}$, for all $0 \leq l \leq k$. Then, by [KT12, Lemma 3.3], $h_\lambda(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\binom{n_S-d-2+\lambda}{\lambda}$, for $\lambda = k-1, k$, which gives:
$$\begin{aligned}
g_k(\mathcal{F}_R) = h_k(\mathcal{F}_R) - h_{k-1}(\mathcal{F}_R) &= \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\tbinom{n_S-d-2+k}{k} - \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\tbinom{n_S-d-2+k-1}{k-1} \\
&= \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\left[\tbinom{n_S-d-2+k}{k} - \tbinom{n_S-d-2+k-1}{k-1}\right] = \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\tbinom{n_S-d-3+k}{k}.
\end{aligned}$$

Suppose now that $g_k(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\binom{n_S-d-3+k}{k}$. By relation (39), we conclude that $h_{k-1}(\mathcal{F}_R)$ must be equal to its upper bound (cf. [KT12, Lemma 3.3]), since, otherwise, $g_k(\mathcal{F}_R)$

would not be maximal, which contradicts our assumption on the value of $g_k(\mathcal{F}_R)$. This gives:

$$h_k(\mathcal{F}_R) = g_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\tbinom{n_S-d-3+k}{k} + \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\tbinom{n_S-d-2+k-1}{k-1}$$
$$= \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\left[\tbinom{n_S-d-2+k-1}{k} + \tbinom{n_S-d-2+k-1}{k-1}\right] = \sum_{\emptyset\subset S\subseteq R} (-1)^{|S|}\tbinom{n_S-d-2+k}{k}.$$

Now the fact that $h_k(\mathcal{F}_R)$ is maximal, implies that $h_l(\mathcal{F}_R)$ must be equal to its maximal value for all $0 \leq l < k$. To see this suppose that $h_l(\mathcal{F}_R)$ is not maximal for some l, with $0 \leq l < k$, and among all such l choose the largest one. Then, Lemmas 3.2 and 3.3 in [KT12] imply that $h_{l+1}(\mathcal{F}_R)$ cannot be maximal, which contradicts the maximality of l. Summarizing, we deduce that if $g_k(\mathcal{F}_R)$ is equal to its upper bound in (37), so is $h_l(\mathcal{F}_R)$ for all $0 \leq l \leq k$. By Lemma 3.3 in [KT12], this implies that $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\tbinom{n_S}{l}$, for all $0 \leq l \leq k$. □

Lemma 9. *For all $0 \leq k \leq d+2$, we have:*

$$h_k(\mathcal{F}_{[3]}) \leq \sum_{\emptyset\subset S\subseteq[3]} (-1)^{3-|S|}\binom{n_S-d-3+k}{k}, \qquad n_S = \sum_{i\in S} n_i. \tag{40}$$

Equality holds for some $0 \leq k \leq \lfloor\frac{d+2}{2}\rfloor$, if and only if $f_{l-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S}{l}$, for all $0 \leq l \leq k$.

Proof. We are going to prove relation (40) by induction on k. The result clearly holds for $k = 0$, since

$$h_0(\mathcal{F}_{[3]}) = 1 = 1-3+3 = \tbinom{n_{[3]}-d-3}{0} - \sum_{i=1}^{3}\tbinom{n_{[3]\setminus\{i\}}-d-3}{0} + \sum_{i=1}^{3}\tbinom{n_i-d-3}{0}.$$

Suppose the bound holds for some $k \geq 0$. We will show that it holds for $k+1$. Using relation (36), Lemma 8, and the fact that, for any $k \geq 0$, $n_{[3]}-d-2+k \geq 3(d+1)-d-2 = 2d+1 > 0$, we have:

$$\begin{aligned}
h_{k+1}(\mathcal{F}_{[3]}) &\leq \tfrac{n_{[3]}-d-2+k}{k+1}h_k(\mathcal{F}_{[3]}) + \sum_{i=1}^{3}\tfrac{n_i}{k+1}g_k(\mathcal{F}_{[3]\setminus\{i\}})\\
&\leq \tfrac{n_{[3]}-d-2+k}{k+1}\sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S-d-3+k}{k} + \sum_{i=1}^{3}\tfrac{n_i}{k+1}\sum_{\emptyset\subset S\subseteq[3]\setminus\{i\}}(-1)^{|S|}\tbinom{n_S-d-3+k}{k}\\
&= \tfrac{n_{[3]}-d-2+k}{k+1}\tbinom{n_{[3]}-d-3+k}{k} - \sum_{i=1}^{3}\tfrac{n_{[3]}-d-2+k}{k+1}\tbinom{n_{[3]\setminus\{i\}}-d-3+k}{k} + \sum_{i=1}^{3}\tfrac{n_{[3]}-d-2+k}{k+1}\tbinom{n_i-d-3+k}{k}\\
&\quad + \sum_{i=1}^{3}\tfrac{n_i}{k+1}\tbinom{n_{[3]\setminus\{i\}}-d-3+k}{k} - \sum_{i=1}^{3}\tfrac{n_i}{k+1}\sum_{j\in[3]\setminus\{i\}}\tbinom{n_j-d-3+k}{k}\\
&= \tbinom{n_{[3]}-d-2+k}{k+1} - \sum_{i=1}^{3}\tfrac{n_{[3]}-d-2+k-n_i}{k+1}\tbinom{n_{[3]\setminus\{i\}}-d-3+k}{k} + \sum_{i=1}^{3}\tfrac{n_{[3]}-d-2+k-n_{[3]\setminus\{i\}}}{k+1}\tbinom{n_i-d-3+k}{k}\\
&= \tbinom{n_{[3]}-d-2+k}{k+1} - \sum_{i=1}^{3}\tfrac{n_{[3]\setminus\{i\}}-d-2+k}{k+1}\tbinom{n_{[3]\setminus\{i\}}-d-3+k}{k} + \sum_{i=1}^{3}\tfrac{n_i-d-2+k}{k+1}\tbinom{n_i-d-3+k}{k}\\
&= \tbinom{n_{[3]}-d-2+k}{k+1} - \sum_{i=1}^{3}\tbinom{n_{[3]\setminus\{i\}}-d-2+k}{k+1} + \sum_{i=1}^{3}\tbinom{n_i-d-2+k}{k+1}\\
&= \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S-d-2+k}{k+1},
\end{aligned}$$

where we used the fact that:

$$\sum_{i=1}^{3} \tfrac{n_{[3]\setminus\{i\}}}{k+1}\tbinom{n_i-d-3+k}{k} = \sum_{i=1}^{3}\left(\sum_{j\in[3]\setminus\{i\}} \tfrac{n_j}{k+1}\right)\tbinom{n_i-d-3+k}{k} = \sum_{i=1}^{3}\sum_{j\in[3]\setminus\{i\}} \tfrac{n_j}{k+1}\tbinom{n_i-d-3+k}{k}$$
$$= \sum_{i=1}^{3}\sum_{j\in[3]\setminus\{i\}} \tfrac{n_i}{k+1}\tbinom{n_j-d-3+k}{k} = \sum_{i=1}^{3}\tfrac{n_i}{k+1}\sum_{j\in[3]\setminus\{i\}}\tbinom{n_j-d-3+k}{k}.$$

The rest of the proof is concerned with the equality claim. Assume first that $f_{l-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\binom{n_S}{l}$, for all $0 \le l \le k$. Then we have:

$$h_k(\mathcal{F}_{[3]}) = \sum_{i=0}^{d+2}(-1)^{k-i}\tbinom{d+2-i}{d+2-k}f_{i-1}(\mathcal{F}_{[3]}) = (-1)^k\sum_{i=0}^{d+2}(-1)^i\tbinom{d+2-i}{d+2-k}\sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S}{i}$$
$$= (-1)^k\sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\sum_{i=0}^{d+2}(-1)^i\tbinom{d+2-i}{d+2-k}\tbinom{n_S}{i} = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S-d-3+k}{k}.$$

In the above relation we used the combinatorial identity (cf. [GKP89, eq. (5.25)]):

$$\sum_{0\le k\le l}\binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{l+m}\binom{s-m-1}{l-m-n},$$

where $k \leftarrow i$, $l \leftarrow d+2$, $m \leftarrow d+2-k$, $n \leftarrow 0$, and $s \leftarrow n_S$.

Suppose now that $h_k(\mathcal{F}_{[3]}) = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\binom{n_S-d-3+k}{k}$. Since relation (36) holds for all $k \ge 0$, we conclude that $h_l(\mathcal{F}_{[3]})$ must be equal to its upper bound in (40), for all $0 \le l < k$. To see this suppose that (40) is not tight for some l, with $0 \le l < k$, and among all such l choose the largest one. Then, relation (36) implies that $h_{l+1}(\mathcal{F}_{[3]})$ cannot be equal to its upper bound from (40), which contradicts the maximality of l. Hence, if $h_k(\mathcal{F}_{[3]})$ is equal to its upper bound in (40), so is $h_l(\mathcal{F}_{[3]})$ for all $0 \le l < k$, which gives, for all l with $0 \le l \le k$:

$$f_{l-1}(\mathcal{F}_{[3]}) = \sum_{i=0}^{d+2}\tbinom{d+2-i}{l-i}h_i(\mathcal{F}_{[3]}) = \sum_{i=0}^{d+2}\tbinom{d+2-i}{l-i}\sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S-d-3+i}{i}$$
$$= \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\sum_{i=0}^{d+2}\tbinom{d+2-i}{l-i}\tbinom{n_S-d-3+i}{i} = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\sum_{i=0}^{d+2}\tbinom{d+2-i}{d+2-l}\tbinom{n_S-d-3+i}{n_S-d-3} \tag{41}$$
$$= \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S}{n_S-l} = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S}{l}, \tag{42}$$

where, in order to get from (41) to (42), we used the combinatorial identity (cf. [GKP89, eq. (5.26)]):

$$\sum_{0\le k\le l}\binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1},$$

with $k \leftarrow i$, $l \leftarrow d+2$, $m \leftarrow d+2-l$, $q \leftarrow n_S-d-3$, and $n \leftarrow n_S-d-3$. □

We are now going to bound the elements of the h-vector of $\mathcal{K}_{[3]}$. More precisely:

Lemma 10. *For all $0 \le k \le d+2$, we have:*

$$h_k(\mathcal{K}_{[3]}) \le \binom{n_{[3]}-d-3+k}{k}. \tag{43}$$

Furthermore, for $d \geq 3$ and d odd, we have:

$$h_{\lfloor\frac{d}{2}\rfloor+1}(\mathcal{K}_{[3]}) \leq \binom{n_{[3]}-\lfloor\frac{d}{2}\rfloor-3}{\lfloor\frac{d}{2}\rfloor+1} - \sum_{i=1}^{3}\binom{n_i-\lfloor\frac{d}{2}\rfloor-2}{\lfloor\frac{d}{2}\rfloor+1}. \tag{44}$$

Equality holds for some k, where $0 \leq k \leq \lfloor\frac{d+1}{2}\rfloor$, if and only if, for all $\emptyset \subset R \subseteq [3]$, $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S}{l}$, for all $0 \leq l \leq \min\{k, \lfloor\frac{d+|R|-1}{2}\rfloor\}$.

Sketch of proof. The complete proof can be found in Section A.3 of Appendix A. To prove the upper bound for $h_k(\mathcal{K}_{[3]})$, we distinguish between two cases: (1) the case $k = 0$, where the result follows by a straightforward calculation from relation (12) with $R = [3]$, and (2) the case $k \geq 1$, where again we use (12) with $R = [3]$ and substitute $g_k(\mathcal{F}_R)$ by its upper bound from relation (39) in Lemma 8. We, thus, obtain a bound for $h_k(\mathcal{K}_{[3]})$ expressed in terms of $h_k(\mathcal{F}_{[3]})$, $h_{k-1}(\mathcal{F}_R), R \in \mathfrak{S}_2$, and $g_\lambda(\partial P_i)$, $\lambda = k, k-1$. Combining the upper bounds from Lemma 9, Lemma 3.3 in [KT12], along with the upper bounds for the g-vector of a d-polytope (cf. [Zie95, Corollary 8.38]), respectively, gives the upper bound in the statement of the lemma.

For the equality claim we assume that $h_k(\mathcal{K}_{[3]})$ attains its maximal value. Then, the expression bounding $h_k(\mathcal{K}_{[3]})$ used above, in conjunction with Lemmas 7, 8, 9, and [KT12, Lemma 3.3], yields the equality conditions in the statement of the lemma. In the opposite direction, we assume that these conditions hold and, using Lemma 9 and [KT12, Lemma 3.3], we show that the quantities in the right hand side of relation (12) with $R = [3]$, attain their maximal values. The conclusion then follows from an easy calculation. □

We are now ready to state and prove the main theorem of the paper concerning upper bounds on the number of k-faces of the Minkowski sum of three convex d-polytopes.

Theorem 11. *Let P_1, P_2 and P_3 be three d-polytopes in $\mathbb{R}^d$, $d \geq 2$, with $n_i \geq d+1$ vertices, $1 \leq i \leq 3$. Then, for all $1 \leq k \leq d$, we have:*

$$\begin{aligned} f_{k-1}(P_1+P_2+P_3) \leq f_{k+1}(C_{d+2}(n_{[3]})) &- \sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\binom{d+2-i}{k+2-i}\sum_{\emptyset\subset S\subset[3]}(-1)^{|S|}\binom{n_S-d-3+i}{i} \\ &- \delta\binom{\lfloor\frac{d}{2}\rfloor+1}{k-\lfloor\frac{d}{2}\rfloor}\sum_{i=1}^{3}\binom{n_i-\lfloor\frac{d}{2}\rfloor-2}{\lfloor\frac{d}{2}\rfloor+1}, \end{aligned} \tag{45}$$

where $\delta = d - 2\lfloor\frac{d}{2}\rfloor$, and $n_S = \sum_{i\in S} n_i$. Equality holds for all $1 \leq k \leq d$, if and only if

$$f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S}{l}, \qquad 0 \leq l \leq \lfloor\tfrac{d+|R|-1}{2}\rfloor, \qquad \emptyset \subset R \subseteq [3]. \tag{46}$$

Proof. If suffices to establish upper bounds for $f_k(\mathcal{F}_{[3]})$ for all $0 \leq k \leq d+1$. Indeed, writing the f-vector of $\mathcal{F}_{[3]}$ in terms of its h-vector, and using relation (14), along with Lemmas 9 and 10 we get:

$$\begin{aligned} f_{k-1}(\mathcal{F}_{[3]}) &= \sum_{i=0}^{d+2}\tbinom{d+2-i}{k-i}h_i(\mathcal{F}_{[3]}) = \sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}h_i(\mathcal{F}_{[3]}) + \sum_{i=\lfloor\frac{d+2}{2}\rfloor+1}^{d+2}\tbinom{d+2-i}{k-i}h_i(\mathcal{F}_{[3]}) \\ &= \sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}h_i(\mathcal{F}_{[3]}) + \sum_{j=0}^{\lfloor\frac{d+1}{2}\rfloor}\tbinom{j}{k-d-2+j}h_{d+2-j}(\mathcal{F}_{[3]}) \\ &= \sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}h_i(\mathcal{F}_{[3]}) + \sum_{j=0}^{\lfloor\frac{d+1}{2}\rfloor}\tbinom{j}{k-d-2+j}h_j(\mathcal{K}_{[3]}). \end{aligned} \tag{47}$$

From Lemma 9 we have:

$$\sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}h_i(\mathcal{F}_{[3]})\le\sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}\sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S-d-3+i}{i},$$

whereas from Lemma 10 we get

$$\sum_{j=0}^{\lfloor\frac{d+1}{2}\rfloor}\tbinom{j}{k-d-2+j}h_j(\mathcal{K}_{[3]})\le\sum_{i=0}^{\lfloor\frac{d+1}{2}\rfloor}\tbinom{n_{[3]}-d-3+j}{j}-\delta\tbinom{\lfloor\frac{d}{2}\rfloor+1}{k-\lfloor\frac{d}{2}\rfloor-2}\sum_{i=1}^{3}\tbinom{n_i-\lfloor\frac{d}{2}\rfloor-2}{\lfloor\frac{d}{2}\rfloor+1},$$

where $\delta=d-2\lfloor\frac{d}{2}\rfloor$. Hence:

$$\begin{aligned}
f_{k-1}(\mathcal{F}_{[3]})&\le\sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}\sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S-d-3+i}{i}+\sum_{j=0}^{\lfloor\frac{d+1}{2}\rfloor}\tbinom{j}{k-d-2+j}\tbinom{n_{[3]}-d-3+j}{j}\\
&\quad-\delta\tbinom{\lfloor\frac{d}{2}\rfloor+1}{k-\lfloor\frac{d}{2}\rfloor-2}\sum_{i=1}^{3}\tbinom{n_i-\lfloor\frac{d}{2}\rfloor-2}{\lfloor\frac{d}{2}\rfloor+1}\\
&=\sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}\tbinom{n_{[3]}-d-3+i}{i}+\sum_{i=0}^{\lfloor\frac{d+1}{2}\rfloor}\tbinom{i}{k-d-2+i}\tbinom{n_{[3]}-d-3+i}{i}\\
&\quad-\sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}\sum_{\emptyset\subset S\subset[3]}(-1)^{|S|}\tbinom{n_S-d-3+i}{i}-\delta\tbinom{\lfloor\frac{d}{2}\rfloor+1}{k-\lfloor\frac{d}{2}\rfloor-2}\sum_{i=1}^{3}\tbinom{n_i-\lfloor\frac{d}{2}\rfloor-2}{\lfloor\frac{d}{2}\rfloor+1}\\
&=\sum_{i=0}^{\frac{d+2}{2}}{}^{*}\Big(\tbinom{d+2-i}{k-i}+\tbinom{i}{k-d-2+i}\Big)\tbinom{n_{[3]}-d-3+i}{i}-\sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}\sum_{\emptyset\subset S\subset[3]}(-1)^{|S|}\tbinom{n_S-d-3+i}{i}\\
&\quad-\delta\tbinom{\lfloor\frac{d}{2}\rfloor+1}{k-\lfloor\frac{d}{2}\rfloor-2}\sum_{i=1}^{3}\tbinom{n_i-\lfloor\frac{d}{2}\rfloor-2}{\lfloor\frac{d}{2}\rfloor+1}\\
&=f_{k-1}(C_{d+2}(n_{[3]}))-\sum_{i=0}^{\lfloor\frac{d+2}{2}\rfloor}\tbinom{d+2-i}{k-i}\sum_{\emptyset\subset S\subset[3]}(-1)^{|S|}\tbinom{n_S-d-3+i}{i}\\
&\quad-\delta\tbinom{\lfloor\frac{d}{2}\rfloor+1}{k-\lfloor\frac{d}{2}\rfloor-2}\sum_{i=1}^{3}\tbinom{n_i-\lfloor\frac{d}{2}\rfloor-2}{\lfloor\frac{d}{2}\rfloor+1},
\end{aligned}$$

where:

$$\sum_{i=0}^{\frac{m}{2}}{}^{*}T_i=\sum_{i=0}^{\lfloor\frac{m}{2}\rfloor-1}T_i+\tfrac{1}{2}\left(1+m-2\lfloor\tfrac{m}{2}\rfloor\right)T_{\lfloor\frac{m}{2}\rfloor}.$$

Our upper bounds follow from the fact that $f_{k-1}(P_1+P_2+P_3)=f_{k+1}(\mathcal{F}_{[3]})$, $1\le k\le d$.

