CHROMATIC POLYNOMIALS OF SIMPLICIAL COMPLEXES

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Abstract. In this note we consider s-chromatic polynomials of finite simplicial complexes. The s-chromatic polynomials of simplicial complexes are higher dimensional analogues of chromatic polynomials for graphs.

1. Introduction

Let K be a finite simplicial complex with vertex set $V(K) \neq \emptyset$ and let $r \geq 1$ and $s \geq 1$ be two natural numbers. A map $\mathrm{col}\colon V(K) \to \{1, 2, \ldots, r\}$ is an (r,s)-*coloring* of K if there are no monochrome s-simplices in K [5]. We write $\chi^s(K,r)$ for the number of (r,s)-colorings of K.

Definition 1.1. *The* s-chromatic polynomial *of K is the function $\chi^s(K,r)$ of r. The* s-chromatic number *of K, $\mathrm{chr}^s(K)$, is the minimal $r \geq 1$ with $\chi^s(K,r) > 0$.*

The theorem below shows that $\chi^s(K,r)$ is indeed polynomial in r for fixed K and s. (By notational convention, $[r]_i = r(r-1)\cdots(r-i+1)$ is the ith falling factorial in r.)

Theorem 1.2. *The s-chromatic polynomial of K is*

$$\chi^s(K,r) = \sum_{i=\mathrm{chr}^s(K)}^{|V(K)|} S(K,i,s)[r]_i$$

where $S(K,i,s)$ is the number of partitions of $V(K)$ into i blocks containing no s-simplex of K.

For $s = 1$, an $(r,1)$-coloring of K is a usual graph coloring, $\chi^1(K,r)$ is the usual chromatic polynomial, and $\mathrm{chr}^1(K)$ the usual chromatic number of the 1-skeleton of K. In general, $\chi^s(K,r)$ depends only on the s-skeleton of K. Although the higher s-chromatic polynomials for simplicial complexes are analogues of 1-chromatic polynomials for graphs we shall shortly see that there are structural differences between the cases $s =$ and $s > 1$.

Figure 1 shows a triangulation MB of the Möbius band. To the left is a $(5,1)$- and to the right a $(2,2)$-coloring of MB. The chromatic polynomials and chromatic numbers [1] of MB are

$$\chi^s(\mathrm{MB},r) = \begin{cases} r^5 - 10r^4 + 35r^3 - 50r^2 + 24r & s = 1 \\ r^5 - 5r^3 + 5r^2 - r & s = 2 \\ r^5 & s \geq 3 \end{cases} \qquad \mathrm{chr}^s(\mathrm{MB}) = \begin{cases} 5 & s = 1 \\ 2 & s = 2 \\ 1 & s \geq 3 \end{cases}$$

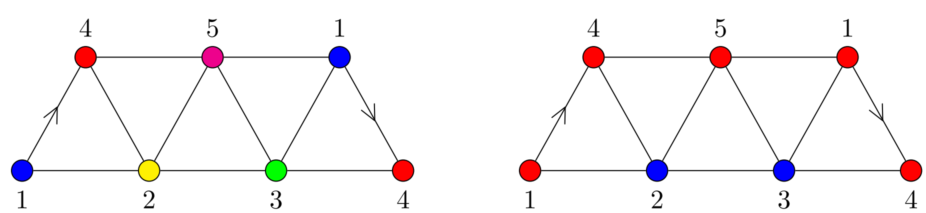

Figure 1. A $(5,1)$-coloring and a $(2,2)$-coloring of a 5-vertex triangulated Möbius band MB

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[1] The computations behind the examples of this note were carried out in the computer algebra system Magma [3].

1.1. **Notation.** We shall use the following notation throughout the paper:

K**:** a finite simplicial complex
K^s**:** the s-skeleton of K
$F^s(K)$**:** the set of s-simplices K
$\#V$ **or** $|V|$**:** the number of elements in the finite set V
$V(K)$**:** the vertex set $\bigcup K$ of K and $m(K) = |V(K)$ is the number of vertices in K
$D[V]$**:** the complete simplicial complex of *all* subsets of the finite set V
$[m]$**:** the finite set $\{1, \dots, m\}$ of cardinality m
$[r]_i$**:** the $i$the falling factorial polynomial $[r]_i = i!\binom{r}{i}$ in r
$P(a, b)$**:** the open interval (a, b) in the poset P

2. Three ways to the s-chromatic polynomial of a simplicial complex

In this section we present three different to approaches to the s-chromatic polynomial $\chi^s(K, r)$:

- Theorem 2.5 via 1-chromatic polynomials of graphs;
- Theorem 2.25 via the Möbius function for the s-chromatic lattice;
- Theorem 1.2 via the simplicial s-Stirling numbers of the second kind.

2.1. **Block-connected s-independent vertex partitions.** Let $s \geq 1$ be a natural number.

Definition 2.1. *Let $B \subset V(K)$ be a set of vertices of K. Then*

- *B is* s-independent *if B contains no s-simplex of K;*
- *B is* connected *if $K \cap D[B]$ is a connected simplicial complex;*
- *the* connected components *of B are the maximal connected subsets of B.*

Definition 2.2. *Let P be a partition of $V(K)$.*

- *The* graph $G_0(P)$ *of P is the simple graph whose vertices are the blocks of P and with two blocks connected by and edge if their union is connected;*
- *The* block-connected refinement P_0 *of P is the refinement whose blocks are the connected components of the blocks of P;*
- *P is* block-connected *if the blocks of P are connected (ie if $P = P_0$).*

Lemma 2.3. *Let P be a partition of $V(K)$. If two different blocks of the block-connected refinement P_0 are connected by an edge in the graph $G_0(P_0)$ of P then they lie in different blocks of P.*

Proof. The connected components of the blocks of P are maximal with respect to connectedness. □

Definition 2.4. $\mathrm{BCP}^s(K)$ *is the set of all block-connected s-independent partitions of $V(K)$.*

Recall that $\chi^1(G_0(P), r)$ is the 1-chromatic polynomial of the simple graph $G_0(P)$ of the partition P.

Theorem 2.5. *The s-chromatic polynomial for K is the sum*

$$\chi^s(K, r) = \sum_{P \in \mathrm{BCP}^s(K)} \chi^1(G_0(P), r)$$

of the 1-chromatic polynomials and the s-chromatic number of K is the minimum

$$\mathrm{chr}^s(K) = \min_{P \in \mathrm{BCP}^s(K)} \mathrm{chr}^1(G_0(P))$$

of the 1-chromatic numbers for the graphs of all the block-connected s-independent partitions of $V(K)$.

Proof. Let $\mathrm{col}\colon V(K) \to [r]$ be an (r, s)-coloring of K. The monochrome partition $P(\mathrm{col})$ of $V(K)$ is the s-independent partition whose blocks are the nonempty monochrome sets of vertices $\{\mathrm{col} = i\}$ for $i \in [r]$. The block-connected refinement $P(\mathrm{col})_0$ of the monochrome partition is a block-connected s-independent partition of K. The original coloring col of K is also a coloring of the graph $G_0(P(\mathrm{col})_0)$ of $P(\mathrm{col})_0$ for, by Lemma 2.3, distinct vertices of 1-simplices of this graph have distinct colors. We have shown that any (r, s)-coloring col of K induces an $(r, 1)$-coloring col_0 of the graph $G_0(P(\mathrm{col})_0)$ of the block-connected refinement of the monochrome partition.

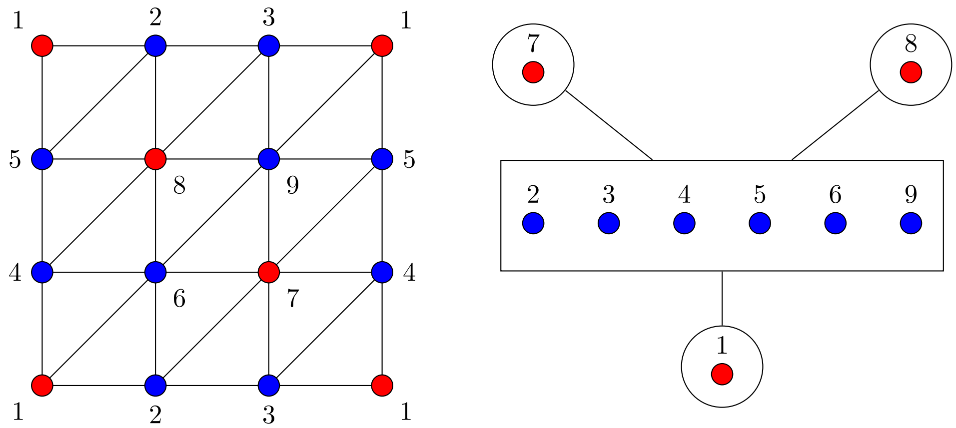
Let $P \in \mathrm{BCP}^s(K)$ be a block-connected s-independent partition of $V(K)$ and $\mathrm{col}_0\colon P \to \{1, \dots, r\}$ an $(r, 1)$-coloring of its graph $G_0(P)$. Then col_0 determines a map $\mathrm{col}\colon V(K) \to [r]$ that is constant on the blocks of P. An s-simplex of K can not be monochrome under col as it intersects at least two different blocks of P connected by an edge of $G_0(P)$. Thus col is an (r, s)-coloring of K.

These two constructions are inverses of each other. □

Remark 2.6 (The minimal block-connected s-independent partition)**.** Let $C_0 = \{\{v\} \mid v \in V(K)\}$ be the block-connected s-independent partition of $V(K)$ whose blocks are singletons. The graph $G_0(C_0) = K^1$ is the 1-skeleton of K. Thus the 1-chromatic polynomial of the 1-skeleton of K is always one of the polynomials in the sum of Theorem 2.5. If K is 1-dimensional, $\mathrm{BCP}^1(K)$ consists only of the partition C_0 and Theorem 2.5 simply says that the 1-chromatic polynomial of a simplicial complex is the 1-chromatic polynomial of its 1-skeleton.

