DENSITIES OF THE RANEY DISTRIBUTIONS

WOJCIECH MŁOTKOWSKI, KAROL A. PENSON, KAROL ŻYCZKOWSKI

ABSTRACT. We prove that if $p \ge 1$ and $0 < r \le p$ then the sequence $\binom{mp+r}{m} \frac{r}{mp+r}$, $m = 0, 1, 2, \ldots$, is positive definite, more precisely, is the moment sequence of a probability measure $\mu(p, r)$ with compact support contained in $[0, +\infty)$. This family of measures encompasses the multiplicative free powers of the Marchenko-Pastur distribution as well as the Wigner's semicircle distribution centered at x = 2. We show that if p > 1 is a rational number, $0 < r \le p$, then $\mu(p, r)$ is absolutely continuous and its density $W_{p,r}(x)$ can be expressed in terms of the Meijer and the generalized hypergeometric functions. In some cases, including the multiplicative free square and the multiplicative free square root of the Marchenko-Pastur measure, $W_{p,r}(x)$ turns out to be an elementary function.

August 7, 2018

INTRODUCTION

For $p, r \in \mathbb{R}$ we define the *Raney numbers* (or *two-parameter Fuss-Catalan numbers*) by

(1)
$$A_m(p,r) := \frac{r}{m!} \prod_{i=1}^{m-1} (mp+r-i).$$

 $A_0(p,r) := 1$. For m = 0, 1, 2, ... we can also write

(2)
$$A_m(p,r) = \binom{mp+r}{m} \frac{r}{mp+r},$$

(provided $mp + r \neq 0$) where the generalized binomial is defined by

$$\binom{a}{m} := \frac{a(a-1)\dots(a-m+1)}{m!}$$

Let $\mathcal{B}_p(z)$ denote the generating function of the sequence $\{A_m(p,1)\}_{m=0}^{\infty}$, the Fuss numbers of order p:

(3)
$$\mathcal{B}_p(z) := \sum_{m=0}^{\infty} A_m(p,1) z^m,$$

convergent in some neighborhood of 0. For example

(4)
$$\mathcal{B}_2(z) = \frac{2}{1 + \sqrt{1 - 4z}}$$

²⁰¹⁰ Mathematics Subject Classification. Primary 44A60; Secondary 33C20.

Key words and phrases. Mellin convolution, free convolution, Meijer function.

K. Ż. is supported by the Grant DEC-2011/02/A/ST1/00119 of Polish National Centre of Science. K. A. P. acknowledges support from PAN/CNRS under Project PICS No. 4339 and from Agence Nationale de la Recherche (Paris, France) under Program PHYSCOMB No. ANR-08-BLAN-0243-2.

Lambert showed that

(5)
$$\mathcal{B}_p(z)^r = \sum_{m=0}^{\infty} A_m(p,r) z^m,$$

see [8]. These generating functions also satisfy

(6)
$$\mathcal{B}_p(z) = 1 + z \mathcal{B}_p(z)^p,$$

which reflects the identity $A_m(p, p) = A_{m+1}(p, 1)$, and

(7)
$$\mathcal{B}_p(z) = \mathcal{B}_{p-r}(z\mathcal{B}_p(z)^r).$$

It was shown in [13] that if $p \ge 1$ and $0 \le r \le p$ then the sequence $\{A_m(p,r)\}_{m=0}^{\infty}$ is positive definite, i.e. is the moment sequence of a probability measure $\mu(p,r)$ on \mathbb{R} . Moreover, $\mu(p,r)$ has compact support (and therefore is unique) contained in the positive half-line $[0,\infty)$ (for example $\mu(p,0) = \delta_0$). The proof involved methods from the free probability theory (see [23, 15, 5]). In particular, for $p \ge 1$

(8)
$$\mu(p,1) = \mu(2,1)^{\boxtimes (p-1)},$$

where \boxtimes denotes the multiplicative free power, and $\mu(2, 1)$ is known as the *Marchenko-Pastur* (called also the *free Poisson*) distribution. It is given by

(9)
$$\mu(2,1) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} \, dx \qquad \text{on } [0,4],$$

and plays an important role in the theory of random matrices, see [24, 9, 10, 2, 1, 4]. It was proved in [1] that the measure $\mu(2, 1)^{\boxtimes n} = \mu(n+1, 1)$ is the limit of the distribution of squared singular values of the power G^n of a random matrix G, when the size of the matrix G goes to infinity.

In this paper we are going to prove positive definiteness of $\{A_m(p,r)\}_{m=0}^{\infty}$ using more classical methods. Namely, we show that if $p > 1, 0 < r \leq p$ and if p is a rational number then $\mu(p,r)$ is absolutely continuous and can be represented as Mellin convolution of modified beta measures. Next we provide a formula for the density $W_{p,r}(x)$ of $\mu(p,r)$ in terms of the Meijer function and consequently, of the generalized hypergeometric functions (cf. [25, 18], where p was assumed to be an integer). This allows us to draw graphs of these densities and, in some particular cases, to express $W_{p,r}(x)$ as an elementary function. It is worth to point out that for r = 1 an alternative description of the densities $W_{p,1}(x)$ has been recently given by Haagerup and Möller, see Corollary 3 in [11].

Finally let us also mention that the measures $\mu(p, r)$ satisfy a peculiar relation:

(10)
$$\mu(p,r) \triangleright \mu(p+s,s) = \mu(p+s,r+s),$$

for $p \ge 1$, $0 < r \le p$ and s > 0, see [13], involving monotonic convolution " \triangleright ", an associative, noncommutative operation on probability measures on \mathbb{R} , introduced by Muraki [14].

1. Preliminaries

For probability measures μ_1 , μ_2 on the positive half-line $[0, \infty)$ the Mellin convolution is defined by

(11)
$$(\mu_1 \circ \mu_2)(A) := \int_0^\infty \int_0^\infty \mathbf{1}_A(xy) d\mu_1(x) d\mu_2(y)$$

for every Borel set $A \subseteq [0, \infty)$. This is the distribution of product $X_1 \cdot X_2$ of two independent nonnegative random variables with $X_i \sim \mu_i$. In particular, if c > 0 then $\mu \circ \delta_c$ is the *dilation* of μ :

$$(\mu \circ \delta_c)(A) = \mathbf{D}_c \mu(A) := \mu\left(\frac{1}{c}A\right).$$

Note that if μ has density f(x) then $\mathbf{D}_c(\mu)$ has density f(x/c)/c.

