ON INTERVALS $(kn, (k+1)n)$ CONTAINING A PRIME FOR ALL $n > 1$

VLADIMIR SHEVELEV, CHARLES R. GREATHOUSE IV,
AND PETER J. C. MOSES

ABSTRACT. We study values of k for which the interval $(kn, (k+1)n)$ contains a prime for every $n > 1$. We prove that the list of such integers k includes $k = 1, 2, 3, 5, 9, 14$, and no others, at least for $k \leq 50,000,000$. For every known k of this list, we give a good upper estimate of the smallest $N_k(m)$, such that, if $n \geq N_k(m)$, then the interval $(kn, (k+1)n)$ contains at least m primes.

1. INTRODUCTION AND MAIN RESULTS

In 1850, P. L. Chebyshev proved the famous Bertrand postulate (1845) that every interval $[n, 2n]$ contains a prime (for a very elegant version of his proof, see Theorem 9.2 in [10]). Other nice proofs were given by S. Ramamujan in 1919 [8] and P. Erdős in 1932 (reproduced in [4], pp.171-173). In 2006, M. El. Bachraoui [1] proved that every interval $[2n, 3n]$ contains a prime, while A. Loo [6] proved the same statement for every interval $[3n, 4n]$. Moreover, A. Loo found a lower estimate for the number of primes in the interval $[3n, 4n]$. Note also that already in 1952 J. Nagura [7] proved that, for $n \geq 25$, there is always a prime between n and $\frac{6}{5}n$. From his result it follows that the interval $[5n, 6n]$ always contains a prime. In this paper we prove the following.

Theorem 1. *The list of integers k for which every interval $(kn, (k+1)n)$, $n > 1$, contains a prime includes $k = 1, 2, 3, 5, 9, 14$ and no others, at least for $k \leq 50,000,000$.*

Besides, in this paper, for every $k = 1, 2, 3, 5, 9, 14$, we give an algorithm for finding the smallest $N_k(m)$, such that, for $n \geq N_k(m)$, the interval $(kn, (k+1)n)$ contains at least m primes.

2. Case $k = 1$

Ramanujan [8] not only proved Bertrand's postulate but also indicated the smallest integers $\{R(m)\}$, such that, if $x \geq R(m)$, then the interval $\left(\frac{x}{2}, x\right]$ contains at least m primes, or, the same, $\pi(x) - \pi(x/2) \geq m$. It is easy to see that here it is sufficient to consider *integer* x and it is evident that every term of $\{R(m)\}$ is prime. The numbers $\{R(m)\}$ are called *Ramanujan primes* [14]. It is the sequence (A104272 in [13]):

$$(1) \qquad 2, 11, 17, 29, 41, 47, 59, 67, 71, 97, ...$$

Since $\pi(x) - \pi(x/2)$ is not a monotonic function, to calculate the Ramanujan numbers one should have an effective upper estimate of $R(m)$. In [8] Ramanujan showed that

$$(2) \qquad \pi(x) - \pi(x/2) > \frac{1}{\ln x}\left(\frac{x}{6} - 3\sqrt{x}\right), \quad x > 300.$$

In particular, for $x \geq 324$, the left hand side is positive and thus ≥ 1. Using direct descent, he found that $\pi(x) - \pi(x/2) \geq 1$ already from $x \geq 2$. Thus $R(1) = 2$ which proves the Bertrand postulate. Further, e.g., for $x \geq 400$, the left hand side of (2) is more than 1 and thus ≥ 2. Again, using direct descent, he found that $\pi(x) - \pi(x/2) \geq 2$ already from $x \geq 11$. Thus $R(2) = 11$, etc. Sondow [14] found that $R(m) < 4m \ln(4m)$ and conjectured that

$$(3) \qquad R(m) < p_{3m}$$

which was proved by Laishram [5]. Since, for $n \geq 2$, $p_n \leq en \ln n$ (cf. [3], Section 4), then (3) yields $R(m) \leq 3em \ln(3m)$, $m \geq 1$. Set $x = 2n$. Then, if $2n \geq R(m)$, then $\pi(2n) - \pi(n) \geq m$. Thus the interval $(n, 2n)$ contains at least m primes, if

$$n \geq \left\lceil \frac{R(m)+1}{2} \right\rceil = \begin{cases} 2, & if \ m = 1, \\ \frac{R(m)+1}{2}, & if \ m \geq 2. \end{cases}$$