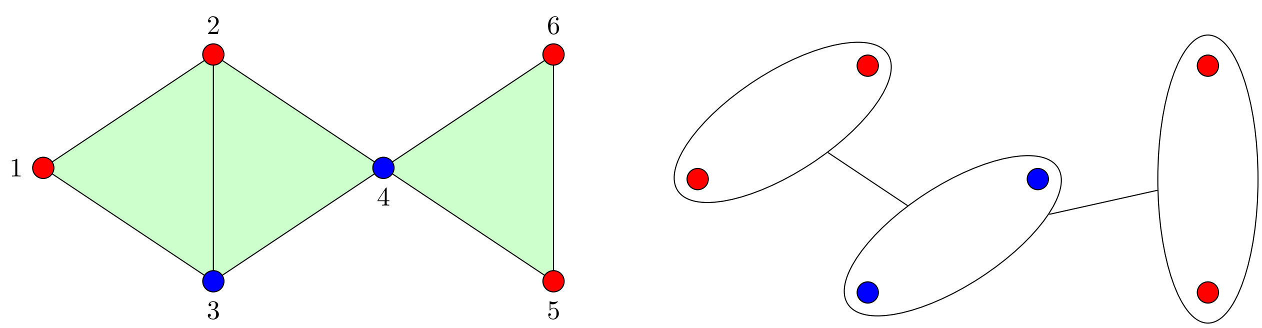
Example 2.7 (The block-connected 2-independent partitions for $D[3]$)**.** The 2-simplex $D[3]$ has 4 block-connected 2-independent partitions C_0, $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, and $\{\{3\},\{1,3\}\}$. The graph of C_0 is the complete graph K_3, the 1-skeleton of $D[3]$. The graphs of the other three partitions are all the complete graph K_2. Thus the 2-chromatic polynomial of $D[3]$ is $\chi^2(D[3], r) = \chi^1(K_3, r) + 3\chi^1(K_2, r) = [r]_3 + 3[r]_2 = [r]_2(r+1) = r^3 - r$ and the 2-chromatic number is $\mathrm{chr}^2(D[3]) = 2$.

Example 2.8 (A $(2,2)$-coloring and the graph of the block-connected refinement of its monochrome partition)**.** The picture below illustrates a $(2,2)$-coloring of a 9-vertex triangulation of the torus



and its corresponding graph. There are 6937 block-connected partitions of the vertex set, and 3 of them has the graph shown above. The 2-chromatic polynomial is $21[r]_2 + 742[r]_3 + 3747[r]_4 + 4908[r]_5 + 2295[r]_6 + 444[r]_7 + 36[r]_8 + [r]_9 = [r]_2(r^7 + r^6 - 17r^5 + 10r^4 + 82r^3 - 116r^2 - 23r + 67)$ and the 2-chromatic number is 2.

Example 2.9 (The $(r,2)$-colorings of a simplicial complex K)**.** Let K be the pure 2-dimensional complex with facets $F^2(K) = \{\{1,2,3\},\{2,3,4\},\{4,5,6\}\}$.



The picture shows a $(2,2)$-coloring of K and the corresponding $(2,1)$-coloring of the associated graph, $G_0(P_0)$, the block connected refinement of the monochrome partition $P = \{\{1,2,5,6\},\{3,4\}\}$. Table 1 shows the graphs $G_0(P)$ for all block connected partitions $P \in \mathrm{BCP}^2(K)$. For each graph, the table records its 1-chromatic polynomial and its 1-chromatic number. The 2-chromatic polynomial of K is $\chi^2(K, 2) = 15[r]_2 + 73[r]_3 + 62[r]_4 + 15[r]_5 + [r]_6 = [r]_2(r-1)(r+1)(r^2+r-1)$ and the 2-chromatic number is $\mathrm{chr}^2(K) = 2$.

Example 2.10 (The $(r,2)$-colorings of the Möbius band)**.** The set $\mathrm{BCP}^2(\mathrm{MB})$ of block-connected 2-independent partitions of the triangulated Möbius band MB (Figure 1) has 36 elements. There are $5, 5, 15, 10, 1$ partitions in $\mathrm{BCP}^2(\mathrm{MB})$ realizing the partitions $[3,2]$, $[3,1,1]$, $[2,2,1]$, $[2,1,1,1]$, $[1,1,1,1,1]$ of the integer $|V(\mathrm{MB})| = 5$. All associated graphs are complete graphs. This yields the 2-chromatic polynomial $\chi^2(\mathrm{MB}, r) = 5[r]_2 + 20[r]_3 + 10[r]_4 + [r]_5 = [r]_2(r^3 + r^2 - 4r + 1) = r^5 - 5r^3 + 5r^2 - r$ and the 2-chromatic number is $\mathrm{chr}^2(\mathrm{MB}) = 2$.

Remark 2.11 (The $\mathcal{S}$-chromatic polynomial of K)**.** Let $\mathcal{S}$ be a set of connected subcomplexes of K. A set $B \subset V(K)$ of vertices is $\mathcal{S}$-independent if B is not a superset of any member of $\mathcal{S}$. Let $\mathrm{BCP}^{\mathcal{S}}(K)$ be the set of

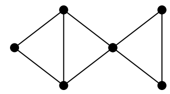
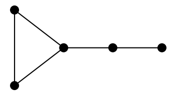
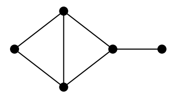
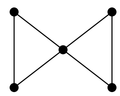
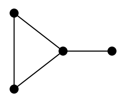
# in $\mathrm{BCP}^2(K)$	$G_0(P)$	$\chi^1(G_0(P), r)$	$\mathrm{chr}^1(G_0(P))$
1		$r(r-1)^2(r-2)^3$	3
1		$r(r-1)^3(r-2)$	3
3		$r(r-1)^2(r-2)^2$	3
4		$r(r-1)^2(r-2)^2$	3
16		$r(r-1)^2(r-2)$	3
3		$r(r-1)^3$	2
12		$r(r-1)^2$	2

TABLE 1. The graphs for the block-connected partitions in $\mathrm{BCP}^2(K)$

$\mathcal{S}$-independent partitions of $V(K)$. An $(r, \mathcal{S})$-coloring is a map $V(K) \to \{1, \dots, r\}$ such that $\#\mathrm{col}(S) > 1$ for all $S \in \mathcal{S}$. The number of $(r, \mathcal{S})$-colorings of K is

$$\chi^{\mathcal{S}}(K, r) = \sum_{P \in \mathrm{BCP}^{\mathcal{S}}(K)} \chi^1(G_0(P), r)$$

as one sees by an obvious generalization of Theorem 2.5. An (r, s)-coloring of K is an $(r, \mathcal{S})$-coloring of K where $\mathcal{S} = F^s(K)$ is the set of s-simplices.

2.2. **The s-chromatic linear program.** Read [9, §10] explains how to construct a linear program with minimal value equal to the s-chromatic number $\mathrm{chr}^s(K)$ of K.

Definition 2.12. *$M^s(K)$ is the set of all maximal s-independent subsets of $V(K)$.*

Let A be the $(m(K) \times |M^s(K)|)$-matrix

$$A(v, M) = \begin{cases} 1 & v \in M \\ 0 & v \notin M \end{cases}$$

recording which vertices $v \in V(K)$ belong to which maximal s-independent sets $M \in M^s(K)$. Now the s-chromatic number

$$\mathrm{chr}^s(K) = \min\{ \sum_{M \in M^s(K)} x(M) \mid x \colon M^s(K) \to \{0, 1\}, \forall v \in V(K) \colon \sum_{M \in M^s(K)} A(v, M) x(M) \geq 1\}$$

is the minimal value of the objective function $\sum_{M \in M^s(K)} x(M)$ in $|M^s(K)|$ variables $x \colon M^s(K) \to \{0, 1\}$, taking values 0 or 1, and $m(K)$ constraints $\sum_{M \in M^s(K)} A(v, M) x(M) \geq 1$, $v \in V(K)$.

2.3. **The s-chromatic lattice.** Our approach here simply follows Rota's classical method for computing chromatic polynomials from Möbius functions of lattices [10, §9]. We need some terminology in order to characterize the monochrome loci for colorings of K. Recall that $F^s(K)$ is the set of s-simplices of K.

Definition 2.13. *Let $S \subset F^s(K)$ be a set of s-simplices of K.*

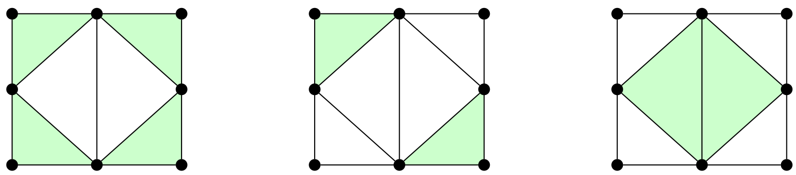
- *The equivalence relation $\sim$ is the smallest equivalence relation in S such that $s_1 \cap s_2 \neq \emptyset \implies s_1 \sim s_2$ for all $s_1, s_2 \in S$;*
- *the connected components of S are the equivalence classes under $\sim$;*
- *$\pi_0(S)$ is the set of connected components of S;*
- *S is* connected *if it has at most one component;*
- *$V(S) = \bigcup S$ is the* vertex set *of S*
- *$\pi(S)$ is the partition of $V(K)$ whose blocks are the vertex sets of the connected components of S together with the singleton blocks $\{v\}$, $v \in V(K) - V(S)$, of vertices not in any simplex in S;*

- *S is* closed *if S contains any s-simplex in K contained in the vertex set of S, ie if*
$$\{\sigma \in F^s(K) \mid \sigma \subset V(S)\} = S$$
- *the* closure *of S is the smallest closed set of s-simplices containing S.*

For instance, the empty set $S = \emptyset$ of 0 s-simplices is connected with 0 connected components. If $K = D[4]$, the set $\{\{1,2\},\{2,4\}\}$ of 1-simplices is connected while $\{\{1,2\},\{3,4\}\}$ has the two components $\{\{1,2\}\}$ and $\{\{3,4\}\}$.