If both the measures μ_1, μ_2 have all moments

$$s_m(\mu_i) := \int_0^\infty x^m \, d\mu_i(x)$$

finite then so has $\mu_1 \circ \mu_2$ and

$$s_m\left(\mu_1\circ\mu_2\right) = s_m(\mu_1)\cdot s_m(\mu_2)$$

for all m.

If μ_1, μ_2 are absolutely continuous, with densities f_1, f_2 respectively, then so is $\mu_1 \circ \mu_2$ and its density is given by the Mellin convolution:

$$(f_1 \circ f_2)(x) := \int_0^\infty f_1(x/y) f_2(y) \frac{dy}{y}.$$

We will need the following *modified beta distributions*:

Lemma 1.1. Let u, v, l > 0. Then

$$\left\{\frac{\Gamma(u+n/l)\Gamma(u+v)}{\Gamma(u+v+n/l)\Gamma(u)}\right\}_{n=0}^{\infty}$$

is the moment sequence of the probability measure

(12)
$$\mathbf{b}(u+v,u,l) := \frac{l}{\mathbf{B}(u,v)} x^{lu-1} \left(1-x^l\right)^{v-1} dx$$

on [0, 1], where B is the Euler beta function.

Proof. Using the substitution $t = x^l$ we obtain:

$$\frac{\Gamma(u+n/l)\Gamma(u+v)}{\Gamma(u+v+n/l)\Gamma(u)} = \frac{\mathcal{B}(u+n/l,v)}{\mathcal{B}(u,v)} = \frac{1}{\mathcal{B}(u,v)} \int_0^1 t^{u+n/l-1} (1-t)^{v-1} dt$$
$$= \frac{l}{\mathcal{B}(u,v)} \int_0^1 x^{lu+n-1} \left(1-x^l\right)^{v-1} dx.$$

Note that if X is a positive random variable whose distribution has density f(x) and if l > 0 then the distribution of $X^{1/l}$ has density $lx^{l-1}f(x^l)$. In particular, if X has beta distribution $\mathbf{b}(u+v,u,1)$ then $X^{1/l}$ has distribution $\mathbf{b}(u+v,u,l)$.

For u, l > 0 we also define

(13)
$$\mathbf{b}(u, u, l) := \delta_1.$$

2. Applying Mellin Convolution

From now on we assume that p > 1 is a rational number, say p = k/l, with $1 \le l < k$, and that $0 < r \le p$. We will show, that then the sequence $A_m(p, r)$ is a moment sequence of a probability measure $\mu(p, r)$, which can be represented as Mellin convolution of modified beta distributions. In particular, $\mu(p, r)$ is absolutely continuous and we will denote its density by $W_{p,r}$. The case when p is an integer was studied in [18, 25].

First we need to express the numbers $A_m(p,r)$ in a special form.

Lemma 2.1. If p = k/l, where k, l are integers, $1 \le l < k$ and $0 < r \le p$ then

(14)
$$A_m(p,r) = \frac{r}{\sqrt{2\pi k l(k-l)}} \left(\frac{p}{p-1}\right)^r \frac{\prod_{j=1}^k \Gamma(\beta_j + m/l)}{\prod_{j=1}^k \Gamma(\alpha_j + m/l)} c(p)^m,$$

where $c(p) = p^p (p-1)^{1-p}$,

(15)
$$\alpha_j = \begin{cases} \frac{j}{l} & \text{if } 1 \le j \le l, \\ \frac{r+j-l}{k-l} & \text{if } l+1 \le j \le k, \end{cases}$$

(16)
$$\beta_j = \frac{r+j-1}{k}, \qquad 1 \le j \le k.$$

Proof. First we write:

(17)
$$\binom{mp+r}{m}\frac{r}{mp+r} = \frac{r\Gamma(mp+r)}{\Gamma(m+1)\Gamma(mp-m+r+1)}$$

Now we apply the Gauss's multiplication formula:

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \Gamma(z) \Gamma\left(z+\frac{1}{n}\right) \Gamma\left(z+\frac{2}{n}\right) \dots \Gamma\left(z+\frac{n-1}{n}\right)$$

to get:

$$\Gamma(mp+r) = \Gamma\left(k\left(\frac{m}{l} + \frac{r}{k}\right)\right) = (2\pi)^{(1-k)/2} k^{mk/l+r-1/2} \prod_{j=1}^{k} \Gamma\left(\frac{m}{l} + \frac{r+j-1}{k}\right),$$

$$\Gamma(m+1) = \Gamma\left(l\frac{m+1}{l}\right) = (2\pi)^{(1-l)/2} l^{m+1/2} \prod_{j=1}^{l} \Gamma\left(\frac{m}{l} + \frac{j}{l}\right)$$

and

$$\begin{split} \Gamma(mp - m + r + 1) &= \Gamma\left((k - l)\left(\frac{m}{l} + \frac{r + 1}{k - l}\right)\right) \\ &= (2\pi)^{(1 - k + l)/2}(k - l)^{m(k - l)/l + r + 1/2} \prod_{j = l + 1}^{k} \Gamma\left(\frac{m}{l} + \frac{r + j - l}{k - l}\right). \end{split}$$

It remains to apply them to (17).

Now we need to modify enumeration of α 's.

Lemma 2.2. For $1 \le i \le l+1$ denote

$$j_i := \left\lfloor \frac{(i-1)k}{l} \right\rfloor + 1,$$

where $\lfloor \cdot \rfloor$ is the floor function, so that

$$1 = j_1 < j_2 < \ldots < j_l < k < k + 1 = j_{l+1}.$$

For $1 \leq j \leq k$ define

(18)
$$\widetilde{\alpha}_{j} = \begin{cases} \frac{i}{l} & \text{if } j = j_{i}, \ 1 \le i \le l, \\ \frac{r+j-i}{k-l} & \text{if } j_{i} < j < j_{i+1}. \end{cases}$$

Then the sequence $\{\widetilde{\alpha}_j\}_{j=1}^k$ is a rearrangement of $\{\alpha_j\}_{j=1}^k$. Moreover, if $0 < r \le p = k/l$ then we have $\beta_j \le \widetilde{\alpha}_j$ for all $j \le k$.

Proof. It is easy to verify the first statement.

Assume that $j = j_i$ for some $i \leq l$. Then we have to show that

$$\frac{r+j_i-1}{k} \le \frac{i}{l},$$

which is equivalent to

$$lr + l\left\lfloor \frac{k(i-1)}{l} \right\rfloor \le ki,$$

and the latter is a consequence of the fact that $\lfloor x \rfloor \leq x$ and the assumption $r \leq p = k/l$.

