Counting Fixed-Length Permutation Patterns

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Abstract

We consider the problem of packing fixed-length patterns into a permutation, and develop a connection between the number of large patterns and the number of bonds in a permutation. Improving upon a result of Kaplansky and Wolfowitz, we obtain exact values for the expectation and variance for the number of large patterns in a random permutation. Finally, we are able to generalize the idea of bonds to obtain results on fixed-length patterns of any size, and present a construction that maximizes the number of distinct large patterns.

1 Background

Two sequences of distinct integers $a_1a_2\dots a_n$ and $b_1b_2\dots b_n$ are *order isomorphic* if, for all $1 \leq i, j \leq$n, we have that $a_i < a_j$ if and only if $b_i < b_j$. Let $q = q_1q_2\dots q_k$ be a permutation in the symmetric group S_k written in one-line notation. We say that a permutation $p = p_1p_2\dots p_n \in S_n$ *contains* q *as a pattern* if there is a subsequence $p_{i_1}p_{i_2}\dots p_{i_k}$ which is in the same relative order as the entries of q. If p does not contain q as a pattern, we say that p *avoids* q.

For example, the permutation $p = 4732615$ contains the pattern $q = 213$ because the 1st, 4th, and 7th entries of p are order isomorphic to the permutation 213. This permutation avoids the pattern 123, however, because p contains no increasing subsequence of length 3. For another example, a permutation p avoids the pattern $q = 21$ if and only if it is strictly increasing, since otherwise p would contain an inversion, and an inversion is precisely a 21 pattern.

As a relation, pattern containment is transitive, reflexive, and anti-symmetric. Therefore the set of all permutations equipped with this ordering forms a graded partially ordered set (poset), which is referred to in the literature as the *pattern poset*. Given a permutation p, the set of all patterns contained in p forms a downset (also referred to as an ideal) of this poset.

The area of permutation patterns has received considerable attention in recent years. The majority of work has been focused on enumerating infinite downsets in the pattern poset (known as permutation classes), particularly those which arise as sets of permutations avoiding specified patterns. An early result in the area, due to Knuth [5], is that the 231 avoiding permutations are counted by the Catalan numbers $\frac{1}{n}\binom{2n}{n}$, and these are exactly the stack sortable permutations. A more comprehensive introduction to the subject can be found in [2].

Interesting questions are raised, however, even if we restrict ourselves to finite downsets of the pattern poset. We focus our attention here on examining the downset of a single permutation. In 2003, Herb Wilf raised the question of finding the maximum number of distinct patterns which can be contained in a permutation of length n, and classifying those permutations which achieve this maximum. Translated to the language of posets, Wilf's question asks to find which permutations

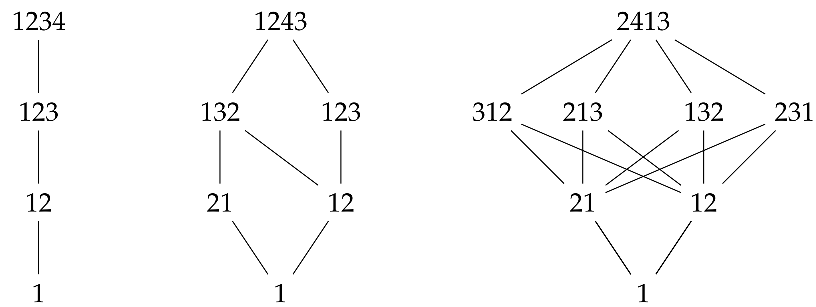

Figure 1: Downsets of 1234, 1243, and 2413

maximize the size of their downset in the pattern poset. In [1], the authors showed that the maximum number of patterns that can be contained in a permutation of length n is asymptotic to 2^n. However, the exact value of the maximum is unknown.

2 Preliminaries

This paper can be divided into two parts: in the first, we examine the number of $(n-1)$-patterns in a random n-permutation, and obtain exact values for both the expectation and variance of this statistic by extending a 1945 result of Kaplansky and Wolfowitz. In the second part, we examine the number of patterns of a fixed size in a given permutation, and provide a partial answer Herb Wilf's question.

In counting the total number of patterns contained in a permutation, it is most useful to use a top-down approach, enumerating all of the largest patterns and working down the downset level by level. We introduce an alternate (but equivalent) definition of permutation patterns which better suits this approach.

Definition 1. Let $p = p_1p_2\ldots p_n \in S_n$, $n \geq 2$. Let $0 \leq k \leq n-1$. We say that a permutation $q \in S_{n-k}$ is an $(n-k)$*-pattern* of p if q can be obtained by deleting k entries of p and then relabelling the remaining entries 1 through $n-k$ with respect to order. Let $D_k(p)$ denote the set of $(n-k)$-patterns of a permutation p, and $\mathcal{D}(p) = \bigcup_k D_k(p)$ denote the set of *all* patterns contained in p.