A set of s-simplices is closed if and only if it equals its closure. For instance in $F^2(D[5])$, the set $\{\{1,2,3\},\{3,4,5\}\}$ is not closed because its closure is the set of all 2-simplices in $D[5]$. The empty set of s-simplices, any set of just one s-simplex, and any set of disjoint s-simplices are closed.

In this picture the green set of 2-simplices is



connected and not closed, closed and not connected, closed and connected, respectively.

The partition $\pi(S)$ has $|\pi(S)| = |\pi_0(S)| + m(K) - |V(S)|$ blocks.

Lemma 2.14. *Let S be a set of s-simplices in K and S_0 a connected component of S. Then S_0 is closed if and only if*
$$\{\sigma \in F^s(K) \mid \sigma \subset V(S_0)\} \subset S$$

Proof. Since the condition is certainly necessary we only need to see that it is sufficient. Let σ be an s-simplex in K with all its vertices in $V(S_0)$. Then σ lies in S by assumption. But σ is equivalent to all elements of the equivalence class S_0. Thus $\sigma \in S_0$. □

Lemma 2.15. *Let S and T be sets of s-simplices in K.*
(1) *If S and T are closed, so is $S \cap T$.*
(2) *If S and T have closed connected components, so does $S \cap T$*

Proof. (1) Let σ be an s-simplex of K and suppose that $\sigma \subset V(S \cap T)$. Then $\sigma \subset V(S)$ an $\sigma \subset V(T)$ so that $\sigma \in S$ and $\sigma \in T$ as S and T are closed.
(2) Let R be a connected component of $S \cap T$. Let S_0 be the connected component of S containing R and T_0 be the connected component of T containing R. Then $R \subset S_0 \cap T_0$. Suppose that $\sigma \in F^s(K)$ is an s-simplex with $\sigma \subset V(R)$. Then $\sigma \subset V(S_0 \cap T_0)$ so $\sigma \in S_0 \cap T_0$ by (1) as the connected components S_0 and T_0 are assumed to be closed. In particular, $\sigma \in S \cap T$. According to Lemma 2.14, the connected component R is closed. □

Definition 2.16. *The s-chromatic lattice of K is the set $L^s(K)$ of all subsets of $F^s(K)$ with closed connected components. $L^s(K)$ is a partially ordered by set inclusion.*

The set $L^s(K)$ contains the empty set $\emptyset$ of s-simplices and the set $F^s(K)$ of all s-simplices. These two elements of $L^s(K)$ are distinct when K has dimension at least s.

Corollary 2.17. *$L^s(K)$ is a finite lattice with $\widehat{0} = \emptyset$, $\widehat{1} = F^s(K)$, and meet $S \wedge T = S \cap T$.*

Proof. If $S, T \in L^s(K)$ then $S \cap T$ is also in $L^s(K)$ by Lemma 2.15 and this is clearly the greatest lower bound of S and T. It is now a standard result that $L^s(K)$ is a finite lattice [12, Proposition 3.3.1]. The join $S \vee T$ of $S, T \in L^s(K)$ is the intersection of all supersets $U \in L^s(K)$ of $S \cup T$. □

Example 2.18 (The s-chromatic lattice $L^s(D[m])$)**.** The closed and connected elements of the s-chromatic lattice $L^s(D[m])$ of the complete simplex $D[m]$ on $m > s$ vertices are $\emptyset$ and the $\binom{m}{k}$ sets $F^s(D[k])$ of all s-simplices in the subcomplexes $D[k]$ for $s < k \le m$. The map $S \to \pi(S)$ is an isomorphism between the lattice $L^s(D[m])$ and the lattice, ordered by refinement, of all partitions of the set $[m]$ into blocks of size $> s$ or 1. The least element, $\widehat{0} = (1)\cdots(m)$, is the partition with m blocks and the greatest element, $\widehat{1} = (1\cdots m)$, the partition with 1 block. $L^s(D[m])$ is not a graded lattice [12, p 99] in general when $s \ge 2$. To see this, observe that the 2-chromatic lattices $L^2(D[3])$, $L^2(D[4])$, and $L^2(D[4])$ are graded but the lattice $L^2(D[6])$ is not graded as it contains two maximal chains
$$\widehat{0} = (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (1234)(5)(6) < (12345)(6) < (123456) = \widehat{1}$$
$$\widehat{0} = (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (123)(456) < (123456) = \widehat{1}$$

of unequal length. In contrast, the 1-chromatic lattice of any finite simplicial complex is always graded and even geometric [10, §9, Lemma 1].

Remark 2.19 (The Möbius function for the s-chromatic lattices $L^s(D[m])$)**.** Our discussion of the Möbius function for the lattice $L^s(D[m])$ echoes the exposition of the Möbius function for the geometric lattice $L^1(D[m])$ of *all* partitions from [12, Example 3.10.4].

Let $w\colon [m] \to \mathbf{N}$ be a function that to every element of $[m]$ associates a natural number, thought of as a weight function. We write $w = 1^{i_1}2^{i_2}\cdots r^{i_r}$, or something similar, for the weight function w defined on the set $[m]$ of cardinality $m = \sum_j i_j$ and mapping i_j elements to j for $1 \leq j \leq r$. The map w extends to a map, also called w, defined on the set of all nonempty subsets X of $[m]$ given by $w(X) = \sum_{x\in X} w(x)$. Let $L^s_m(w)$ be the lattice of all partitions of the set $[m]$ into blocks X that are singletons or have weight $w(X) > s$. The non-singleton blocks of the meet $\sigma \wedge \tau$ of two partitions $\sigma, \tau \in L^s_m(w)$ are the subsets of weight $> s$ of the form $S \cap T$ where S is a block in σ and T a block in τ. Write $\mu^s_m(w)$ for the Möbius function of $L^s_m(w)$.

In particular, $L^s_m(1^m)$ is a synonym for $L^s(D[m])$ and we are primarily interested in the Möbius function $\mu^s_m(1^m)$ of the uniform weight $w = 1^m$. However, the computation of this Möbius function will involve the Möbius functions of other weights as well. We shall therefore discuss the Möbius functions $\mu^s_m(w)$ for general weight functions w.

Suppose that $\sigma \in L^s_m(w)$, $\sigma < \widehat{1}$, is a partition of $[m]$ into singleton blocks or blocks of weight $> s$. Let $w(\sigma)$ be the restriction of w to the set of blocks of σ. Thus $w(\sigma)(X) = \sum_{x\in X} w(x)$ for any block X of σ. Then the interval

$$L^s_m(w) \supset [\sigma, \widehat{1}] = L^s_{|\sigma|}(w(\sigma))$$

so that $\mu^s_m(w)(\sigma, \widehat{1}) = \mu^s_{|\sigma|}(w(\sigma))(\widehat{0}, \widehat{1})$. More generally, suppose that $\sigma < \tau$ for some $\tau \in L^s_m(w)$. Assume that the partition τ has blocks τ_j. Let σ_j be the set of those blocks of σ that intersect the block τ_j of τ. Let $w(\sigma_j)$ be the restriction of $w(\sigma)$ to σ_j. Then the interval

$$L^s_m(w) \supset [\sigma, \tau] = \prod_j L^s_{|\sigma_j|}(w(\sigma_j))$$

and therefore the value of the Möbius function on the pair (σ, τ)

$$\mu^s_m(w)(\sigma, \tau) = \prod_j \mu^s_{|\sigma_j|}(w(\sigma_j))(\widehat{0}, \widehat{1})$$

by the product theorem for Möbius functions [12, Proposition 3.8.2]. We conclude that the complete Möbius functions on all the lattices $L^s_m(w)$, are actually determined by the values $\mu^s_m(w)(\widehat{0}, \widehat{1})$ of these Möbius functions on just $(\widehat{0}, \widehat{1})$. See Equation (2.36) for more information about these Euler characteristics.