Now assume that $j_i < j < j_{i+1}$. We ought to show that $r+j-1 \ , r+j-i$

$$\frac{r+j-1}{k} \le \frac{r+j-i}{k-l},$$

which is equivalent to

$$lr + lj + k - l - ki \ge 0.$$

Using the inequality |x| + 1 > x we obtain

$$lj + k - l - ki \ge l(j_i + 1) + k - l - ki = lj_i + k - ki > k(i - 1) + k - ki = 0,$$

which completes the proof, as r > 0.

Now we are ready to prove the main theorem of this section.

Theorem 2.3. Suppose that p = k/l, where k, l are integers such that $1 \le l < k$, and that r is a real number, $0 < r \le p$. Then there exists a unique probability measure $\mu(p,r)$ such that (1) is its moment sequence. Moreover $\mu(p,r)$ can be represented as the following Mellin convolution:

$$\mu(p,r) = \mathbf{b}(\widetilde{\alpha}_1,\beta_1,l) \circ \ldots \circ \mathbf{b}(\widetilde{\alpha}_k,\beta_k,l) \circ \delta_{c(p)},$$

where

$$c(p) := \frac{p^p}{(p-1)^{p-1}}$$

Consequently, $\mu(p, r)$ is absolutely continuous and its support is [0, c(p)].

Note that the representation of densities in the form of Mellin convolution of modified beta distributions was used in different context in [7], see its Appendix A.

Example. For the Marchenko-Pastur measure we get the following decomposition:

(19)
$$\mu(2,1) = \mathbf{b}(1,1/2,1) \circ \mathbf{b}(2,1,1) \circ \delta_4$$

where $\mathbf{b}(1, 1/2, 1)$ has density $1/(\pi\sqrt{x-x^2})$ on [0, 1], the arcsine distribution with the moment sequence $\binom{2m}{m}4^{-m}$, and $\mathbf{b}(2, 1, 1)$ is the Lebesgue measure on [0, 1] with the moment sequence 1/(m+1).

Proof. In view of Lemma 2.1 and Lemma 2.2 we can write

$$A_m(p,r) = D \prod_{j=1}^k \frac{\Gamma(\beta_j + m/l)\Gamma(\widetilde{\alpha}_j)}{\Gamma(\widetilde{\alpha}_j + m/l)\Gamma(\beta_j)} \cdot c(p)^m$$

for some constant D. Taking m = 0 we see that D = 1.

Note that a part of the theorem illustrates a result of Kargin [12], who proved that if μ is a compactly supported probability measure on $[0, \infty)$, with expectation 1 and variance V, and if L_n denotes the supremum of the support of the multiplicative free convolution power $\mu^{\boxtimes n}$, then

(20)
$$\lim_{n \to \infty} \frac{L_n}{n} = eV,$$

where e = 2.71... is the Euler's number. The Marchenko-Pastur measure $\mu(2, 1)$ has expectation and variance equal to 1 and $\mu(2, 1)^{\boxtimes n} = \mu(n + 1, 1)$, so in this case $L_n = (n + 1)^{n+1}/n^n$ (this was also proved in [24] and [10]) and (20) holds.

The density function for $\mu(p,r)$ will be denoted by $W_{p,r}(x)$. Since $A_m(p,p) = A_{m+1}(p,1)$, we have

(21)
$$W_{p,p}(x) = x \cdot W_{p,1}(x),$$

for example

(22)
$$W_{2,2}(x) = \frac{1}{2\pi}\sqrt{x(4-x)}$$
 on $[0,4],$

which is the famous semicircle Wigner distribution with radius 2, centered at x = 2. Now we can reprove the main result of [13].

Theorem 2.4. Suppose that p, r are real numbers such that $p \ge 1$ and $0 \le r \le p$. Then there exists a unique probability measure $\mu(p, r)$, with support contained in [0, c(p)], such that $\{A_m(p, r)\}_{m=0}^{\infty}$ is its moment sequence.

Proof. It follows from the fact that the class of positive definite sequence is closed under pointwise limits. \Box

3. Applying Meijer G-function

The aim of this section is to describe the density function $W_{p,r}(x)$ of $\mu(p,r)$ in terms of the Meijer *G*-function (see [16] for example) and consequently, as a linear combination of generalized hypergeometric functions. We will see that in some particular cases $W_{p,r}$ can be represented as an elementary function.

Theorem 3.1. Let p = k/l > 1, where k, l are integers such that $1 \le l < k$, and let r be a positive real number, $r \le p$. Then the density $W_{p,r}$ of the probability measure $\mu(p,r)$ can be expressed as

(23)
$$W_{p,r}(x) = \frac{rp^r}{x(p-1)^{r+1/2}\sqrt{2k\pi}} G_{k,k}^{k,0} \begin{pmatrix} \alpha_1, \dots, \alpha_k \\ \beta_1, \dots, \beta_k \end{pmatrix} \frac{x^l}{c(p)^l},$$

 $x \in (0, c(p))$, where $c(p) = p^p (p-1)^{1-p}$ and the parameters α_j, β_j are given by (15) and (16).

Proof. Define

$$\phi_{p,r}(\sigma) = \frac{r\Gamma(\sigma p - p + r)}{\Gamma(\sigma)\Gamma(\sigma p - \sigma - p + r + 2)}$$

If m is a natural number then

$$\phi_{p,r}(m+1) = \binom{mp+r}{m} \frac{r}{mp+r}$$

so $\phi_{p,r}$ is the Mellin transform of the density function $W_{p,r}$ of $\mu(p,r)$:

$$\phi_{p,r}(\sigma) = \int_0^\infty x^{\sigma-1} W_{p,r}(x) \, dx.$$

In order to reconstruct $W_{p,r}$ we apply the inverse Mellin transform:

$$W_{p,r}(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} x^{-\sigma} \phi_{p,r}(\sigma) \, d\sigma,$$

see [3, 16, 19] for details. Putting $m = \sigma - 1$ in (14) we get

$$\phi_{p,r}(\sigma) = \frac{r(p-1)^{p-1-r}}{p^{p-r}\sqrt{2\pi k l(k-l)}} \frac{\prod_{j=1}^{k} \Gamma(\beta_j - 1/l + \sigma/l)}{\prod_{j=1}^{k} \Gamma(\alpha_j - 1/l + \sigma/l)} c(p)^{\sigma}.$$

