# DENSITIES OF THE RANEY DISTRIBUTIONS 

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#### Abstract

We prove that if $p \geq 1$ and $0<r \leq p$ then the sequence $\binom{m p+r}{m} \frac{r}{m p+r}$, $m=0,1,2, \ldots$, is positive definite, more precisely, is the moment sequence of a probability measure $\mu(p, r)$ with compact support contained in $[0,+\infty)$. This family of measures encompasses the multiplicative free powers of the Marchenko-Pastur distribution as well as the Wigner's semicircle distribution centered at $x=2$. We show that if $p>1$ is a rational number, $0<r \leq p$, then $\mu(p, r)$ is absolutely continuous and its density $W_{p, r}(x)$ can be expressed in terms of the Meijer and the generalized hypergeometric functions. In some cases, including the multiplicative free square and the multiplicative free square root of the Marchenko-Pastur measure, $W_{p, r}(x)$ turns out to be an elementary function.


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## Introduction

For $p, r \in \mathbb{R}$ we define the Raney numbers (or two-parameter Fuss-Catalan numbers) by

$$
\begin{equation*}
A_{m}(p, r):=\frac{r}{m!} \prod_{i=1}^{m-1}(m p+r-i) \tag{1}
\end{equation*}
$$

$A_{0}(p, r):=1$. For $m=0,1,2, \ldots$ we can also write

$$
\begin{equation*}
A_{m}(p, r)=\binom{m p+r}{m} \frac{r}{m p+r} \tag{2}
\end{equation*}
$$

(provided $m p+r \neq 0$ ) where the generalized binomial is defined by

$$
\binom{a}{m}:=\frac{a(a-1) \ldots(a-m+1)}{m!} .
$$

Let $\mathcal{B}_{p}(z)$ denote the generating function of the sequence $\left\{A_{m}(p, 1)\right\}_{m=0}^{\infty}$, the Fuss numbers of order $p$ :

$$
\begin{equation*}
\mathcal{B}_{p}(z):=\sum_{m=0}^{\infty} A_{m}(p, 1) z^{m}, \tag{3}
\end{equation*}
$$

convergent in some neighborhood of 0 . For example

$$
\begin{equation*}
\mathcal{B}_{2}(z)=\frac{2}{1+\sqrt{1-4 z}} . \tag{4}
\end{equation*}
$$

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Lambert showed that

$$
\begin{equation*}
\mathcal{B}_{p}(z)^{r}=\sum_{m=0}^{\infty} A_{m}(p, r) z^{m}, \tag{5}
\end{equation*}
$$

see [8]. These generating functions also satisfy

$$
\begin{equation*}
\mathcal{B}_{p}(z)=1+z \mathcal{B}_{p}(z)^{p} \tag{6}
\end{equation*}
$$

which reflects the identity $A_{m}(p, p)=A_{m+1}(p, 1)$, and

$$
\begin{equation*}
\mathcal{B}_{p}(z)=\mathcal{B}_{p-r}\left(z \mathcal{B}_{p}(z)^{r}\right) \tag{7}
\end{equation*}
$$

It was shown in [13] that if $p \geq 1$ and $0 \leq r \leq p$ then the sequence $\left\{A_{m}(p, r)\right\}_{m=0}^{\infty}$ is positive definite, i.e. is the moment sequence of a probability measure $\mu(p, r)$ on $\mathbb{R}$. Moreover, $\mu(p, r)$ has compact support (and therefore is unique) contained in the positive half-line $[0, \infty)$ (for example $\mu(p, 0)=\delta_{0}$ ). The proof involved methods from the free probability theory (see [23, [15, [5]). In particular, for $p \geq 1$

$$
\begin{equation*}
\mu(p, 1)=\mu(2,1)^{\boxtimes(p-1)}, \tag{8}
\end{equation*}
$$

where $\boxtimes$ denotes the multiplicative free power, and $\mu(2,1)$ is known as the MarchenkoPastur (called also the free Poisson) distribution. It is given by

$$
\begin{equation*}
\mu(2,1)=\frac{1}{2 \pi} \sqrt{\frac{4-x}{x}} d x \quad \text { on }[0,4], \tag{9}
\end{equation*}
$$

and plays an important role in the theory of random matrices, see [24, 9, 10, 2, 1, 4, 4. It was proved in [1] that the measure $\mu(2,1)^{\boxtimes n}=\mu(n+1,1)$ is the limit of the distribution of squared singular values of the power $G^{n}$ of a random matrix $G$, when the size of the matrix $G$ goes to infinity.

In this paper we are going to prove positive definiteness of $\left\{A_{m}(p, r)\right\}_{m=0}^{\infty}$ using more classical methods. Namely, we show that if $p>1,0<r \leq p$ and if $p$ is a rational number then $\mu(p, r)$ is absolutely continuous and can be represented as Mellin convolution of modified beta measures. Next we provide a formula for the density $W_{p, r}(x)$ of $\mu(p, r)$ in terms of the Meijer function and consequently, of the generalized hypergeometric functions (cf. [25, 18], where $p$ was assumed to be an integer). This allows us to draw graphs of these densities and, in some particular cases, to express $W_{p, r}(x)$ as an elementary function. It is worth to point out that for $r=1$ an alternative description of the densities $W_{p, 1}(x)$ has been recently given by Haagerup and Möller, see Corollary 3 in [11.

Finally let us also mention that the measures $\mu(p, r)$ satisfy a peculiar relation:

$$
\begin{equation*}
\mu(p, r) \triangleright \mu(p+s, s)=\mu(p+s, r+s) \tag{10}
\end{equation*}
$$

for $p \geq 1,0<r \leq p$ and $s>0$, see [13], involving monotonic convolution " $\square$ ", an associative, noncommutative operation on probability measures on $\mathbb{R}$, introduced by Muraki (14].