For the following it is convenient to name the elements of the domain $[m]$ of w so that the element m carries minimal weight. Assume that $a_m = (1 \cdots m-1)(m)$ is an element of $L^s_m(w)$, ie that $w(1) + \cdots + w(m-1) > s$. We shall determine the set of lattice elements x with $x \wedge a_m = \widehat{0}$. There is only one solution to this equation with $x \leq a_m$ and that is $x = \widehat{0}$. As the other solutions satisfy $x \not\leq a_m$, they must have a block that contains m and at least one other element. It follows that the solutions $x \neq \widehat{0}$ are all elements of the form

$$x = (x_1 \cdots x_t m)(\cdot)\cdots(\cdot) \text{ with } \begin{cases} w(x_1) > s - w(m) & t = 1 \\ s \geq w(x_1) + \cdots + w(x_t) > s - w(m) & t > 1 \end{cases}$$

where all blocks but the unique block containing m are singletons. There are $t+1$ elements in the block containing m where t is some number in the range $1 \leq t \leq s$. (All the solutions $x \neq \widehat{0}$ are atoms in the lattice $L^s_m(w)$.) Since we are in a lattice, the Möbius function satisfies the equation [12, Corollary 3.9.3]

$$\mu^s_m(w)(\widehat{0}, \widehat{1}) = -\sum_{\substack{x\wedge a_m = \widehat{0} \\ x \neq \widehat{0}}} \mu^s_m(w)(x, \widehat{1})$$

which translates to

$$\mu^s_m(w)(\widehat{0}, \widehat{1}) = -\sum_{\substack{x\wedge a_m = \widehat{0} \\ x \neq \widehat{0}}} \mu^s_{|x|}(w(x))(\widehat{0}, \widehat{1}) = \tag{2.20}$$

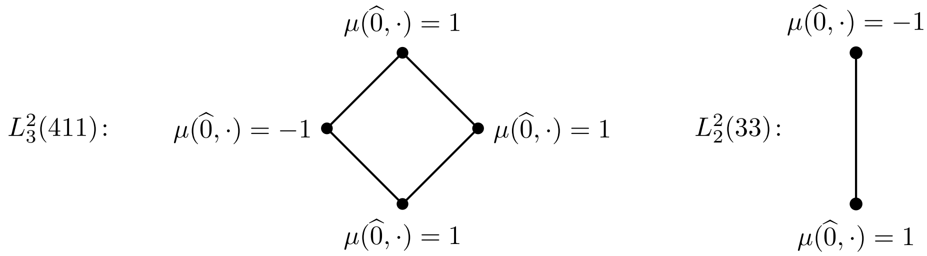
$$-\sum_{\substack{1\leq x_1 \leq m-1 \\ w(x_1) > s-w(m)}} \mu^s_{m-1}(w(x_1m)w(\cdot)\cdots w(\cdot))(\widehat{0}, \widehat{1}) - \sum_{1<t\leq s} \sum_{\substack{1\leq x_1,\ldots,x_t\leq m-1 \\ s\geq w(x_1)+\cdots+w(x_t) > s-w(m)}} \mu^s_{m-t}(w(x_1\cdots x_t m))w(\cdot)\cdots w(\cdot))(\widehat{0}, \widehat{1})$$

This describes a recursive procedure for computing all values of the Möbius function on the weight lattices $L^s_m(w)$.

As an illustration we compute $\mu_6^2(1^6)(\widehat{0},\widehat{1})$. Using (2.20) twice gives

$$\mu_6^2(1^6)(\widehat{0},\widehat{1}) = -10\mu_4^2(3111)(\widehat{0},\widehat{1}) = 10(\mu_3^2(411)(\widehat{0},\widehat{1}) + \mu_2^2(33)(\widehat{0},\widehat{1}))$$

The lattices $L_4^2(411)$ and $L_2^2(33)$ have 4 and 2 elements, respectively, and they look like



so that $\mu_3^2(411)(\widehat{0},\widehat{1}) = 1$ and $\mu_2^2(33)(\widehat{0},\widehat{1}) = -1$. Therefore $\mu_6^2(1^6)(\widehat{0},\widehat{1}) = 0$.

We remind the reader of the well-known fact that $\mu_m^s(w)(\widehat{0},\widehat{1})$ is the reduced Euler characteristic of the open interval $L_m^s(w)(\widehat{0},\widehat{1})$ between $\widehat{0}$ and $\widehat{1}$ in the lattice $L_m^s(w)$.

Proposition 2.21. [10, §6] [12, Proposition 3.8.5] *Let $x < y$ be two elements in a finite poset. The value of the Möbius function on the pair (x, y) is the reduced Euler characteristic of the open interval (x, y).*

Proof. Write μ be the Möbius function of P and E for Euler characteristic. The closed interval from x to y has Euler characteristic 1 since it has a smallest element. Thus

$$\begin{aligned}1 = \mathrm{E}([x,y]) = \sum_{a,b\in[x,y]}\mu(a,b) &= \sum_{a,b\in(x,y)}\mu(a,b) + \sum_{a\in[x,y]}\mu(a,y) + \sum_{b\in[x,y]}\mu(x,b) - \mu(x,y)\\ &= \mathrm{E}((x,y)) + 0 + 0 - \mu(x,y) = \mathrm{E}((x,y)) - \mu(x,y)\end{aligned}$$

or $\mu(x,y) = \widetilde{\mathrm{E}}((x,y))$. □

For $1 \leq s \leq m+1$ let $B(m,s)$ be the graded poset of nonempty subsets of $[m]$ of cardinality less than s.

Lemma 2.22. *The reduced Euler characteristic of $B(m,s)$ is*

$$\widetilde{E}(B(m,s)) = (-1)^s\binom{m-1}{s-1}, \qquad 1 \leq s \leq m+1$$

Proof. It is rather easy to get the recurrence relation

$$\begin{aligned}&E(B(m,2)) = m\\ &E(B(m,s)) = E(B(m,s-1)) + \binom{m}{s-1}\sum_{j=1}^{s-1}(-1)^{s-1-j}\binom{s-1}{j}, \qquad 2 < s < 2+m\end{aligned}$$

Since the sum of binomial coefficients has value $(-1)^s$, we get the recurrence relation

$$\begin{aligned}&\widetilde{E}(B(m,2)) = m-1\\ &\widetilde{E}(B(m,s)) = \widetilde{E}(B(m,s-1)) + (-1)^s\binom{m}{s-1}, \qquad 2 < s < 2+m\end{aligned}$$

for the reduced Euler characteristic. The claim of the lemma follows immediately. □

Example 2.23 (Reduced Euler characteristics of the s-chromatic lattice intervals $L_m^s(w)(\widehat{0},\widehat{1})$)**.** The reduced Euler characteristics $\mu_m^s(1^m)(\widehat{0},\widehat{1}) = \widetilde{E}(L_m^s(1^m)(\widehat{0},\widehat{1}))$, $m \geq s+2$, for $s = 1, 2, \ldots, 8$ are

$2, -6, 24, -120, 720, -5040, 40320, -362880, 3628800, -39916800, 479001600, -6227020800, 87178291200, \ldots$

$3, -6, 0, 90, -630, 2520, 0, -113400, 1247400, -7484400, 0, 681080400, -10216206000, 81729648000, \ldots$

$4, -10, 20, -70, 560, -4200, 25200, -138600, 924000, -8408400, 84084000, -798798000, 7399392000, \ldots$

$5, -15, 35, -70, 0, 2100, -23100, 173250, -1051050, 5255250, -15765750, -105105000, 2858856000, \ldots$

$6, -21, 56, -126, 252, -924, 11088, -126126, 1093092, -7693686, 46414368, -254438184, 1492322832, \ldots$

$7, -28, 84, -210, 462, -924, 0, 42042, -630630, 6390384, -51459408, 351639288, -2118412296, 11406835440\ldots$

$8, -36, 120, -330, 792, -1716, 3432, -12870, 205920, -3150576, 35706528, -322583976, 2460949920\ldots$

$9, -45, 165, -495, 1287, -3003, 6435, -12870, 0, 787644, -14965236, 191222460, -1920538620\ldots$

The first sequence, $\mu^1_m(1^m)(\widehat{0},\widehat{1})$, $m \geq 2$, is the sequence $(-1)^{m-1}(m-1)!$ of reduced Euler characteristics of the lattice of partitions of $[m]$ [12, Example 3.10.4]. The second sequence, $\mu^2_m(1^m)(\widehat{0},\widehat{1})$, $m \geq 3$, seems to coincide with first terms of the sequence A009014 from The On-Line Encyclopedia of Integer Sequences (OES). The remaining 6 sequences apparently do not match any sequences of the OES.

The first s terms of these sequences are signed binomial coefficients. This is because the interval $(\widehat{0},\widehat{1})$ in $L^s(D[m])$ is isomorphic to the opposite of the poset $B(m, m-s)$ when $s+2 \leq m \leq 2s+1$. Thus the reduced Euler characteristic

$$\mu^s_m(1^m)(\widehat{0},\widehat{1}) = \widetilde{E}(B(m, m-s)) = (-1)^{m-s}\binom{m-1}{s}, \qquad s+2 \leq m \leq 2s+1,$$

according to Lemma 2.22.

The first terms of the sequence $\mu^2_m(3^1 1^{m-1})(\widehat{0},\widehat{1})$, $m \geq 3$, of reduced Euler characteristics of the weighted lattice intervals $L^2_m(3^1 1^{m-1})(\widehat{0},\widehat{1})$,

$$1, 0, -6, 30, -90, 0, 2520, -22680, 113400, 0, -7484400, 97297200, -681080400, 0, 81729648000, -1389404016000, \dots$$

seem to coincide up to sign with first terms of the sequence A009775 from OES. The sequence of reduced Euler characteristics $\mu^2_m(3^2 1^{m-2})(\widehat{0},\widehat{1})$, $m \geq 3$, of the lattice interval $L^2_m(3^2 1^{m-2})(\widehat{0},\widehat{1})$,

$$2, -4, 6, 6, -120, 720, -2520, -2520, 136080, -1360800, 7484400, 7484400, \\ -778377600, 10897286400, -81729648000, -81729648000, 13894040160000, \dots$$

apparently does not match any sequence in the OES.

Define the *s-monochrome set* of a map $\mathrm{col}\colon V(K) \to [r] = \{1, \dots, r\}$ to be the set

$$M^s(\mathrm{col}) = \{\sigma \in F^s(K) \mid |\mathrm{col}(\sigma)| = 1\}$$

of all monochrome s-simplices in K. The map col is an (r, s)-coloring of K if and only if $M^s(\mathrm{col}) = \emptyset$.