In what follows we concentrate on the necessary and sufficient conditions for the upper bounds in (45) to hold as equalities. From the derivation of the upper bounds above (see also relation (47)), it is clear that the bounds are tight if and only if:

(1) $h_k(\mathcal{F}_{[3]})$ is maximal, for all $0\le k\le\lfloor\frac{d+2}{2}\rfloor$, and

(2) $h_k(\mathcal{K}_{[3]})$ is maximal, for all $0\le k\le\lfloor\frac{d+1}{2}\rfloor$.

According to Lemma 9 and Lemma 10, these conditions are, respectively, equivalent to requiring that:

(1) $f_{l-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\binom{n_S}{l}$, for all $0 \leq l \leq \lfloor\frac{d+2}{2}\rfloor$, and

(2) $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S}{l}$, for all $0 \leq l \leq \min\{\lfloor\frac{d+1}{2}\rfloor, \lfloor\frac{d+|R|-1}{2}\rfloor\}$, and for all $\emptyset \subset R \subseteq [3]$.

For $R \equiv [3]$, condition (1) implies condition (2), while for $R \subset [3]$, $\min\{\lfloor\frac{d+1}{2}\rfloor, \lfloor\frac{d+|R|-1}{2}\rfloor\} = \lfloor\frac{d+|R|-1}{2}\rfloor$. We, therefore, conclude that the bounds in (45) are attained if and only if, conditions (46) hold true for all $0 \leq k \leq \lfloor\frac{d-|R|+1}{2}\rfloor$ and for all $\emptyset \subset R \subseteq [3]$. □

6 Tightness of upper bounds

In this section we show that the bounds in Theorem 11 are tight. We distinguish between the cases $d = 2$, $d = 3$ and $d \geq 4$. For $d = 2$, it is easy to verify that for $k = 1, 2$, the right-hard side of inequality (45) evaluates to $n_1 + n_2 + n_3$, which is known to be tight.

6.1 Three dimensions

For $d = 3$, the upper bounds in Theorem 11 are as follows:

$$\begin{aligned} f_0(P_1 + P_2 + P_3) &\leq n_1n_2 + n_2n_3 + n_1n_3 - n_1 - n_2 - n_3 + 2, \\ f_1(P_1 + P_2 + P_3) &\leq 2n_1n_2 + 2n_2n_3 + 2n_1n_3 - n_1 - n_2 - n_3 - 6, \\ f_2(P_1 + P_2 + P_3) &\leq n_1n_2 + n_2n_3 + n_1n_3 - 6. \end{aligned} \tag{48}$$

In order to prove that these bounds are tight, we exploit two results: one by Fukuda and Weibel [FW07] and one by Weibel [Wei12]. Weibel [Wei12] has shown that the number of k-faces of the Minkowski sum of r d-polytopes $P_1, \ldots, P_r$ in $\mathbb{R}^d$, where $r \geq d$, is related to the number of k-faces of the Minkowski sum of subsets of these polytopes of size at most $d-1$ as follows:

$$f_k(P_1 + P_2 + \cdots + P_r) - \alpha = \sum_{j=1}^{d-1}(-1)^{d-1-j}\binom{r-1-j}{d-1-j}\sum_{S\in\mathfrak{S}_j^r}(f_k(P_S) - \alpha), \tag{49}$$

where $\mathfrak{S}_j^r$ is the family of subsets of $[r]$ of size j, P_S is the Minkowski sum of the polytopes in S, and $\alpha = 2$ if $k = 0$ and d is odd, and $\alpha = 0$ otherwise. For $d = r = 3$, equation (49) simplifies to:

$$\begin{aligned} f_k(P_1 + P_2 + P_3) &= \alpha + \sum_{j=1}^{2}(-1)^{2-j}\tbinom{2-j}{2-j}\sum_{S\in\mathfrak{S}_j^3}(f_k(P_S) - \alpha) \\ &= \alpha - \sum_{i=1}^{3}(f_k(P_i) - \alpha) + \sum_{i=1}^{3}(f_k(P_{[3]\setminus\{i\}}) - \alpha) \\ &= \alpha - \sum_{i=1}^{3} f_k(P_i) + 3\alpha + \sum_{i=1}^{3} f_k(P_{[3]\setminus\{i\}}) - 3\alpha \\ &= \alpha + \sum_{1\leq i<j\leq 3} f_k(P_i + P_j) - \sum_{i=1}^{3} f_k(P_i). \end{aligned} \tag{50}$$

Besides relation (49), Weibel [Wei12] also presented a construction of r simplicial d-polytopes, such that any subset S of these polytopes of size at most $d-1$ has the maximum possible number of vertices, namely, $f_0(P_S) = \prod_{i\in S} n_i$. Specializing this construction in our case, i.e., for $r = d = 3$, we deduce that it is possible to construct three simplicial 3-polytopes P_1, P_2, P_3

in $\mathbb{R}^3$, such that $f_0(P_i) = n_i$, $1 \leq i \leq 3$, and $f_0(P_i + P_j) = n_i n_j$, $1 \leq i < j \leq 3$. Substituting in (50) for $k = 0$, we get:

$$f_0(P_1 + P_2 + P_3) = 2 + \sum_{1 \leq i < j \leq 3} n_i n_j - \sum_{i=1}^{3} n_i = n_1 n_2 + n_2 n_3 + n_1 n_3 - n_1 - n_2 - n_3 + 2,$$

i.e., the upper bound in (48) is tight for $k = 0$[3]. Since all P_i's are simplicial, we have

$$f_1(P_i) = 3n_i - 6, \quad f_2(P_i) = 2n_i - 4, \qquad 1 \leq i \leq 3. \tag{51}$$

On the other hand, since $f_0(P_i + P_j)$ is maximal, for all $1 \leq i < j \leq 3$, we get, by [FW07, Corollary 4], that $f_k(P_i + P_j)$ is also maximal for $k = 1, 2$, and for all $1 \leq i < j \leq 3$. Hence:

$$f_1(P_i + P_j) = 2n_i n_j + n_i + n_j - 8, \qquad f_2(P_i + P_j) = n_i n_j + n_i + n_j - 6. \tag{52}$$

Substituting from (51) and (52) in (50), and recalling that $\alpha = 0$ for $k > 0$, we get:

$$\begin{aligned} f_1(P_1 + P_2 + P_3) = &\sum_{1 \leq i < j \leq 3} (2n_i n_j + n_i + n_j - 8) - \sum_{i=1}^{3} (3n_i - 6) \\ &= [2(n_1 n_2 + n_2 n_3 + n_1 n_3) + 2(n_1 + n_2 + n_3) - 24] - [3(n_1 + n_2 + n_3) - 18] \\ &= 2n_1 n_2 + 2n_2 n_3 + 2n_1 n_3 - n_1 - n_2 - n_3 - 6, \end{aligned}$$

and

$$\begin{aligned} f_2(P_1 + P_2 + P_3) = &\sum_{1 \leq i < j \leq 3} (n_i n_j + n_i + n_j - 6) - \sum_{i=1}^{3} (2n_i - 4) \\ &= [n_1 n_2 + n_2 n_3 + n_1 n_3 + 2(n_1 + n_2 + n_3) - 18] - [2(n_1 + n_2 + n_3) - 12] \\ &= n_1 n_2 + n_2 n_3 + n_1 n_3 - 6, \end{aligned}$$

i.e., the upper bounds in (48) are tight for $k = 1, 2$.

6.2 Four or more dimensions

We now focus on the case $d \geq 4$. We shall construct three d-polytopes P_1, P_2 and P_3 in $\mathbb{R}^d$, such that they satisfy the conditions in relation (46). Consequently, as Theorem 11 asserts, these polytopes attain the upper bounds in (45).

Consider the following d-dimensional moment-like curves in $\mathbb{R}^d$:

$$\begin{aligned} \boldsymbol{\gamma}_1(t) &= (t, \zeta t^2, \zeta t^3, t^4, t^5, \ldots, t^d), \\ \boldsymbol{\gamma}_2(t) &= (\zeta t, t^2, \zeta t^3, t^4, t^5, \ldots, t^d), \\ \boldsymbol{\gamma}_3(t) &= (\zeta t, \zeta t^2, t^3, t^4, t^5, \ldots, t^d), \end{aligned}$$

where $t > 0$, and $\zeta \geq 0$. Let $\boldsymbol{e}_{1,1} = (0), \boldsymbol{e}_{1,2} = (1)$ be the standard affine basis of $\mathbb{R}$ and recall that $\boldsymbol{e}_{2,1} = (0,0), \boldsymbol{e}_{2,2} = (1,0), \boldsymbol{e}_{2,3} = (0,1)$ is the standard affine basis of $\mathbb{R}^2$. We shall define three polytopes as the convex hulls of points, chosen appropriately on each of these d-curves. We then proceed to show that $\mathcal{F}_R$, $R \in \mathfrak{S}_2$, and $\mathcal{F}_{[3]}$, have the following property: every set of $k = \lfloor \frac{d+1}{2} \rfloor$ vertices from $\mathcal{F}_R$, or $k \leq \lfloor \frac{d+2}{2} \rfloor$ vertices from $\mathcal{F}_{[3]}$, defines a $(k-1)$-face of $\mathcal{F}_R$ or $\mathcal{F}_{[3]}$, respectively. This property readily yields the necessary and sufficient conditions establishing the tightness of the upper bounds (cf. rel. (46)).

[3] This is essentially the result of Theorem 3 in [Wei12] for $d = r = 3$; however, we recapitulate this result in order to show that Weibel's construction yields tights bounds for $k = 1, 2$ also.

Let $x_{i,j}$, $1 \le j \le n_i$, $1 \le i \le 3$, be $n_{[3]}$ positive real numbers, such that $x_{i,j} < x_{i,j+1}$, $1 \le j \le n_i - 1$, and let τ be a positive real parameter. Let $x^{\epsilon}_{i,j} = x_{i,j} + \epsilon$, $t_{i,j} = x_{i,j}\tau^{\nu_i}$, $t^{\epsilon}_{i,j} = x^{\epsilon}_{i,j}\tau^{\nu_i}$, where $1 \le j \le n_i$, $1 \le i \le 3$, $\epsilon > 0$, and $\nu_i = 3 - i$, $1 \le i \le 3$. The value of ϵ is chosen such that $x^{\epsilon}_{i,j} < x_{i,j+1}$, for all $1 \le j < n_i$, and for all $1 \le i \le 3$. Finally, we set $\zeta = \tau^M$, where $M \ge d(d+1)$. We are going to define three vertex sets V_i as follows:

$$V_i = \{\boldsymbol{\gamma}_i(t_{i,1}), \boldsymbol{\gamma}_i(t_{i,2}), \dots \boldsymbol{\gamma}_i(t_{i,n_i})\} \qquad 1 \le i \le 3. \tag{53}$$

Call P_i the d-polytope we get as the convex hull of the vertices in V_i, and let $\mathcal{V}_i$ be the image of V_i via the Cayley embedding. As in Section 2, call $\mathcal{C}$ the Cayley polytope of the P_i's in $\mathbb{R}^{d+2}$, and $\mathcal{F}_R$, $\emptyset \subset R \subseteq [3]$, the set of faces of $\mathcal{C}$ with at least one vertex from each $\mathcal{V}_i$, $i \in R$. Note that, by construction, P_i is a $\lfloor \frac{d}{2} \rfloor$-neighborly polytope in $\mathbb{R}^d$ with n_i vertices, which immediately implies that conditions (46) hold for $R \in \mathfrak{S}_1$ and for all $0 \le l \le \lfloor \frac{d}{2} \rfloor$. Hence, it suffices to show that:

$$f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S}{l}, \qquad 0 \le l \le \lfloor \tfrac{d+|R|-1}{2} \rfloor, \qquad 2 \le |R| \le 3, \tag{54}$$

which we will succeed by choosing a sufficiently small value for τ.

To prove that the constructed polytopes have the desired properties (see Lemmas 12 and 13, bellow), we adopt the key idea used in the proofs of [Zie95, Theorem 0.7 & Corollary 0.8] on basic properties of cyclic d-polytopes, and adapt this idea to our setting, where we view the faces the Minkowski sum of the polytopes P_i, $i \in R$, via the face set $\mathcal{F}_R$ of their Cayley polytope, where $2 \le |R| \le 3$.

We start off with subsets R of size two. To show that $f_{k-1}(\mathcal{F}_R)$ is according to relation (54), recall (cf. Section 2) that the polytope $\mathcal{C}$ contains the Cayley polytope $\mathcal{C}_R$ of the polytopes in R as a d-subcomplex embedded in $\mathbb{R}^{d+2}$. Thus, in order to prove relation (54) for $\mathcal{F}_R$, we may consider $\mathcal{C}_R$ and $\mathcal{F}_R$ independently of $\mathcal{C}$, i.e., we can disassociate the polytopes P_i, $i \in R$, from the Cayley polytope $\mathcal{C}$. In other words, we think of the polytopes P_i, $i \in R$, as d-polytopes in $\mathbb{R}^d$, while their Cayley polytope $\mathcal{C}_R$ is seen as a $(d+1)$-polytope in $\mathbb{R}^{d+1}$. We exploit this observation in order to prove the following lemma.

Lemma 12. *There exists a sufficiently small positive value $\hat{\tau}_R$ for τ such that, for all $\tau \in (0, \hat{\tau}_R)$,*

$$f_{k-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{2-|S|} \binom{n_S}{k}, \qquad 2 \le k \le \lfloor \tfrac{d+1}{2} \rfloor, \quad R \in \mathfrak{S}_2.$$

Proof. Without loss of generality let $R = \{1,3\}$. The rest of the cases are analogous. The condition in the statement of the lemma is equivalent to the requirement that $\mathcal{C}_{\{1,3\}}$ is a $(\mathcal{V}_1, \lfloor \frac{d+1}{2} \rfloor)$-bineighborly polytope (see [KT12] for definitions and details), which in turn is equivalent to the requirement that

$$f_{\lfloor \frac{d+1}{2} \rfloor - 1}(\mathcal{F}_{\{1,3\}}) = \sum_{\emptyset \subset S \subseteq \{1,3\}} (-1)^{2-|S|} \binom{n_S}{\lfloor \frac{d+1}{2} \rfloor}. \tag{55}$$

We shall prove that condition (55) holds true for the Cayley polytope $\mathcal{C}_{\{1,3\}}$ of the polytopes P_1, P_3, and for sufficiently small values of τ, as described in the statement of the lemma.