## 1. Preliminaries

For probability measures $\mu_{1}, \mu_{2}$ on the positive half-line $[0, \infty)$ the Mellin convolution is defined by

$$
\begin{equation*}
\left(\mu_{1} \circ \mu_{2}\right)(A):=\int_{0}^{\infty} \int_{0}^{\infty} \mathbf{1}_{A}(x y) d \mu_{1}(x) d \mu_{2}(y) \tag{11}
\end{equation*}
$$

for every Borel set $A \subseteq[0, \infty)$. This is the distribution of product $X_{1} \cdot X_{2}$ of two independent nonnegative random variables with $X_{i} \sim \mu_{i}$. In particular, if $c>0$ then $\mu \circ \delta_{c}$ is the dilation of $\mu$ :

$$
\left(\mu \circ \delta_{c}\right)(A)=\mathbf{D}_{c} \mu(A):=\mu\left(\frac{1}{c} A\right) .
$$

Note that if $\mu$ has density $f(x)$ then $\mathbf{D}_{c}(\mu)$ has density $f(x / c) / c$.
If both the measures $\mu_{1}, \mu_{2}$ have all moments

$$
s_{m}\left(\mu_{i}\right):=\int_{0}^{\infty} x^{m} d \mu_{i}(x)
$$

finite then so has $\mu_{1} \circ \mu_{2}$ and

$$
s_{m}\left(\mu_{1} \circ \mu_{2}\right)=s_{m}\left(\mu_{1}\right) \cdot s_{m}\left(\mu_{2}\right)
$$

for all $m$.
If $\mu_{1}, \mu_{2}$ are absolutely continuous, with densities $f_{1}, f_{2}$ respectively, then so is $\mu_{1} \circ \mu_{2}$ and its density is given by the Mellin convolution:

$$
\left(f_{1} \circ f_{2}\right)(x):=\int_{0}^{\infty} f_{1}(x / y) f_{2}(y) \frac{d y}{y} .
$$

We will need the following modified beta distributions:
Lemma 1.1. Let $u, v, l>0$. Then

$$
\left\{\frac{\Gamma(u+n / l) \Gamma(u+v)}{\Gamma(u+v+n / l) \Gamma(u)}\right\}_{n=0}^{\infty}
$$

is the moment sequence of the probability measure

$$
\begin{equation*}
\mathbf{b}(u+v, u, l):=\frac{l}{\mathrm{~B}(u, v)} x^{l u-1}\left(1-x^{l}\right)^{v-1} d x \tag{12}
\end{equation*}
$$

on $[0,1]$, where B is the Euler beta function.
Proof. Using the substitution $t=x^{l}$ we obtain:

$$
\begin{aligned}
\frac{\Gamma(u+n / l) \Gamma(u+v)}{\Gamma(u+v+n / l) \Gamma(u)} & =\frac{\mathrm{B}(u+n / l, v)}{\mathrm{B}(u, v)}=\frac{1}{\mathrm{~B}(u, v)} \int_{0}^{1} t^{u+n / l-1}(1-t)^{v-1} d t \\
& =\frac{l}{\mathrm{~B}(u, v)} \int_{0}^{1} x^{l u+n-1}\left(1-x^{l}\right)^{v-1} d x
\end{aligned}
$$

Note that if $X$ is a positive random variable whose distribution has density $f(x)$ and if $l>0$ then the distribution of $X^{1 / l}$ has density $l x^{l-1} f\left(x^{l}\right)$. In particular, if $X$ has beta distribution $\mathbf{b}(u+v, u, 1)$ then $X^{1 / l}$ has distribution $\mathbf{b}(u+v, u, l)$.

For $u, l>0$ we also define

$$
\begin{equation*}
\mathbf{b}(u, u, l):=\delta_{1} . \tag{13}
\end{equation*}
$$

## 2. Applying Mellin convolution

From now on we assume that $p>1$ is a rational number, say $p=k / l$, with $1 \leq l<k$, and that $0<r \leq p$. We will show, that then the sequence $A_{m}(p, r)$ is a moment sequence of a probability measure $\mu(p, r)$, which can be represented as Mellin convolution of modified beta distributions. In particular, $\mu(p, r)$ is absolutely continuous and we will denote its density by $W_{p, r}$. The case when $p$ is an integer was studied in [18, 25].

First we need to express the numbers $A_{m}(p, r)$ in a special form.
Lemma 2.1. If $p=k / l$, where $k, l$ are integers, $1 \leq l<k$ and $0<r \leq p$ then

$$
\begin{equation*}
A_{m}(p, r)=\frac{r}{\sqrt{2 \pi k l(k-l)}}\left(\frac{p}{p-1}\right)^{r} \frac{\prod_{j=1}^{k} \Gamma\left(\beta_{j}+m / l\right)}{\prod_{j=1}^{k} \Gamma\left(\alpha_{j}+m / l\right)} c(p)^{m}, \tag{14}
\end{equation*}
$$

where $c(p)=p^{p}(p-1)^{1-p}$,

$$
\begin{align*}
& \alpha_{j}= \begin{cases}\frac{j}{l} & \text { if } 1 \leq j \leq l, \\
\frac{r+j-l}{k-l} & \text { if } l+1 \leq j \leq k,\end{cases}  \tag{15}\\
& \beta_{j}=\frac{r+j-1}{k}, \quad 1 \leq j \leq k . \tag{16}
\end{align*}
$$

Proof. First we write:

$$
\begin{equation*}
\binom{m p+r}{m} \frac{r}{m p+r}=\frac{r \Gamma(m p+r)}{\Gamma(m+1) \Gamma(m p-m+r+1)} \tag{17}
\end{equation*}
$$

Now we apply the Gauss's multiplication formula:

$$
\Gamma(n z)=(2 \pi)^{(1-n) / 2} n^{n z-1 / 2} \Gamma(z) \Gamma\left(z+\frac{1}{n}\right) \Gamma\left(z+\frac{2}{n}\right) \ldots \Gamma\left(z+\frac{n-1}{n}\right)
$$

to get:

$$
\begin{aligned}
\Gamma(m p+r) & =\Gamma\left(k\left(\frac{m}{l}+\frac{r}{k}\right)\right)=(2 \pi)^{(1-k) / 2} k^{m k / l+r-1 / 2} \prod_{j=1}^{k} \Gamma\left(\frac{m}{l}+\frac{r+j-1}{k}\right), \\
\Gamma(m+1) & =\Gamma\left(l \frac{m+1}{l}\right)=(2 \pi)^{(1-l) / 2} l^{m+1 / 2} \prod_{j=1}^{l} \Gamma\left(\frac{m}{l}+\frac{j}{l}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma(m p-m+r+1) & =\Gamma\left((k-l)\left(\frac{m}{l}+\frac{r+1}{k-l}\right)\right) \\
& =(2 \pi)^{(1-k+l) / 2}(k-l)^{m(k-l) / l+r+1 / 2} \prod_{j=l+1}^{k} \Gamma\left(\frac{m}{l}+\frac{r+j-l}{k-l}\right) .
\end{aligned}
$$

It remains to apply them to (17).
Now we need to modify enumeration of $\alpha$ 's.