Lemma 2.24. *The s-monochrome set $M^s(\mathrm{col})$ of any map $\mathrm{col}\colon V(K) \to [r]$ is an element of the s-chromatic lattice $L^s(K)$.*

Proof. Let S be a connected component of $M^s(\mathrm{col})$. Since S is connected, all vertices in S have the *same* color. Let $\sigma \in F^s(K)$ be an s-simplex of K such that $\sigma \subset V(S)$. The σ is monochrome: $\sigma \in M^s(\mathrm{col})$. By Lemma 2.14, S is closed. ☐

Theorem 2.25. *The number of (r, s)-colorings of K is*

$$\chi^s(K, r) = \sum_{T \in L^s(K)} \mu(\widehat{0}, T) r^{|\pi(T)|}$$

where μ the Möbius function for the s-chromatic lattice $L^s(K)$.

Proof. For any $B \in L^s(K)$, let $\chi(K, r, s, B)$ be the number of maps $\mathrm{col}\colon V(K) \to [r]$ with $M^s(\mathrm{col}) = B$. We want to determine $\chi(K, r, s, \emptyset) = \chi^r(s, K)$. For any $A \in L^s(K)$,

$$r^{|\pi(A)|} = \sum_{A \leq B} \chi(K, r, s, B)$$

because there are $r^{|\pi_0(A)|} r^{m(K)-|V(A)|} = r^{|\pi(A)|}$ maps $\mathrm{col}\colon V(K) \to [r]$ with $A \leq M^s(\mathrm{col})$. Equivalently,

$$\sum_{A \leq B} \mu(A, B) r^{|\pi(B)|} = \chi(K, r, s, A)$$

by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A = \widehat{0}$. ☐

The defining rules for the Möbius function of the poset $L^s(K)$ [12, 3.7]

- $\mu(S, S) = 1$ for all $S \in L^s(K)$
- $\sum_{R \leq S \leq T} \mu(R, S) = 0$ when $R \lneqq T$
- $\mu(R, S) = 0$ when $R \not\leq S$

imply that $\mu(\widehat{0}, \widehat{0}) = 1$ and $\mu(\widehat{0}, \{\sigma\}) = -1$ for every s-simplex $\sigma \in F^s(K)$.

Corollary 2.26. *The highest degree terms of the s-chromatic polynomial are*

$$\chi^s(K, r) = r^{m(K)} - f_s(K) r^{m(K)-s} + \cdots$$

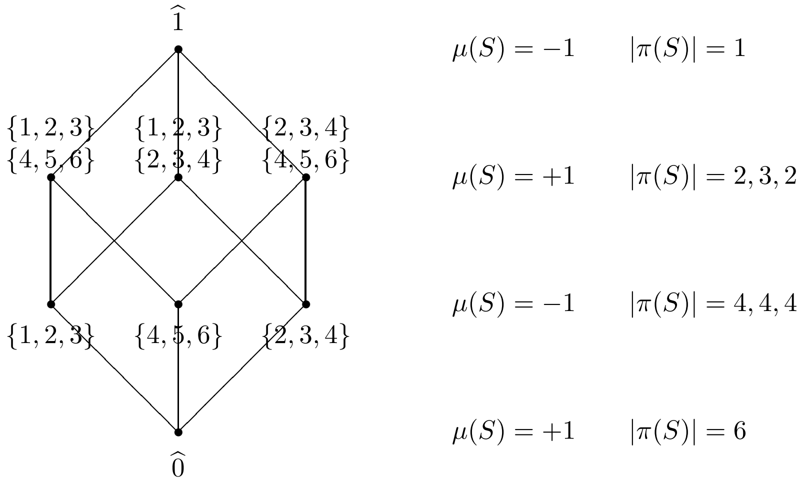
Thus the s-chromatic polynomial determines $f_0(K)$ and $f_s(K)$.

Proof. The s-chromatic polynomial is

$$\chi^s(K,r) = \mu(\widehat{0},\widehat{0})r^{f_0(K)} + \sum_{\sigma\in F^s(K)} \mu(\widehat{0},\{\sigma\})r^{f_0(K)-s} + \cdots$$

where $\mu(\widehat{0},\widehat{0}) = 1$ and $\mu(\widehat{0},\{\sigma\}) = -1$ for all s-simplices σ of K. □

Example 2.27. Consider the 2-dimensional complex K from Example 2.9. The 2-chromatic lattice $L^2(K)$ of K

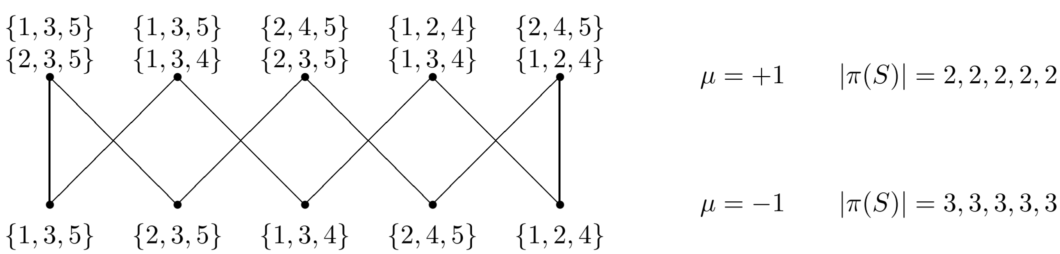


consists of *all* subsets of $F^2(K)$. The 2-chromatic polynomial is

$$\chi^2(K,r) = r^6 - r^4 - r^4 - r^4 + r^2 + r^3 + r^2 - r = r^6 - 3r^4 + r^3 + 2r^2 - r$$

K has $\chi^2(K,2) = 30$ $(2,2)$-colorings and $\chi^2(K,3) = 528$ $(3,2)$-colorings.

Example 2.28. The triangulation MB of the Möbius band with f-vector $f(\mathrm{MB}) = (5,10,5)$ shown in Figure 1 has the following (reduced) 2-chromatic lattice $L^2(\mathrm{MB}) - \{\widehat{0},\widehat{1}\}$



and 2-chromatic polynomial

$$\chi^2(\mathrm{MB},r) = r^5 - 5r^3 + 5r^2 - r$$

The lattice $L^2(\mathrm{MB})$ is graded but it is still not semi-modular [12, Proposition 3.3.2]: The meet and join of $a = \{\{2,3,5\}\}$ and $b = \{\{1,3,4\}\}$ are $a \wedge b = \widehat{0}$ and $a \vee b = \widehat{1}$. Thus a and b cover $a \wedge b$ but $a \vee b$ covers neither a nor b.

Example 2.29. Let MT be Möbius's minimal triangulation of the torus with f-vector $f(\mathrm{MT}) = (7,21,14)$ and P2 the triangulation of the projective plane with f-vector $f(\mathrm{P2}) = (1,6,15,10)$ shown in Figure 2 (decorated with $(3,2)$-colorings). The chromatic polynomials of these two simplicial complexes are

$$\chi^1(\mathrm{MT},r) = [r]_7, \qquad \chi^2(\mathrm{MT},r) = r^7 - 14r^5 + 21r^4 + 7r^3 - 21r^2 + 6r$$
$$\chi^1(\mathrm{P2},r) = [r]_6, \qquad \chi^2(\mathrm{P2},r) = r^6 - 10r^4 + 15r^3 - 6r^2$$

In both cases, the 1-skeleton is the complete graph on the vertex set. The chromatic numbers are $\mathrm{chr}^1(\mathrm{MT}) = 7$, $\mathrm{chr}^1(\mathrm{P2}) = 6$, and $\mathrm{chr}^2(\mathrm{MT}) = 3 = \mathrm{chr}^2(\mathrm{P2})$.

The chromatic polynomials of simple graphs (the 1-chromatic polynomials of simplicial complexes) are known to have these properties:

- The coefficients are sign-alternating [10, §7, Corollary]
- The coefficients are log-concave (Definition 2.43) in absolute value [7]
- There are no negative roots and no roots between 0 and 1 [14]

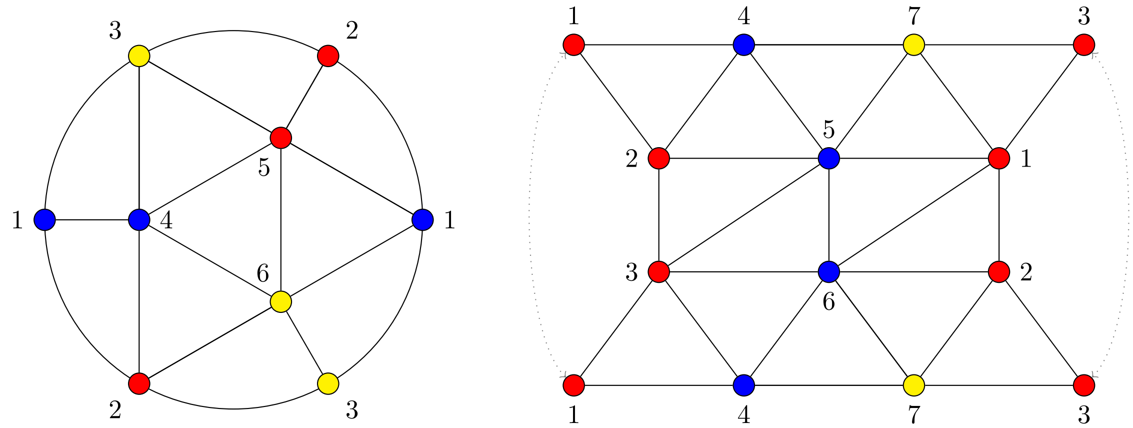


FIGURE 2. $(3,2)$-colorings of P2 and MT

In contrast, the coefficients of the 2-chromatic polynomial

$$\chi^2(\mathrm{MT}, r) = r^7 - 14r^5 + 21r^4 + 7r^3 - 21r^2 + 6r = [r]_3(r+1)(r^3 + 2r^2 - 9r + 3)$$

are not sign-alternating, not log-concave in absolute value, and the polynomial has a negative root and a root between 0 and 1.