Define $\delta := d + 1 - 2\lfloor \frac{d+1}{2} \rfloor$. Let X be a positive real number such that $X > x^{\epsilon}_{3,n_3}$, and let[4] $T = X\tau^{\nu_3}$. Choose a set U of $k_m \neq 0$ vertices $\boldsymbol{\gamma}_m(t_{m,j_{m,1}}), \boldsymbol{\gamma}_m(t_{m,j_{m,2}}), \dots, \boldsymbol{\gamma}_m(t_{m,j_{m,k_m}})$ from the set V_m, such that $j_{m,1} < j_{m,2} < \dots < j_{m,k_m}$, for $m \in \{1,3\}$, and $k_1 + k_3 = \lfloor \frac{d+1}{2} \rfloor$. Let $\mathcal{U} = \{\boldsymbol{\beta}_m(t_{m,j_{m,1}}), \boldsymbol{\beta}_m(t_{m,j_{m,2}}), \dots, \boldsymbol{\beta}_m(t_{m,j_{m,k_m}}) \mid m \in \{1,3\}\}$, be the Cayley embedding of U in $\mathbb{R}^{d+1}$ (using the affine basis $\boldsymbol{e}_{1,1}, \boldsymbol{e}_{1,2}$). For a vector $\boldsymbol{x} = (x_1, x_2, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$, we define the $(d+2) \times (d+2)$ determinant $H_{\mathcal{U}}(\boldsymbol{x})$ as follows:

[4] Although we have set $\nu_3 = 0$, we keep ν_3 as is in the proof, so as to make more profound the analogy of the proof presented here for $R = \{1,3\}$ with the cases $R = \{1,2\}$ and $R = \{2,3\}$.

$$(-1)^6 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \boldsymbol{x} & \boldsymbol{\beta}_1(t_{1,j_{1,1}}) & \boldsymbol{\beta}_1(t^{\epsilon}_{1,j_{1,1}}) & \cdots & \boldsymbol{\beta}_1(t_{1,j_{1,k_1}}) & \boldsymbol{\beta}_1(t^{\epsilon}_{1,j_{1,k_1}}) & \boldsymbol{\beta}_3(t_{3,j_{3,1}}) & \boldsymbol{\beta}_3(t^{\epsilon}_{3,j_{3,1}}) & \cdots & \boldsymbol{\beta}_3(t_{3,j_{3,k_3}}) & \boldsymbol{\beta}_3(t^{\epsilon}_{3,j_{3,k_3}}) & \boldsymbol{\beta}_3(\delta T) \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ x_1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ x_2 & t_{1,j_{1,1}} & t^{\epsilon}_{1,j_{1,1}} & \cdots & t_{1,j_{1,k_1}} & t^{\epsilon}_{1,j_{1,k_1}} & \tau^M t_{3,j_{3,1}} & \tau^M t^{\epsilon}_{3,j_{3,1}} & \cdots & \tau^M t_{3,j_{3,k_3}} & \tau^M t^{\epsilon}_{3,j_{3,k_3}} & \tau^M \delta T \\ x_3 & \tau^M (t_{1,j_{1,1}})^2 & \tau^M (t^{\epsilon}_{1,j_{1,1}})^2 & \cdots & \tau^M (t_{1,j_{1,k_1}})^2 & \tau^M (t^{\epsilon}_{1,j_{1,k_1}})^2 & \tau^M (t_{3,j_{3,1}})^2 & \tau^M (t^{\epsilon}_{3,j_{3,1}})^2 & \cdots & \tau^M (t_{3,j_{3,k_3}})^2 & \tau^M (t^{\epsilon}_{3,j_{3,k_3}})^2 & \tau^M (\delta T)^2 \\ x_4 & \tau^M (t_{1,j_{1,1}})^3 & \tau^M (t^{\epsilon}_{1,j_{1,1}})^3 & \cdots & \tau^M (t_{1,j_{1,k_1}})^3 & \tau^M (t^{\epsilon}_{1,j_{1,k_1}})^3 & (t_{3,j_{3,1}})^2 & (t^{\epsilon}_{3,j_{3,1}})^3 & \cdots & (t_{3,j_{3,k_3}})^3 & (t^{\epsilon}_{3,j_{3,k_3}})^3 & (\delta T)^3 \\ x_5 & (t_{1,j_{1,1}})^4 & (t^{\epsilon}_{1,j_{1,1}})^4 & \cdots & (t_{1,j_{1,k_1}})^4 & (t^{\epsilon}_{1,j_{1,k_1}})^4 & (t_{3,j_{3,1}})^2 & (t^{\epsilon}_{3,j_{3,1}})^4 & \cdots & (t_{3,j_{3,k_3}})^4 & (t^{\epsilon}_{3,j_{3,k_3}})^4 & (\delta T)^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{d+1} & (t_{1,j_{1,1}})^d & (t^{\epsilon}_{1,j_{1,1}})^d & \cdots & (t_{1,j_{1,k_1}})^d & (t^{\epsilon}_{1,j_{1,k_1}})^d & (t_{3,j_{3,1}})^d & (t^{\epsilon}_{3,j_{3,1}})^d & \cdots & (t_{3,j_{3,k_3}})^d & (t^{\epsilon}_{3,j_{3,k_3}})^d & (\delta T)^d \end{vmatrix}.$$

Notice that for d odd the last column $\binom{1}{\boldsymbol{\beta}_3(\delta T)}$ of $H_{\mathcal{U}}(x)$ does not exist. The equation $H_{\mathcal{U}}(\boldsymbol{x}) = 0$ is the equation of a hyperplane in $\mathbb{R}^{d+1}$ that passes through the points in $\mathcal{U}$. We are going to show that, for any choice of $\mathcal{U}$, and for all vertices $\boldsymbol{v}$ in $\mathcal{V}_{\{1,3\}} \setminus \mathcal{U}$, $\mathcal{V}_{\{1,3\}} = \mathcal{V}_1 \cup \mathcal{V}_3$, we have $H_{\mathcal{U}}(\boldsymbol{v}) > 0$ for sufficiently small values of τ.

Suppose we have some vertex $\boldsymbol{v} \in \mathcal{V}_{\{1,3\}} \setminus \mathcal{U}$. Then, $\boldsymbol{v} = \boldsymbol{\beta}_s(t_{s,\lambda})$, $t_{s,\lambda} = x_{s,\lambda}\tau^{\nu_s}$, where $1 \leq \lambda \leq n_s$, s is either 1 or 3, and $\lambda \notin \{j_{s,1}, j_{s,2}, \ldots, j_{s,k_s}\}$. We perform the following determinant transformations on $H_{\mathcal{U}}(\boldsymbol{v})$: initially we subtract its second row from its first, and then we shift its first column to the right via an even number of column swaps. More precisely, we need to shift the first column of $H_{\mathcal{U}}(\boldsymbol{v})$ to the right so that the values $t_{s,\lambda}, t_{s,j_{s,1}}, t^{\epsilon}_{s,j_{s,1}}, t_{s,j_{s,2}}, t^{\epsilon}_{s,j_{s,2}}, \ldots, t_{s,j_{s,k_s}}$, $t^{\epsilon}_{s,j_{s,k_s}}$ appear consecutively in the columns of $H_{\mathcal{U}}(\boldsymbol{v})$ and in increasing order. To do that we always need an even number of column swaps, due to the way we have chosen ϵ.

Consider the case where $s = 1$ and suppose that all necessary operations on $H_{\mathcal{U}}(\boldsymbol{v})$ have been performed. Then $H_{\mathcal{U}}(\boldsymbol{v})$ is in the form of the determinant $D_{n,m}(\tau; I, J, \boldsymbol{\mu})$ of Lemma 16 (multiplied by τ^M), with $n \leftarrow 2k_1 + 1$, $m \leftarrow 2k_3$, $l \leftarrow d + 2$, $\boldsymbol{\mu} \leftarrow (0, 0, 1, 2, \ldots, d)$, $\alpha \leftarrow \nu_1$, $\beta \leftarrow \nu_3$, $I \leftarrow 3$, and $J \leftarrow 5$. Note that the requirement for M in Lemma 16 is satisfied by our choice of M. According to Lemma 16, $H_{\mathcal{U}}(\boldsymbol{v})$ has the following asymptotic expansion in terms of τ:

$$H_{\mathcal{U}}(\boldsymbol{v}) = \tau^M(C\tau^{\xi} + \Theta(\tau^{\xi+1})), \qquad \xi = \nu_1(-2 + \sum_{i=4}^{2k_1+3}(i-2)) + \nu_3(3 + \sum_{i=2k_1+4}^{d+2}(i-2)), \tag{56}$$

where C is a positive constant independent of τ. The asymptotic expansion in (56) implies that there exists a positive value $\hat{\tau}_{\boldsymbol{v},\mathcal{U}}$ for τ such that for all $\tau \in (0, \hat{\tau}_{\boldsymbol{v},\mathcal{U}})$, $H_{\mathcal{U}}(\boldsymbol{v}) > 0$. The case $s = 3$ is completely analogous.

Since the number of the subsets $\mathcal{U}$ is finite, while for each such subset $\mathcal{U}$ we need to consider a finite number of vertices in $\mathcal{V}_{\{1,3\}} \setminus \mathcal{U}$, it suffices to consider a positive value $\hat{\tau}_{\{1,3\}}$ for τ that is small enough, so that all possible determinants $H_{\mathcal{U}}(\boldsymbol{v})$ are strictly positive for any $\tau \in (0, \hat{\tau}_{\{1,3\}})$. For $\tau \in (0, \hat{\tau}_{\{1,3\}})$, our analysis above immediately implies that *for each* set $\mathcal{U}$ the equation $H_{\mathcal{U}}(\boldsymbol{x}) = 0$, $\boldsymbol{x} \in \mathbb{R}^{d+1}$, is the equation of a supporting hyperplane of $\mathcal{C}_R$ passing through the vertices of $\mathcal{U}$, and those only. In other words, every set $\mathcal{U}$, where $|\mathcal{U}| = \lfloor \frac{d+1}{2} \rfloor$, $|\mathcal{U} \cap \mathcal{V}_1| = k_1 \neq 0$, and $|\mathcal{U} \cap \mathcal{V}_3| = k_3 \neq 0$, defines a $(\lfloor \frac{d+1}{2} \rfloor - 1)$-face of $\mathcal{C}_R$. Taking into account that the number of such subsets $\mathcal{U}$ is $\sum_{i=1}^{\lfloor \frac{d+1}{2} \rfloor - 1} \binom{n_1}{i} \binom{n_3}{\lfloor \frac{d+1}{2} \rfloor - i}$, we deduce that

$$\begin{aligned} f_{\lfloor \frac{d+1}{2} \rfloor - 1}(\mathcal{F}_{\{1,3\}}) &= \sum_{i=1}^{\lfloor \frac{d+1}{2} \rfloor - 1} \binom{n_1}{i} \binom{n_3}{\lfloor \frac{d+1}{2} \rfloor - i} = \binom{n_1+n_3}{\lfloor \frac{d+1}{2} \rfloor} - \binom{n_1}{\lfloor \frac{d+1}{2} \rfloor} - \binom{n_3}{\lfloor \frac{d+1}{2} \rfloor} \\ &= \sum_{\emptyset \subset S \subseteq \{1,3\}} (-1)^{2-|S|} \binom{n_S}{\lfloor \frac{d+1}{2} \rfloor}. \end{aligned}$$

Hence, condition (55) is satisfied for all $\tau \in (0, \hat{\tau}_{\{1,3\}})$. $\square$

We now consider the case $R = [3]$. In this case we can show that:

Lemma 13. *There exists a sufficiently small positive value $\hat{\tau}_{[3]}$ for τ such that, for all $\tau \in (0, \hat{\tau}_{[3]})$,*

$$f_{k-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \binom{n_S}{k}, \qquad 3 \leq k \leq \lfloor \tfrac{d+2}{2} \rfloor, \tag{57}$$

Proof. Define $\delta := d + 2 - 2k$ and let T be a positive real number such that $T > t^{\epsilon}_{3,n_3} (= x^{\epsilon}_{3,n_3})$. Choose a set U of $k_i \neq 0$ vertices from V_i, $1 \leq i \leq 3$, such that $k_1 + k_2 + k_3 = k$, and denote by $\mathcal{U}$ the Cayley embedding of U in $\mathbb{R}^{d+2}$ (using the affine basis $\boldsymbol{e}_{2,i}$, $1 \leq i \leq 3$). Let $\boldsymbol{\gamma}_i(t_{i,j_{i,1}})$,$\boldsymbol{\gamma}(t_{i,j_{i,2}})$,..., $\boldsymbol{\gamma}_i(t_{i,j_{i,k_i}})$, be the vertices in U, and $\boldsymbol{\beta}_i(t_{i,j_{i,1}}), \boldsymbol{\beta}_i(t_{i,j_{i,2}}), \ldots, \boldsymbol{\beta}_i(t_{i,j_{i,k_i}})$, be their corresponding vertices in $\mathcal{U}$, where $j_{i,1} < j_{i,2} < \ldots < j_{i,k_i}$ for all $1 \leq i \leq 3$. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_{d+2})$ and define the $(d+3) \times (d+3)$ determinant $H_{\mathcal{U}}(\boldsymbol{x})$ as follows:

$$-\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \boldsymbol{x} & \boldsymbol{\beta}_1(t_{1,j_{1,1}}) & \cdots & \boldsymbol{\beta}_1(t^{\epsilon}_{1,j_{1,k_1}}) & \boldsymbol{\beta}_2(t_{2,j_{2,1}}) & \cdots & \boldsymbol{\beta}_2(t^{\epsilon}_{j_{2,k_2}}) & \boldsymbol{\beta}_3(t^{\epsilon}_{3,j_{3,1}}) & \cdots & \boldsymbol{\beta}_3(t^{\epsilon}_{3,j_{3,k_3}}) & \boldsymbol{\beta}_3(T) & \cdots & \boldsymbol{\beta}_3(\delta T) \end{vmatrix} =$$

$$-\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_3 & t_{1,j_{1,1}} & \cdots & t^{\epsilon}_{1,j_{1,k_1}} & \tau^M t_{2,j_{2,1}} & \cdots & \tau^M t^{\epsilon}_{2,j_{2,k_2}} & \tau^M t_{3,j_{3,1}} & \cdots & \tau^M t^{\epsilon}_{3,j_{3,k_3}} & \tau^M T & \cdots & \tau^M \delta T \\ x_4 & \tau^M (t_{1,j_{1,1}})^2 & \cdots & \tau^M (t^{\epsilon}_{1,j_{1,k_1}})^2 & (t_{2,j_{2,1}})^2 & \cdots & (t^{\epsilon}_{2,j_{2,k_2}})^2 & \tau^M (t_{3,j_{3,1}})^2 & \cdots & \tau^M (t^{\epsilon}_{3,j_{3,k_3}})^2 & \tau^M T^2 & \cdots & \tau^M (\delta T)^2 \\ x_5 & \tau^M (t_{1,j_{1,1}})^3 & \cdots & \tau^M (t^{\epsilon}_{1,j_{1,k_1}})^3 & \tau^M (t_{2,j_{2,1}})^3 & \cdots & \tau^M (t^{\epsilon}_{2,j_{2,k_2}})^3 & (t_{3,j_{3,1}})^3 & \cdots & (t^{\epsilon}_{3,j_{3,k_3}})^3 & T^3 & \cdots & (\delta T)^3 \\ x_6 & (t_{1,j_{1,1}})^4 & \cdots & (t^{\epsilon}_{1,j_{1,k_1}})^4 & (t_{2,j_{2,1}})^4 & \cdots & (t^{\epsilon}_{2,j_{2,k_2}})^4 & (t_{3,j_{3,1}})^4 & \cdots & (t^{\epsilon}_{3,j_{3,k_3}})^4 & T^4 & \cdots & (\delta T)^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{d+2} & (t_{1,j_{1,1}})^d & \cdots & (t^{\epsilon}_{1,j_{1,k_1}})^d & (t_{2,j_{2,1}})^d & \cdots & (t^{\epsilon}_{2,j_{2,k_2}})^d & (t_{3,j_{3,1}})^d & \cdots & (t^{\epsilon}_{3,j_{3,k_3}})^d & T^d & \cdots & (\delta T)^d \end{vmatrix}.$$

We can alternatively describe $H_{\mathcal{U}}(\boldsymbol{x})$ as follows:

(i) The first column of $H_{\mathcal{U}}(\boldsymbol{x})$ is $\binom{1}{\boldsymbol{x}}$.

(ii) For i ranging from 1 to 3, and for λ ranging from 1 to k_i, the next k_i *pairs of columns* of $H_{\mathcal{U}}(\boldsymbol{x})$ are $\binom{1}{\boldsymbol{\beta}_i(t_{i,j_{i,\lambda}})}$ and $\binom{1}{\boldsymbol{\beta}_i(t^{\epsilon}_{i,j_{i,\lambda}})}$.

(iii) For λ ranging from 1 to δ, the last δ columns of $H_{\mathcal{U}}(\boldsymbol{x})$ are $\binom{1}{\boldsymbol{\beta}_3(\lambda T)}$. Notice that if $k = \lfloor \frac{d+2}{2} \rfloor$ and d is even, this category of columns of $H_{\mathcal{U}}(\boldsymbol{x})$ does not exist.