Lemma 2.2. For $1 \leq i \leq l+1$ denote

$$
j_{i}:=\left\lfloor\frac{(i-1) k}{l}\right\rfloor+1,
$$

where $\lfloor\cdot\rfloor$ is the floor function, so that

$$
1=j_{1}<j_{2}<\ldots<j_{l}<k<k+1=j_{l+1} .
$$

For $1 \leq j \leq k$ define

$$
\widetilde{\alpha}_{j}= \begin{cases}i & \text { if } j=j_{i}, 1 \leq i \leq l,  \tag{18}\\ \frac{r+j-i}{k-l} & \text { if } j_{i}<j<j_{i+1}\end{cases}
$$

Then the sequence $\left\{\widetilde{\alpha}_{j}\right\}_{j=1}^{k}$ is a rearrangement of $\left\{\alpha_{j}\right\}_{j=1}^{k}$. Moreover, if $0<r \leq p=k / l$ then we have $\beta_{j} \leq \widetilde{\alpha}_{j}$ for all $j \leq k$.
Proof. It is easy to verify the first statement.
Assume that $j=j_{i}$ for some $i \leq l$. Then we have to show that

$$
\frac{r+j_{i}-1}{k} \leq \frac{i}{l}
$$

which is equivalent to

$$
l r+l\left\lfloor\frac{k(i-1)}{l}\right\rfloor \leq k i
$$

and the latter is a consequence of the fact that $\lfloor x\rfloor \leq x$ and the assumption $r \leq p=k / l$.
Now assume that $j_{i}<j<j_{i+1}$. We ought to show that

$$
\frac{r+j-1}{k} \leq \frac{r+j-i}{k-l}
$$

which is equivalent to

$$
l r+l j+k-l-k i \geq 0
$$

Using the inequality $\lfloor x\rfloor+1>x$ we obtain

$$
l j+k-l-k i \geq l\left(j_{i}+1\right)+k-l-k i=l j_{i}+k-k i>k(i-1)+k-k i=0
$$

which completes the proof, as $r>0$.
Now we are ready to prove the main theorem of this section.
Theorem 2.3. Suppose that $p=k / l$, where $k, l$ are integers such that $1 \leq l<k$, and that $r$ is a real number, $0<r \leq p$. Then there exists a unique probability measure $\mu(p, r)$ such that (1) is its moment sequence. Moreover $\mu(p, r)$ can be represented as the following Mellin convolution:

$$
\mu(p, r)=\mathbf{b}\left(\widetilde{\alpha}_{1}, \beta_{1}, l\right) \circ \ldots \circ \mathbf{b}\left(\widetilde{\alpha}_{k}, \beta_{k}, l\right) \circ \delta_{c(p)},
$$

where

$$
c(p):=\frac{p^{p}}{(p-1)^{p-1}} .
$$

Consequently, $\mu(p, r)$ is absolutely continuous and its support is $[0, c(p)]$.

Note that the representation of densities in the form of Mellin convolution of modified beta distributions was used in different context in [7], see its Appendix A.

Example. For the Marchenko-Pastur measure we get the following decomposition:

$$
\begin{equation*}
\mu(2,1)=\mathbf{b}(1,1 / 2,1) \circ \mathbf{b}(2,1,1) \circ \delta_{4}, \tag{19}
\end{equation*}
$$

where $\mathbf{b}(1,1 / 2,1)$ has density $1 /\left(\pi \sqrt{x-x^{2}}\right)$ on $[0,1]$, the arcsine distribution with the moment sequence $\binom{2 m}{m} 4^{-m}$, and $\mathbf{b}(2,1,1)$ is the Lebesgue measure on $[0,1]$ with the moment sequence $1 /(m+1)$.

Proof. In view of Lemma 2.1 and Lemma 2.2 we can write

$$
A_{m}(p, r)=D \prod_{j=1}^{k} \frac{\Gamma\left(\beta_{j}+m / l\right) \Gamma\left(\widetilde{\alpha}_{j}\right)}{\Gamma\left(\widetilde{\alpha}_{j}+m / l\right) \Gamma\left(\beta_{j}\right)} \cdot c(p)^{m}
$$

for some constant $D$. Taking $m=0$ we see that $D=1$.
Note that a part of the theorem illustrates a result of Kargin [12], who proved that if $\mu$ is a compactly supported probability measure on $[0, \infty)$, with expectation 1 and variance $V$, and if $L_{n}$ denotes the supremum of the support of the multiplicative free convolution power $\mu^{\boxtimes n}$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{L_{n}}{n}=e V \tag{20}
\end{equation*}
$$

where $e=2.71 \ldots$ is the Euler's number. The Marchenko-Pastur measure $\mu(2,1)$ has expectation and variance equal to 1 and $\mu(2,1)^{\boxtimes n}=\mu(n+1,1)$, so in this case $L_{n}=$ $(n+1)^{n+1} / n^{n}$ (this was also proved in [24] and [10]) and (20) holds.