Where Wilf's problem asks which permutations maximize the total size of the downset, our focus will be on finding those permutations which maximize the width of a *specified* level of this downset.

First, we investigate the number of coatoms. That is, we will fix an $n \geq 2$ and focus our attention on patterns of size $(n-1)$ contained in a given n-permutation p. We limit our attention to $(n-1)$-patterns not only because they are easier to work with, but because these results can in some cases be extended to results for $(n-k)$-patterns, simply by working our way down the downset level by level. To start, we formalize our notion of $(n-1)$-patterns with a function. To simplify the notation, we use $[n]$ to denote the set $\{1, 2, 3, \ldots n\}$.

Definition 2. Let $del : S_n \times [n] \to S_{n-1}$ be the function where $del(p, i)$ is defined by deleting the ith entry of p, and relabelling the remaining entries 1 through $n-1$ with respect to order.

Since any $(n-1)$-pattern q of a permutation p uses all but one entry of p, we see that $q = del(p, k)$ for some $k \in [n]$. Also, it is clear that if $q = del(p, k)$ for some $k \in [n]$, then q is contained in p as a pattern. This implies that $D_1(p) = \{del(p, k) : k \in [n]\}$.

Inversely, we can build up an $(n-1)$-permutation into an n-permutation by inserting an extra entry. We define another function to formalize this idea.

Definition 3. Let $ins : S_{n-1} \times [n] \times [n] \to S_n$ be the function where $ins(q, j, k)$ is defined by inserting the entry $k - 1/2$ immediately to the left of the jth entry of q, and then relabelling 1 through n with respect to order. Let $I_1(q) = \bigcup_i \bigcup_j ins(q, i, j)$ denote the set of all n-permutations which can be obtained by inserting one entry to q.

The function *ins* can best be understood graphically:

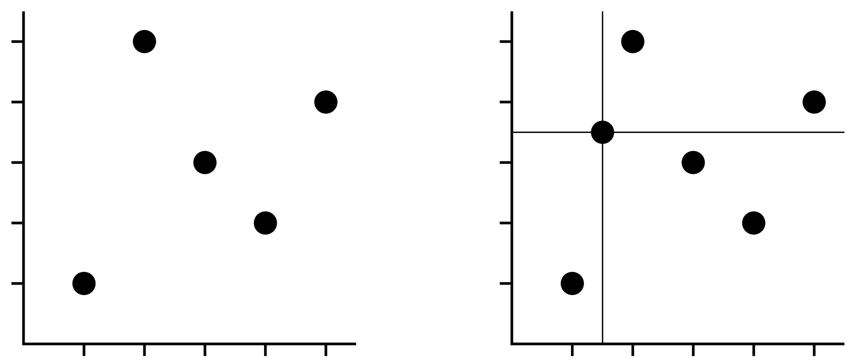

Figure 2: $ins(15324, 2, 4) = 146325$.

Now, from the definitions of these two functions, we see that they satisfy the following inverse relationship:

$$del(ins(q, j, k), j) = q \text{ and } ins(del(p, i), i, p_i) = p.$$

This relationship, along with the fact that $D_1(p)$ is the set of all $(n-1)$-patterns contained in p, implies that $I_1(q)$ is exactly the set of all n-permutations which contain q as a pattern.

3 The size of $D_1(p)$

It follows directly from the definition that given any permutation $p \in S_n$, $|D_1(p)| \leq n$, and that $|D_1(p)| = n$ if and only if $del(p, i) = del(p, j)$ implies that $i = j$. Before investigating further, we introduce another definition.

Definition 4. Let $p = p_1 p_2 \dots p_n$ be a permutation, and let $i \in [n-1]$. Say that the pair (p_i, p_{i+1}) is a *bond*, of entries of p if $p_i - p_{i+1} = \pm 1$. We say that the sequence $(p_i, p_{i+1}, \dots p_{i+k-1})$ is a *run* of length k if, for $1 \leq j \leq k-2$, the pair (p_{i+j}, p_{i+j+1}) is a bond. Denote by $C(p)$ the number of bonds contained in a permutation p.

Note that runs are necessarily either increasing or decreasing, and that a run of length k contains $k-1$ bonds. We can now establish a fundamental relationship between bonds and $(n-1)$-patterns.

Lemma 5. *Let $p = p_1 p_2 \dots p_n$, and $1 \leq j < k \leq n$. Then $del(p, j) = del(p, k)$ if and only if p_j and p_k are part of the same run.*

Proof. The forward direction is clear, since removing any element of a run and relabelling simply results in a shorter run.