Denote by $N_1(m)$ the smallest number such that, if $n \geq N_1(m)$, then the interval $(n, 2n)$ contains at least m primes. It is clear, that $N_1(1) = R(1) = 2$. If $m \geq 2$, formally the condition $x = 2n \geq 2N_1(m)$ is not stronger than the condition $x \geq R(m)$, since the latter holds for x even and odd. Therefore, for $m \geq 2$, we have $N_1(m) \leq \frac{R(m)+1}{2}$. Let us show that in fact we have here the equality.

Proposition 2. *For $m \geq 2$,*

(4) $$N_1(m) = \frac{R(m) + 1}{2}.$$

Proof. Note that the interval $\left(\frac{R(m)-1}{2}, R(m) - 1\right)$ cannot contain more than $m-1$ primes. Indeed, it is an interval of type $(\frac{x}{2}, x)$ for integer x and the following such interval is $\left(\frac{R(m)}{2}, R(m)\right)$. By the definition, $R(m)$ is the *smallest* number such that if $x \geq R(m)$, then $\{(\frac{x}{2}, x)\}$ contains $\geq m$ primes. Therefore, the supposition that already interval $\left(\frac{R(m)-1}{2}, R(m) - 1\right)$ contains $\geq m$ primes contradicts the minimality of $R(m)$. Since the following interval of type $(y, \, 2y)$ with integer $y \geq \frac{R(m)-1}{2}$ is $\left(\frac{R(m)+1}{2}, \, R(m) + 1\right)$, then (4) follows. $\square$

So the sequence $\{N_1(m)\}$, by (1), is (A084140 in [13])

(5) $$2, 6, 9, 15, 21, 24, 30, 34, 36, 49, ...$$

3. Generalized Ramanujan numbers

Further our research is based on a generalization of Ramanujan's method. With this aim, we define generalized Ramanujan numbers (cf. [12], Section 10, and earlier (2009) comment in A164952 [13]).

Definition 3. Let $v > 1$ be a real number. A v-Ramanujan number $(R_v(m))$, is the smallest integer such that if $x \geq R_v(m)$, then $\pi(x) - \pi(x/v) \geq m$.

It is known [10] that all v-Ramanujan numbers are primes. In particular, $R_2(m) = R(m)$, $m = 1, 2, ...$, are the proper Ramanujan primes.

Definition 4. For a real number $v > 1$ the v-*Chebyshev number* $C_v(m)$ is the smallest integer, such that if $x \geq C_v(m)$, then $\vartheta(x) - \vartheta(x/v) \geq m \ln x$, where $\vartheta(x) = \sum_{p \leq x} \ln p$ is the Chebyshev function.

Since $\frac{\vartheta(x) - \vartheta(x/v)}{\ln x}$ can enlarge on 1 only when x is prime, then all v-Chebyshev numbers $C_v(m)$ are primes.

Proposition 5. *We have*

(6) $$R_v(m) \leq C_v(m).$$

Proof. Let $x \geq C_v(m)$. Then we have

$$(7) \qquad m \leq \frac{\vartheta(x) - \vartheta(x/v)}{\ln x} = \sum_{\frac{x}{v} < p \leq x} \frac{\ln p}{\ln x} \leq \sum_{\frac{x}{v} < p \leq x} 1 = \pi(x) - \pi(x/v).$$

Thus, if $x \geq C_v(m)$, then *always* $\pi(x) - \pi(x/v) \geq m$. By the Definition 3, this means that $R_v(m) \leq C_v(m)$. $\qquad \square$

Now we give an upper estimates for $C_v(m)$ and $R_v(m)$.