The density function for $\mu(p, r)$ will be denoted by $W_{p, r}(x)$. Since $A_{m}(p, p)=$ $A_{m+1}(p, 1)$, we have

$$
\begin{equation*}
W_{p, p}(x)=x \cdot W_{p, 1}(x), \tag{21}
\end{equation*}
$$

for example

$$
\begin{equation*}
W_{2,2}(x)=\frac{1}{2 \pi} \sqrt{x(4-x)} \quad \text { on }[0,4], \tag{22}
\end{equation*}
$$

which is the famous semicircle Wigner distribution with radius 2 , centered at $x=2$.
Now we can reprove the main result of [13].
Theorem 2.4. Suppose that $p, r$ are real numbers such that $p \geq 1$ and $0 \leq r \leq p$. Then there exists a unique probability measure $\mu(p, r)$, with support contained in $[0, c(p)]$, such that $\left\{A_{m}(p, r)\right\}_{m=0}^{\infty}$ is its moment sequence.

Proof. It follows from the fact that the class of positive definite sequence is closed under pointwise limits.

## 3. Applying Meijer $G$-function

The aim of this section is to describe the density function $W_{p, r}(x)$ of $\mu(p, r)$ in terms of the Meijer $G$-function (see [16] for example) and consequently, as a linear combination of generalized hypergeometric functions. We will see that in some particular cases $W_{p, r}$ can be represented as an elementary function.

Theorem 3.1. Let $p=k / l>1$, where $k, l$ are integers such that $1 \leq l<k$, and let $r$ be a positive real number, $r \leq p$. Then the density $W_{p, r}$ of the probability measure $\mu(p, r)$ can be expressed as

$$
W_{p, r}(x)=\frac{r p^{r}}{x(p-1)^{r+1 / 2} \sqrt{2 k \pi}} G_{k, k}^{k, 0}\left(\left.\begin{array}{c}
\alpha_{1}, \ldots, \alpha_{k}  \tag{23}\\
\beta_{1}, \ldots, \beta_{k}
\end{array} \right\rvert\, \frac{x^{l}}{c(p)^{l}}\right),
$$

$x \in(0, c(p))$, where $c(p)=p^{p}(p-1)^{1-p}$ and the parameters $\alpha_{j}, \beta_{j}$ are given by (15) and (16).

Proof. Define

$$
\phi_{p, r}(\sigma)=\frac{r \Gamma(\sigma p-p+r)}{\Gamma(\sigma) \Gamma(\sigma p-\sigma-p+r+2)} .
$$

If $m$ is a natural number then

$$
\phi_{p, r}(m+1)=\binom{m p+r}{m} \frac{r}{m p+r}
$$

so $\phi_{p, r}$ is the Mellin transform of the density function $W_{p, r}$ of $\mu(p, r)$ :

$$
\phi_{p, r}(\sigma)=\int_{0}^{\infty} x^{\sigma-1} W_{p, r}(x) d x
$$

In order to reconstruct $W_{p, r}$ we apply the inverse Mellin transform:

$$
W_{p, r}(x)=\frac{1}{2 \pi \mathrm{i}} \int_{C-\mathrm{i} \infty}^{C+\mathrm{i} \infty} x^{-\sigma} \phi_{p, r}(\sigma) d \sigma,
$$

see [3, 16, 19] for details. Putting $m=\sigma-1$ in (14) we get

$$
\phi_{p, r}(\sigma)=\frac{r(p-1)^{p-1-r}}{p^{p-r} \sqrt{2 \pi k l(k-l)}} \frac{\prod_{j=1}^{k} \Gamma\left(\beta_{j}-1 / l+\sigma / l\right)}{\prod_{j=1}^{k} \Gamma\left(\alpha_{j}-1 / l+\sigma / l\right)} c(p)^{\sigma} .
$$

Writing the right hand side as $\Phi(\sigma / l) c(p)^{\sigma}$, using the substitution $\sigma=l u$ and the definition of the Meijer $G$-function (see [16] for example) we obtain

$$
\begin{gathered}
W_{p, r}(x)=\frac{1}{2 \pi \mathrm{i}} \int_{C-\mathrm{i} \infty}^{C+\mathrm{i} \infty} \Phi(\sigma / l) c(p)^{\sigma} x^{-\sigma} d \sigma=\frac{l}{2 \pi \mathrm{i}} \int_{C-\mathrm{i} \infty}^{C+\mathrm{i} \infty} \Phi(u)\left(x^{l} / c(p)^{l}\right)^{-u} d u \\
=\frac{r(p-1)^{p-r-3 / 2}}{p^{p-r} \sqrt{2 \pi k}} G_{k, k}^{k, 0}\left(\left.\begin{array}{c}
\alpha_{1}^{-}, \ldots, \alpha_{k}^{-} \\
\beta_{1}^{-}, \ldots, \beta_{k}^{-}
\end{array} \right\rvert\, z\right),
\end{gathered}
$$

where $z=x^{l} / c(p)^{l}, \alpha_{j}^{-}=\alpha_{j}-1 / l, \beta_{j}^{-}=\beta_{j}-1 / l$. Finally we use formula (16.19.2) in [16] and obtain

$$
W_{p, r}(x)=\frac{r(p-1)^{p-r-3 / 2}}{z^{1 / l} p^{p-r} \sqrt{2 \pi k}} G_{k, k}^{k, 0}\left(\left.\begin{array}{c}
\alpha_{1}, \ldots, \alpha_{k}  \tag{24}\\
\beta_{1}, \ldots, \beta_{k}
\end{array} \right\rvert\, z\right),
$$

which is equivalent to (23).
Now applying Slater's theorem (see (16.17.2) in [16]) we can represent $W_{p, r}$ as a linear combination of hypergeometric functions.