Writing the right hand side as $\Phi(\sigma/l)c(p)^{\sigma}$, using the substitution $\sigma = lu$ and the definition of the Meijer *G*-function (see [16] for example) we obtain

$$W_{p,r}(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Phi(\sigma/l) c(p)^{\sigma} x^{-\sigma} d\sigma = \frac{l}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Phi(u) \left(x^{l}/c(p)^{l} \right)^{-u} du$$
$$= \frac{r(p-1)^{p-r-3/2}}{p^{p-r}\sqrt{2\pi k}} G_{k,k}^{k,0} \left(\begin{array}{c} \alpha_{1}^{-}, \dots, \alpha_{k}^{-} \\ \beta_{1}^{-}, \dots, \beta_{k}^{-} \end{array} \right| z \right),$$

where $z = x^l/c(p)^l$, $\alpha_j^- = \alpha_j - 1/l$, $\beta_j^- = \beta_j - 1/l$. Finally we use formula (16.19.2) in [16] and obtain

(24)
$$W_{p,r}(x) = \frac{r(p-1)^{p-r-3/2}}{z^{1/l}p^{p-r}\sqrt{2\pi k}} G_{k,k}^{k,0} \begin{pmatrix} \alpha_1, \dots, \alpha_k \\ \beta_1, \dots, \beta_k \\ \end{pmatrix} z \end{pmatrix},$$

which is equivalent to (23).

Now applying Slater's theorem (see (16.17.2) in [16]) we can represent $W_{p,r}$ as a linear combination of hypergeometric functions.

Theorem 3.2. For for p = k/l, with $1 \le l < k$, r > 0 and $x \in (0, c(p))$ we have

(25)
$$W_{p,r}(x) = \gamma(k,l,r) \sum_{h=1}^{k} c(h,k,l,r) {}_{k}F_{k-1} \begin{pmatrix} \mathbf{a}(h,k,l,r) \\ \mathbf{b}(h,k,l,r) \\ \end{pmatrix} z^{(r+h-1)/k-1/l}$$

where $z = x^l/c(p)^l$,

(26)
$$\gamma(k,l,r) = \frac{r(p-1)^{p-r-3/2}}{p^{p-r}\sqrt{2\pi k}},$$

(27)
$$c(h,k,l,r) = \frac{\prod_{j=1}^{h-1} \Gamma\left(\frac{j-h}{k}\right) \prod_{j=h+1}^{k} \Gamma\left(\frac{j-h}{k}\right)}{\prod_{j=1}^{l} \Gamma\left(\frac{j}{l} - \frac{r+h-1}{k}\right) \prod_{j=l+1}^{k} \Gamma\left(\frac{r+j-l}{k-l} - \frac{r+h-1}{k}\right)}$$

and the parameter vectors of the hypergeometric functions are

(28)
$$\mathbf{a}(h,k,l,r) = \left(\left\{ \frac{r+h-1}{k} - \frac{j-l}{l} \right\}_{j=1}^{l}, \left\{ \frac{r+h-1}{k} - \frac{r+j-k}{k-l} \right\}_{j=l+1}^{k} \right),$$

(29)
$$\mathbf{b}(h,k,l,r) = \left(\left\{\frac{k+h-j}{k}\right\}_{j=1}^{h-1}, \left\{\frac{k+h-j}{k}\right\}_{j=h+1}^{k}\right)$$

The most tractable case is p = 2.

Corollary 3.3. For $p = 2, 0 < r \le 2$, the density function is

(30)
$$W_{2,r}(x) = \frac{\sin\left(r \cdot \arccos\sqrt{x/4}\right)}{\pi x^{1-r/2}},$$

 $x \in (0,4)$. In particular for r = 1/2 we have

(31)
$$W_{2,1/2}(x) = \frac{\sqrt{2} - \sqrt{x}}{2\pi x^{3/4}},$$

and for r = 3/2

(32)
$$W_{2,3/2}(x) = \frac{(\sqrt{x}+1)\sqrt{2}-\sqrt{x}}{2\pi x^{1/4}}.$$

Note that if r > 2 then $W_{2,r}(x) < 0$ for some values of $x \in (0, 4)$.

Proof. We take k = 2, l = 1 so that c(2) = 4, z = x/4 and $\gamma(2, 1, r) = r2^r/(8\sqrt{\pi})$. Using the Euler's reflection formula and the identity $\Gamma(1 + r/2) = \Gamma(r/2)r/2$ we get

$$c(1,2,1,r) = \frac{\Gamma(1/2)}{\Gamma(1-r/2)\Gamma(1+r/2)} = \frac{2\sin(\pi r/2)}{r\sqrt{\pi}},$$

$$c(2,2,1,r) = \frac{\Gamma(-1/2)}{\Gamma((1-r)/2)\Gamma((1+r)/2)} = \frac{-2\cos(\pi r/2)}{\sqrt{\pi}}$$

We also need formulas for two hypergeometric functions, namely

$${}_{2}F_{1}\left(\frac{r}{2}, \frac{-r}{2}; \frac{1}{2} \middle| z\right) = \cos(r \arcsin\sqrt{z}),$$
$${}_{2}F_{1}\left(\frac{1+r}{2}, \frac{1-r}{2}; \frac{3}{2} \middle| z\right) = \frac{\sin(r \arcsin\sqrt{z})}{r\sqrt{z}},$$

see 15.4.12 and 15.4.16 in [16]. Now we can write

$$W_{2,r}(x) = \frac{\sin(\pi r/2)\cos\left(r \arcsin\sqrt{x/4}\right) - \cos(\pi r/2)\sin\left(r \arcsin\sqrt{x/4}\right)}{\pi x^{1-r/2}}$$
$$= \frac{\sin\left(\pi r/2 - r \arcsin\sqrt{x/4}\right)}{\pi x^{1-r/2}} = \frac{\sin\left(r \arccos\sqrt{x/4}\right)}{\pi x^{1-r/2}},$$
concludes the proof.

which concludes the proof.

Remark. Note that

(33)
$$\frac{W_{2,1}(\sqrt{x})}{2\sqrt{x}} = \frac{1}{4}W_{2,1/2}\left(\frac{x}{4}\right).$$

It means that if X, Y are random variables such that $X \sim \mu(2,1)$ and $Y \sim \mu(2,1/2)$ then $X^2 \sim 4Y$. This can be also derived from the relation $A_m(2, 1/2)4^m = A_{2m}(2, 1)$.