Proposition 6. *Let $x = x_v(m) \geq 2$ be any number for which*

$$(8) \qquad \frac{x}{\ln x}\left(1 - \frac{1300}{\ln^4 x}\right) \geq \frac{vm}{v-1}.$$

Then

$$(9) \qquad R_v(m) \leq C_v(m) \leq x_v(m).$$

Proof. We use the following inequality of Dusart [3] (see his Theorem 5.2):

$$|\vartheta(x) - x| \leq \frac{1300x}{\ln^4 x}, \quad x \geq 2.$$

Thus we have

$$\vartheta(x) - \vartheta(x/v) \geq x\left(1 - \frac{1}{v} - 1300\left(\frac{1}{\ln^4 x} - \frac{1}{v\ln^4 \frac{x}{v}}\right)\right)$$

$$\geq x\left(1 - \frac{1}{v}\right)\left(1 - \frac{1300}{\ln^4 x}\right).$$

If now

$$x\left(1 - \frac{1}{v}\right)\left(1 - \frac{1300}{\ln^4 x}\right) \geq m\ln x, \quad x \geq x_v(m),$$

then

$$\vartheta(x) - \vartheta(x/v) \geq m\ln x, \quad x \geq x_v(m)$$

and, by the Definition 4, $C_v(m) \leq x_v(m)$. So, according to (6), we conclude that $R_v(m) \leq x_v(m)$. $\qquad \square$

Remark 7. In fact, in Theorem 5.2 [3] Dusart gives several inequalities of the form

$$|\vartheta(x) - x| \leq \frac{ax}{\ln^b x}, \quad x \geq x_0(a, b).$$

In the proof we used the maximal value $b = 4$. However, with the computer point of view, the values $a = 1300$, $b = 4$ from Dusart's theorem not always are the best. The analysis for $x \geq 25$ shows that the condition

$$x(1 - \frac{1}{v})\left(1 - \frac{ax}{\ln^b x}\right) \geq m\ln x$$

is the weakest and thus satisfies for the smallest $x_v(x) = x_v(a, b)$, if to use the following values of a and b from Dusart's theorem:

$a = 3.965, \ b = 2$ for x in range $(25, \ 7 \cdot 10^7]$;

$a = \ 1300, \ b = 4$ for x in range $(7 \cdot 10^7, \ 10^9]$;

$a = 0.001, \ b = 1$ for x in range $(10^9, \ 8 \cdot 10^9]$;

$a = \ \ 0.78, \ b = 3$ for x in range $(8 \cdot 10^9, \ 7 \cdot 10^{33}]$;

$a = \ 1300, \ b = 4$ for $x > 7 \cdot 10^{33}$.

Proposition 6 gives the terms of sequences $\{C_v(m)\}$, $\{R_v(m)\}$ for every $v > 1$, $m \geq 1$. In particular, if $k = 1$ we find $\{C_2(m)\}$:

$$11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167, 179, 223,$$

$$229, 233, 239, 241, 263, 269, 281, 307, 311, 347, 349, 367, 373, 401, 409,$$

$$(10) \qquad\qquad\qquad 419, 431, 433, 443, \dots .$$

This sequence requires a separate comment. We observe that up to $C_2(100) = 1489$ only two terms of this sequence ($C_2(17) = 223$ and $C_2(36) = 443$) are not Ramanujan numbers, and the sequence is missing only the following Ramanujan numbers: 181,227,439,491,1283,1301 and no others up to 1489. The latter observation shows how much the ratio $\frac{\vartheta(x)}{\ln x}$ exactly approximates $\pi(x)$.

Further, for $v = \frac{k+1}{k}$, we find the following sequences:

for $k = 2$, $\{C_v(m)\}$,

$$(11) \qquad 13, 37, 41, 67, 73, 97, 127, 137, 173, 179, 181, 211, 229, 239, \dots ;$$

for $k = 2$, $\{R_v(m)\}$,

$$(12) \qquad 2, 13, 37, 41, 67, 73, 97, 127, 137, 173, 179, 181, 211, 229, 239, \dots ;$$

for $k = 3$, $\{C_v(m)\}$,

$$(13) \qquad 29, 59, 67, 101, 149, 157, 163, 191, 227, 269, 271, 307, 379, \dots ;$$

for $k = 3$, $\{R_v(m)\}$,

$$(14) \qquad 11, 29, 59, 67, 101, 149, 157, 163, 191, 227, 269, 271, 307, 379, \dots ;$$

for $k = 5$, $\{C_v(m)\}$,

$$(15) \qquad 59, 137, 139, 149, 223, 241, 347, 353, 383, 389, 563, 569, 593, \dots ;$$

for $k = 5$, $\{R_v(m)\}$,

$$(16) \qquad 29, 59, 137, 139, 149, 223, 241, 347, 353, 383, 389, 563, 569, 593, \dots ;$$