Theorem 3.2. For for $p=k / l$, with $1 \leq l<k, r>0$ and $x \in(0, c(p))$ we have

$$
W_{p, r}(x)=\gamma(k, l, r) \sum_{h=1}^{k} c(h, k, l, r)_{k} F_{k-1}\left(\left.\begin{array}{l}
\mathbf{a}(h, k, l, r)  \tag{25}\\
\mathbf{b}(h, k, l, r)
\end{array} \right\rvert\, z\right) z^{(r+h-1) / k-1 / l}
$$

where $z=x^{l} / c(p)^{l}$,

$$
\begin{align*}
\gamma(k, l, r) & =\frac{r(p-1)^{p-r-3 / 2}}{p^{p-r} \sqrt{2 \pi k}},  \tag{26}\\
c(h, k, l, r) & =\frac{\prod_{j=1}^{h-1} \Gamma\left(\frac{j-h}{k}\right) \prod_{j=h+1}^{k} \Gamma\left(\frac{j-h}{k}\right)}{\prod_{j=1}^{l} \Gamma\left(\frac{j}{l}-\frac{r+h-1}{k}\right) \prod_{j=l+1}^{k} \Gamma\left(\frac{r+j-l}{k-l}-\frac{r+h-1}{k}\right)}, \tag{27}
\end{align*}
$$

and the parameter vectors of the hypergeometric functions are

$$
\begin{align*}
\mathbf{a}(h, k, l, r) & =\left(\left\{\frac{r+h-1}{k}-\frac{j-l}{l}\right\}_{j=1}^{l},\left\{\frac{r+h-1}{k}-\frac{r+j-k}{k-l}\right\}_{j=l+1}^{k}\right)  \tag{28}\\
\mathbf{b}(h, k, l, r) & =\left(\left\{\frac{k+h-j}{k}\right\}_{j=1}^{h-1},\left\{\frac{k+h-j}{k}\right\}_{j=h+1}^{k}\right)
\end{align*}
$$

The most tractable case is $p=2$.
Corollary 3.3. For $p=2,0<r \leq 2$, the density function is

$$
\begin{equation*}
W_{2, r}(x)=\frac{\sin (r \cdot \arccos \sqrt{x / 4})}{\pi x^{1-r / 2}} \tag{30}
\end{equation*}
$$

$x \in(0,4)$. In particular for $r=1 / 2$ we have

$$
\begin{equation*}
W_{2,1 / 2}(x)=\frac{\sqrt{2-\sqrt{x}}}{2 \pi x^{3 / 4}} \tag{31}
\end{equation*}
$$

and for $r=3 / 2$

$$
\begin{equation*}
W_{2,3 / 2}(x)=\frac{(\sqrt{x}+1) \sqrt{2-\sqrt{x}}}{2 \pi x^{1 / 4}} \tag{32}
\end{equation*}
$$

Note that if $r>2$ then $W_{2, r}(x)<0$ for some values of $x \in(0,4)$.
Proof. We take $k=2, l=1$ so that $c(2)=4, z=x / 4$ and $\gamma(2,1, r)=r 2^{r} /(8 \sqrt{\pi})$. Using the Euler's reflection formula and the identity $\Gamma(1+r / 2)=\Gamma(r / 2) r / 2$ we get

$$
\begin{aligned}
c(1,2,1, r) & =\frac{\Gamma(1 / 2)}{\Gamma(1-r / 2) \Gamma(1+r / 2)}=\frac{2 \sin (\pi r / 2)}{r \sqrt{\pi}} \\
c(2,2,1, r) & =\frac{\Gamma(-1 / 2)}{\Gamma((1-r) / 2) \Gamma((1+r) / 2)}=\frac{-2 \cos (\pi r / 2)}{\sqrt{\pi}}
\end{aligned}
$$

We also need formulas for two hypergeometric functions, namely

$$
\begin{aligned}
{ }_{2} F_{1}\left(\frac{r}{2}, \frac{-r}{2} ; \left.\frac{1}{2} \right\rvert\, z\right) & =\cos (r \arcsin \sqrt{z}), \\
{ }_{2} F_{1}\left(\frac{1+r}{2}, \frac{1-r}{2} ; \left.\frac{3}{2} \right\rvert\, z\right) & =\frac{\sin (r \arcsin \sqrt{z})}{r \sqrt{z}}
\end{aligned}
$$

see 15.4.12 and 15.4.16 in [16]. Now we can write

$$
\begin{aligned}
W_{2, r}(x)= & \frac{\sin (\pi r / 2) \cos (r \arcsin \sqrt{x / 4})-\cos (\pi r / 2) \sin (r \arcsin \sqrt{x / 4})}{\pi x^{1-r / 2}} \\
& =\frac{\sin (\pi r / 2-r \arcsin \sqrt{x / 4})}{\pi x^{1-r / 2}}=\frac{\sin (r \arccos \sqrt{x / 4})}{\pi x^{1-r / 2}},
\end{aligned}
$$

which concludes the proof.
Remark. Note that

$$
\begin{equation*}
\frac{W_{2,1}(\sqrt{x})}{2 \sqrt{x}}=\frac{1}{4} W_{2,1 / 2}\left(\frac{x}{4}\right) . \tag{33}
\end{equation*}
$$

It means that if $X, Y$ are random variables such that $X \sim \mu(2,1)$ and $Y \sim \mu(2,1 / 2)$ then $X^{2} \sim 4 Y$. This can be also derived from the relation $A_{m}(2,1 / 2) 4^{m}=A_{2 m}(2,1)$.

## 4. Some particular cases

In this part we will see that for $k=3$ some densities still can be represented as elementary functions. We will need two families of formulas (cf. 15.4.17 in [16]).

Lemma 4.1. For $c \neq 0,-1,-2, \ldots$ we have

$$
\begin{align*}
{ }_{2} F_{1}\left(\frac{c}{2}, \frac{c-1}{2} ; c \mid z\right) & =2^{c-1}(1+\sqrt{1-z})^{1-c},  \tag{34}\\
{ }_{2} F_{1}\left(\frac{c+1}{2}, \frac{c-2}{2} ; c \mid z\right) & =\frac{2^{c-1}}{c}(1+\sqrt{1-z})^{1-c}(c-1+\sqrt{1-z}) . \tag{35}
\end{align*}
$$

Proof. We know that ${ }_{2} F_{1}(a, b ; c \mid z)$ is the unique function $f$ which is analytic at $z=0$, with $f(0)=1$, and satisfies the hypergeometric equation:

$$
z(1-z) f^{\prime \prime}(z)+[c-(a+b+1) z] f^{\prime}(z)-a b f(z)=0
$$

(see [3]). Now one can check that this equation is satisfied by the right hand sides of (34) and (35) for given parameters $a, b, c$.