2.4. **The s-chromatic polynomial in falling factorial form.** Theorem 1.2 provides an interpretation of the coefficients of the falling factorial $[r]_i$ in the s-chromatic polynomial of the simplicial complex K.

Definition 2.30. *$S(K, r, s)$ is the number of partitions of $V(K)$ into r s-independent blocks.*

We think of $S(K, r, s)$ as an s-Stirling number of the second kind for the simplicial complex K. If $s > \dim(K)$, then there are no s-simplices in K and all partitions of $V(K)$ are s-independent, so that $S(K, r, s)$ is the Stirling number of the second kind $S(m(K), r)$ [12, p 33]. We now explain the general relation between these simplicial Stirling numbers $S(K, r, s)$ and the usual Stirling numbers of the second kind.

Define the *s-monochrome set* of a partition P of $V(K)$ to be the set

$$M^s(P) = \{\sigma \in F^s(K) \mid \sigma \text{ is contained in a block of } P\}$$

of all s-simplices entirely contained in one of the blocks of P. The set $M^s(P)$ is an element of the s-chromatic lattice $L^s(K)$ by Lemma 2.24.

Theorem 2.31. *The number of partitions of $V(K)$ into r s-independent blocks is*

$$S(K, r, s) = \sum_{T \in L^s(K)} \mu(\widehat{0}, T) S(|\pi(T)|, r)$$

where μ the Möbius function for the s-chromatic lattice $L^s(K)$.

Proof. For any $B \in L^s(K)$, let $S(K, r, s, B)$ be the number of partitions P of $V(K)$ into r blocks with monochrome set $M^s(P) = B$. We want to determine $S(K, r, s, \emptyset) = S(K, r, s)$. For any $A \in L^s(K)$,

$$S(|\pi(A)|, r) = \sum_{A \leq B} S(K, r, s, B)$$

because there are $S(|\pi(A)|, r)$ partitions P of $V(K)$ into r blocks with $A \leq M^s(P)$. Equivalently,

$$\sum_{A \leq B} \mu(A, B) S(|\pi(B)|, r) = S(K, r, s, A)$$

by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A = \widehat{0}$. □

Proof of Theorem 1.2. We simply follow the proof of the similar statement for chromatic polynomials for graphs [9, Theorem 15]. When $r \geq i$ we can get an (r, s)-coloring out of one of the $S(K, i, s)$ partitions of $V(K)$ into i s-independent blocks by choosing i out of the r colors and assigning them to the i blocks. There are $\binom{r}{i}$ ways of choosing the i out of r colors and $i!$ ways of coloring i blocks in i colors. The number of (r,s)-colorings of K in exactly i colors is thus

$$S(K,i,s)\binom{r}{i}i! = S(K,i,s)[r]_i$$

so that

$$\chi^s(K,r) = \sum_{i=1}^{m(K)} S(K,i,s)[r]_i$$

is the total number of (r,s)-colorings of K. □

Corollary 2.32. *The reduced Euler characteristic of the open interval $(\widehat{0},\widehat{1})$ in s-chromatic lattice $L^s(K)$ is*

$$\mu(L^s(K))(\widehat{0},\widehat{1}) = \sum_{i=\mathrm{chr}^s(K)}^{m(K)} (-1)^{i-1}(i-1)!S(K,i,s)$$

Proof. Equate the terms of degree 1 of the two expressions

$$\sum_{T\in L^s(K)} \mu(\widehat{0},T)r^{|\pi(T)|} = \sum_{i=\mathrm{chr}^s(K)}^{m(K)} S(K,i,s)[r]_i \tag{2.33}$$

from Theorem 2.25 and Theorem 1.2 for the s-chromatic polynomial of K. □

We observe that

$$\sum_i S(K,i,s)[r]_i = \sum_i\sum_T \mu(\widehat{0},T)S(|\pi(T)|,i)[r]_i = \sum_T \mu(\widehat{0},T)\sum_i S(|\pi(T)|,i)[r]_i = \sum_T \mu(\widehat{0},T)r^{|\pi(T)|}$$

so that Theorem 2.31 implies Theorem 1.2.

The s-chromatic number of K is immediately visible with the s-chromatic polynomial in factorial form because

$$\mathrm{chr}^s(K) = \min\{i \mid S(K,i,s)\neq 0\}$$

is the lowest degree of the nonzero terms. The positive integer sequence

$$\chi^s(K,\mathrm{chr}^s(K)),\dots,\chi^s(K,m(K)) = 1$$

has no internal zeros. (If there is a partition of $V(K)$ into r blocks not containing any s-simplex of K and $r < m(K)$, then split one of the blocks with more than one vertex into two sub-blocks to get a partition of $V(K)$ into $r+1$ blocks containing no s-simplices of K.)

The simplicial Stirling numbers satisfy the recurrence relations

$$S(K,r,s) = \sum_{\substack{\emptyset\subsetneq U\subseteq V(K)-\{v_0\}\\ V(K)-U\ s\text{-independent}}} S(K\cap D[U],r-1,s), \qquad S(K,1,s) = \begin{cases}1 & s>\dim(K)\geq 0\\ 0 & \text{otherwise}\end{cases}$$

To see this, fix a vertex v_0 of K. Let P be partition of $V(K)$ into r s-independent subsets. Let U_0 be the block containing v_0. The other blocks in P form a partition P_0 of $K\cap D[V(K)-U_0]$ into $r-1$ s-independent subsets. The map $P\leftrightarrow (P_0,U_0)$ is a bijection.

The familiar recurrence relation $S(m,r) = S(m-1,r-1)+rS(m-1,r)$ for Stirling numbers of the second kind does not readily apply to simplicial Stirling numbers. The closest analogue may be

$$S(K,r,s) = S(K\cap D[V(K)-\{v_0\}],r-1,s) + \sum_{P\in\mathcal{S}(K\cap D[V(K)-\{v_0\}],r,s)} |\{B\in P \mid B\cup\{v_0\} \text{ is } s\text{-independent in } K\}|$$

where v_0 is a vertex of K and $\mathcal{S}(K\cap D[V(K)-\{v_0\},r,s)$ is the set of partitions P of the vertex set of $K\cap D[V(K)-\{v_0\}$ into r s-independent subsets.

Proposition 2.34. *Let K be a subcomplex of L and assume that $V(K)=V(L)$.*

(1) $S(K,r,s)\geq S(L,r,s)$ for all r.
(2) If $S(K,r,s) = S(L,r,s)$ for some r with $\frac{1}{s}(|V|-1)\leq r\leq |V|-s$, then $K^s = L^s$.

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 6 & 1 \\ 0 & 7 & 6 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 15 & 10 & 1 \\ 0 & 10 & 25 & 10 & 1 \\ 0 & 15 & 25 & 10 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 15 & 45 & 15 & 1 \\ 0 & 10 & 75 & 65 & 15 & 1 \\ 0 & 25 & 90 & 65 & 15 & 1 \\ 0 & 31 & 90 & 65 & 15 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 105 & 105 & 21 & 1 \\ 0 & 0 & 175 & 315 & 140 & 21 & 1 \\ 0 & 35 & 280 & 350 & 140 & 21 & 1 \\ 0 & 56 & 301 & 350 & 140 & 21 & 1 \\ 0 & 63 & 301 & 350 & 140 & 21 & 1 \end{pmatrix}$$

TABLE 2. Chromatic tables for complete simplices $D[m]$ for $m = 2, \dots, 7$

Proof. (1) Let V be the vertex set of K and L. Write $\mathcal{S}(K,r,s)$ and $\mathcal{S}(L,r,s)$ for the set of partitions of V into r blocks containing no s-simplex of K or L, respectively. Then $\mathcal{S}(L,r,s) \subseteq \mathcal{S}(K,r,s)$ for all r and s. Thus $S(L,r,s) \leq S(K,r,s)$.
(2) Suppose that $\sigma \in F^s(L) - F^s(K)$ is an s-simplex of L that is not an s-simplex of K. Any partition of the form

$$\{\sigma\} \cup \tau, \qquad \tau \in \mathcal{S}(D[V-\sigma], r-1, s),$$

in $\mathcal{S}(K,r,s) - \mathcal{S}(L,r,s)$. The set $\mathcal{S}(D[V-\sigma], r-1, s)$ is nonempty when

$$\mathrm{chr}^s(D[V-\sigma]) = \left\lceil \frac{|V|-s-1}{s} \right\rceil \leq r-1 \leq |V|-s-1$$

and thus $S(K,r,s)$ is strictly greater than $S(L,r,s)$ when $\frac{|V|-1}{s} \leq r \leq |V|-s$. □

Remark 2.35 ($S(K,r,s)$ for the complete simplex $K = D[m]$)**.** For any finite set M, let $S(M,r,s)$ stand for $S(D[M],r,s)$ (Definition 2.30), the number of partitions of the set M into r blocks containing at most s elements. Let us even write $S(m,r,s)$ in case $M = [m]$, $m \geq 1$, $r,s \geq 0$. Clearly, $S(m,r,s)$ is nonzero only when $m/s \leq r \leq m$. Also, $S(m,r,s) = S(m,r)$ when r is among the s numbers $m-s+1, \dots, m$. The recurrence relation

$$S(m,r,s) = \sum_{j=m-s}^{m-1} \binom{m-1}{j} S(j, r-1, s)$$

can be used to compute these numbers. Table 2 shows $S(m,r,s)$ for small m; the number $S(m,r,s)$ is in row s and column r in the chromatic table (Definition 2.39) for $D[m]$. All the red numbers are usual Stirling numbers of the second kind.