The other direction takes a bit more work. Suppose that there are j, k with $1 \leq j < k \leq n$ with $del(p, j) = del(p, k)$. We proceed by induction on $k - j$.

Suppose that $k = j + 1$. Assume first that $p_j < p_{j+1}$, and consider the jth entry of $del(p, j) = del(p, j+1)$. By the definition of *del*, the jth entry of $del(p, j)$ is $p_{j+1} - 1$, and the same entry in $del(p, j+1)$ is p_j. Therefore, we see that $p_{j+1} - 1 = p_j$, which means that (p_j, p_{j+1}) is a bond. Again, the case where $p_{j+1} < p_j$ follows similarly.

Now assume by way of induction that the statement holds when $k = j + m - 1$, and suppose there exists $1 \le j < k \le n$ such that $k - j = m$ and $del(p, j) = del(p, k)$. Assume first that $p_j < p_k$. $del(p, j) = del(p, k)$ implies, in particular, that the $(k-1)$st entries on both sides of the equality are equal. By definition, the $k-1$ entry of $del(p, j)$ is $p_k - 1$, while the $k-1$ entry of $del(p, k)$ is either p_{k-1} or $p_{k-1} - 1$. The latter case would imply that $p_{k-1} = p_k$, a contradiction, and so it follows that $p_{k-1} = p_k$.

By what has already been proved, $del(p, k-1) = del(p, k)$ since these entries form a bond. But then $del(p, j) = del(p, k) = del(p, k-1)$, and so by the induction hypothesis the entries $(p_j p_{j+1} \ldots p_{k-1})$ form a run. Finally, $p_k - 1 = p_{k-1}$ implies that $(p_j p_{j+1} \ldots p_{k-1} p_k)$ is a length m run. Once more, the case where $p_j > p_k$ follows similarly, and the lemma is proved.

□

The simplest examples of permutations with runs are the ascending and descending permutations. Removing any element from the ascending (descending) permutation of length n and renumbering results in the ascending (descending) permutation of length $n-1$. In other words, the $(n-1)$-pattern set of either of these permutations has size 1, and the lemma shows that these are the only permutations with this property.

We can now establish our connection between the number of bonds and the number of $(n-1)$-patterns. Lemma 5 directly implies the following theorem.

Theorem 6. *Let $p \in S_n$. Then $|D_1(p)| = n - C(p)$.*

This leads to a number of useful corollaries. The first is clear, and provides motivation for generalization.

Corollary 7. *A permutation has the maximum number $(n-1)$-patterns if and only if it contains no bonds.*

Theorem 6 also provides a simple proof of the following local property of the permutation pattern poset.

Corollary 8. *If $q \in S_{n-1}$, then $|I_1(q)| = n^2 - 2n + 2$. In other words, every $(n-1)$-permutation is contained in exactly $(n-1)^2 + 1$ n-permutations.*

Proof. By definition, the set $I_1(q) = \{ins(q, j, k) : 1 \le j, k \le n\}$, so we see that $|I_1(q)| \le n^2$.

Now, a permutation $p \in S_n$ is contained in $I_1(q)$ more than once exactly when q can be obtained in more than one way by deleting a entry of p. It follows that q is contained in a permutation $p \in S_n$ more than once exactly when $ins(q, j, k) = ins(q, j', k')$ where $(j, k) \ne (j', k')$. By the lemma, this happens exactly when the jth entry of $ins(q, j, k)$ is a part of the same run as the j' entry of $(ins(q, j', k'))$. We can prevent this from occuring by never inserting an element just to the right and directly above or below an existing element of q, as this ensures that any new bonds can be created in exactly one way.

This eliminates exactly $2(n-1)$ choices for inserting an entry into q, and so therefore $|I_1(q)| = n^2 - 2(n-1) = (n-1)^2 + 1$, and the proof is complete. □

4 Expectation and Variance of $|D_1(p)|$

We now examine the distribution of the number of $(n-1)$-patterns in a randomly chosen n-permutation p by first examining the distribution of bonds. Kaplansky and Wolfowitz presented in [4] and [8] the asymptotic distribution of the number of bonds in a random permutation. Using more modern techniques of generating function analysis we are able to improve upon their results and obtain *exact* formulas for the expectation and the variance for the number of bonds in a random permutation. Theorem 6 allows us to translate these into the results on fixed-length patterns.