The equation $H_{\mathcal{U}}(\boldsymbol{x}) = 0$ is the equation of a hyperplane in $\mathbb{R}^{d+2}$ that passes through the points in $\mathcal{U}$. Recall that $\mathcal{V}_{[3]} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$. We are going to show that, for any choice of $\mathcal{U}$, and for all vertices $\boldsymbol{v}$ in $\mathcal{V}_{[3]} \setminus \mathcal{U}$, we have $H_{\mathcal{U}}(\boldsymbol{v}) > 0$ for sufficiently small τ.

Suppose we have some vertex $\boldsymbol{v} \in \mathcal{V}_{[3]} \setminus \mathcal{U}$. Then, $\boldsymbol{v} = \boldsymbol{\beta}_s(t_{s,\lambda})$, $t_{s,\lambda} = x_{s,\lambda}\tau^{\nu_s}$, for some $1 \leq \lambda \leq n_s$ and $1 \leq s \leq 3$, such that $\lambda \notin \{j_{s,1}, j_{s,2}, \ldots, j_{s,k_s}\}$. Then we can transform $H_{\mathcal{U}}(\boldsymbol{v})$ in the form of the determinant $E_{n,m,k}(\tau; \boldsymbol{\mu})$ of Lemma 17, by subtracting the second and third row of $H_{\mathcal{U}}(\boldsymbol{v})$ from its first row and shifting the first column of $H_{\mathcal{U}}(\boldsymbol{v})$ to the right via an even number of column swaps. More precisely, we need to shift the first column of $H_{\mathcal{U}}(\boldsymbol{v})$ to the right so that the values $t_{s,\lambda}, t_{s,j_{s,1}}, t^{\epsilon}_{s,j_{s,1}}, t_{s,j_{s,2}}, t^{\epsilon}_{s,j_{s,2}}, \ldots, t_{s,j_{s,k_s}}, t^{\epsilon}_{s,j_{s,k_s}}$, appear consecutively in the columns of $H_{\mathcal{U}}(\boldsymbol{v})$ and in increasing order. To do that we always need an even number of column swaps, due to the way we have chosen ϵ.

Now, suppose that $\boldsymbol{v} \in \mathcal{V}_1$. Then $H_{\mathcal{U}}(\boldsymbol{v})$ is in the form of the determinant $E_{n,m,k}(\tau; \boldsymbol{\mu})$ of Lemma 17, where $n \leftarrow 2k_1 + 1$, $m \leftarrow 2k_2$, $k \leftarrow 2k_3 + \delta$, $l \leftarrow d + 3$, and $\boldsymbol{\mu} \leftarrow (0, 0, 0, 1, 2, \ldots, d)$. Obviously, $M \geq 2|\boldsymbol{\mu}| = d(d+1)$. Applying now Lemma 17, we deduce that $H_{\mathcal{U}}(\boldsymbol{v})$ can be written as:

$$H_{\mathcal{U}}(\boldsymbol{v}) = C'\tau^{\xi} + \Theta(\tau^{\xi+1}), \qquad \xi = 4 + 2\sum_{i=7}^{2k_1+5} (i-3) + \sum_{i=2k_1+6}^{2k_1+2k_2+3} (i-3),$$

where C' is a positive constant independent of τ. The asymptotic estimate above implies that $H_{\mathcal{U}}(\boldsymbol{v}) > 0$, for sufficiently small τ.

The remaining cases, i.e., the cases $\boldsymbol{v} \in \mathcal{V}_2$ and $\boldsymbol{v} \in \mathcal{V}_3$, are completely analogous and we omit them. We thus conclude that, for any specific choice of U, and for any specific vertex $\boldsymbol{v} \in \mathcal{V}_{[3]} \setminus \mathcal{U}$, there exists some $\tau_{\boldsymbol{v},\mathcal{U}} > 0$ (cf. Lemma 17) that depends on $\boldsymbol{v}$ and $\mathcal{U}$, such that for all $\tau \in (0, \tau_{\boldsymbol{v},\mathcal{U}})$ we have $H_{\mathcal{U}}(\boldsymbol{v}) > 0$. For each k with $3 \leq k \leq \lfloor \frac{d+2}{2} \rfloor$, the number of the sets $\mathcal{U}$ of size k containing at least one vertex from each $\mathcal{V}_i$, $1 \leq i \leq 3$, is

$$\tbinom{n_1+n_2+n_3}{k} - \tbinom{n_1+n_2}{k} - \tbinom{n_1+n_3}{k} - \tbinom{n_2+n_3}{k} + \tbinom{n_1}{k} + \tbinom{n_2}{k} + \tbinom{n_3}{k} = \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \tbinom{n_S}{k}.$$

For each such subset $\mathcal{U}$ we need to consider the $(n_1 + n_2 + n_3 - k)$ vertices in $\mathcal{V}_{[3]} \setminus \mathcal{U}$, therefore it suffices to consider a positive value $\hat{\tau}_{[3]}$ for τ that is small enough, so that all

$$\sum_{k=2}^{\lfloor \frac{d+1}{2} \rfloor} (n_1 + n_2 + n_3 - k) \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \tbinom{n_S}{k},$$

possible determinants $H_{\mathcal{U}}(\boldsymbol{v})$ are strictly positive. For $\tau \leftarrow \hat{\tau}_{[3]}$, our analysis above immediately implies that *for each* set $\mathcal{U}$ the equation $H_{\mathcal{U}}(\boldsymbol{x}) = 0$, $\boldsymbol{x} \in \mathbb{R}^{d+2}$, is the equation of a supporting hyperplane for $\mathcal{C}$ passing through the vertices of $\mathcal{U}$, and those only. In other words, every set $\mathcal{U}$, of k vertices, for $3 \leq k \leq \lfloor \frac{d+2}{2} \rfloor$, with at least one vertex from each $\mathcal{V}_i$, $1 \leq i \leq 3$, defines a $(k-1)$-face of $\mathcal{C}$, which means that

$$f_{k-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset \subset S \subseteq R} (-1)^{3-|S|} \tbinom{n_S}{k}, \quad \text{for all } 3 \leq k \leq \lfloor \tfrac{d+2}{2} \rfloor. \qquad \square$$

Relation (54) now immediately follows from Lemmas 12 and 13. First choose a value $\tau^\star$ for τ, smaller that $\hat{\tau}_R$, for all $2 \leq |R| \leq 3$. Then for this value of τ, the results of both Lemma 12 and Lemma 13 hold true. Moreover, since P_1, P_2 and P_3 are $\lfloor \frac{d}{2} \rfloor$-neighborly for any $\tau > 0$, and since $f_{-1}(\mathcal{F}_R) = (-1)^{|R|-1}$, for all $\emptyset \subset R \subseteq [3]$, while $f_{k-1}(\mathcal{F}_R) = 0$, for all $1 \leq k \leq |R|$, we conclude that, for $\tau \equiv \tau^\star$, relations (54) hold.

Based on the analysis above, as well as the analysis in Section 6.1, we conclude that the upper bounds stated in Theorem 11 are actually tight for any $d \geq 2$. We can, thus, restate Theorem 11 in its complete and definitive form:

Theorem 14. *Let P_1, P_2 and P_3 be three d-polytopes in $\mathbb{R}^d$, $d \geq 2$, with $n_i \geq d+1$ vertices, $1 \leq i \leq 3$. Then, for all $1 \leq k \leq d$, we have:*

$$f_{k-1}(P_1 + P_2 + P_3) \leq f_{k+1}(C_{d+2}(n_{[3]})) - \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k+2-i} \sum_{\emptyset \subset S \subset [3]} (-1)^{|S|} \binom{n_S - d - 3 + i}{i}$$
$$- \delta \binom{\lfloor \frac{d}{2} \rfloor + 1}{k - \lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{3} \binom{n_i - \lfloor \frac{d}{2} \rfloor - 2}{\lfloor \frac{d}{2} \rfloor + 1},$$

where $\delta = d - 2\lfloor \frac{d}{2} \rfloor$, and $n_S = \sum_{i \in S} n_i$. Moreover, for any $d \geq 2$, there exist three d-polytopes in $\mathbb{R}^d$ for which the bounds above are attained for all $1 \leq k \leq d$.

7 Summary and open problems

In this paper we have computed the maximum number of k-faces, $f_k(P_1+P_2+P_3)$, $0 \leq k \leq d-1$, of the Minkowski sum of three d-polytopes P_1, P_2 and P_3 in $\mathbb{R}^d$ as a function of the number of their vertices n_1, n_2 and n_3. When $d = 2$ our expressions reduce to known tight bounds, while for $d = 3$ we show the tightness of our upper bounds by exploiting results from [FW07] and [Wei12]. In four or more dimensions we present a novel construction that achieves the upper bounds: we consider the d-dimensional moment-like curves $\boldsymbol{\gamma}_1(t) = (t, \zeta t^2, \zeta t^3, t^4, \ldots, t^d)$, $\boldsymbol{\gamma}_2(t) = (\zeta t, t^2, \zeta t^3, t^4, \ldots, t^d)$, and $\boldsymbol{\gamma}_3(t) = (\zeta t, \zeta t^2, t^3, t^4, \ldots, t^d)$, and we show that our maximal values are attained when P_i is the d-polytope with vertex set

$$V_i = \{\boldsymbol{\gamma}_i(x_{i,1}\tau^\star), \boldsymbol{\gamma}_i(x_{i,2}\tau^\star), \ldots, \boldsymbol{\gamma}_i(x_{i,n_i}\tau^\star)\}, \qquad i = 1, 2, 3,$$

with $0 < x_{i,1} < x_{i,2} < \cdots < x_{i,n_i}$ and $\zeta = (\tau^\star)^M$. The parameter value $\tau^\star$ is a sufficiently small positive number, while M is chosen sufficiently large.

Our ultimate goal is to extend our results for the Minkowski sum of r d-polytopes in $\mathbb{R}^d$, for $r \geq 4$ and $d \geq 3$. Towards this direction, we can extend our methodology and tools so as to prove relations for r polytopes that generalize certain relations that hold true for two or three polytopes. For example, relation (12) in Lemma 3 generalizes to:

$$h_k(\mathcal{K}_R) = \sum_{\emptyset \subset S \subseteq R} g_k^{(|R|-|S|)}(\mathcal{F}_S), \qquad 0 \leq k \leq d + |R| - 1,$$

while the Dehn-Sommerville-like equations in the same lemma (cf. rel. (14)), generalize to:

$$h_{d+r-1-k}(\mathcal{F}_{[r]}) = h_k(\mathcal{K}_{[r]}), \qquad 0 \leq k \leq d + r - 1, \tag{58}$$

where $[r] = \{1, 2, \ldots, r\}$, while $\mathcal{F}_R$ and $\mathcal{K}_R$, $\emptyset \subset R \subseteq [r]$, are defined as in Section 2. Notice that, since for $r = 1$ we have $\mathcal{F}_{[1]} \equiv \mathcal{K}_{[1]} \equiv \partial P_1$, the equations in (58) reduce to the well-known Dehn-Sommerville equations for a simplicial d-polytope. We can also obtain a generalization of relation (13). Let $\mathcal{Q}$ be the simplicial $(d+r-1)$-sphere we get by performing stellar subdivisions on the non-simplicial faces of the Cayley polytope of the r polytopes. For all $0 \leq k \leq d + r - 1$, we can obtain the following two expressions relating the h-vector elements of $\partial\mathcal{Q}$ with those of $\mathcal{F}_S$ and $\mathcal{K}_S$, $\emptyset \subset S \subseteq [r]$:

$$h_k(\partial Q) = h_k(\mathcal{F}_{[r]}) + \sum_{\emptyset \subset S \subset [r]} \sum_{i=0}^{r-|S|-1} E_{r-|S|,i}\, h_{k-i}(\mathcal{F}_S),$$

$$h_k(\partial Q) = h_k(\mathcal{F}_{[r]}) + \sum_{\emptyset \subset S \subset [r]} \sum_{i=0}^{r-|S|-1} E_{r-|S|,i}\, h_{k-1-i}(\mathcal{K}_S),$$

where $E_{m,k}$, $m \geq k + 1 > 0$, are the Eulerian numbers [GKP89, A00]:

$$E_{m,k} = \sum_{i=0}^{k} (-1)^i \binom{m+1}{i} (k + 1 - i)^m, \qquad m \geq k + 1 > 0.$$

A recurrence relation similar to (36) in Lemma 7 is not as straightforward to obtain. However, we conjecture that the following recurrence relation holds for all $0 \leq k \leq d + r - 2$:

$$h_{k+1}(\mathcal{F}_{[r]}) \leq \frac{n_{[r]} - d - r + 1 + k}{k+1} h_k(\mathcal{F}_{[r]}) + \sum_{i=1}^{r} \frac{n_i}{k+1} g_k(\mathcal{F}_{[r]\setminus\{i\}}), \qquad n_{[r]} = \sum_{i=1}^{r} n_i.$$

The bounds presented in this paper refer to polytopes of the same dimension. We would like to derive similar bounds for two or more polytopes when the dimensions of these polytopes differ, as well as in the special case of simple polytopes. Finally, a similar problem is to express the number of k-faces of the Minkowski sum of r d-polytopes in terms of the number of facets of these polytopes. Results in this direction are known for $d = 2$ and $d = 3$ only. We would like to derive such expressions for any $d \geq 4$ and any number, r, of summands.

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A Omitted & full proofs

A.1 Omitted & full proofs of Section 3

Proof of Lemma 1. The result clearly holds for $m = 0$, since:

$$g_k^{(0)}(\mathcal{Y}) = h_k(\mathcal{Y}) = \sum_{i=0}^{0}(-1)^i\tbinom{0}{i}h_{k-i}(\mathcal{Y}).$$

Suppose the relation holds for some $m \geq 0$. We will show it holds for $m+1$. Indeed:

$$\begin{aligned}
g_k^{(m+1)}(\mathcal{Y}) &= g_k^{(m)}(\mathcal{Y}) - g_{k-1}^{(m)}(\mathcal{Y}) \\
&= \sum_{i=0}^{m}(-1)^i\tbinom{m}{i}h_{k-i}(\mathcal{Y}) - \sum_{i=0}^{m}(-1)^i\tbinom{m}{i}h_{k-1-i}(\mathcal{Y}) \\
&= \sum_{i=0}^{m+1}(-1)^i\tbinom{m}{i}h_{k-i}(\mathcal{Y}) - \sum_{j=1}^{m+1}(-1)^{j-1}\tbinom{m}{j-1}h_{k-j}(\mathcal{Y}) \\
&= \sum_{i=0}^{m+1}(-1)^i\tbinom{m}{i}h_{k-i}(\mathcal{Y}) - \sum_{j=0}^{m+1}(-1)^{j-1}\tbinom{m}{j-1}h_{k-j}(\mathcal{Y}) \\
&= \sum_{i=0}^{m+1}(-1)^i\tbinom{m}{i}h_{k-i}(\mathcal{Y}) + \sum_{i=0}^{m+1}(-1)^i\tbinom{m}{i-1}h_{k-i}(\mathcal{Y}) \\
&= \sum_{i=0}^{m+1}(-1)^i\left[\tbinom{m}{i} + \tbinom{m}{i-1}\right]h_{k-i}(\mathcal{Y}) \\
&= \sum_{i=0}^{m+1}(-1)^i\tbinom{m+1}{i}h_{k-i}(\mathcal{Y}).
\end{aligned}$$

□

Proof of Lemma 2. By replacing $h_{k-\nu-j}(\mathcal{Y})$ from its defining equation, we get:

$$\begin{aligned}
g_{k-\nu}^{(D-\delta-\nu)}(\mathcal{Y}) &= \sum_{j=0}^{D-\delta-\nu}(-1)^j\tbinom{D-\delta-\nu}{j}h_{k-\nu-j}(\mathcal{Y}) \\
&= \sum_{j=0}^{D-\delta-\nu}(-1)^j\tbinom{D-\delta-\nu}{j}\sum_{i=0}^{\delta+1}(-1)^{k-\nu-j-i}\tbinom{\delta+1-i}{\delta+1-k+\nu+j}f_{i-1}(\mathcal{Y}) && (59) \\
&= \sum_{j=0}^{D-\delta-\nu}(-1)^j\tbinom{D-\delta-\nu}{j}\sum_{i=0}^{D+1}(-1)^{k-\nu-j-i}\tbinom{\delta+1-i}{\delta+1-k+\nu+j}f_{i-1}(\mathcal{Y}) && (60) \\
&= \sum_{i=0}^{D+1}(-1)^{k-\nu-i}f_{i-1}(\mathcal{Y})\sum_{j=0}^{D-\delta-\nu}\tbinom{D-\delta-\nu}{j}\tbinom{\delta+1-i}{k-\nu-i-j} && (61) \\
&= \sum_{i=0}^{D+1}(-1)^{k-\nu-i}\tbinom{D+1-\nu-i}{k-\nu-i}f_{i-1}(\mathcal{Y}) && (62)
\end{aligned}$$

$$= \sum_{i=0}^{k-\nu} (-1)^{k-\nu-i} \tbinom{D+1-\nu-i}{D+1-k} f_{i-1}(\mathcal{Y}) \tag{63}$$

$$= \sum_{j=\nu}^{k} (-1)^{k-j} \tbinom{D+1-j}{D+1-k} f_{j-\nu-1}(\mathcal{Y}) \tag{64}$$

$$= \sum_{j=0}^{D+1} (-1)^{k-j} \tbinom{D+1-j}{D+1-k} f_{j-\nu-1}(\mathcal{Y}) \tag{65}$$

$$= \mathcal{S}_k(\mathcal{Y}; D, \nu),$$

where:

- in order to go from (59) to (60), we used that $\binom{\delta+1-i}{\delta+1-k+\nu+j} = 0$ for $i > \delta + 1$,
- in order to go from (61) to (62), we used the combinatorial identity:

$$\sum_{i=0}^{n} \tbinom{n}{i}\tbinom{m}{k-i} = \tbinom{n+m}{k} = \sum_{i=0}^{k} \tbinom{n}{i}\tbinom{m}{k-i} = \tbinom{n+m}{k},$$

  with $n \leftarrow D - \delta - \nu$, $m \leftarrow \delta + 1 - i$, $i \leftarrow j$, $k \leftarrow k - \nu - i$,
- in order to go from (62) to (63), we used that $\binom{D+1-\nu-i}{k-\nu-i} = 0$ for $i > k - \nu$, and that $\binom{D+1-\nu-i}{k-\nu-i} = \binom{D+1-\nu-i}{(D+1-\nu-i)-(k-\nu-i)} = \binom{D+1-\nu-i}{D+1-k}$, and, finally,
- in order to go from (64) to (65), we used that $f_{j-\nu-1}(\mathcal{Y}) = 0$ for $j < \nu$ (i.e., for $j - \nu - 1 < -1$), and that $\binom{D+1-j}{D+1-k} = 0$ for $j > k$. □

A.2 Omitted & full proofs of Section 4

Proof of Lemma 5. Using relation (13), and after rearranging the terms, the left hand side of relation (18) becomes:

$$\overbrace{(k+1)h_{k+1}(\mathcal{F}_{[3]}) + (d+2-k)h_k(\mathcal{F}_{[3]})}^{T_1} + \overbrace{\sum_{R\in\mathfrak{S}_2} [(k+1)h_{k+1}(\mathcal{F}_R) + (d+2-k)h_k(\mathcal{F}_R)]}^{T_2}$$
$$+ \overbrace{\sum_{R\in\mathfrak{S}_1} [(k+1)h_{k+1}(\mathcal{F}_R) + (d+2-k)h_k(\mathcal{F}_R)]}^{T_3} + \overbrace{\sum_{R\in\mathfrak{S}_1} [(k+1)h_k(\mathcal{F}_R) + (d+2-k)h_{k-1}(\mathcal{F}_R)]}^{T_4}. \tag{66}$$

We are going to analyze each term in the expression above separately. For any $R \in \mathfrak{S}_2$: (i) the relation at the top of page 18 in [KT12, Lemma 3.2], (ii) relations (12), with $R \in \mathfrak{S}_2$, and (iii) relation (3.9) in [KT12], give:

$$(k+1)h_{k+1}(\mathcal{F}_R) + (d+1-k)h_k(\mathcal{F}_R) = \sum_{i\in R}\sum_{v\in\mathcal{V}_i} [h_k(\mathcal{K}_R/v) - g_k(\mathcal{K}_{\{i\}}/v)]$$
$$= \sum_{v\in\mathcal{V}_R} h_k(\mathcal{K}_R/v) - \sum_{\emptyset\subset S\subset R}\sum_{v\in\mathcal{V}_S} g_k(\mathcal{K}_S/v).$$

Hence term T_2 can be rewritten as:

$$T_2 = \sum_{R\in\mathfrak{S}_2} h_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_2}\sum_{v\in\mathcal{V}_R} h_k(\mathcal{K}_R/v) - \sum_{R\in\mathfrak{S}_2}\sum_{\emptyset\subset S\subset R}\sum_{v\in\mathcal{V}_S} g_k(\mathcal{K}_S/v)$$
$$= \overbrace{\sum_{R\in\mathfrak{S}_2} h_k(\mathcal{F}_R)}^{T_5} + \overbrace{\sum_{R\in\mathfrak{S}_2}\sum_{v\in\mathcal{V}_R} h_k(\mathcal{K}_R/v)}^{T_6} - \overbrace{2\sum_{R\in\mathfrak{S}_1}\sum_{v\in\mathcal{V}_R} g_k(\mathcal{K}_R/v)}^{T_7}. \tag{67}$$

Applying relation (17) to the $(d-1)$-complex $\mathcal{F}_R$, $R \in \mathfrak{S}_1$, and using the identity $\mathcal{F}_R \equiv \mathcal{K}_R (\equiv \partial P_R)$, we derive the following expressions:

$$
\begin{aligned}
(k+1)h_{k+1}(\mathcal{F}_R) + (d-k)h_k(\mathcal{F}_R) &= \sum_{v \in \mathcal{V}_R} h_k(\mathcal{K}_R/v), \\
kh_k(\mathcal{F}_R) + (d-(k-1))h_{k-1}(\mathcal{F}_R) &= \sum_{v \in \mathcal{V}_R} h_{k-1}(\mathcal{K}_R/v),
\end{aligned}
$$

which, in turn yield the following expansions for T_3 and T_4:

$$
T_3 = \overbrace{\sum_{R \in \mathfrak{S}_1} \sum_{v \in \mathcal{V}_R} h_k(\mathcal{K}_R/v)}^{T_8} + \overbrace{2 \sum_{R \in \mathfrak{S}_1} h_k(\mathcal{F}_R)}^{T_9}, \tag{68}
$$

$$
T_4 = \overbrace{\sum_{R \in \mathfrak{S}_1} \sum_{v \in \mathcal{V}_R} h_{k-1}(\mathcal{K}_R/v)}^{T_{10}} + \overbrace{\sum_{R \in \mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)]}^{T_{11}}. \tag{69}
$$

On the other hand, utilizing the expressions in Lemma 4, we arrive at the following expansion for the right-hand side of (18):

$$
\begin{aligned}
&\sum_{i=1}^{3} \sum_{v \in \mathcal{V}_i} \left[h_k(\mathcal{K}_{[3]}/v) + \sum_{\{i\} \subseteq R \subset [3]} h_{k-1}(\mathcal{K}_R/v) + h_{k-2}(\mathcal{K}_{\{i\}}/v) \right] \\
&+ \sum_{R \in \mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)] + \sum_{R \in \mathfrak{S}_2} \sum_{\emptyset \subset S \subseteq R} h_k(\mathcal{F}_S).
\end{aligned} \tag{70}
$$

Since

$$
\begin{aligned}
\sum_{i=1}^{3} \sum_{v \in \mathcal{V}_i} h_k(\mathcal{K}_{[3]}/v) &= \sum_{v \in \mathcal{V}_{[3]}} h_k(\mathcal{K}_{[3]}/v), \\
\sum_{i=1}^{3} \sum_{v \in \mathcal{V}_i} \sum_{\{i\} \subseteq R \subset [3]} h_{k-1}(\mathcal{K}_R/v) &= \sum_{R \in \mathfrak{S}_2} \sum_{v \in \mathcal{V}_R} h_{k-1}(\mathcal{K}_R/v) + \sum_{R \in \mathfrak{S}_1} \sum_{v \in \mathcal{V}_R} h_{k-1}(\mathcal{K}_R/v), \\
\sum_{i=1}^{3} \sum_{v \in \mathcal{V}_i} h_{k-2}(\mathcal{K}_{\{i\}}/v) &= \sum_{R \in \mathfrak{S}_1} \sum_{v \in \mathcal{V}_R} h_{k-2}(\mathcal{K}_R/v),
\end{aligned}
$$

and

$$
\sum_{R \in \mathfrak{S}_2} \sum_{\emptyset \subset S \subseteq R} h_k(\mathcal{F}_S) = \sum_{R \in \mathfrak{S}_2} h_k(\mathcal{F}_R) + 2 \sum_{R \in \mathfrak{S}_1} h_k(\mathcal{F}_R),
$$

the expression in (70) can be rewritten in the following more convenient form:

$$
\begin{aligned}
&\overbrace{\sum_{v \in \mathcal{V}_{[3]}} h_k(\mathcal{K}_{[3]}/v)}^{T_{12}} + \overbrace{\sum_{R \in \mathfrak{S}_2} \sum_{v \in \mathcal{V}_R} h_{k-1}(\mathcal{K}_R/v)}^{T_{13}} + \overbrace{\sum_{R \in \mathfrak{S}_1} \sum_{v \in \mathcal{V}_R} [h_{k-1}(\mathcal{K}_R/v) + h_{k-2}(\mathcal{K}_R/v)]}^{T_{14}} \\
&+ \overbrace{\sum_{R \in \mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)] + \sum_{R \in \mathfrak{S}_2} h_k(\mathcal{F}_R) + 2 \sum_{R \in \mathfrak{S}_1} h_k(\mathcal{F}_R)}^{T_{15}}.
\end{aligned}
$$

Solving relation (18) in terms of the term T_1, we get:

$$
T_1 = T_{12} + T_{13} + T_{14} + T_{15} - (T_2 + T_3 + T_4)
$$

$$
\begin{aligned}
&= T_{12} + T_{13} + T_{14} + T_{15} - [(T_5 + T_6 - T_7) + (T_8 + T_9) + (T_{10} + T_{11})] \\
&= T_{12} + (T_{13} - T_6) + (T_{14} + T_7 - T_8 - T_{10}) + (T_{15} - T_5 - T_9 - T_{11}) \\
&= T_{12} + (T_{13} - T_6) + (T_{14} + T_7 - T_8 - T_{10}),
\end{aligned}
$$

where we used the fact that the terms T_5, T_9 and T_{11} cancel-out with the term T_{15}. Observe now that:

$$
T_{13} - T_6 = \sum_{R\in\mathfrak{S}_2}\sum_{v\in\mathcal{V}_R} h_{k-1}(\mathcal{K}_R/v) - \sum_{R\in\mathfrak{S}_2}\sum_{v\in\mathcal{V}_R} h_k(\mathcal{K}_R/v) = -\sum_{R\in\mathfrak{S}_2}\sum_{v\in\mathcal{V}_R} g_k(\mathcal{K}_R/v),
$$

while

$$
\begin{aligned}
T_{14} + T_7 - T_8 - T_{10} &= \sum_{R\in\mathfrak{S}_1}\sum_{v\in\mathcal{V}_R} [h_{k-1}(\mathcal{K}_R/v) + h_{k-2}(\mathcal{K}_R/v)] + 2\sum_{R\in\mathfrak{S}_1}\sum_{v\in\mathcal{V}_R} g_k(\mathcal{K}_R/v) \\
&\quad - \sum_{R\in\mathfrak{S}_1}\sum_{v\in\mathcal{V}_R} h_k(\mathcal{K}_R/v) - \sum_{R\in\mathfrak{S}_1}\sum_{v\in\mathcal{V}_R} h_{k-1}(\mathcal{K}_R/v) \\
&= \sum_{R\in\mathfrak{S}_1}\sum_{v\in\mathcal{V}_R} \{h_{k-1}(\mathcal{K}_R/v) + h_{k-2}(\mathcal{K}_R/v) + 2[h_k(\mathcal{K}_R/v) - h_{k-1}(\mathcal{K}_R/v)] \\
&\qquad\qquad - h_k(\mathcal{K}_R/v) - h_{k-1}(\mathcal{K}_R/v)\} \\
&= \sum_{R\in\mathfrak{S}_1}\sum_{v\in\mathcal{V}_R} [h_k(\mathcal{K}_R/v) - 2h_{k-1}(\mathcal{K}_R/v) + h_{k-2}(\mathcal{K}_R/v)] \\
&= \sum_{R\in\mathfrak{S}_1}\sum_{v\in\mathcal{V}_R} g_k^{(2)}(\mathcal{K}_R/v).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
T_1 &= \sum_{v\in\mathcal{V}_{[3]}} h_k(\mathcal{K}_{[3]}/v) - \sum_{R\in\mathfrak{S}_2}\sum_{v\in\mathcal{V}_R} g_k(\mathcal{K}_R/v) + \sum_{R\in\mathfrak{S}_1}\sum_{v\in\mathcal{V}_R} g_k^{(2)}(\mathcal{K}_R/v) \\
&= \sum_{\emptyset\subset R\subseteq[3]} (-1)^{3-|R|} \sum_{v\in\mathcal{V}_R} g_k^{(3-|R|)}(\mathcal{K}_R/v). \qquad \square
\end{aligned}
$$

Proof of Lemma 7. By Lemma 6, relation (25) yields:

$$
\begin{aligned}
(k+1)h_{k+1}(\mathcal{F}_{[3]}) + (d+2-k)h_k(\mathcal{F}_{[3]}) &\le \sum_{\emptyset\subset R\subseteq[3]} (-1)^{3-|R|} \sum_{v\in\mathcal{V}_R} g_k^{(3-|R|)}(\mathcal{K}_R) \\
&= n_{[3]}h_k(\mathcal{K}_{[3]}) - \sum_{R\in\mathfrak{S}_2} n_R g_k(\mathcal{K}_R) + \sum_{R\in\mathfrak{S}_1} n_R g_k^{(2)}(\mathcal{K}_R)
\end{aligned} \tag{71}
$$

By relation (12) with $R \equiv [3]$, we can write $h_k(\mathcal{K}_{[3]})$ as:

$$
h_k(\mathcal{K}_{[3]}) = h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} g_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_1} g_k^{(2)}(\mathcal{F}_R), \tag{72}
$$

whereas from relation (12) for all $R \in \mathfrak{S}_2$ we easily get:

$$
g_k(\mathcal{K}_R) = g_k(\mathcal{F}_R) + \sum_{\emptyset\subset S\subset R} g_k^{(2)}(\mathcal{F}_S). \tag{73}
$$

Since $\mathcal{K}_R \equiv \mathcal{F}_R$, for any $R \in \mathfrak{S}_1$, we can employ relations (72) and (73) to rewrite the right hand side of (71) as follows:

$$\begin{aligned}
n_{[3]} h_k(\mathcal{K}_{[3]}) &- \sum_{R\in\mathfrak{S}_2} n_R\, g_k(\mathcal{K}_R) + \sum_{R\in\mathfrak{S}_1} n_R g_k^{(2)}(\mathcal{K}_R) \\
&= n_{[3]} h_k(\mathcal{F}_{[3]}) + n_{[3]} \sum_{R\in\mathfrak{S}_2} g_k(\mathcal{F}_R) + n_{[3]} \sum_{R\in\mathfrak{S}_1} g_k^{(2)}(\mathcal{F}_R) \\
&\quad - \left[\sum_{R\in\mathfrak{S}_2} n_R g_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_2} n_R \sum_{\emptyset\subset S\subset R} g_k^{(2)}(\mathcal{F}_S) \right] + \sum_{R\in\mathfrak{S}_1} n_R g_k^{(2)}(\mathcal{F}_R) \\
&= n_{[3]} h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} (n_{[3]} - n_R) g_k(\mathcal{F}_R) \\
&\quad + \overbrace{\left[n_{[3]} \sum_{i=1}^{3} g_k^{(2)}(\mathcal{F}_{\{i\}}) - \sum_{R\in\mathfrak{S}_2} n_R \sum_{\emptyset\subset S\subset R} g_k^{(2)}(\mathcal{F}_S) + \sum_{i=1}^{3} n_i g_k^{(2)}(\mathcal{F}_{\{i\}}) \right]}^{T}.
\end{aligned}$$

Using the identity:

$$\sum_{R\in\mathfrak{S}_2} n_R \sum_{\emptyset\subset S\subset R} g_k^{(2)}(\mathcal{F}_S) = 2\sum_{i=1}^{3} n_i g_k^{(2)}(\mathcal{F}_{\{i\}}) + \sum_{i=1}^{3} n_{[3]\setminus\{i\}}\, g_k^{(2)}(\mathcal{F}_{\{i\}}),$$

we see that the last term (term T) in the relation above vanishes:

$$\begin{aligned}
n_{[3]} &\sum_{i=1}^{3} g_k^{(2)}(\mathcal{F}_{\{i\}}) - \sum_{R\in\mathfrak{S}_2} n_R \sum_{\emptyset\subset S\subset R} g_k^{(2)}(\mathcal{F}_S) + \sum_{i=1}^{3} n_{\{i\}} g_k^{(2)}(\mathcal{F}_{\{i\}}) \\
&= n_{[3]} \sum_{i=1}^{3} g_k^{(2)}(\mathcal{F}_{\{i\}}) - \left[2\sum_{i=1}^{3} n_i g_k^{(2)}(\mathcal{F}_{\{i\}}) + \sum_{i=1}^{3} n_{[3]\setminus\{i\}}\, g_k^{(2)}(\mathcal{F}_{\{i\}}) \right] + \sum_{i=1}^{3} n_i g_k^{(2)}(\mathcal{F}_{\{i\}}) \\
&= \sum_{i=1}^{3} (n_{[3]} - 2n_i - n_{[3]\setminus\{i\}} + n_i)\, g_k^{(2)}(\mathcal{F}_{\{i\}}) = 0.
\end{aligned}$$