Now consider $p=3 / 2$.
Theorem 4.2. Assume that $p=3 / 2$. Then for $r=1 / 2$ we have

$$
\begin{equation*}
W_{3 / 2,1 / 2}(x)=\frac{\left(1+\sqrt{1-4 x^{2} / 27}\right)^{2 / 3}-\left(1-\sqrt{1-4 x^{2} / 27}\right)^{2 / 3}}{2^{5 / 3} 3^{-1 / 2} \pi x^{2 / 3}}, \tag{36}
\end{equation*}
$$

for $r=1$ :

$$
\begin{gather*}
W_{3 / 2,1}(x)=3^{1 / 2} \frac{\left(1+\sqrt{1-4 x^{2} / 27}\right)^{1 / 3}-\left(1-\sqrt{1-4 x^{2} / 27}\right)^{1 / 3}}{2^{4 / 3} \pi x^{1 / 3}}  \tag{37}\\
+3^{1 / 2} x^{1 / 3} \frac{\left(1+\sqrt{1-4 x^{2} / 27}\right)^{2 / 3}-\left(1-\sqrt{1-4 x^{2} / 27}\right)^{2 / 3}}{2^{5 / 3} \pi}
\end{gather*}
$$

and, finally, $W_{3 / 2,3 / 2}(x)=x \cdot W_{3 / 2,1}(x)$, with $x \in(0,3 \sqrt{3} / 2)$.

Proof. For arbitrary $r$ we have

$$
\begin{gathered}
W_{3 / 2, r}(x)=\frac{2^{1-2 r / 3} \sin (2 \pi r / 3)}{3^{3 / 2-r} \pi}{ }_{3} F_{2}\left(\frac{3+2 r}{6}, \frac{r}{3}, \frac{-2 r}{3} ; \frac{2}{3}, \left.\frac{1}{3} \right\rvert\, z\right) z^{r / 3-1 / 2} \\
-\frac{2^{(4-2 r) / 3} r \sin ((1-2 r) \pi / 3)}{3^{3 / 2-r} \pi}{ }_{3} F_{2}\left(\frac{5+2 r}{6}, \frac{1+r}{3}, \frac{1-2 r}{3} ; \frac{4}{3}, \left.\frac{2}{3} \right\rvert\, z\right) z^{(r+1) / 3-1 / 2} \\
- \\
-\frac{r(1+2 r) \sin ((1+2 r) \pi / 3)}{2^{(1+2 r) / 3} 3^{3 / 2-r} \pi}{ }_{3} F_{2}\left(\frac{7+2 r}{6}, \frac{2+r}{3}, \frac{2-2 r}{3} ; \frac{5}{3}, \left.\frac{4}{3} \right\rvert\, z\right) z^{(r+2) / 3-1 / 2},
\end{gathered}
$$

where $z=4 x^{2} / 27$. If $r=1 / 2$ or $r=1$ then one term vanishes and in the two others the hypergeometric functions reduce to ${ }_{2} F_{1}$.

For $r=1 / 2$ we apply (34) to obtain:

$$
\begin{gathered}
W_{3 / 2,1 / 2}(x)=\frac{z^{-1 / 3}}{2^{1 / 3} 3^{1 / 2} \pi}{ }_{2} F_{1}\left(\frac{1}{6}, \frac{-1}{3} ; \left.\frac{1}{3} \right\rvert\, z\right)-\frac{z^{1 / 3}}{2^{5 / 3} 3^{1 / 2} \pi}{ }_{2} F_{1}\left(\frac{5}{6}, \frac{1}{3} ; \left.\frac{5}{3} \right\rvert\, z\right) \\
=\frac{z^{-1 / 3}}{2^{1 / 3} 3^{1 / 2} \pi} 2^{-2 / 3}(1+\sqrt{1-z})^{2 / 3}-\frac{z^{1 / 3}}{2^{5 / 3} 3^{1 / 2} \pi} 2^{2 / 3}(1+\sqrt{1-z})^{-2 / 3} \\
=\frac{z^{-1 / 3}}{2 \cdot 3^{1 / 2} \pi}(1+\sqrt{1-z})^{2 / 3}-\frac{z^{1 / 3}}{2 \cdot 3^{1 / 2} \pi}\left(\frac{1-\sqrt{1-z}}{z}\right)^{2 / 3} \\
=\frac{z^{-1 / 3}}{2 \cdot 3^{1 / 2} \pi}(1+\sqrt{1-z})^{2 / 3}-\frac{z^{-1 / 3}}{2 \cdot 3^{1 / 2} \pi}(1-\sqrt{1-z})^{2 / 3}
\end{gathered}
$$

and this yields (36).
For $r=1$ we use (35):

$$
\begin{aligned}
& W_{3 / 2,1}(x)=\frac{z^{-1 / 6}}{2^{2 / 3} \pi}{ }_{2} F_{1}\left(\frac{5}{6}, \frac{-2}{3} ; \left.\frac{2}{3} \right\rvert\, z\right)+\frac{z^{1 / 6}}{2^{1 / 3} \pi}{ }_{2} F_{1}\left(\frac{7}{6}, \frac{-1}{3} ; \left.\frac{4}{3} \right\rvert\, z\right) \\
= & \frac{z^{-1 / 6}}{4 \pi}(1+\sqrt{1-z})^{1 / 3}(3 \sqrt{1-z}-1)+\frac{z^{1 / 6}}{4 \pi}(1+\sqrt{1-z})^{-1 / 3}(3 \sqrt{1-z}+1) \\
= & \frac{z^{-1 / 6}}{4 \pi}(1+\sqrt{1-z})^{1 / 3}(3 \sqrt{1-z}-1)+\frac{z^{-1 / 6}}{4 \pi}(1-\sqrt{1-z})^{1 / 3}(3 \sqrt{1-z}+1) .
\end{aligned}
$$