Throughout this section, we will let $\varphi : S_n \to \mathbb{Z}^{\geq 0}$ be the variable indicating the number of distinct $(n-1)$-patterns of an n-permutation, and $\chi : S_n \to \mathbb{Z}^{\geq 0}$ be the variable indicating the number of bonds. Our main tool will be multivariate generating functions, but first we note that the $\mathbb{E}(\varphi)$ can be obtained directly using our connection to pattern containment.

Proposition 9. *The expectation $\mathbb{E}(\varphi) = n - \frac{2(n-1)}{n}$, which approaches $n-2$ as n increases.*

Proof. This follows immediately from Corollary 8 and the identity

$$(n-1)!(n^2-2n+2) = n!\left(n - \frac{2(n-1)}{n}\right). \qquad \square$$

Generating functions, however, allow us to go several steps further. It follows from Theorem 6 and the linearity of expectation that $\mathbb{E}(\varphi) = n - \mathbb{E}(\chi)$, which allows us to easily translate results about bonds into results about distinct $(n-1)$-patterns. We can now begin the construction of our multivariate generating function, using a technique similar to the cluster method of Goulden and Jackson.

Theorem 10. *Let $a_{n,k}$ be the number of permutations of length n which contain exactly k bonds, and set $a_{0,0} = 1$. Then we have that*

$$F(z,u) := \sum_{n\geq 0}\sum_{k\geq 0} a_{n,k}x^n u^k = \sum_{m\geq 0} m!\left(z + \frac{2z^2(u-1)}{1-z(u-1)}\right)^m.$$

Proof. First, we construct a generating function $G(z,u) = \sum_{n\geq 0}\sum_{k\geq 0} b_{n,k}x^n u^k$, where $b_{n,k}$ is the number of permutations of length n with k *distinguished* bonds. For example, $b_{n,0} = n!$, as every permutation can be written with no bonds distinguished, and no permutation is counted more than once.

The function $G(z,u)$ is easier to construct, as we can build an n-permutation with k distinguished bonds by first specifying our distinguished ascending and descending runs, then permuting these runs with the remaining entries. Now, a run of length j contains $j-1$ bonds, and we have the option of making each run either increasing or decreasing. This leads to

$$G(z,u) = \sum_{m\geq 0} m!\left(z + \frac{2z^2u}{1-zu}\right)^m.$$

Now we can use the function G to obtain a formula for F. Since G counts only the distinguished bonds and F counts every bond, we see that F and G are related by the transformation $F(z,u+1) = G(z,u)$. Therefore $F(z,u) = G(z,u-1)$, and the theorem is proved. $\square$

A simple transformation can be used to obtain a multivariate generating function which indicates the number of distinct $(n-1)$-patterns of a permutation. However, the function F is more useful to work with, as we will see soon.

Corollary 11. *Let $d_{n,k}$ be the number of permutations of length n with exactly k distinct $(n-1)$-patterns. Then*

$$H(z,u) = 1 + \sum_{n\geq k\geq 1} d_{n,k}x^n u^k = \sum_{m\geq 0} m!\left(zu + \frac{2zu^2(u^{-1}-1)}{1-zu(u^{-1}-1)}\right)^m.$$

Proof. Since $|M(p)| = n - C(p)$, it follows immediately that $H(z,u) = F(zu, u^{-1})$. $\square$

We can now coax several results out of the function $F(z,u)$. To start, plugging in $u = 0$ gives the generating function for permutations with no bonds. Expanding, we see that

$$F(z,0) = 1 + z + 2z^4 + 14z^5 + 90z^6 + 646z^7 + 5242z^8 \dots.$$

The sequence, $1, 1, 0, 0, 2, 14, 90, 646, 5242, \dots$ is A002464 in the OEIS, and is easily seen to be equal to the number of ways to place n non-attacking kings on an $n \times n$ chessboard with one king in each row and each column. It was shown in [7] that this sequence is asymptotic to $n!/e^2$, and so Corollary 7 implies the following.

Proposition 12. *The probability that a randomly selected n-permutation has all distinct $(n-1)$-patterns tends to $1/e^2$ as $n \to \infty$.*

We can take this a step further, and use the function $F(z,u)$ to determine the expected number of bonds in a random n-permutation. As described in [3], we have that

$$\mathbb{E}(\chi) = \frac{[z^n]\partial_u F(z,u)|_{u=1}}{n!} = \frac{1}{n!}[z^n]\partial_u \left(\sum_{m \geq 0} m! \left(z + \frac{2z^2(u-1)}{1-z(u-1)} \right)^m \right).$$