Hence, relation (71) simplifies to:

$$\begin{aligned}
(k+1)h_{k+1}(\mathcal{F}_{[3]}) + (d+2-k)h_k(\mathcal{F}_{[3]}) &\le n_{[3]} h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} (n_{[3]} - n_R) g_k(\mathcal{F}_R) \\
&= n_{[3]} h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} n_{[3]\setminus R} g_k(\mathcal{F}_R) = n_{[3]} h_k(\mathcal{F}_{[3]}) + \sum_{i=1}^{3} n_i g_k(\mathcal{F}_{[3]\setminus\{i\}}),
\end{aligned}$$

from which we obtain the relation in the statement of the lemma. □

A.3 Omitted & full proofs of Section 5

Proof of Lemma 10. The bound for $h_k(\mathcal{K}_{[3]})$ holds as equality for $k = 0$, since by relation (12) with $R = [3]$, (see also (72)), we have

$$h_0(\mathcal{K}_{[3]}) = h_0(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} g_0(\mathcal{F}_R) + \sum_{i=1}^{3} g_0^{(2)}(\partial P_i)$$

$$= 1 + \sum_{R\in\mathfrak{S}_2} [h_0(\mathcal{F}_R) - h_{-1}(\mathcal{F}_R)] + \sum_{i=1}^{3}[h_0(\partial P_i) - 2h_{-1}(\partial P_i) + h_{-2}(\partial P_i)]$$

$$= 1 + \sum_{R\in\mathfrak{S}_2} [(-1) - 0] + \sum_{i=1}^{3}[1 - 2\cdot 0 + 0] = 1.$$

Suppose now that $k \geq 1$. Then, using relation (39), we get, for $k \geq 1$:

$$\begin{aligned}
h_k(\mathcal{K}_{[3]}) &= h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2} g_k(\mathcal{F}_R) + \sum_{R\in\mathfrak{S}_1} g_k^{(2)}(\mathcal{F}_R)\\
&\leq h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2}\left[\tfrac{n_R-d-3}{k}h_{k-1}(\mathcal{F}_R) + \sum_{i\in R}\tfrac{n_{R\setminus\{i\}}}{k}g_{k-1}(\partial P_i)\right] + \sum_{i=1}^{3} g_k^{(2)}(\partial P_i)\\
&= h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2}\tfrac{n_R-d-3}{k}h_{k-1}(\mathcal{F}_R) + \sum_{i=1}^{3}\left[\tfrac{n_{[3]\setminus\{i\}}}{k}g_{k-1}(\partial P_i) + g_k(\partial P_i) - g_{k-1}(\partial P_i)\right],
\end{aligned}$$

which finally yields:

$$h_k(\mathcal{K}_{[3]}) \leq h_k(\mathcal{F}_{[3]}) + \sum_{R\in\mathfrak{S}_2}\tfrac{n_R-d-3}{k}h_{k-1}(\mathcal{F}_R) + \sum_{i=1}^{3}\left[\tfrac{n_{[3]\setminus\{i\}}-k}{k}g_{k-1}(\partial P_i) + g_k(\partial P_i)\right]. \tag{74}$$

Since $n_R-d-3 \geq 2(d+1)-d-3 = d-1 > 0$, for $R \in \mathfrak{S}_2$, and $n_{[3]\setminus\{i\}}-k \geq 2(d+1)-(d+2) = d > 0$ for any $0 \leq k \leq d+2$, we can use the upper bounds for $h_k(\mathcal{F}_{[3]})$ and $h_{k-1}(\mathcal{F}_R)$, $R \in \mathfrak{S}_2$ from Lemma 9 and [KT12, Lemma 3.3], respectively, in conjunction with the known upper bounds for the elements of the g-vector of a d-polytope (cf. [Zie95, Corollary 8.38]). More precisely:

$$\begin{aligned}
h_k(\mathcal{K}_{[3]}) \leq{}& \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S-d-3+k}{k} + \sum_{R\in\mathfrak{S}_2}\tfrac{n_R-d-3}{k}\left[\tbinom{n_R-d-2+k-1}{k-1} - \sum_{i\in R}\tbinom{n_i-d-2+k-1}{k-1}\right]\\
&+ \sum_{i=1}^{3}\left[\tfrac{n_{[3]\setminus\{i\}}-k}{k}\tbinom{n_i-d-2+k-1}{k-1} + g_k(\partial P_i)\right]\\
={}& \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\tbinom{n_S-d-3+k}{k} + \sum_{R\in\mathcal{R}_2}\tfrac{n_R-d-3}{k}\left[\tbinom{n_R-d-3+k}{k-1} - \sum_{i\in R}\tbinom{n_i-d-3+k}{k-1}\right]\\
&+ \sum_{i=1}^{3}\left[\tfrac{n_{[3]\setminus\{i\}}}{k}\tbinom{n_i-d-3+k}{k-1} - \tbinom{n_i-d-3+k}{k-1} + \tbinom{n_i-d-2+k}{k} + g_k(\partial P_i) - \tbinom{n_i-d-2+k}{k}\right]\\
={}& \tbinom{n_{[3]}-d-3+k}{k} - \sum_{i=1}^{3}\tbinom{n_{[3]\setminus\{i\}}-d-3+k}{k} + \sum_{i=1}^{3}\tbinom{n_i-d-3+k}{k}\\
&+ \sum_{R\in\mathfrak{S}_2}\tfrac{n_R-d-3}{k}\left[\tbinom{n_R-d-3+k}{k-1} - \sum_{i\in R}\tbinom{n_i-d-3+k}{k-1}\right]\\
&+ \sum_{i=1}^{3}\tfrac{n_{[3]\setminus\{i\}}}{k}\tbinom{n_i-d-3+k}{k-1} + \sum_{i=1}^{3}\tbinom{n_i-d-3+k}{k} + \sum_{i=1}^{3}\left[g_k(\partial P_i) - \tbinom{n_i-d-2+k}{k}\right].
\end{aligned}$$

From the proof of Lemma 8 it is easy to see that:

$$\sum_{R\in\mathfrak{S}_2}\tfrac{n_R-d-3}{k}\left[\tbinom{n_R-d-3+k}{k-1} - \sum_{i\in R}\tbinom{n_i-d-3+k}{k-1}\right] + \sum_{i=1}^{3}\tfrac{n_{[3]\setminus\{i\}}}{k}\tbinom{n_i-d-3+k}{k-1}$$

$$= \sum_{R\in\mathfrak{S}_2}\left[\binom{n_R-d-3+k}{k} - \sum_{i\in R}\binom{n_i-d-3+k}{k}\right] = \sum_{i=1}^{3}\binom{n_{[3]\setminus\{i\}}-d-3+k}{k} - 2\sum_{i=1}^{3}\binom{n_i-d-3+k}{k}$$

Hence we have:

$$\begin{aligned} h_k(\mathcal{K}_{[3]}) \le& \binom{n_{[3]}-d-3+k}{k} - \sum_{i=1}^{3}\binom{n_{[3]\setminus\{i\}}-d-3+k}{k} + \sum_{i=1}^{3}\binom{n_i-d-3+k}{k} + \sum_{i=1}^{3}\binom{n_{[3]\setminus\{i\}}-d-3+k}{k} \\ &- 2\sum_{i=1}^{3}\binom{n_i-d-3+k}{k} + \sum_{i=1}^{3}\binom{n_i-d-3+k}{k} + \sum_{i=1}^{3}\left[g_k(\partial P_i) - \binom{n_i-d-2+k}{k}\right] \\ =& \binom{n_{[3]}-d-3+k}{k} + \sum_{i=1}^{3}\left[g_k(\partial P_i) - \binom{n_i-d-2+k}{k}\right]. \end{aligned}$$

Since $g_k(\partial P_i) - \binom{n_i-d-2+k}{k} \le 0$, for all $k \ge 0$, we get the sought-for bound in (43) for $0 \le k \le d+2$. Furthermore, for d odd and $k = \lfloor\frac{d}{2}\rfloor + 1$, we have $g_k(\partial P_i) = 0$, which yields the bound in (44).

To prove the equality claim, we distinguish between the cases $k \le \lfloor\frac{d}{2}\rfloor$, and $k = \lfloor\frac{d}{2}\rfloor + 1$ with d odd. Consider the case $k \le \lfloor\frac{d}{2}\rfloor$ first, and assume that $h_k(\mathcal{K}_{[3]}) = \binom{n_{[3]}-d-3+k}{k}$. From relation (74) we deduce that both $h_k(\mathcal{F}_{[3]})$ and $g_k(\partial P_i)$, $1 \le i \le 3$, must be equal to their maximum values, since otherwise we would have that $h_k(\mathcal{K}_{[3]}) < \binom{n_{[3]}-d-3+k}{k}$. In view of Lemma 9, the maximality of $h_k(\mathcal{F}_{[3]})$ implies that $f_{l-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\binom{n_S}{l}$, for all $0 \le l \le k$, whereas the maximality of $g_k(\partial P_i)$ implies that P_i is k-neighborly, for all $1 \le i \le 3$, i.e., for all $1 \le i \le 3$, $f_{l-1}(\partial P_i) = f_{l-1}(\mathcal{F}_{\{i\}}) = \binom{n_i}{l}$, for all $0 \le l \le k$. But then we also have that $g_{k-1}(\partial P_i) = \binom{n_i-d-2+k-1}{k-1}$, which gives:

$$g_k^{(2)}(\partial P_i) = g_k(\partial P_i) - g_{k-1}(\partial P_i) = \binom{n_i-d-2+k}{k} - \binom{n_i-d-2-k-1}{k-1} = \binom{n_i-d-3+k}{k}. \qquad (75)$$

By relation (36), the maximality of $h_k(\mathcal{F}_{[3]})$ implies that $g_{k-1}(\mathcal{F}_{[3]\setminus\{i\}})$ attains its maximum value for all $1 \le i \le 3$. By following the argumentation in the proof of Lemma 8, the maximality of $g_{k-1}(\mathcal{F}_{[3]\setminus\{i\}})$ further implies that $h_l(\mathcal{F}_{[3]\setminus\{i\}})$ is maximal, for all $0 \le l \le k-1$. Solving, now, equation (12) (for $R \equiv [3]$) in terms of the sum of the $h_k(\mathcal{F}_{[3]\setminus\{i\}})$'s we get:

$$\sum_{i=1}^{3} h_k(\mathcal{F}_{[3]\setminus\{i\}}) = h_k(\mathcal{K}_{[3]}) - h_k(\mathcal{F}_{[3]}) + \sum_{i=1}^{3} h_{k-1}(\mathcal{F}_{[3]\setminus\{i\}}) - \sum_{i=1}^{3} g_k^{(2)}(\partial P_i).$$

Substituting in the above equation the values for $h_k(\mathcal{K}_{[3]})$, $h_k(\mathcal{F}_{[3]})$, $h_{k-1}(\mathcal{F}_{[3]\setminus\{i\}})$ and $g_k^{(2)}(\partial P_i)$, it is easy to verify that

$$\sum_{i=1}^{3} h_k(\mathcal{F}_{[3]\setminus\{i\}}) = \sum_{i=1}^{3}\left[\binom{n_{[3]\setminus\{i\}}-d-2+k}{k} - \sum_{j\in[3]\setminus\{i\}}\binom{n_j-d-2+k}{k}\right].$$

In other words, the sum of the $h_k(\mathcal{F}_{[3]\setminus\{i\}})$'s attains its maximum value, which implies that each of the summands attains its maximum value. We thus conclude that $h_l(\mathcal{F}_{[3]\setminus\{i\}})$ is maximal, for all $0 \le l \le k$, which, by [KT12, Lemma 3.3], implies that, for all $R \in \mathfrak{S}_2$, $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{2-|S|}\binom{n_S}{l}$, for all $0 \le l \le k$.

Let us now consider the reverse direction and assume that for all $\emptyset \subset R \subseteq [3]$, $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S}{l}$, for all $0 \le l \le k$ (for $k \le \lfloor\frac{d}{2}\rfloor$, $\min\{k, \lfloor\frac{d+|R|-1}{2}\rfloor\} = k$). Using Lemma 9, the condition above, for $R = [3]$, implies that $h_l(\mathcal{F}_{[3]})$ attains its upper bound value for all $0 \le l \le k$. Using [KT12, Lemma 3.3], the condition above, for $R \in \mathfrak{S}_2$, implies that

$h_l(\mathcal{F}_R)$ attains its upper bound value for all $0 \le l \le k$, and thus $g_k(\mathcal{F}_R)$ attains its upper bound value. Finally, the condition above, for $1 \le i \le 3$, implies that P_i is k-neighborly, which means that $g_l(\partial P_i) = g_l(\mathcal{F}_{\{i\}}) = \binom{n_i-d-2+l}{l}$, for all $0 \le l \le k$, and thus (cf. (75)) $g_k^{(2)}(\partial P_i) = g_k^{(2)}(\mathcal{F}_{\{i\}}) = \binom{n_i-d-3+k}{k}$. Appealing now to relation (12) for $R \equiv [3]$, it is easy to verify that $h_k(\mathcal{K}_{[3]}) = \binom{n_{[3]}-d-3+k}{k}$.

We end the equality claim proof by considering the case $k = \lfloor \frac{d}{2} \rfloor + 1$, for d odd. Since for d odd, $g_{\lfloor \frac{d}{2} \rfloor+1}(\partial P_i) = 0$, relation (74), simplifies to:

$$h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{K}_{[3]}) \le h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{F}_{[3]}) + \sum_{R \in \mathfrak{S}_2} \tfrac{n_R-d-3}{\lfloor \frac{d}{2} \rfloor+1} h_{\lfloor \frac{d}{2} \rfloor}(\mathcal{F}_R) + \sum_{i=1}^{3} \tfrac{n_{[3]\setminus\{i\}}-\lfloor \frac{d}{2} \rfloor-1}{\lfloor \frac{d}{2} \rfloor+1} g_{\lfloor \frac{d}{2} \rfloor}(\partial P_i), \tag{76}$$

while relation (12) (with $R \equiv [3]$) simplifies to:

$$h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{K}_{[3]}) = h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{F}_{[3]}) + \sum_{i=1}^{3} g_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{F}_{[3]\setminus\{i\}}) - \sum_{i=1}^{3} g_{\lfloor \frac{d}{2} \rfloor}(\partial P_i). \tag{77}$$

The argument in this case is essentially the same as before. Assuming that $h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{K}_{[3]})$ is maximal, we deduce, from (76), that both $h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{F}_{[3]})$ and $g_{\lfloor \frac{d}{2} \rfloor}(\partial P_i)$ are maximal, which, imply, respectively, that $f_{l-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \binom{n_S}{l}$, for all $0 \le l \le \lfloor \frac{d}{2} \rfloor + 1 = \lfloor \frac{d+2}{2} \rfloor$, and that, for all $1 \le i \le 3$, $f_{l-1}(\mathcal{F}_{\{i\}}) = \binom{n_i}{l}$, for all $0 \le l \le \lfloor \frac{d}{2} \rfloor$. The maximality of $h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{F}_{[3]})$ implies also the maximality of $g_l(\mathcal{F}_R)$, for all $R \in \mathfrak{S}_2$, and for all $0 \le l \le \lfloor \frac{d}{2} \rfloor$, and thus the maximality of $h_l(\mathcal{F}_R)$, for all $R \in \mathfrak{S}_2$, and for all $0 \le l \le \lfloor \frac{d}{2} \rfloor$. By solving equation (77) in terms of the sum of the $h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{F}_R)$'s, we also deduce that $h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{F}_R)$ is maximal, for all $R \in \mathfrak{S}_2$. Hence, we have that $h_l(\mathcal{F}_R)$ is maximal, for all $R \in \mathfrak{S}_2$, and for all $0 \le l \le \lfloor \frac{d}{2} \rfloor + 1 = \lfloor \frac{d+1}{2} \rfloor$, which, by [KT12, Lemma 3.3], gives that $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{2-|S|} \binom{n_S}{l}$, for all $R \in \mathfrak{S}_2$, and for all $0 \le l \le \lfloor \frac{d+1}{2} \rfloor$.
Assuming now that, for all $\emptyset \subset R \subseteq [3]$, $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S}{l}$, for all $0 \le l \le \min\{\lfloor \frac{d}{2} \rfloor + 1, \lfloor \frac{d+|R|-1}{2} \rfloor\}$, we deduce, from Lemma 9, that that $h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{F}_{[3]})$ attains its upper bound value for all $0 \le l \le \lfloor \frac{d+2}{2} \rfloor = \lfloor \frac{d}{2} \rfloor + 1$. Furthermore, Lemma 3.3 in [KT12], implies that, for all $R \in \mathfrak{S}_2$, $h_l(\mathcal{F}_R)$ attains its upper bound value for all $0 \le l \le \lfloor \frac{d+1}{2} \rfloor = \lfloor \frac{d}{2} \rfloor + 1$, which means that $g_{\lfloor \frac{d+1}{2} \rfloor}(\mathcal{F}_R)$ attains its upper bound value, for all $R \in \mathfrak{S}_2$. Finally, our assumption above, implies that, for all $1 \le i \le 3$, P_i is neighborly, which means that $g_{\lfloor \frac{d}{2} \rfloor}(\partial P_i) = \binom{n_i-\lfloor \frac{d}{2} \rfloor-3}{\lfloor \frac{d}{2} \rfloor}$. Appealing to relation (77) above, it is easy to verify that $h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{K}_{[3]})$ attains its upper bound in (44). $\square$

B Asymptotic analysis of Vandermonde-like determinants

We start by introducing what is known as *Laplace's Expansion Theorem* for determinants (see [Gan60, HK71] for details and proofs). Consider a $n \times n$ matrix A. Let $\boldsymbol{r} = (r_1, r_2, \ldots, r_k)$, be a vector of k row indices for A, where $1 \le k < n$ and $1 \le r_1 < r_2 < \ldots < r_k \le n$. Let $\boldsymbol{c} = (c_1, c_2, \ldots, c_k)$ be a vector of k column indices for A, where $1 \le k < n$ and $1 \le c_1 < c_2 < \ldots < c_k \le n$. We denote by $S(A; \boldsymbol{r}, \boldsymbol{c})$ the $k \times k$ submatrix of A constructed by keeping the entries of A that belong to a row in $\boldsymbol{r}$ and a column in $\boldsymbol{c}$. The *complementary submatrix* for $S(A; \boldsymbol{r}, \boldsymbol{c})$, denoted by $\bar{S}(A; \boldsymbol{r}, \boldsymbol{c})$, is the $(n-k) \times (n-k)$ submatrix of A constructed by removing the rows and columns of A in $\boldsymbol{r}$ and $\boldsymbol{c}$, respectively. Then, the determinant of A can be computed by expanding in terms of the k columns of A in $\boldsymbol{c}$ according to the following theorem.