Now we have

$$
\begin{gathered}
(1+\sqrt{1-z})^{1 / 3}(3 \sqrt{1-z}-1)=-(1+\sqrt{1-z})^{1 / 3}(3-3 \sqrt{1-z}-2) \\
=-3 z^{1 / 3}(1-\sqrt{1-z})^{2 / 3}+2(1+\sqrt{1-z})^{1 / 3}
\end{gathered}
$$

and similarly

$$
(1-\sqrt{1-z})^{1 / 3}(3 \sqrt{1-z}+1)=3 z^{1 / 3}(1+\sqrt{1-z})^{2 / 3}-2(1-\sqrt{1-z})^{1 / 3} .
$$

Therefore

$$
\begin{gathered}
W_{3 / 2,1}(x)=\frac{z^{-1 / 6}}{2 \pi}\left((1+\sqrt{1-z})^{1 / 3}-(1-\sqrt{1-z})^{1 / 3}\right) \\
+\frac{3 z^{1 / 6}}{4 \pi}\left((1+\sqrt{1-z})^{2 / 3}-(1-\sqrt{1-z})^{2 / 3}\right)
\end{gathered}
$$

which entails (37). The last statement is a consequence of (21).

The dilation $\mathbf{D}_{2} \mu(3 / 2,1 / 2)$, with the density $W_{3 / 2,1 / 2}(x / 2) / 2$, is known as the Bures distribution, see (4.4) in [21]. Moreover, the integer sequence $4^{m} A_{m}(3 / 2,1 / 2)$, moments of $\mathbf{D}_{4} \mu(3 / 2,1 / 2)$, appears as $A 078531$ in [20] and counts the number of symmetric noncrossing connected graphs on $2 n+1$ equidistant nodes on a circle. The axis of symmetry is a diameter of a circle passing through a given node, see [6].

The measure $\mu(3 / 2,1)$ is equal to $\mu(2,1)^{\boxtimes 1 / 2}$, the multiplicative free square root of the Marchenko-Pastur distribution.

For the sake of completeness we also include the cases $p=3, r=1$ and $p=3, r=2$, which have already appeared in [17, 18].

Theorem 4.3. Assume that $p=3$. Then for $r=1$ we have

$$
\begin{equation*}
W_{3,1}(x)=\frac{3(1+\sqrt{1-4 x / 27})^{2 / 3}-2^{2 / 3} x^{1 / 3}}{2^{4 / 3} 3^{1 / 2} \pi x^{2 / 3}(1+\sqrt{1-4 x / 27})^{1 / 3}}, \tag{38}
\end{equation*}
$$

for $r=2$ :

$$
\begin{equation*}
W_{3,2}(x)=\frac{9(1+\sqrt{1-4 x / 27})^{4 / 3}-2^{4 / 3} x^{2 / 3}}{2^{5 / 3} 3^{3 / 2} \pi x^{1 / 3}(1+\sqrt{1-4 x / 27})^{2 / 3}} \tag{39}
\end{equation*}
$$

and, finally, $W_{3,3}(x)=x \cdot W_{3,1}(x)$, with $x \in(0,27 / 4)$.
Proof. For arbitrary $r$ we have

$$
\begin{array}{r}
W_{3, r}(x)=\frac{2^{(6-2 r) / 3} \sin (\pi r / 3)}{3^{3-r} \pi}{ }_{3} F_{2}\left(\frac{r}{3}, \frac{3-r}{6}, \frac{-r}{6} ; \frac{2}{3}, \left.\frac{1}{3} \right\rvert\, z\right) z^{(r-3) / 3} \\
-\frac{2^{(4-2 r) / 3} r \sin ((1-2 r) \pi / 3)}{3^{3-r} \pi}{ }_{3} F_{2}\left(\frac{1+r}{3}, \frac{5-r}{6}, \frac{2-r}{6} ; \frac{4}{3}, \left.\frac{2}{3} \right\rvert\, z\right) z^{(r-2) / 3} \\
+\frac{r(r-1) \sin ((1-r) \pi / 3)}{2^{(1+2 r) / 3} 3^{3-r} \pi}{ }_{3} F_{2}\left(\frac{2+r}{3}, \frac{7-r}{6}, \frac{4-r}{6} ; \frac{5}{3}, \left.\frac{4}{3} \right\rvert\, z\right) z^{(r-1) / 3},
\end{array}
$$

where $z=4 x / 27$. For $r=1$ and $r=2$ we have similar reduction as in the previous proof. Here we will be using only (34).

Taking $r=1$ we get

$$
\begin{gathered}
W_{3,1}(x)=\frac{2^{1 / 3} z^{-2 / 3}}{3^{3 / 2} \pi}{ }_{2} F_{1}\left(\frac{1}{3}, \frac{-1}{6} ; \left.\frac{2}{3} \right\rvert\, z\right)-\frac{z^{-1 / 3}}{2^{1 / 3} 3^{3 / 2} \pi}{ }_{2} F_{1}\left(\frac{2}{3}, \frac{1}{6} ; \left.\frac{4}{3} \right\rvert\, z\right) \\
=\frac{z^{-2 / 3}}{3^{3 / 2} \pi}(1+\sqrt{1-z})^{1 / 3}-\frac{z^{-1 / 3}}{3^{3 / 2} \pi}(1+\sqrt{1-z})^{-1 / 3} \\
=\frac{(1+\sqrt{1-z})^{2 / 3}-z^{1 / 3}}{3^{3 / 2} \pi z^{2 / 3}(1+\sqrt{1-z})^{1 / 3}},
\end{gathered}
$$

which implies (38).
Now we take $r=2$ :

$$
W_{3,2}(x)=\frac{z^{-1 / 3}}{2^{1 / 3} 3^{1 / 2} \pi}{ }_{2} F_{1}\left(\frac{1}{6}, \frac{-1}{3} ; \left.\frac{1}{3} \right\rvert\, z\right)-\frac{z^{1 / 3}}{2^{5 / 3} 3^{1 / 2} \pi}{ }_{2} F_{1}\left(\frac{5}{6}, \frac{1}{3} ; \left.\frac{5}{3} \right\rvert\, z\right)
$$

$$
\begin{gathered}
=\frac{z^{-1 / 3}}{2 \cdot 3^{1 / 2} \pi}(1+\sqrt{1-z})^{2 / 3}-\frac{z^{1 / 3}}{2 \cdot 3^{1 / 2} \pi}(1+\sqrt{1-z})^{-2 / 3} \\
=\frac{(1+\sqrt{1-z})^{4 / 3}-z^{2 / 3}}{2 \cdot 3^{1 / 2} \pi z^{1 / 3}(1+\sqrt{1-z})^{2 / 3}},
\end{gathered}
$$

and this gives us (39). Finally we apply (21).
Note that the measure $\mu(3,1)$ is equal to $\mu(2,1)^{\boxtimes 2}$, the multiplicative free square of the Marchenko-Pastur distribution.