4. Some particular cases

In this part we will see that for k = 3 some densities still can be represented as elementary functions. We will need two families of formulas (cf. 15.4.17 in [16]).

Lemma 4.1. For $c \neq 0, -1, -2, ...$ we have

(34)
$$_{2}F_{1}\left(\frac{c}{2}, \frac{c-1}{2}; c \middle| z\right) = 2^{c-1} (1 + \sqrt{1-z})^{1-c},$$

(35)
$$_{2}F_{1}\left(\frac{c+1}{2}, \frac{c-2}{2}; c \mid z\right) = \frac{2^{c-1}}{c} \left(1 + \sqrt{1-z}\right)^{1-c} \left(c - 1 + \sqrt{1-z}\right).$$

Proof. We know that ${}_{2}F_{1}(a,b;c|z)$ is the unique function f which is analytic at z=0, with f(0) = 1, and satisfies the hypergeometric equation:

$$z(1-z)f''(z) + [c - (a+b+1)z]f'(z) - abf(z) = 0$$

(see [3]). Now one can check that this equation is satisfied by the right hand sides of (34) and (35) for given parameters a, b, c.

Now consider p = 3/2.

Theorem 4.2. Assume that p = 3/2. Then for r = 1/2 we have

(36)
$$W_{3/2,1/2}(x) = \frac{\left(1 + \sqrt{1 - 4x^2/27}\right)^{2/3} - \left(1 - \sqrt{1 - 4x^2/27}\right)^{2/3}}{2^{5/3} 3^{-1/2} \pi x^{2/3}},$$

for r = 1:

(37)
$$W_{3/2,1}(x) = 3^{1/2} \frac{\left(1 + \sqrt{1 - 4x^2/27}\right)^{1/3} - \left(1 - \sqrt{1 - 4x^2/27}\right)^{1/3}}{2^{4/3}\pi x^{1/3}}$$

$$+3^{1/2}x^{1/3}\frac{\left(1+\sqrt{1-4x^2/27}\right)^{2/3}-\left(1-\sqrt{1-4x^2/27}\right)^{2/3}}{2^{5/3}\pi}$$

and, finally, $W_{3/2,3/2}(x) = x \cdot W_{3/2,1}(x)$, with $x \in (0, 3\sqrt{3}/2)$.

Proof. For arbitrary r we have

$$W_{3/2,r}(x) = \frac{2^{1-2r/3} \sin\left(2\pi r/3\right)}{3^{3/2-r}\pi} {}_{3}F_{2}\left(\frac{3+2r}{6}, \frac{r}{3}, \frac{-2r}{3}; \frac{2}{3}, \frac{1}{3} \middle| z\right) z^{r/3-1/2}$$
$$-\frac{2^{(4-2r)/3} r \sin\left((1-2r)\pi/3\right)}{3^{3/2-r}\pi} {}_{3}F_{2}\left(\frac{5+2r}{6}, \frac{1+r}{3}, \frac{1-2r}{3}; \frac{4}{3}, \frac{2}{3} \middle| z\right) z^{(r+1)/3-1/2}$$
$$-\frac{r(1+2r) \sin\left((1+2r)\pi/3\right)}{2^{(1+2r)/3} 3^{3/2-r}\pi} {}_{3}F_{2}\left(\frac{7+2r}{6}, \frac{2+r}{3}, \frac{2-2r}{3}; \frac{5}{3}, \frac{4}{3} \middle| z\right) z^{(r+2)/3-1/2},$$

where $z = 4x^2/27$. If r = 1/2 or r = 1 then one term vanishes and in the two others the hypergeometric functions reduce to $_2F_1$.

For r = 1/2 we apply (34) to obtain:

$$W_{3/2,1/2}(x) = \frac{z^{-1/3}}{2^{1/3}3^{1/2}\pi} {}_{2}F_{1}\left(\frac{1}{6}, \frac{-1}{3}; \frac{1}{3} \middle| z\right) - \frac{z^{1/3}}{2^{5/3}3^{1/2}\pi} {}_{2}F_{1}\left(\frac{5}{6}, \frac{1}{3}; \frac{5}{3} \middle| z\right)$$
$$= \frac{z^{-1/3}}{2^{1/3}3^{1/2}\pi} 2^{-2/3} \left(1 + \sqrt{1-z}\right)^{2/3} - \frac{z^{1/3}}{2^{5/3}3^{1/2}\pi} 2^{2/3} \left(1 + \sqrt{1-z}\right)^{-2/3}$$
$$= \frac{z^{-1/3}}{2 \cdot 3^{1/2}\pi} \left(1 + \sqrt{1-z}\right)^{2/3} - \frac{z^{1/3}}{2 \cdot 3^{1/2}\pi} \left(\frac{1 - \sqrt{1-z}}{z}\right)^{2/3}$$
$$= \frac{z^{-1/3}}{2 \cdot 3^{1/2}\pi} \left(1 + \sqrt{1-z}\right)^{2/3} - \frac{z^{-1/3}}{2 \cdot 3^{1/2}\pi} \left(1 - \sqrt{1-z}\right)^{2/3}$$

and this yields (36).

For r = 1 we use (35):

$$W_{3/2,1}(x) = \frac{z^{-1/6}}{2^{2/3}\pi} {}_{2}F_{1}\left(\frac{5}{6}, \frac{-2}{3}; \frac{2}{3} \middle| z\right) + \frac{z^{1/6}}{2^{1/3}\pi} {}_{2}F_{1}\left(\frac{7}{6}, \frac{-1}{3}; \frac{4}{3} \middle| z\right)$$
$$= \frac{z^{-1/6}}{4\pi} \left(1 + \sqrt{1-z}\right)^{1/3} \left(3\sqrt{1-z} - 1\right) + \frac{z^{1/6}}{4\pi} \left(1 + \sqrt{1-z}\right)^{-1/3} \left(3\sqrt{1-z} + 1\right)$$
$$= \frac{z^{-1/6}}{4\pi} \left(1 + \sqrt{1-z}\right)^{1/3} \left(3\sqrt{1-z} - 1\right) + \frac{z^{-1/6}}{4\pi} \left(1 - \sqrt{1-z}\right)^{1/3} \left(3\sqrt{1-z} + 1\right).$$

Now we have

$$(1+\sqrt{1-z})^{1/3} (3\sqrt{1-z}-1) = -(1+\sqrt{1-z})^{1/3} (3-3\sqrt{1-z}-2)$$
$$= -3z^{1/3} (1-\sqrt{1-z})^{2/3} + 2(1+\sqrt{1-z})^{1/3}$$

and similarly

$$\left(1 - \sqrt{1-z}\right)^{1/3} \left(3\sqrt{1-z} + 1\right) = 3z^{1/3} \left(1 + \sqrt{1-z}\right)^{2/3} - 2\left(1 - \sqrt{1-z}\right)^{1/3}.$$

Therefore

$$W_{3/2,1}(x) = \frac{z^{-1/6}}{2\pi} \left(\left(1 + \sqrt{1-z}\right)^{1/3} - \left(1 - \sqrt{1-z}\right)^{1/3} \right) + \frac{3z^{1/6}}{4\pi} \left(\left(1 + \sqrt{1-z}\right)^{2/3} - \left(1 - \sqrt{1-z}\right)^{2/3} \right),$$

which entails (37). The last statement is a consequence of (21).