Taking the partial derivative with respect to u, we find that

$$\partial_u F(z,u) = \partial_u \left(\sum_{m \geq 0} m! \left(z + \frac{2z^2(u-1)}{1-z(u-1)} \right)^m \right) =$$

$$\sum_{m \geq 0} m \cdot m! \frac{\left(z + \frac{2z^2(u-1)}{1-z(u-1)} \right)^m \left(\frac{2z^2}{1-z(u-1)} + \frac{2z^3(u-1)}{(1-z(u-1))^2} \right)}{z + \frac{2z^2(u-1)}{1-z(u-1)}}.$$

Plugging in $u = 1$ simplifies this expression greatly, leaving

$$\partial_u F(z,u)|_{u=1} = \sum_{m \geq 0} 2m! \cdot mz^{m+1} = \sum_{m \geq 1} 2(m-1)! \cdot (m-1)z^m.$$

From this it follows that

$$\mathbb{E}(\chi) = \frac{[z^n]\partial_u F(z,u)|_{u=1}}{n!} = 2\frac{(n-1)! \cdot (n-1)}{n!} = 2\frac{(n-1)}{n}.$$

Finally, by using linearity of expectation and the fact that $\varphi = n - \chi$, we find that $\mathbb{E}(\varphi) = n - \mathbb{E}(\chi) = n - 2\frac{n-1}{n}$, in agreement with Proposition 9.

The variance is given by $\mathbb{V}(\chi) = \mathbb{E}(\chi^2) - \mathbb{E}(\chi)^2$, and so we find that

$$\mathbb{V}(\chi) = \mathbb{E}(\chi(\chi-1)) + \mathbb{E}(\chi) - (\chi)^2.$$

The factorial moment can be computed directly from the bivariate generating function F as follows:

$$\mathbb{E}(\chi(\chi-1)) = \frac{[z^n]\partial_u^2 F(z,u)|_{u=1}}{n!}.$$

This leads to

$$\begin{aligned} \mathbb{V}(\chi) &= \frac{[z^n]\partial_u^2 F(z,u)|_{u=1}}{n!} + \frac{[z^n]\partial_u F(z,u)|_{u=1}}{n!} - \left(\frac{[z^n]\partial_u F(z,u)|_{u=1}}{n!} \right)^2 \\ &= \frac{[z^n]\partial_u^2 F(z,u)|_{u=1}}{n!} + 2\frac{n-1}{n} - \left(2\frac{n-1}{n} \right)^2. \end{aligned}$$

We begin by taking the second derivative of $F(z,u)$ with respect to u, which gives:

$$\partial_u F(z,u) = \sum_{m\geq 0} m! \cdot m \left(\frac{m\left(z+\frac{2z^2(u-1)}{1-z(u-1)}\right)^m \left(\frac{2z^2}{1-z(u-1)}+\frac{2z^3(u-1)}{(1-z(u-1))^2}\right)^2}{\left(z+\frac{2z^2(u-1)}{1-z(u-1)}\right)^2} \right.$$
$$+ \frac{\left(z+\frac{2z^2(u-1)}{1-z(u-1)}\right)^m \left(\frac{4z^3}{(1-z(u-1))^2}+\frac{4z^4(u-1)}{(1-z(u-1))^3}\right)}{z+\frac{2z^2(u-1)}{1-z(u-1)}}$$
$$\left. - \frac{\left(z+\frac{2z^2(u-1)}{1-z(u-1)}\right)^m \left(\frac{2z^2}{1-z(u-1)}+\frac{2z^3(u-1)}{(1-z(u-1))^2}\right)^2}{\left(z+\frac{2z^2(u-1)}{1-z(u-1)}\right)^2} \right).$$

Once again, setting $u=1$ simplifies this expression immensely:

$$\partial_u^2 F(z,u)|_{u=1} = \sum_{m\geq 0} 4m! \cdot m^2 z^{m+2} = \sum_{m\geq 2} 4(m-2)!(m-2)^2 z^m.$$

Which produces:

$$\begin{aligned}\mathbb{V}(\chi) &= \frac{[z^n]\partial_u^2 F(z,u)|_{u=1}}{n!} + 2\frac{n-1}{n} - \left(2\frac{n-1}{n}\right)^2 \\ &= \frac{4(n-2)!(n-2)^2}{n!} + 2\frac{n-1}{n} - \left(2\frac{n-1}{n}\right)^2 \\ &= 4\frac{(n-2)^2}{n(n-1)} + 2\frac{n-1}{n} - 4\frac{(n-1)^2}{n^2}.\end{aligned}$$

Which converges to 2 for large n. From the fact that $\varphi = n - \chi$, it follows that $\mathbb{V}(\varphi) = \mathbb{V}(\chi)$. We summarize this in the following theorem.