Theorem 15 (Laplace's Expansion Theorem). *Let A be a $n \times n$ matrix. Let $\boldsymbol{c} = (c_1, c_2, \ldots, c_k)$ be a vector of k column indices for A, where $1 \leq k < n$ and $1 \leq c_1 < c_2 < \ldots < c_k \leq n$. Then:*

$$\det(A) = \sum_{\boldsymbol{r}} (-1)^{|\boldsymbol{r}|+|\boldsymbol{c}|} \det(S(A; \boldsymbol{r}, \boldsymbol{c})) \det(\bar{S}(A; \boldsymbol{r}, \boldsymbol{c})), \tag{78}$$

where $|\boldsymbol{r}| = r_1 + r_2 + \ldots + r_k$, $|\boldsymbol{c}| = c_1 + c_2 + \ldots + c_k$, and the summation is taken over all row vectors $\boldsymbol{r} = (r_1, r_2, \ldots, r_k)$ of k row indices for A, where $1 \leq r_1 < r_2 < \ldots < r_k \leq n$.

In what follows we recall some facts concerning generalized Vandermonde determinants that will be in use to us later. Let $n \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, where we require that $0 \leq \mu_1 < \mu_2 < \ldots < \mu_n$. The *generalized Vandermonde determinant*, denoted by $\mathrm{GVD}(\boldsymbol{x}; \boldsymbol{\mu})$, is the $n \times n$ determinant whose i-th row is the vector $\boldsymbol{x}$ with all its entries raised to μ_i. While there is no general formula for the generalized Vandermonde determinant, it is a well-known fact that, if the elements of $\boldsymbol{x}$ are in strictly increasing order, then $\mathrm{GVD}(\boldsymbol{x}; \boldsymbol{\mu}) > 0$ (for example, see [Gan05] for a proof of this fact).

In the remainder of this section we consider two determinants that are parameterized by a positive parameter τ, and we study their asymptotic behavior with respect to τ. These determinants are generalizations of the determinants that arise in the proofs of Lemmas 12 and 13 in Section 6, and are directly associated with the equations of some appropriately defined supporting hyperplanes for the faces of $\mathcal{F}_R$ where $R \in \mathfrak{S}_2$ or $R \equiv [3]$ (recall that $\mathcal{F}_R$ stands for the set of faces of the Cayley polytope of $|R|$ polytopes P_i, $i \in R$, with the property that each face in $\mathcal{F}_R$ has at least one vertex from each polytope P_i). The two determinants that we study are generalized-Vandermonde-like determinants that are polynomial functions of τ, and correspond, respectively, to the two cases $R \in \mathfrak{S}_2$ and $R \equiv [3]$ mentioned above. Since in Section 6 we are interested in small values of τ, our asymptotic analysis in the two lemmas below is targeted towards revealing the term of τ of minimal exponent.

We start-off with the generalized version of the determinant that arises in the upper bound tightness construction in Section 6 when $R \in \mathfrak{S}_2$.

Lemma 16. *Fix two integers $m \geq 2$ and $n \geq 2$, with $n + m \geq 5$. Let $D_{n,m}(\tau; I, J, \boldsymbol{\mu})$ be the $(n+m) \times (n+m)$ determinant:*

$$(-1)^{J+1} \begin{vmatrix} (x_1\tau^\alpha)^{\mu_1} & \cdots & (x_n\tau^\alpha)^{\mu_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (y_1\tau^\beta)^{\mu_2} & \cdots & (y_m\tau^\beta)^{\mu_2} \\ f_3(\tau)(x_1\tau^\alpha)^{\mu_3} & \cdots & f_3(\tau)(x_n\tau^\alpha)^{\mu_3} & g_3(\tau)(y_1\tau^\beta)^{\mu_3} & \cdots & g_3(\tau)(y_m\tau^\beta)^{\mu_3} \\ f_4(\tau)(x_1\tau^\alpha)^{\mu_4} & \cdots & f_4(\tau)(x_n\tau^\alpha)^{\mu_4} & g_4(\tau)(y_1\tau^\beta)^{\mu_4} & \cdots & g_4(\tau)(y_m\tau^\beta)^{\mu_4} \\ f_5(\tau)(x_1\tau^\alpha)^{\mu_5} & \cdots & f_5(\tau)(x_n\tau^\alpha)^{\mu_5} & g_5(\tau)(y_1\tau^\beta)^{\mu_5} & \cdots & g_5(\tau)(y_m\tau^\beta)^{\mu_3} \\ (x_1\tau^\alpha)^{\mu_6} & \cdots & (x_n\tau^\alpha)^{\mu_6} & (y_1\tau^\beta)^{\mu_6} & \cdots & (y_m\tau^\beta)^{\mu_6} \\ (x_1\tau^\alpha)^{\mu_7} & \cdots & (x_n\tau^\alpha)^{\mu_7} & (y_1\tau^\beta)^{\mu_7} & \cdots & (y_m\tau^\beta)^{\mu_7} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (x_1\tau^\alpha)^{\mu_\ell} & \cdots & (x_n\tau^\alpha)^{\mu_\ell} & (y_1\tau^\beta)^{\mu_\ell} & \cdots & (y_m\tau^\beta)^{\mu_\ell} \end{vmatrix}$$

where $0 < x_1 < x_2 < \ldots < x_n$, $0 < y_1 < y_2 < \ldots < y_m$, $\ell = n + m$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_\ell)$, with $0 \leq \mu_1 \leq \mu_2 < \mu_3 < \ldots < \mu_\ell$, $(I, J) \in \{(3,4), (3,5), (4,5)\}$, $f_I(\tau) = g_J(\tau) = 1$, $f_i(\tau) = g_j(\tau) = \tau^M$, for $i \neq I$ and $j \neq J$, $\alpha > \beta \geq 0$, $M \geq \alpha|\boldsymbol{\mu}|$ and $\tau > 0$. Then:

$$D_{n,m}(\tau; I, J, \boldsymbol{\mu}) = C\tau^\xi + \Theta(\tau^{\xi+1}), \quad \xi = \alpha\left(\mu_1 + \mu_3 + \sum_{i=4}^{n+2} \mu_i - \mu_J\right) + \beta\left(\mu_2 + \mu_J + \sum_{i=n+3}^{\ell} \mu_i\right),$$

where C is a positive constant independent of τ.

Proof. For simplicity, we write $D_{n;m}(\;)$ instead of $D_{n;m}(\;;I;J;\;)$, suppressing I, J and in the notation. We denote by $_{n;m}(\;)$ the matrix corresponding to the determinant $(\;1)^{J+1}D_{n;m}(\;)$. If we apply Laplace's expansion with respect to the first n columns, i.e., when $c = (1;2;:::;n)$, we get:

$$D_{n;m}(\;) = (\;1)^{J+1} \sum_{\substack{r=(r_1;r_2;:::;r_n)\\ 1\; r_1<r_2<\;<r_n\; n+m}} (\;1)^{jrj+jcj} \det(S(\;_{n;m}(\;);r;c))\det(S(\;_{n;m}(\;);r;c))$$

$$= \sum_{\substack{r=(r_1;r_2;:::;r_n)\\ 1\; r_1<r_2<\;<r_n\; n+m}} (\;1)^{jrj+\frac{n(n+1)}{2}+J+1} \det(S(\;_{n;m}(\;);r;c))\det(S(\;_{n;m}(\;);r;c)): \qquad (79)$$

The above sum consists of $\binom{n+m}{n}$ terms. Among these terms:

(i) all those for which r contains the second row vanish (in this case the corresponding row of $S(\;_{n;m}(\;);r;c)$ consists of zeros), and

(ii) all those for which r does not contain the first row vanish (in this case at least two rows of $S(\;_{n;m}(\;);r;c)$ consist of zeros).

The remaining terms of the expansion are the $\binom{n+m\;2}{n\;1}$ terms for which r contains 1 but not 2, i.e., $r = (1;r_2;r_3;:::;r_n)$, with $3\; r_2 < r_3 < ::: < r_n\; n+m$. For any given r, we denote by r the vector of the m, among the $n+m$, row indices for $_{n;m}(\;)$ that do not belong to r (recall that 2 always belongs to r). Notice that the elements of the k-th row of $_{n;m}(\;)$ have exponent $_k$. Denoting by $_r$ the vector the i-th element of which is $_{r_i}$, we have that:

(i) $\det(S(\;_{n;m}(\;);r;c))$ is the $n\;n$ generalized Vandermonde determinant $GVD(\;x;\;_r)$, multiplied by ^{M} if $J\;2\;r$.

(ii) $\det(S(\;_{n;m}(\;);r;c))$ is the $m\;m$ generalized Vandermonde determinant $GVD(\;y;\;_r)$, multiplied by ^{M} if $I\;2\;r$.

We can, thus, simplify the expansion in (79) to get:

$$D_{n;m}(\;) = \sum_{f\,r\,j1 2 r\,;2 62 g} (\;1)^{jrj+\frac{n(n+1)}{2}+J+1} h(r;\;;I;J)GVD(\;x;\;_r)GVD(\;y;\;_r)$$

$$= \sum_{f\,r\,j1 2 r\,;2 62 g} (\;1)^{jrj+\frac{n(n+1)}{2}+J+1} h(r;\;;I;J)\;^{j\;_rj+j\;_rj}GVD(x;\;_r)GVD(y;\;_r); \qquad (80)$$

where

$$h(r;\;;I;J) = \begin{cases} 1; & I\;2\;r \text{ and } J\;62 r; \\ ^{2M}; & I\;62 r \text{ and } J\;2\;r; \\ ^{M}; & \text{otherwise}: \end{cases}$$

In the remainder of the proof we seek to find the unique term in the expansion (80) that corresponds to the minimum order of , or, equivalently, the minimum exponent for . Since $>\; 0$, for any r, with $I\;2\;r$ and $J\;62 r$, the exponent of is:

$$j\;_rj + j\;_rj < j\;_rj + j\;_rj = j\;j\;M;$$

where we used the fact that:

$$j\;_rj + j\;_rj = \sum_{i=1}^{n}\;_{r_i} + \sum_{i=1}^{m}\;_{r_i} = \sum_{i\,2\,r}\;_i + \sum_{i\,2\,r}\;_i = \sum_{i=1}^{`}\;_i = j\;j:$$

This implies that the terms in (80) that correspond to the row vectors $\boldsymbol{r}$ that contain J cannot be the terms of minimal order of τ, since for these terms the exponent of τ is at least

$$\alpha|\boldsymbol{\mu_r}| + \beta|\boldsymbol{\mu_{\bar{r}}}| + M > \beta|\boldsymbol{\mu_r}| + \beta|\boldsymbol{\mu_{\bar{r}}}| + M = \beta|\boldsymbol{\mu}| + M \geq M.$$

For the remaining terms, i.e., for those $\boldsymbol{r}$ that do not contain J, we have $h(\boldsymbol{r}, \tau; I, J) = 1$. For these terms the exponent of τ is $\alpha|\boldsymbol{\mu_r}| + \beta|\boldsymbol{\mu_{\bar{r}}}|$. Since $\alpha > \beta$, we may write $\alpha = \beta + \theta$ for some $\theta > 0$. This gives:

$$\alpha|\boldsymbol{\mu_r}| + \beta|\boldsymbol{\mu_{\bar{r}}}| = (\beta + \theta)|\boldsymbol{\mu_r}| + \beta|\boldsymbol{\mu_{\bar{r}}}| = \beta|\boldsymbol{\mu}| + \theta|\boldsymbol{\mu_r}|.$$

Clearly, in this case, the quantity $\alpha|\boldsymbol{\mu_r}| + \beta|\boldsymbol{\mu_{\bar{r}}}|$ attains its minimum when $|\boldsymbol{\mu_r}|$ is minimal. We distinguish between the following cases:

- $(I, J) = (3, 4)$. In this case $|\boldsymbol{\mu_r}|$ attains its minimal value if and only if $\boldsymbol{r}$ is equal to $\boldsymbol{\rho} = (1, 3, 5, 6, \ldots, n+2)$. Furthermore,

$$\begin{aligned} |\boldsymbol{\mu_\rho}| &= \mu_1 + \mu_3 + \sum_{i=5}^{n+2} \mu_i = \mu_1 + \mu_3 + \sum_{i=4}^{n+2} \mu_i - \mu_J, \\ |\boldsymbol{\mu_{\bar{\rho}}}| &= \mu_2 + \mu_4 + \sum_{i=n+3}^{\ell} \mu_i = \mu_2 + \mu_J + \sum_{i=n+3}^{\ell} \mu_i \end{aligned}$$

  and

$$\begin{aligned} |\boldsymbol{\rho}| + \frac{n(n+1)}{2} + J + 1 &= \sum_{i=1}^{n+2} i - (2+4) + \frac{n(n+1)}{2} + 4 + 1 \\ &= \frac{(n+2)(n+3)}{2} + \frac{n(n+1)}{2} - 1 \\ &= n^2 + 3n + 3 - 1 \\ &= (n+1)(n+2), \end{aligned}$$

  which is even for any $n \geq 2$.

- $I \in \{3, 4\}$ and $J = 5$. In this case $|\boldsymbol{\mu_r}|$ attains its minimal value if and only if $\boldsymbol{r}$ is equal to $\boldsymbol{\rho} = (1, 3, 4, 6, \ldots, n+2)$. Furthermore,

$$\begin{aligned} |\boldsymbol{\mu_\rho}| &= \mu_1 + \mu_3 + \mu_4 + \sum_{i=6}^{n+2} = \mu_1 + \mu_3 + \sum_{i=4}^{n+2} -\mu_J, \\ |\boldsymbol{\mu_{\bar{\rho}}}| &= \mu_2 + \mu_5 + \sum_{i=n+3}^{\ell} = \mu_2 + \mu_J + \sum_{i=n+3}^{\ell} \mu_i, \end{aligned}$$

  and

$$\begin{aligned} |\boldsymbol{\rho}| + \frac{n(n+1)}{2} + J + 1 &= \sum_{i=1}^{n+1} i - (2+5) + \frac{n(n+1)}{2} + 5 + 1 \\ &= \frac{(n+2)(n+3)}{2} + \frac{n(n+1)}{2} - 1 \\ &= n^2 + 3n + 3 - 1 \\ &= (n+1)(n+2), \end{aligned}$$

  which is again even for any $n \geq 2$.

We can thus rewrite (80) in the following form:

$$D_{n,m}(\tau) = \tau^{\alpha|\boldsymbol{\mu}_{\boldsymbol{\rho}}|+\beta|\boldsymbol{\mu}_{\bar{\boldsymbol{\rho}}}|}\,\mathrm{GVD}(\boldsymbol{x};\boldsymbol{\mu}_{\boldsymbol{\rho}})\mathrm{GVD}(\boldsymbol{y};\boldsymbol{\mu}_{\bar{\boldsymbol{\rho}}}) + \Theta(\tau^{\alpha|\boldsymbol{\mu}_{\boldsymbol{\rho}}|+\beta|\boldsymbol{\mu}_{\bar{\boldsymbol{\rho}}}|+1}).$$

The lemma immediately follows from the positivity of the generalized Vandermonde determinants $\mathrm{GVD}(\boldsymbol{x};\boldsymbol{\mu}_{\boldsymbol{\rho}})$ and $\mathrm{GVD}(\boldsymbol{y};\boldsymbol{\mu}_{\bar{\boldsymbol{\rho}}})$. □

We end with the following lemma, where we perform the asymptotic analysis of the generalized version of the determinant that arises in the upper bound tightness construction in Section 6 when $R \equiv [3]$.