The dilation $\mathbf{D}_2\mu(3/2, 1/2)$, with the density $W_{3/2,1/2}(x/2)/2$, is known as the *Bures* distribution, see (4.4) in [21]. Moreover, the integer sequence $4^m A_m(3/2, 1/2)$, moments of $\mathbf{D}_4\mu(3/2, 1/2)$, appears as A078531 in [20] and counts the number of symmetric noncrossing connected graphs on 2n + 1 equidistant nodes on a circle. The axis of symmetry is a diameter of a circle passing through a given node, see [6].

The measure $\mu(3/2, 1)$ is equal to $\mu(2, 1)^{\boxtimes 1/2}$, the multiplicative free square root of the Marchenko-Pastur distribution.

For the sake of completeness we also include the cases p = 3, r = 1 and p = 3, r = 2, which have already appeared in [17, 18].

Theorem 4.3. Assume that p = 3. Then for r = 1 we have

(38)
$$W_{3,1}(x) = \frac{3\left(1 + \sqrt{1 - 4x/27}\right)^{2/3} - 2^{2/3}x^{1/3}}{2^{4/3}3^{1/2}\pi x^{2/3}\left(1 + \sqrt{1 - 4x/27}\right)^{1/3}},$$

for r = 2:

(39)
$$W_{3,2}(x) = \frac{9\left(1 + \sqrt{1 - 4x/27}\right)^{4/3} - 2^{4/3}x^{2/3}}{2^{5/3}3^{3/2}\pi x^{1/3}\left(1 + \sqrt{1 - 4x/27}\right)^{2/3}}$$

and, finally, $W_{3,3}(x) = x \cdot W_{3,1}(x)$, with $x \in (0, 27/4)$.

Proof. For arbitrary r we have

$$W_{3,r}(x) = \frac{2^{(6-2r)/3} \sin(\pi r/3)}{3^{3-r}\pi} {}_{3}F_{2}\left(\frac{r}{3}, \frac{3-r}{6}, \frac{-r}{6}; \frac{2}{3}, \frac{1}{3} \middle| z\right) z^{(r-3)/3}$$
$$-\frac{2^{(4-2r)/3} r \sin\left((1-2r)\pi/3\right)}{3^{3-r}\pi} {}_{3}F_{2}\left(\frac{1+r}{3}, \frac{5-r}{6}, \frac{2-r}{6}; \frac{4}{3}, \frac{2}{3} \middle| z\right) z^{(r-2)/3}$$
$$+\frac{r(r-1) \sin\left((1-r)\pi/3\right)}{2^{(1+2r)/3}3^{3-r}\pi} {}_{3}F_{2}\left(\frac{2+r}{3}, \frac{7-r}{6}, \frac{4-r}{6}; \frac{5}{3}, \frac{4}{3} \middle| z\right) z^{(r-1)/3},$$

where z = 4x/27. For r = 1 and r = 2 we have similar reduction as in the previous proof. Here we will be using only (34).

Taking r = 1 we get

$$W_{3,1}(x) = \frac{2^{1/3}z^{-2/3}}{3^{3/2}\pi} {}_2F_1\left(\frac{1}{3}, \frac{-1}{6}; \frac{2}{3} \middle| z\right) - \frac{z^{-1/3}}{2^{1/3}3^{3/2}\pi} {}_2F_1\left(\frac{2}{3}, \frac{1}{6}; \frac{4}{3} \middle| z\right)$$
$$= \frac{z^{-2/3}}{3^{3/2}\pi} \left(1 + \sqrt{1-z}\right)^{1/3} - \frac{z^{-1/3}}{3^{3/2}\pi} \left(1 + \sqrt{1-z}\right)^{-1/3}$$
$$= \frac{\left(1 + \sqrt{1-z}\right)^{2/3} - z^{1/3}}{3^{3/2}\pi z^{2/3} \left(1 + \sqrt{1-z}\right)^{1/3}},$$

which implies (38).

Now we take r = 2:

$$W_{3,2}(x) = \frac{z^{-1/3}}{2^{1/3}3^{1/2}\pi} {}_{2}F_{1}\left(\frac{1}{6}, \frac{-1}{3}; \frac{1}{3} \middle| z\right) - \frac{z^{1/3}}{2^{5/3}3^{1/2}\pi} {}_{2}F_{1}\left(\frac{5}{6}, \frac{1}{3}; \frac{5}{3} \middle| z\right)$$

$$= \frac{z^{-1/3}}{2 \cdot 3^{1/2} \pi} \left(1 + \sqrt{1-z}\right)^{2/3} - \frac{z^{1/3}}{2 \cdot 3^{1/2} \pi} \left(1 + \sqrt{1-z}\right)^{-2/3}$$
$$= \frac{\left(1 + \sqrt{1-z}\right)^{4/3} - z^{2/3}}{2 \cdot 3^{1/2} \pi z^{1/3} \left(1 + \sqrt{1-z}\right)^{2/3}},$$

and this gives us (39). Finally we apply (21).

Note that the measure $\mu(3,1)$ is equal to $\mu(2,1)^{\boxtimes 2}$, the multiplicative free square of the Marchenko-Pastur distribution.