Theorem 13. *Let $m : S_n \to \mathbb{Z}^+$ be the variable indicating the number of distinct $(n-1)$-patterns of a permutation $p \in S_n$. Then we have:*

$$\mathbb{E}(\varphi) = n - 2\frac{n-1}{n} \text{ and } \mathbb{V}(\varphi) = 4\frac{(n-2)^2}{n(n-1)} + 2\frac{n-1}{n} - 4\frac{(n-1)^2}{n^2}.$$

An immediate consequence, we see that for large n these approach $\mathbb{E}(\varphi) = n-2$ and $\mathbb{V}(\varphi) = 2$ respectively, implying as a special case the results of [4] and [8]. These same techniques can be applied to recursively calculate higher moments.

5 Patterns of other sizes

We turn our attention now to determining the number $|D_k(p)|$ of distinct $(n-k)$-patterns of a permutation, for $k > 1$. In particular, we seek to determine which permutations (if any) have the property that $|D_k(p)| = \binom{n}{k}$, the maximum number of possible $(n-k)$-patterns. To start, we generalize our notion of bonds with the following definition.

Definition 14. Let $p = p_1 p_2 \ldots p_n \in S_n$ be any permutation. Define a metric on the entries of p by $d_p(i,j) = |i-j| + |p_i - p_j|$. Define the *minimum gap* of a permutation p to be $mg(p) = min\{d_p(i,j) : 1 \leq i < j \leq n\}$.

If the permutation is plotted on a lattice, then the metric d is just the taxicab metric on $\mathbb{Z}^2$. It is easy to see that (p_i, p_j) is a bond if and only if $d(i,j) = 2$. Therefore, we see that p has all distinct $(n-1)$-patterns if and only if $mg(p) \geq 3$. This motivates a generalization of Corollary 7, after we establish some suitable notation.

Definition 15. Let $S = \{a_1, a_2, \ldots a_k\} \subseteq [n]$, with $1 \leq a_1 < a_2 < \ldots < a_k \leq n$. We denote $del(\ldots del(del(del(p, a_k), a_{k-1}), a_{k-2}), \ldots, a_1)$ by $del(p; S)$. In other words, to obtain $del(p; S)$ we remove $p_{a_1}, p_{a_2}, \ldots p_{a_k}$ from p and renumber the remaining entries with respect to order.

Definition 16. Let $p = p_1 p_2 \ldots p_n$ be a permutation, and $1 \leq i < j \leq n$ then the *span* of the entries p_i and p_j is denoted $span_p(i,j)$ and is defined to be the set of indices for the entries that lie between p_i and p_j either vertically or horizontally.

Formally, if $p_i < p_j$, then $span_p(i,j) = \{k : i < k < j \text{ or } p_i < p_k < p_j\}$, with a similar definition when $p_i > p_j$.

Lemma 17. *If $p = p_1 p_2 \ldots p_n$ is a permutation with $mg(p) = k$, and if $1 \leq i < j \leq n$ are such that $d_p(i,j) = k$, then $|span_p(i,j)| = k - 2$.*

Proof. It is clear that $|span_p(i,j)| \leq k-2$. The only way in which $|span_p(i,j)| < k-2$ would hold is if there existed an entry p_m which was in between p_i and p_j both vertically and horizontally. However, this p_m would contradict the minimality of k, so $|span_p(i,j)| = k-2$. □

Corollary 18 now follows immediately from the lemma.

Corollary 18. *If $p = p_1 p_2 \ldots p_n$ is an n-permutation with $mg(p) = k$, then $mg(del(p,i)) \geq k-1$ for all $i \in [n]$.*

We are now able to prove our generalization of Corollary 7.

Theorem 19. *Let $p \in S_n$. Then p has all distinct $(n-k)$-patterns if and only if $mg(p) \geq k+2$.*

Proof. We start with the forward direction. Let $p = p_1 p_2 \ldots p_n$ be a permutation with the maximum number of $(n-k)$-patterns. Assume, by way of contradiction that $mg(p) = m < k+2$. Let $i < j$ be such that $d_p(i,j) = m$. By Lemma 17, we have that $span_p(i,j) = \{a_1, a_2, \ldots a_{m-2}\}$. Now set $q = del(p; \{a_1, a_2, \ldots a_{m-2}\}) \in S_{n-m+2}$. It follows $mg(q) = 2$, and so q has a consecutive pair, which implies that q does not the maximum number $(n-1)$-patterns. Since $m-2 < k$, p cannot have the maximum number of $(n-k)$-patterns, a contradiction.

For the reverse implication, let $p = p_1 p_2 \ldots p_n$ be a permutation with $mg(p) \geq k+2$. We use induction on k. We have already seen that the statement is true for $k = 1$, so assume the statement holds for all positive integer less than k. Let $p = p_1 p_2 \ldots p_n$ be a permutation with $mg(p) = k+2$. By induction, we know that this permutation has all distinct $(n-m)$-patterns for all $1 \leq m < k$.