Lemma 17. *Fix three integers $m \geq 2$, $n \geq 2$ and $k \geq 2$, with $n+m+k \geq 7$. Let $E_{n,m,k}(\tau;\boldsymbol{\mu})$ be the $(n+m+k)\times(n+m+k)$ determinant:*

$$-\begin{vmatrix}
(x_1\tau^2)^{\mu_1} & \cdots & (x_n\tau^2)^{\mu_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & (y_1\tau)^{\mu_2} & \cdots & (y_m\tau)^{\mu_2} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & z_1^{\mu_3} & \cdots & z_k^{\mu_3} \\
(x_1\tau^2)^{\mu_4} & \cdots & (x_n\tau^2)^{\mu_4} & \tau^M(y_1\tau)^{\mu_4} & \cdots & \tau^M(y_n\tau)^{\mu_4} & \tau^M z_1^{\mu_4} & \cdots & \tau^M z_n^{\mu_4} \\
\tau^M(x_1\tau^2)^{\mu_5} & \cdots & \tau^M(x_n\tau^2)^{\mu_5} & (y_1\tau)^{\mu_5} & \cdots & (y_m\tau)^{\mu_5} & \tau^M z_1^{\mu_5} & \cdots & \tau^M z_n^{\mu_5} \\
\tau^M(x_1\tau^2)^{\mu_6} & \cdots & \tau^M(x_n\tau^2)^{\mu_6} & \tau^M(y_1\tau)^{\mu_6} & \cdots & \tau^M(y_m\tau)^{\mu_6} & z_1^{\mu_6} & \cdots & z_m^{\mu_6} \\
(x_1\tau^2)^{\mu_7} & \cdots & (x_n\tau^2)^{\mu_7} & (y_1\tau)^{\mu_7} & \cdots & (y_m\tau)^{\mu_7} & z_1^{\mu_7} & \cdots & z_k^{\mu_7} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(x_1\tau^2)^{\mu_\ell} & \cdots & (x_n\tau^2)^{\mu_\ell} & (y_1\tau)^{\mu_\ell} & \cdots & (y_m\tau)^{\mu_\ell} & z_1^{\mu_\ell} & \cdots & z_k^{\mu_\ell}
\end{vmatrix}$$

where $0 < x_1 < x_2 < \ldots < x_n$, $0 < y_1 < y_2 < \ldots < y_m$, $0 < z_1 < z_2 < \ldots < z_k$, $\ell = n+m+k$, $\boldsymbol{\mu} = (\mu_1,\mu_2,\ldots,\mu_\ell)$, with $0 \leq \mu_1 \leq \mu_2 \leq \mu_3 < \mu_4 < \mu_5 < \ldots < \mu_\ell$, $M \geq 2|\boldsymbol{\mu}|$ and $\tau > 0$. Then,

$$E_{n,m,k}(\tau;\boldsymbol{\mu}) = C'\tau^{\xi} + \Theta(\tau^{\xi+1}), \quad \xi = 2\left(\mu_1+\mu_4+\sum_{i=7}^{n+4}\mu_i\right) + \mu_2+\mu_5+\sum_{i=n+5}^{n+m+2}\mu_i,$$

where C' is a positive constant independent of τ.

Proof. We write $E_{n,m,k}(\tau)$ instead of $E_{n,m,k}(\tau;\boldsymbol{\mu})$, suppressing $\boldsymbol{\mu}$ in the notation. We denote by $\mathcal{E}_{n,m,k}(\tau)$ the matrix corresponding to the determinant $-E_{n,m,k}(\tau)$. If we apply Laplace's expansion theorem with respect to the first n columns, i.e., when $\boldsymbol{c} = (1,2,\ldots,n)$, we get:

$$\begin{aligned}
E_{n,m,k}(\tau) &= -\sum_{\boldsymbol{r}}(-1)^{|\boldsymbol{r}|+|\boldsymbol{c}|}\det(S(\mathcal{E}_{n,m,k}(\tau);\boldsymbol{r},\boldsymbol{c}))\det(\bar{S}(\mathcal{E}_{n,m,k}(\tau);\boldsymbol{r},\boldsymbol{c})) \\
&= \sum_{\boldsymbol{r}}(-1)^{|\boldsymbol{r}|+\frac{n(n+1)}{2}+1}\det(S(\mathcal{E}_{n,m,k}(\tau);\boldsymbol{r},\boldsymbol{c}))\det(\bar{S}(\mathcal{E}_{n,m,k}(\tau);\boldsymbol{r},\boldsymbol{c})). \qquad (81)
\end{aligned}$$

The above sum consists of $\binom{n+m+k}{n}$ terms. Among these terms:

(i) all those for which $\boldsymbol{r}$ contains the second or third row vanish (the corresponding row of $S(\mathcal{E}_{n,m,k}(\tau);\boldsymbol{r},\boldsymbol{c})$ consists of zeros), and

(ii) all those for which $\boldsymbol{r}$ does not contain the first row vanish (in this case there exists a row of $\bar{S}(\mathcal{E}_{n,m,k}(\tau);\boldsymbol{r},\boldsymbol{c})$ that consists of zeros).

The remaining terms of the expansion are the $\binom{n+m+k-3}{n-1}$ terms for which $\boldsymbol{r} = (1,r_2,r_3,\ldots,r_n)$, with $4 \leq r_2 < r_3 < \ldots < r_n \leq n+m+k$. As a result, the expansion in (81) simplifies to:

$$E_{n,m,k}(\tau) = \sum_{\{\boldsymbol{r}|1\in\boldsymbol{r},2,3\notin\boldsymbol{r}\}}(-1)^{|\boldsymbol{r}|+\frac{n(n+1)}{2}+1}\det(S(\mathcal{E}_{n,m,k}(\tau);\boldsymbol{r},\boldsymbol{c}))\det(\bar{S}(\mathcal{E}_{n,m,k}(\tau);\boldsymbol{r},\boldsymbol{c})). \qquad (82)$$

For any given $\boldsymbol{r}$, we denote by $\bar{\boldsymbol{r}}$ the vector of the $m+k$ row indices for $\mathcal{E}_{n,m,k}(\tau)$ that do not belong to $\boldsymbol{r}$. Moreover, $\boldsymbol{\mu_r}$ is the vector the i-th element of which is μ_{r_i}. As in the proof of Lemma 16, we seek to find the unique minimum term in the expansion (82) that corresponds to the minimum order of τ, or, equivalently, the minimum exponent for τ.

Let us denote by $\mathcal{R}$ the set of row vectors $\mathcal{R} = \{\boldsymbol{r} \mid 1,4 \in \boldsymbol{r} \text{ and } 2,3,5,6 \notin \boldsymbol{r}\}$. For any $\boldsymbol{r} \in \mathcal{R}$, observe that:

(i) $\det(S(\mathcal{E}_{n,m,k}(\tau); \boldsymbol{r}, \boldsymbol{c}))$ is the $n \times n$ generalized Vandermonde determinant $\mathrm{GVD}(\tau^2\boldsymbol{x}; \boldsymbol{\mu_r})$.

(ii) $\det(\bar{S}(\mathcal{E}_{n,m,k}(\tau); \boldsymbol{r}, \boldsymbol{c}))$ is the $(m+k)\times(m+k)$ determinant $D_{m,k}(\tau; 3, 4, \boldsymbol{\mu_{\bar{r}}})$ of Lemma 16 multiplied by $(-1)^{4+1} = -1$, with $\boldsymbol{x} \leftarrow \boldsymbol{y}$, $\boldsymbol{y} \leftarrow \boldsymbol{z}$, $\boldsymbol{\mu} \leftarrow \boldsymbol{\mu_{\bar{r}}}$, $(I, J) = (3, 4)$, $\alpha \leftarrow 1$, $\beta \leftarrow 0$ and $M \leftarrow M$ (since $M \geq 2|\boldsymbol{\mu}| > |\boldsymbol{\mu_{\bar{r}}}|$, the condition for M in Lemma 16 is satisfied).

We can, thus, rewrite the expansion in (82) to get:

$$
\begin{aligned}
E_{n,m,k}(\tau) &= \sum_{\boldsymbol{r}\in\mathcal{R}} (-1)^{|\boldsymbol{r}|+\frac{n(n+1)}{2}+1}\, \mathrm{GVD}(\tau^2\boldsymbol{x}; \boldsymbol{\mu_r})\, (-D_{m,k}(\tau; 3, 4, \boldsymbol{\mu_{\bar{r}}})) \\
&\quad + \sum_{\boldsymbol{r}\notin\mathcal{R}} (-1)^{|\boldsymbol{r}|+\frac{n(n+1)}{2}+1}\, \det(S(\mathcal{E}_{n,m,k}(\tau); \boldsymbol{r}, \boldsymbol{c}))\det(\bar{S}(\mathcal{E}_{n,m,k}(\tau); \boldsymbol{r}, \boldsymbol{c})) \\
&= \sum_{\boldsymbol{r}\in\mathcal{R}} (-1)^{|\boldsymbol{r}|+\frac{n(n+1)}{2}}\, \tau^{2|\boldsymbol{\mu_r}|}\, \mathrm{GVD}(\boldsymbol{x}; \boldsymbol{\mu_r}) D_{m,k}(\tau; 3, 4, \boldsymbol{\mu_{\bar{r}}}) \\
&\quad + \sum_{\boldsymbol{r}\notin\mathcal{R}} (-1)^{|\boldsymbol{r}|+\frac{n(n+1)}{2}+1}\, \det(S(\mathcal{E}_{n,m,k}(\tau); \boldsymbol{r}, \boldsymbol{c}))\det(\bar{S}(\mathcal{E}_{n,m,k}(\tau); \boldsymbol{r}, \boldsymbol{c}))
\end{aligned}
\tag{83}
$$

By Lemma 16 we have:

$$
D_{m,k}(\tau; 3, 4, \boldsymbol{\mu_{\bar{r}}}) = C_{\boldsymbol{r}}\, \tau^{1\cdot|\boldsymbol{\mu_{\bar{u}}}|+0\cdot|\boldsymbol{\mu_{\bar{v}}}|} + \Theta(\tau^{1\cdot|\boldsymbol{\mu_{\bar{u}}}|+0\cdot|\boldsymbol{\mu_{\bar{v}}}|+1}) = C_{\boldsymbol{r}}\, \tau^{|\boldsymbol{\mu_{\bar{u}}}|} + \Theta(\tau^{|\boldsymbol{\mu_{\bar{u}}}|+1}),
$$

where $\bar{\boldsymbol{u}} = (2, 5, \bar{r}_5, \ldots, \bar{r}_{m+2})$, $\bar{\boldsymbol{v}} = (3, 6, \bar{r}_{m+3}, \ldots, \bar{r}_{m+k})$ and $C_{\boldsymbol{r}} > 0$. Hence, for any $\boldsymbol{r} \in \mathcal{R}$, the term in the expansion of $E_{n,m,k}(\tau)$ that corresponds to $\boldsymbol{r}$ becomes:

$$
(-1)^{|\boldsymbol{r}|+\frac{n(n+1)}{2}}\, C_{\boldsymbol{r}}\, \tau^{2|\boldsymbol{\mu_r}|+|\boldsymbol{\mu_{\bar{u}}}|}\, \mathrm{GVD}(\boldsymbol{x}; \boldsymbol{\mu_r}) + \Theta(\tau^{2|\boldsymbol{\mu_r}|+|\boldsymbol{\mu_{\bar{u}}}|+1}).
$$

From this expression we deduce that the minimum exponent of τ for any specific $\boldsymbol{r} \in \mathcal{R}$ is:

$$
2|\boldsymbol{\mu_r}| + |\boldsymbol{\mu_{\bar{u}}}| < 2|\boldsymbol{\mu_r}| + |\boldsymbol{\mu_{\bar{r}}}| < 2|\boldsymbol{\mu_r}| + 2|\boldsymbol{\mu_{\bar{r}}}| = 2|\boldsymbol{\mu}| \leq M,
$$

where we used the fact that:

$$
|\boldsymbol{\mu_r}| + |\boldsymbol{\mu_{\bar{r}}}| = \sum_{i=1}^{n} \mu_{r_i} + \sum_{i=1}^{m+k} \mu_{\bar{r}_i} = \sum_{i\in\boldsymbol{r}} \mu_i + \sum_{i\in\bar{\boldsymbol{r}}} \mu_i = \sum_{i=1}^{\ell} \mu_i = |\boldsymbol{\mu}|.
$$

On the other hand, the terms in (83) that correspond to the row vectors $\boldsymbol{r} \notin \mathcal{R}$ cannot be the terms of minimal order of τ, since for these terms the exponent of τ is greater than M. We can thus restrict our attention to the terms for which $\boldsymbol{r} \in \mathcal{R}$, and rewrite (83) as:

$$
E_{n,m,k}(\tau) = \sum_{\boldsymbol{r}\in\mathcal{R}} \Big((-1)^{|\boldsymbol{r}|+\frac{n(n+1)}{2}}\, C_{\boldsymbol{r}}\, \tau^{2|\boldsymbol{\mu_r}|+|\boldsymbol{\mu_{\bar{u}}}|}\, \mathrm{GVD}(\boldsymbol{x}; \boldsymbol{\mu_r}) + \Theta(\tau^{2|\boldsymbol{\mu_r}|+|\boldsymbol{\mu_{\bar{u}}}|+1}) \Big) + \Omega(\tau^M).
$$

From the expression above, we infer that the term of $E_{n,m,k}(\tau)$ for which the exponent of τ is minimal is the term for which the quantity $2|\boldsymbol{\mu_r}| + |\boldsymbol{\mu_{\bar{u}}}|$ is minimized. However, we have that:

$$
2|\boldsymbol{\mu_r}| + |\boldsymbol{\mu_{\bar{u}}}| = 2|\boldsymbol{\mu_r}| + |\boldsymbol{\mu_{\bar{u}}}| + |\boldsymbol{\mu_{\bar{v}}}| - |\boldsymbol{\mu_{\bar{v}}}| = |\boldsymbol{\mu_r}| + |\boldsymbol{\mu_r}| + |\boldsymbol{\mu_{\bar{r}}}| - |\boldsymbol{\mu_{\bar{v}}}| = |\boldsymbol{\mu_r}| + |\boldsymbol{\mu}| - |\boldsymbol{\mu_{\bar{v}}}|.
$$

So, minimizing $2|\boldsymbol{\mu_r}| + |\boldsymbol{\mu_{\bar{u}}}|$ amounts to determining the vectors $\boldsymbol{r}$ and $\bar{\boldsymbol{v}}$ for which the difference $|\boldsymbol{\mu_r}| - |\boldsymbol{\mu_{\bar{v}}}|$ becomes minimal. Let $\boldsymbol{\rho} = (1, 4, 7, 8, \ldots, n+4)$, $\bar{\boldsymbol{\rho}}' = (2, 5, n+5, n+6, \ldots, n+m+2)$ and $\bar{\boldsymbol{\rho}}'' = (3, 6, n+m+3, n+m+4, \ldots, \ell)$. It is trivial to verify that

- $|\boldsymbol{\mu}_{\boldsymbol{r}}| > |\boldsymbol{\mu}_{\boldsymbol{\rho}}|$, for all $\boldsymbol{r} \neq \boldsymbol{\rho}$, and
- $|\boldsymbol{\mu}_{\bar{\boldsymbol{v}}}| < |\boldsymbol{\mu}_{\bar{\boldsymbol{\rho}}''}|$, for all $\bar{\boldsymbol{v}} \neq \bar{\boldsymbol{\rho}}''$.

From this observation we deduce that the unique minimal value for $2|\boldsymbol{\mu}_{\boldsymbol{r}}|+|\boldsymbol{\mu}_{\bar{\boldsymbol{u}}}|$ is attained when $\boldsymbol{r}$, $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}}$ are equal to $\boldsymbol{\rho}$, $\bar{\boldsymbol{\rho}}'$ and $\bar{\boldsymbol{\rho}}''$, respectively. Moreover,

$$|\boldsymbol{\rho}| + \frac{n(n+1)}{2} = \sum_{i=1}^{n+4} i - (2+3+5+6) + \frac{n(n+1)}{2} = \frac{(n+4)(n+5)}{2} - 16 + \frac{n(n+1)}{2}$$
$$= \frac{(n^2+9n+20)+(n^2+n)}{2} - 16 = (n^2+5n+10) - 16 = (n-1)(n+6).$$

Since $(n-1)(n+6)$ is even for any n, the term in the expansion of $E_{n,m,k}(\tau)$ corresponding to the minimum exponent for τ becomes $C_{\boldsymbol{\rho}}\, \tau^{2|\boldsymbol{\mu}_{\boldsymbol{\rho}}|+|\boldsymbol{\mu}_{\bar{\boldsymbol{\rho}}'}|}\, \mathrm{GVD}(\boldsymbol{x}; \boldsymbol{\mu}_{\boldsymbol{\rho}})$. The claim in the statement of the lemma immediately follows from the positivity of $C_{\boldsymbol{\rho}}$ and $\mathrm{GVD}(\boldsymbol{x}; \boldsymbol{\mu}_{\boldsymbol{\rho}})$, and by observing that $2|\boldsymbol{\mu}_{\boldsymbol{\rho}}| + |\boldsymbol{\mu}_{\bar{\boldsymbol{\rho}}'}|$ equals ξ. $\square$