5. Graphical representation of selected cases

The explicit form of $W_{p,r}(x)$ given in Theorem 3.2 permits a graphical visualization for any rational p > 0 and arbitrary r > 0. We shall represent some selected cases in Figures 1–9. These graphs which are partly negative are drawn as dashed curves. In Fig. 1 the graphs of the functions $W_{3/2,r}(x)$ for values of r ranging from 0.9 to 2.3 are given. For r < 3/2 these functions are positive, otherwise they develop a negative part. In Fig. 2 we represent $W_{5/2,r}(x)$ for r ranging from 2 to 3.4. In Fig. 3 we display the densities $W_{p,p}(x)$ for p = 6/5, 5/4, 4/3 and 3/2. All these densities are unimodal and vanish at the extremities of their supports. They can be therefore considered as generalizations of the Wigner's semicircle distribution $W_{2,2}(x)$, see equation (22). In Fig. 4 we depict the functions $W_{4/3,r}(x)$, for values r ranging from 0.8 to 2.4. Here for $r \ge 1.4$ negative contributions clearly appear. In Fig. 5 and 6 we present six densities expressible through elementary functions, namely $W_{3/2,r}(x)$ for r = 1/2, 1, 3/2, see Theorem 4.2 and $W_{3,r}(x)$ for r = 1, 2, 3, see Theorem 4.3. In Fig. 7 the set of densities $W_{p,1}(x)$ for five fractional values of p is presented. The appearance of maximum near x = 1 corresponds to the fact that $\mu(p,1)$ weakly converges to δ_1 as $p \to 1^+$. In Fig. 8 the fine details of densities $W_{p,1}(x)$ for p = 5/2, 7/3, 9/4, 11/5, on a narrower range $2 \le x \le 4.5$ are presented. In Fig. 9 we display the densities $W_{p,1}(x)$ for p = 2, 5/2, 3, 7/2, 4, near the upper edge of their respective supports, for $3 \le x \le 9.5$.

The Fig. 10 summarizes our results in the p > 0, r > 0 quadrant of the (p, r) plane, describing the Raney numbers (c.f. Fig. 5.1 in [13] and Fig. 7 in [18]). The shaded region Σ indicates the probability measures $\mu(p, r)$ (i.e. where $W_{p,r}(x)$ is a nonegative function). The vertical line p = 2 and the stars indicate the pairs (p, r) for which $W_{p,r}(x)$ is an elementary function, see Corollary 3.3, Theorem 4.2 and Theorem 4.3. The points (3/2, 1) and (3, 1) correspond to the multiplicative free powers $MP^{\boxtimes 1/2}$ and $MP^{\boxtimes 2}$ of the Marchenko-Pastur distribution MP. Symbol B at (3/2, 1/2) indicates the Bures distribution and SC at (2, 2) denotes the semicircle law centered at x = 2, with radius 2.

It is our pleasure to thank M. Bożejko, Z. Burda, K. Górska, I. Nechita and M. A. Nowak for fruitful interactions.

References

- N. Alexeev, F. Götze, A. Tikhomirov, Asymptotic distribution of singular values of powers of random matrices, Lith. Math. J. 50 (2010), 121-132.
- [2] G. W. Anderson, A. Guionnet, O. Zeitouni, An introduction to random matrices, Cambridge Studies in Advanced Mathematics, 118. Cambridge University Press, Cambridge, 2010.
- [3] G. E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge 1999.

- [4] T. Banica, S. Belinschi, M. Capitaine, B. Collins, Free Bessel laws, Canad. J. Math. 63 (2011), 3–37.
- [5] P. Biane, *Free probability and combinatorics*, Proceedings of the International Congress of Mathematics, Higher Ed. Press, Beijing 2002, Vol. II (2002), 765–774.
- [6] P. Flajolet and M. Noy, Analytic combinatorics of non-crossing configurations, Discrete Math., 204 (1999) 203–229.
- [7] K. Górska, K. A. Penson, A. Horzela, G. H. E. Duchamp, P. Blasiak, A. I. Solomon, *Quasiclassical asymptotics and coherent states for bounded discrete spectra*, J. Math. Phys. **51** (2010), 122102, 12 pp.
- [8] R. L. Graham, D. E. Knuth, O. Patashnik, Concrete Mathematics. A Foundation for Computer Science, Addison-Wesley, New York 1994.
- [9] U. Haagerup, F. Larsen, Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras, J. Funct. Anal. **176** (2000), 331-367.
- [10] U. Haagerup, S. Thorbjørnsen, A new application of random matrices: $Ext(C^*_{red}(F_2))$ is not a group, Ann. of Math. 162 (2005), 711-775.
- [11] U. Haagerup, S. Möller, The law of large numbers for the free multiplicative convolution, arXiv:1211.4457.
- [12] V. Kargin, On asymptotic growth of the support of free multiplicative convolutions, Elec. Comm. Prob. 13 (2008), 415–421.
- [13] W. Młotkowski, Fuss-Catalan numbers in noncommutative probability, Documenta Math. 15 (2010) 939–955.
- [14] N. Muraki, Monotonic independence, monotonic central limit theorem and monotonic law of small numbers, Inf. Dim. Anal. Quantum Probab. Rel. Topics 4 (2001) 39–58.
- [15] A. Nica, R. Speicher, Lectures on the Combinatorics of Free Probability, Cambridge University Press, 2006.
- [16] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge 2010.
- [17] K. A. Penson, A. I. Solomon, Coherent states from combinatorial sequences, Quantum theory and symmetries, Kraków 2001, World Sci. Publ., River Edge, NJ, 2002, 527–530.
- [18] K. A. Penson, K. Zyczkowski, Product of Ginibre matrices: Fuss-Catalan and Raney distributions Phys. Rev. E 83 (2011) 061118, 9 pp.
- [19] A. D. Polyanin, A. V. Manzhirov, Handbook of Integral Equations, CRC Press, Boca Raton, 1998.
- [20] N. J. A. Sloane, The On-line Encyclopedia of Integer Sequences, (2012), published electronically at: http://oeis.org/.
- [21] H.-J. Sommers, K. Życzkowski, Statistical properties of random density matrices, J. Phys. A: Math. Gen. 37 (2004) 8457–8466.
- [22] R. P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge 1999.
- [23] D. V. Voiculescu, K. J. Dykema, A. Nica, Free random variables, CRM, Montréal, 1992.
- [24] R. Wegmann, The asymptotic eigenvalue-distribution for a certain class of random matrices J. Math. Anal. Appl. 56 (1976) 113-132.
- [25] K. Zyczkowski, K. A. Penson, I. Nechita, B. Collins, Generating random density matrices, J. Math. Phys. 52 (2011) 062201, 20 pp.