Suppose, by way of contradiction, that $q \in S_{n-k}$ is contained in p in two different ways. That is, suppose that $del(p; \{a_1, a_2, \ldots a_k\}) = q = del(p; \{b_1, b_2, \ldots b_k\})$, with $a_i < a_j$ and $b_i < b_j$ when $i < j$, and $A = \{a_1, a_2, \ldots a_k\} \neq \{b_1, b_2, \ldots b_k\} = B$.

Now we claim that $A \cap B = \emptyset$. To see this, we can suppose that $a_i = b_j$. But then q is contained as a pattern in two different ways in $del(p, a_i)$. However this contradicts $mg(del(p, a_i)) \geq k+1$, because by induction it has the maximum number of $((n-1) - (k-1)) = (n-k)$-patterns.

Assume, without loss of generality, that $a_1 < b_1$. Let $j \in [n]$ be the smallest value such that $j > a_1$ but $j \notin A$. Since $del(p; \{a_1, a_2, \ldots a_k\}) = del(p; \{b_1, b_2, \ldots b_k\}) = q = q_1 q_2 \ldots q_{n-k}$, it follows that p_{a_1} and p_j will both move to fulfill the role of q_{a_1} once the entries from A or B are removed and the permutation is renumbered. However, since $|A| = k$, this implies that $d(a_1, j) < k+2$, our final contradiction.

□

Corollary 20. *Let $p \in S_n$. If $|D_k(p)| = \binom{n}{k}$, then $D_j(p) = \binom{n}{j}$ for all $j \in [k]$.*

To see that permutations with arbitrarily large gap sizes exist, we first note that the slanted cube construction presented in [1] creates a permutation of length n^2 minimum gap equal to $n+1$. We will construct a sequence of permutations $\{\pi^{(n)}\}_{n=2}^{\infty}$ which does slightly better, creating the same minimum gap size with shorter length.

To build the permutation $\pi^{(n)}$ with gap size n, we begin with a tiling of the plane with squares with side length n. Then we simply rotate the tiling by 45 degrees and use the centers of the squares as our permutation entries. We define this formally as follows.

Definition 21. Let a_i be defined as

$$a_i = min\{d \in [k-1] : i \leq d(k-1)\} \text{ and } b_i = (i-1 \mod (k-1)) \cdot (k-1).$$

Define $p_i = a_i + b_i$. Now take the permutation $p' = p_1 p_2 \ldots \pi^{(k-1)^2}$, and define $\pi^{(n)} = del(p', 1, (k-1)^2)$, the permutation obtained by deleting the first and last entries of p'.

The permutation $\pi^{(k)}$ for $k = 4, 5$ is shown below, and it is clear that these permutations have minimum gap size equal to 4 and 5 respectively. It is clear that $\pi^{(k)}$ is an involution for all k, and that the complement of $\pi^{(k)}$ is equal to it's own reverse. Therefore, the orbit of $\pi^{(k)}$ under the automorphism group of the pattern poset has order 2.

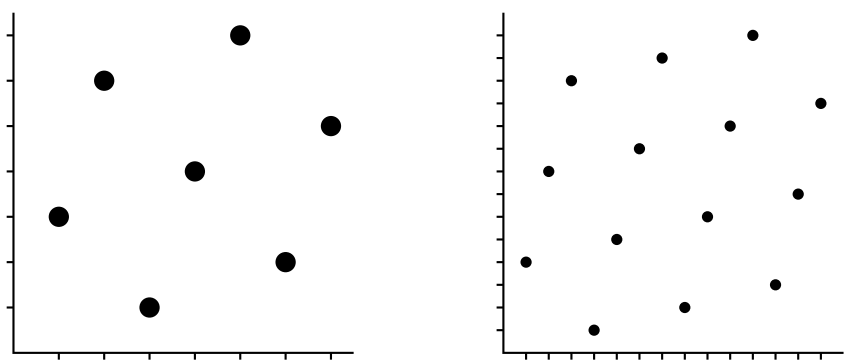

Figure 3: $\pi^{(4)} = 3\ 6\ 1\ 4\ 7\ 2\ 5$, and $\pi^{(5)} = 4\ 8\ 12\ 1\ 5\ 9\ 13\ 2\ 6\ 10\ 14\ 3\ 7\ 11$

A permutation can be embedded in the plane and the metric d_p can be extended to the taxicab metric d_1 on $\mathbb{R}^2$. It follows that any permutation p with minimum gap size equal to k defines a tiling of the plane by tilted squares with side lengths equal to k and centers on points of $\mathbb{Z}^2$. It is clear that a minimal sized permutation with gap size equal to k will produce a maximum tiling of the plane with tilted squares centered on different horizontal and vertical lines. There are exactly two such tilings of the plane, corresponding to the permutations $\pi^{(k)}$ and its reverse. We summarize this in the following theorem.