for $k = 9$, $\{C_v(m)\}$,

$$(17) \qquad 223, 227, 269, 349, 359, 569, 587, 593, 739, 809, 857, 991, 1009, \ldots;$$

for $k = 9$, $\{R_v(m)\}$,

$$(18) \quad 127, 223, 227, 269, 349, 359, 569, 587, 593, 739, 809, 857, 991, 1009, \ldots;$$

for $k = 14$, $\{C_v(m)\}$,

$$(19) \qquad 307, 347, 563, 569, 733, 821, 1427, 1429, 1433, 1439, 1447, 1481, \ldots;$$

for $k = 14$, $\{R_v(m)\}$,

$$(20) \qquad 127, 307, 347, 563, 569, 733, 1423, 1427, 1429, 1433, 1439, 1447, \ldots.$$

4. ESTIMATES OF TYPE (3)

Proposition 8. *We have*

$$(21) \qquad\qquad\qquad C_2(m-1) \leq p_{3m}, \ m \geq 2;$$

$$(22) \qquad R_{\frac{3}{2}}(m) \leq p_{4m}, \ m \geq 1; \ C_{\frac{3}{2}}(m-1) \leq p_{4m}, \ m \geq 2;$$

$$(23) \qquad R_{\frac{4}{3}}(m) \leq p_{6m}, \ m \geq 1; \ C_{\frac{4}{3}}(m-1) \leq p_{6m}, \ m \geq 2;$$

$$(24) \qquad R_{\frac{6}{5}}(m) \leq p_{11m}, \ m \geq 1; \ C_{\frac{6}{5}}(m-1) \leq p_{11m}, \ m \geq 2;$$

$$(25) \qquad R_{\frac{10}{9}}(m) \leq p_{31m}, \ m \geq 1; \ C_{\frac{10}{9}}(m-1) \leq p_{31m}, \ m \geq 2;$$

$$(26) \qquad R_{\frac{15}{14}}(m) \leq p_{32m}, \ m \geq 1; \ C_{\frac{15}{14}}(m-1) \leq p_{32m}, \ m \geq 2.$$

Proof. Firstly let us find some values of $m_0 = m_0(k)$, such that, at least, for $m \geq m_0$ all formulas (21)-(26) hold. According to (8)-(9), it is sufficient to show that, for $m \geq m_0$, we can take p_{tm}, where $t = 3, 4, 6, 11, 31, 32$ for formulas (21)-(26) respectively, in the capacity of $x_v(m)$. As we noted in Remark 7, in order to get possibly smaller values of m_0, we use, instead of (8), the estimate

$$(27) \qquad\qquad \frac{x}{\ln x}\left(1 - \frac{3.965}{\ln^2 x}\right) \geq \frac{vm}{v-1}.$$

In order to get $x = p_{mt}$ satisfying this inequality, note that [11]

$$p_n \geq n \ln n.$$

Therefore, it is sufficient to consider p_{mt} satisfying the inequality

$$\ln p_{tm} \leq \left(1 - \frac{1}{v}\right) t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right).$$

On the other hand, for $n \geq 2$, (see (4.2) in [3])

$$\ln p_n \leq \ln n + \ln \ln n + 1.$$

Thus it is sufficient to choose m so large that the following inequality holds

$$\ln(tm) + \ln \ln(tm) + 1 \leq \left(1 - \frac{1}{v}\right) t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right),$$

or, since $1 - \frac{1}{v} = \frac{1}{k+1}$, that

(28)
$$\frac{\ln(tm) + \ln \ln(tm) + 1}{\ln(tm)(1 - \frac{3.965}{\ln^2(tm \ln(tm))})} \leq \frac{t}{k+1}.$$

Let, e.g., $k = 1$, $t = 3$. We can choose $m_0 = 350$. Then the left hand side of (28) equals $1.4976... < 1.5$. This means that that at least, for $m \geq 350$, the estimate (3) and, for $m \geq 351$, the estimate (21) are valid. Using a computer verification for $m \leq 350$, we obtain both of these estimates. Note that another short proof of (3) was obtained in [12] (see there Remark 32). Other estimates of the proposition are proved in the same way.   □

5. Estimates and formulas for $N_k(m)$

Proposition 9.