Instytut Matematyczny, Uniwersytet Wrocławski, Plac Grunwaldzki 2/4, 50-384 Wrocław, Poland

E-mail address: mlotkow@math.uni.wroc.pl

LABORATOIRE DE PHYSIQUE THÉORIQUE DE LA MATIÈRE CONDENSÉE (LPTMC), UNIVERSITÉ PIERRE ET MARIE CURIE, CNRS UMR 7600, TOUR 13 - 5IÈME ÉT., BOÎTE COURRIER 121, 4 PLACE JUSSIEU, F 75252 PARIS CEDEX 05, FRANCE *E-mail address*: penson@lptl.jussieu.fr

Institute of Physics, Jagiellonian University, Cracow and Center for Theoretical

Physics, Polish Academy of Sciences, Warsaw, Poland

E-mail address: karol@tatry.if.uj.edu.pl

FIGURE 1. Raney distributions $W_{3/2,r}(x)$ with values of the parameter r labeling each curve. For r > p solutions drawn with dashed lines are not positive.

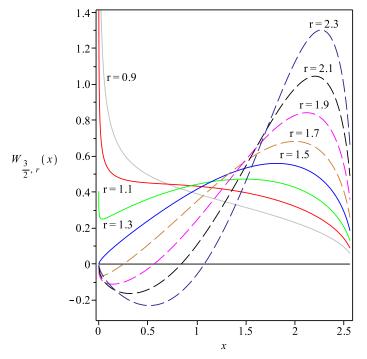


FIGURE 2. As in Fig. 1 for Raney distributions $W_{5/2,r}(x)$.

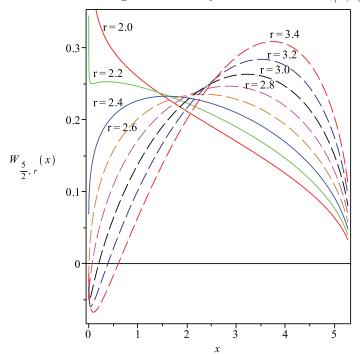


FIGURE 3. Diagonal Raney distributions $W_{p,p}(x)$ with values of the parameter p labeling each curve.

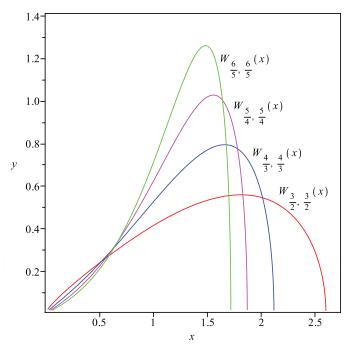


FIGURE 4. The functions $W_{4/3,r}(x)$ for r ranging from 0.8 to 2.4.

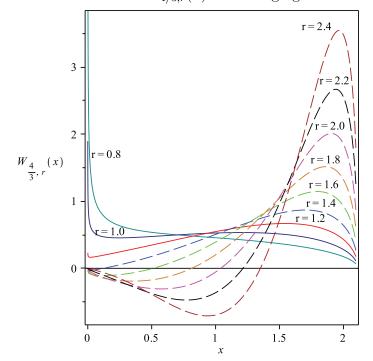


FIGURE 5. Raney distributions $W_{3/2,r}(x)$ with values of the parameter r labeling each curve. The case $W_{3/2,1}(x)$ represents the multiplicative free square root of the Marchenko Pastur distribution.

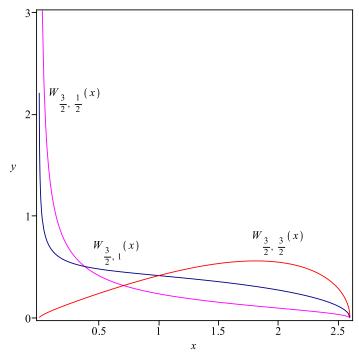


FIGURE 6. Raney distributions $W_{3,r}(x)$ with values of the parameter r labeling each curve. The case $W_{3,1}(x)$ represents the multiplicative free square of the Marchenko Pastur distribution.

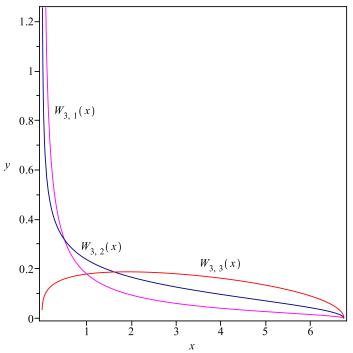


FIGURE 7. Raney distributions $W_{p,1}(x)$ with values of the parameter p labeling each curve. The case $W_{3/2,1}(x)$ represents the multiplicative free square root of the Marchenko–Pastur distribution, $MP^{\boxtimes 1/2}$.

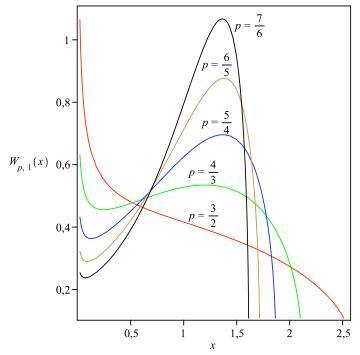
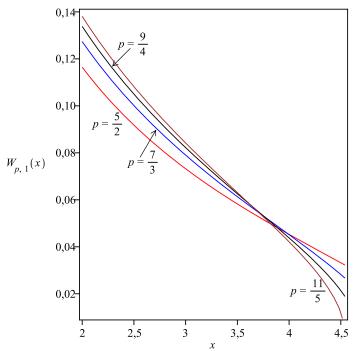


FIGURE 8. Tails of the Raney distributions $W_{p,1}(x)$ with values of the parameter p labeling each curve.



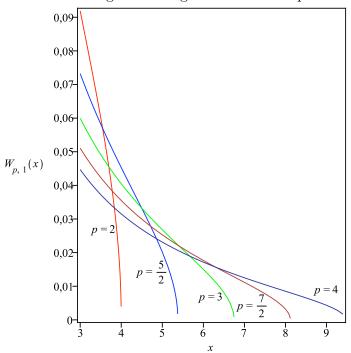


FIGURE 9. As in Fig. 8 for larger values of the parameter p.

FIGURE 10. Parameter plane (p, r) describing the Raney numbers. The shaded set Σ corresponds to nonnegative probability measures $\mu(p, r)$. The vertical line p = 2 and the stars represent values of parameters for which $W_{p,r}(x)$ is an elementary function. Here MP denotes the Marchenko–Pastur distribution, $MP^{\boxtimes s}$ its *s*-th free mutiplicative power, B-the Bures distribution while SC denotes the semicircle law. For p > 1the points (p, p) on the upper edge of Σ represent the generalizations of the Wigner semicircle law, see Fig. 3.