Theorem 22. *The permutation $\pi^{(k)}$ and its reverse are the shortest permutations with minimum gap size equal to k.*

Corollary 23. *Given any $k \in \mathbb{Z}^+$, the permutation $\pi^{(k)} \in S_{(k-1)^2-2}$ has the property that $M_j(p) = \binom{n}{j}$ for all $0 \leq j \leq k-2$. Furthermore, no permutation of length less than $(k-1)^2 - 2$ has this property.*

Proof. Immediate from the construction above, Theorem 19, and Corollary 20. □

We end this section with one last theorem, a generalization of Theorem 6.

Theorem 24. *Let p be an n-permutation with $mg(p) = k+1$, and let w_k be the number of pairs $(i,j) \in [n] \times [n]$ such that $|span_p(i,j)| = k-1$. Then the number of $(n-k)$-patterns in p is $\binom{n}{k} - w_k$.*

Proof. Let $p \in S_n$ with $mg(p) = k+1$, and let $i, j \in [n]$ be such that $d_p(i,j) = k+1$ (that is, $|span_p(i,j)| = k-1$). Then if we let $S = span_p(i,j) \cup i$ and $S' = span_p(i,j) \cup j$, we see that $del(p; S) = del(p; S')$, and so $|D_k(p)| \leq \binom{n}{k} - w_k$.

For equality, we use a modification of the argument used in Theorem 19. Suppose that $del(p; A) = del(p; B)$ for some $A = \{a_1, a_2, \ldots a_k\} \neq B = \{b_1, b_2, \ldots b_k\}$, with $a_i < a_j$ and $b_i < b_j$ for $i < j$. Suppose that $a_1 \neq b_1$, and let $s \in [n]$ be the smallest integer so that $s \notin A$. Then as in the proof of Theorem 19, we must have that $d_p(a_1, s) = k+1$ and $A - a_1 = B - b_1 = span_p(a_1, s)$.

In the case where $a_1 = b_1$, let $p' = del(p, a_1)$, and $A' = A - a_1$, $B' = B - b_1$. By Corollary 18, $mg(p') = k$, since if $mg(p') = k+1$, $del(p', A') = del(p', B')$ would contradict Theorem 19. We now repeat the argument, and find that either $a_2 = b_2$ or $A' - a_2 = B' - b_2$. We repeat as necessary (no more than k times) to conclude that $|A \cap B| = k-1$.

Finally, let i, j be such that $a_i \notin B$ and $b_j \notin A$. It follows that $A - a_i = B - b_j = span_p(i,j)$, and so $d_p(i,j) = k+1$. Thus, for each pair of entries with distance $k+1$, there are exactly two sets A, B for which $del(p; A) = del(p; B)$, and so $|D_k(p)| = \binom{n}{k} - w_k$. □

6 Further Questions

Considering Wilf's pattern packing problem, we would hope that maximizing the large patterns would also maximize the total patterns. For example, having the maximum number of patterns of large sizes seems to maximize the total number of patterns, but there is some subtlety involved. For example, the two permutations in Figure 5 have been verified to have the maximum number of patterns for their size, but not every permutation with the maximum number of patterns has the maximum minimum gap.

For a concrete example, we see that $|\mathcal{D}(3614725)| = |\mathcal{D}(5274136)| = 55$, the maximum number of patterns for permutations of size 7. However, we see that $mg(3614725) = 4$ while $mg(5274136) = 3$. Relaxing the requirement that permutations have the maximum number of patterns for their size allows us to take this a step further. Setting $p = 31462758$ and $q = 36147825$, we find that $|\mathcal{D}(p)| = 75$ while $|\mathcal{D}(q)| = 89$, though $mg(p) = 3$ and $mg(q) = 2$.

The data suggests that while maximizing the number of fixed size patterns requires a large minimum gap size, the total number of patterns is more dependent on the average value of the gaps between pairs of entries. The slanted square construction of [1] yields a permutation with maximum the average gap size between entries, while the construction presented here maximizes the minimum gap size.

Another question which arises is whether or not we can construct a well-behaved multivariate generating function which grants us insight into the distribution for the number of distinct $(n-k)$-patterns of random permutations as we did with the $k = 1$ case. However, even if we had the distribution of the minimum gap size of random permutations, there is no guarantee that this would translate to exact formulas for the distribution of the number of patterns of each length.

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