According to Theorem 1.2, the numbers $S(m,r,s)$ determine the s-chromatic polynomial in falling factorial form of the complete simplex on m vertices

$$\chi^s(D[m], r) = \sum_{i=\lceil m/s \rceil}^{m} S(m,i,s)[r]_i$$

and, according to Corollary 2.32, they also determine the reduced Euler characteristic

$$\mu^s_m(1^m)(\widehat{0}, \widehat{1}) = \sum_{i=\lceil m/s \rceil}^{m} (-1)^{i-1}(i-1)!S(m,i,s)$$

of the s-chromatic lattice $L^s(D[m])$.

More generally, if $w \colon M \to \mathbf{N}$ is a function on M with natural numbers as values, let $S(M,w,r,s)$ be the number of partitions of M into admissible blocks, where we declare a block admissible if it is a singleton or it has weight at most s. (Then $S(m,r,s) = S([m],1^m,r,s)$ occur when $M = [m]$ and $w = 1^m$ places weight 1 on all elements.) Any such partition is a partition of M into blocks of weight at most s, and therefore $S(M,w,r,s) \leq S(\#M,r,s)$. In particular, $S(M,w,r,s)$ is nonzero only when $\#M/s \leq r \leq \#M$. The recurrence relation

$$S(M,w,r,s) = \sum_{\substack{\emptyset \neq J \subset M - \{\max(M)\} \\ M-J \text{ admissible}}} S(J, w|J, r-1, s)$$

provides a means to compute these numbers.

The weighted version of Equation (2.33) for $K = D[m]$,

$$\sum_{\sigma\in L^s_m(w)} \mu^s_m(w)(\widehat{0},\sigma)r^{|\sigma|} = \sum_{i=\lceil m/s\rceil}^{m} S([m],w,i,s)[r]_i$$

implies, by equating coefficients of first degree terms, the expression

$$\mu^s_m(w)(\widehat{0},\widehat{1}) = \sum_{i=\lceil m/s\rceil}^{m} (-1)^{i-1}(i-1)!S([m],w,i,s) \tag{2.36}$$

for the Euler characteristic of the weighted lattice $L^s_m(w)$ from Remark 2.19.

Because any simplicial complex K is a subcomplex of the complete simplex $D[m(K)]$ on its vertex set, we have

$$S(m(K),r) \geq S(K,r,s) \geq S(m(K),r,s), \qquad 1 \leq r \leq m(K) \tag{2.37}$$

Moreover, these inequalities are equalities for the s highest values $m(K)-s+1,\dots,m(K)$ of r. Thus the s terms of highest falling factorial degree in the s-chromatic polynomial of K

$$\chi^s(K,r) = \sum_{i=0}^{m(K)-s} S(K,i,s)[r]_i + \sum_{i=m(K)-s+1}^{m(K)} S(m(K),i)[r]_i$$

are given by the s Stirling numbers $S(m(K),m(K)-s+1),\dots,S(m(K),m(K))$ of the second kind. These coefficients depend only on the size of the vertex set of K. We shall next show that the coefficient number $s+1$ counted from above, $S(K,m(K)-s,s)$, informs about the number $f_s(K)$ of s-simplices in K.

Proposition 2.38. *$S(K,m(K)-s,s) = S(m(K),m(K)-s) - f_s(K)$. If $S(K,m(K)-s,s) = S(m(K),m(K)-s,s)$ then $K^s = D[m(K)]^s$.*

Proof. The only partitions of the $S(m,m-s)$ partitions of $V(K)$ into $m-s$ blocks that are not s-independent are those consisting of one s-simplex of K together with singleton blocks. If $S(K,m(K)-s,s) = S(D[m(K)],m(K)-s,s)$ then $f_s(K) = f_s(D[m(K)])$ so $K^s = D[m(K)]^s$. (This is a special case of Proposition 2.34.(2).) □

Definition 2.39. *The* chromatic table, *$\chi(K)$, of K is the $(\dim(K)\times m(K))$-table with $S(K,r,s)$ in row s and column r.*

This means that row s in the chromatic table lists the coefficients of the s-chromatic polynomial. The chromatic table of a 3-dimensional simplicial complex K, for instance, looks like this

	$r=1$	$r=2$	$\cdots$	$r=m-3$	$r=m-2$	$r=m-1$	$r=m$
$S(K,\cdot,1)$	$S(K,1,1)$	$S(K,2,1)$	$\cdots$	$S(K,m-3,1)$	$S(K,m-2,1)$	$S(m,m-1)-f_1$	$S(m,m)=1$
$S(K,\cdot,2)$	$S(K,1,2)$	$S(K,2,2)$	$\cdots$	$S(K,m-3,2)$	$S(m,m-2)-f_2$	$S(m,m-1)$	$S(m,m)=1$
$S(K,\cdot,3)$	$S(K,1,3)$	$S(K,2,3)$	$\cdots$	$S(m,m-3)-f_3$	$S(m,m-2)$	$S(m,m-1)$	$S(m,m)=1$

where the red entries in row s are Stirling numbers of the second kind $S(m,r)$ for $m-s+1 \leq r \leq m$, and the blue entry in row s is $S(m(K),m(K)-s)-f_s(K)$.

Example 2.40. The chromatic tables of the 2-dimensional simplicial complexes from Examples 2.9, 2.28, and 2.29 are

$$\chi(K) = \begin{pmatrix} 0 & 0 & 2 & 10 & 7 & 1 \\ 0 & 15 & 73 & 62 & 15 & 1 \end{pmatrix} \qquad \chi(\mathrm{MB}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 20 & 10 & 1 \end{pmatrix}$$

$$\chi(\mathrm{MT}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 84 & 231 & 126 & 21 & 1 \end{pmatrix} \qquad \chi(\mathrm{P2}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 45 & 55 & 15 & 1 \end{pmatrix}$$

The red entries in column r are Stirling numbers $S(m,r)$ and they are independent of the row index. The blue entry in row s and column $m-s$, which equals $S(m-s,s)-f_s(K)$, detects if K has maximal s-skeleton by Proposition 3.

Example 2.41. Let $K = \mathrm{AS3}$ be Altshuler's peculiar triangulation of the 3-sphere with f-vector $f = (10,45,70,35)$ [1]. The 1-chromatic polynomial is $\chi^1(\mathrm{AS3},r) = [r]_{10}$ as K^1 is the complete graph on 10 vertices. The chromatic table is

$$\chi(\mathrm{AS3}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1360 & 8475 & 10355 & 4200 & 680 & 45 & 1 \\ 0 & 26 & 4320 & 25915 & 38550 & 22152 & 5845 & 750 & 45 & 1 \end{pmatrix}$$

The blue numbers determine the f-vector

$$f(\mathrm{AS3}) = (10, S(10,9)-\chi(\mathrm{AS3})_{19}, S(10,8)-\chi(\mathrm{AS3})_{28}, S(10,7)-\chi(\mathrm{AS3})_{37})$$

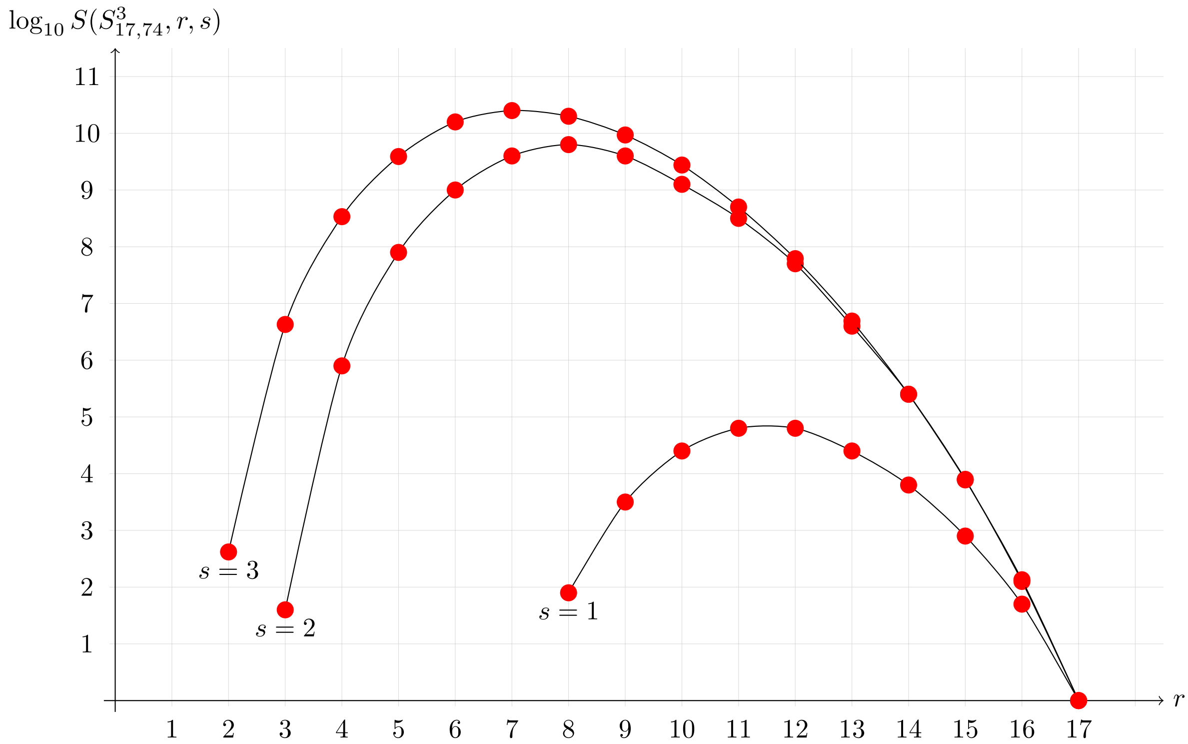


FIGURE 3. The simplicial Stirling numbers for $S^3_{17,74}$

The row numbers of the first nonzero term in each row tell us that $\mathrm{chr}^1(\mathrm{AS3}) = 10$, $\mathrm{chr}^2(\mathrm{AS3}) = 4$, and $\mathrm{chr}^3(\mathrm{AS3}) = 2$.