## 5. Graphical representation of selected cases

The explicit form of $W_{p, r}(x)$ given in Theorem 3.2 permits a graphical visualization for any rational $p>0$ and arbitrary $r>0$. We shall represent some selected cases in Figures $1-9$. These graphs which are partly negative are drawn as dashed curves. In Fig. 1 the graphs of the functions $W_{3 / 2, r}(x)$ for values of $r$ ranging from 0.9 to 2.3 are given. For $r \leq 3 / 2$ these functions are positive, otherwise they develop a negative part. In Fig. 2 we represent $W_{5 / 2, r}(x)$ for $r$ ranging from 2 to 3.4. In Fig. 3 we display the densities $W_{p, p}(x)$ for $p=6 / 5,5 / 4,4 / 3$ and $3 / 2$. All these densities are unimodal and vanish at the extremities of their supports. They can be therefore considered as generalizations of the Wigner's semicircle distribution $W_{2,2}(x)$, see equation (22). In Fig. 4 we depict the functions $W_{4 / 3, r}(x)$, for values $r$ ranging from 0.8 to 2.4. Here for $r \geq 1.4$ negative contributions clearly appear. In Fig. 5 and 6 we present six densities expressible through elementary functions, namely $W_{3 / 2, r}(x)$ for $r=1 / 2,1,3 / 2$, see Theorem 4.2 and $W_{3, r}(x)$ for $r=1,2,3$, see Theorem 4.3. In Fig. 7 the set of densities $W_{p, 1}(x)$ for five fractional values of $p$ is presented. The appearance of maximum near $x=1$ corresponds to the fact that $\mu(p, 1)$ weakly converges to $\delta_{1}$ as $p \rightarrow 1^{+}$. In Fig. 8 the fine details of densities $W_{p, 1}(x)$ for $p=5 / 2,7 / 3,9 / 4,11 / 5$, on a narrower range $2 \leq x \leq 4.5$ are presented. In Fig. 9 we display the densities $W_{p, 1}(x)$ for $p=2,5 / 2,3,7 / 2,4$, near the upper edge of their respective supports, for $3 \leq x \leq 9.5$.

The Fig. 10 summarizes our results in the $p>0, r>0$ quadrant of the ( $p, r$ ) plane, describing the Raney numbers (c.f. Fig. 5.1 in [13] and Fig. 7 in [18]). The shaded region $\Sigma$ indicates the probability measures $\mu(p, r)$ (i.e. where $W_{p, r}(x)$ is a nonegative function). The vertical line $p=2$ and the stars indicate the pairs $(p, r)$ for which $W_{p, r}(x)$ is an elementary function, see Corollary 3.3. Theorem 4.2 and Theorem 4.3 . The points $(3 / 2,1)$ and $(3,1)$ correspond to the multiplicative free powers $M^{\boxtimes 1 / 2}$ and MP ${ }^{\boxtimes 2}$ of the Marchenko-Pastur distribution MP. Symbol B at $(3 / 2,1 / 2)$ indicates the Bures distribution and SC at $(2,2)$ denotes the semicircle law centered at $x=2$, with radius 2.

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Figure 1. Raney distributions $W_{3 / 2, r}(x)$ with values of the parameter $r$ labeling each curve. For $r>p$ solutions drawn with dashed lines are not positive.


Figure 2. As in Fig. 1 for Raney distributions $W_{5 / 2, r}(x)$.


Figure 3. Diagonal Raney distributions $W_{p, p}(x)$ with values of the parameter $p$ labeling each curve.


Figure 4. The functions $W_{4 / 3, r}(x)$ for $r$ ranging from 0.8 to 2.4.


Figure 5. Raney distributions $W_{3 / 2, r}(x)$ with values of the parameter $r$ labeling each curve. The case $W_{3 / 2,1}(x)$ represents the multiplicative free square root of the Marchenko Pastur distribution.


Figure 6. Raney distributions $W_{3, r}(x)$ with values of the parameter $r$ labeling each curve. The case $W_{3,1}(x)$ represents the multiplicative free square of the Marchenko Pastur distribution.


Figure 7. Raney distributions $W_{p, 1}(x)$ with values of the parameter $p$ labeling each curve. The case $W_{3 / 2,1}(x)$ represents the multiplicative free square root of the Marchenko-Pastur distribution, $M P^{\boxtimes 1 / 2}$.


Figure 8. Tails of the Raney distributions $W_{p, 1}(x)$ with values of the parameter $p$ labeling each curve.


Figure 9. As in Fig. 8 for larger values of the parameter $p$.


Figure 10. Parameter plane ( $p, r$ ) describing the Raney numbers. The shaded set $\Sigma$ corresponds to nonnegative probability measures $\mu(p, r)$. The vertical line $p=2$ and the stars represent values of parameters for which $W_{p, r}(x)$ is an elementary function. Here MP denotes the Marchenko-Pastur distribution, $\mathrm{MP}^{\boxtimes s}$ its $s$-th free mutiplicative power, B-the Bures distribution while SC denotes the semicircle law. For $p>1$ the points $(p, p)$ on the upper edge of $\Sigma$ represent the generalizations of the Wigner semicircle law, see Fig. 3.