Example 2.42. The nonconstructible, nonshellable 3-sphere $S^3_{17,74}$, $f = (17, 91, 148, 74)$, found by Lutz [8], has

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$
$s=1$	0	0	0	0	0	0	0	88	3089
$s=2$	0	0	36	702475	82949364	1075420155	3827766587	5493687086	3876597169
$s=3$	0	422	4319865	338438489	3903094622	14292381565	22946854806	19158310796	9202775199

	$r=10$	$r=11$	$r=12$	$r=13$	$r=14$	$r=15$	$r=16$	$r=17$
$s=1$	23017	55285	54973	25941	6210	762	45	1
$s=2$	1507939074	346346664	48855523	4302470	235026	7672	136	1
$s=3$	2708454744	507528561	61784524	4903589	249826	7820	136	1

as its chromatic table. Figure 3 shows a semi-logarithmic plot of the simplicial Stirling numbers $S(S^3_{17,74}, r, s)$. The triangulation Σ^3_{16}, $f = (16, 106, 180, 90)$, of the Poincaré homology 3-sphere constructed by Björner and Lutz [2, Theorem 5] has

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
$s=1$	0	0	0	0	0	0	0	0
$s=2$	0	0	0	4589	2974411	69671411	300475213	442354547
$s=3$	0	3	845561	70005500	701299653	2158716508	2888730959	2000811501

	$r=9$	$r=10$	$r=11$	$r=12$	$r=13$	$r=14$	$r=15$	$r=16$
$s=1$	0	0	0	0	28	44	14	1
$s=2$	292864435	100793551	19546606	2225261	150095	5840	120	1
$s=3$	792553648	190527025	28730056	2750278	165530	6020	120	1

as its chromatic table.

Observe that all the above chromatic tables have strictly log-concave rows.

Definition 2.43. [11] *A finite sequence $a_1, a_2, \dots, a_N$ of $N \geq 3$ nonnegative integers is strictly* log-concave *if $a_{i-1}a_{i+1} < a_i^2$ for $1 < i < N$ (and* log-concave *if $a_{i-1}a_{i+1} \leq a_i^2$).*

It has been conjectured that the sequence of coefficients of the 1-chromatic polynomial of a simple graph in falling factorial form, $r \to S(K,1,r)$, $\mathrm{chr}^1(K) \le r \le m(K)$, is log-concave [4, Conjecture 3.11]. More generally, one may ask

Question 2.44. *Is the finite sequence of simplicial Stirling numbers*

$$r \to S(K,r,s), \qquad \mathrm{chr}^s(K) \le r \le m(K),$$

log-*concave for fixed* K *and* s*?*

This seems to be the right question to ask as it may be true for *all* the chromatic polynomials of a simplicial complex and we have seen that the absolute value of the coefficients of the s-chromatic polynomial are simply not log-concave for $s > 1$.

Note that the Stirling numbers of the second kind, which are upper bounds for the simplicial Stirling numbers $S(K,r,s)$ by the inequalities (2.37), are log-concave in r [11, Corollary 2].

We shall now examine Question 2.44 on two spherical boundary complexes of cyclic n-polytopes.

Definition 2.45. $\partial\mathrm{CP}(m,n)$*,* $m > n$*, is the* $(n-1)$*-dimensional simplicial complex on the ordered set* $[m]$ *with the following facets: An* n*-subset* σ *of* $[m]$ *is a facet if and only if between any two elements of* $[m]-\sigma$ *there is an even number of vertices in* σ*.*

By Gale's Evenness Theorem [6], the simplicial complex $\partial\mathrm{CP}(m,n)$ triangulates the boundary of the cyclic n-polytope on m vertices. Thus $\partial\mathrm{CP}(m,n)$ is a simplicial $(n-1)$-sphere on m vertices and it is $\lfloor n/2\rfloor$-neighborly in the sense that $\partial\mathrm{CP}(m,n)$ has the same s-skeleton as the full simplex on its vertex set when $s < \lfloor n/2\rfloor$.

Example 2.46 (Cyclic polytopes with log-concave simplicial Stirling numbers of the second kind)**.** Let $\partial\mathrm{CP}(m,n)$ be the triangulated boundary of the cyclic polytope on m vertices in $\mathbf{R}^n$. The simplicial complex $\partial\mathrm{CP}(m,n)$ is an m-vertex triangulation of S^{n-1}. The chromatic tables of the simplicial 3-spheres $\partial\mathrm{CP}(m,4)$ on $m = 6,7,8,9,10$ vertices are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 21 & 47 & 15 & 1 \\ 0 & 16 & 81 & 65 & 15 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 28 & 147 & 112 & 21 & 1 \\ 0 & 21 & 238 & 336 & 140 & 21 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 50 & 393 & 582 & 226 & 28 & 1 \\ 0 & 29 & 654 & 1533 & 1030 & 266 & 28 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 94 & 1062 & 2523 & 1719 & 408 & 36 & 1 \\ 0 & 36 & 1729 & 6471 & 6591 & 2619 & 462 & 36 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 180 & 2980 & 10200 & 10777 & 4225 & 680 & 45 & 1 \\ 0 & 46 & 4445 & 25960 & 38550 & 22152 & 5845 & 750 & 45 & 1 \end{pmatrix}$$

All rows are strictly log-concave. As $\partial\mathrm{CP}(m,4)^1 = D[m]^1$, the 1-chromatic number $\mathrm{chr}^1(\partial\mathrm{CP}(m,4)) = m$, and it is not difficult to see that the 2-chromatic number $\mathrm{chr}^2(\partial\mathrm{CP}(m,4))$ is 2 if m is even and 3 if m is odd [5].

Right multiplication with the upper triangular matrix $([j]_i)_{1\le i,j\le m(K)}$ with $[j]_i = \binom{j}{i} i! = \frac{j!}{(i-j)!}$ in row i and column j transforms, by Theorem 1.2, the chromatic table into the $(\dim(K)\times m(K))$-matrix

$$\chi(K)([j]_i)_{1\le i,j\le m(K)} = (\chi^s(K,i))_{\substack{1\le s\le \dim(K)\\ 1\le i\le m(K)}}$$

with the $m(K)$ values $\chi^s(K,i)$, $1 \le i \le m(K)$, of the s-chromatic polynomial in row s. This matrix of chromatic polynomial values appears also to have log-concave rows.

3. Chromatic uniqueness

In this section we briefly discuss to what extent simplicial complexes are determined by their chromatic polynomials. Proposition shows that the chromatic table of a simplicial complex determines its f-vector.

Definition 3.1. K *is* chromatically unique *if it is determined up to isomorphism by its chromatic table.*

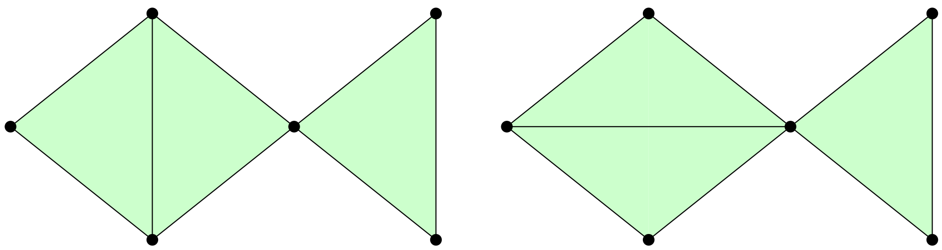
In Lemma 3.2 below, $K \amalg L$ is the disjoint union and $K \vee L$ the one-point union of K and L. The proof is identical to the one for the similar statements about chromatic polynomials for simple graphs.

Lemma 3.2. *If* K *and* L *are finite simplicial complexes then*

$$\chi^s(K \amalg L, r) = \chi^s(K,r)\chi^s(L,r), \qquad \chi^s(K\vee L, r) = \frac{\chi^s(K,r)\chi^s(L,r)}{r}$$

for all r *and all* $s \ge 0$*.*

The two nonisomorphic simplicial complexes



are not chromatically unique as they have identical chromatic tables

$$\begin{pmatrix} 0 & 0 & 2 & 10 & 7 & 1 \\ 0 & 15 & 73 & 62 & 15 & 1 \end{pmatrix}$$

by Lemma 3.2. (These two complexes are, however, PL-isomorphic.)

On the other hand, Proposition 2.34.(2) immediately implies that the s-skeleton of a full simplex is chromatically unique (in a very strong sense).

Proposition 3.3. *If K has the same s-chromatic polynomial as a full simplex $D[N]$, then K and $D[N]$ have isomorphic s-skeleta.*

Proof. If K and $D[N]$ have the same s-chromatic polynomial for some $s \geq 1$, then K has N vertices (Corollary 2.26), and, since $\chi^s(K, N-s) = \chi^s(D[N], N-s)$, the s-skeleton of K is isomorphic to the s-skeleton of the full simplex on N vertices (Proposition 2.34.(2)). □

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