(29)
$$N_k(1) = 2, \ k = 2, 3, 5, 9, 14.$$

For $m \geq 2$,

(30)
$$N_k(m) \leq \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil ;$$

besides, if $R_{\frac{k+1}{k}}(m) \equiv 1 \pmod{k+1}$, then

(31)
$$N_k(m) = \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil = \frac{R_{\frac{k+1}{k}}(m) + k}{k+1}$$

and, if $R_{\frac{k+1}{k}}(m) \equiv 2 \pmod{k+1}$, then

(32)
$$N_k(m) = \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil = \frac{R_{\frac{k+1}{k}}(m) + k - 1}{k+1}.$$

Proof. If $m \geq 2$, formally the condition $x = (k+1)n \geq (k+1)N_k(m)$ is not stronger than the condition $x \geq R_{\frac{k+1}{k}}(m)$, since the first one is valid only for x multiple of $k+1$. Therefore, for $m \geq 2$, (30) holds. It allows to calculate the terms of sequence $\{N_k(m)\}$ for every $k > 1$, $m \geq 2$. Since $N_k(1) \leq N_k(2)$, then, having $N_k(2)$, we also can prove (29), using direct calculations. Now

let $R_{\frac{k+1}{k}}(m) \equiv 1 \pmod{k+1}$. Note that, for $y = (R_{\frac{k+1}{k}}(m) - 1)/(k+1)$ the interval

$$(33) \qquad (ky, \ (k+1)y) = \left(\frac{k}{k+1} \left(R_{\frac{k+1}{k}}(m) - 1 \right), \ R_{\frac{k+1}{k}}(m) - 1 \right)$$

cannot contain more than $m - 1$ primes. Indeed, it is an interval of type $\left(\frac{k}{k+1}x, x \right)$ for integer x and the following such interval is

$$\left(\frac{k}{k+1} \left(R_{\frac{k+1}{k}}(m) \right), R_{\frac{k+1}{k}}(m) \right).$$

By the definition, $R_{\frac{k+1}{k}}(m)$ is the *smallest* number such that if $x \geq R_{\frac{k+1}{k}}(m)$, then $\{(\frac{k}{k+1}x, x)\}$ contains $\geq m$ primes. Therefore, the supposition that already interval (33) contains $\geq m$ primes contradicts the minimality of $R_{\frac{k+1}{k}}(m)$. Since the following interval of type $(ky, \ (k+1)y)$ with integer $y \geq \frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 1)$ is

$$\left(\frac{k}{k+1}(R_{\frac{k+1}{k}}(m) + k), \ R_{\frac{k+1}{k}}(m) + k \right),$$

then (31) follows.

Finally, let $R_{\frac{k+1}{k}}(m) \equiv 2 \pmod{k+1}$. Again show that, for $y = (R_{\frac{k+1}{k}}(m) - 2)/(k+1)$ the interval

$$(34) \qquad (ky, \ (k+1)y) = \left(\frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 2), \ R_{\frac{k+1}{k}}(m) - 2 \right)$$

cannot contain more than $m - 1$ primes. Indeed, comparing interval (34) with interval (33), we see that they contain the same integers except for $R_{\frac{k+1}{k}}(m) - 2$ which is multiple of $k + 1$. Therefore, they contain the same number of primes and this number does not exceed $m - 1$. Again, since the following interval of type $(ky, \ (k+1)y)$ with integer $y \geq \frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 2)$ is

$$\left(\frac{k}{k+1}(R_{\frac{k+1}{k}}(m) + k - 1), \ R_{\frac{k+1}{k}}(m) + k - 1 \right),$$

then (32) follows. $\square$

Remark 10. Obviously formulas (30)-(32) are valid for not only for the considered values of k, but for arbitrary $k \geq 1$.

As a corollary from (29), (31)-(32), we obtain the following formula in case $k = 2$.

Proposition 11.

$$(35) \qquad N_2(m) = \begin{cases} 2, \ if \ m = 1, \\ \left\lceil \dfrac{R_{\frac{3}{2}}(m)}{3} \right\rceil, \ if \ m \geq 2. \end{cases}$$

Formula (35) shows that the case $k = 2$ over its regularity not concedes to a classic case $k = 1$. Note that, if $k \geq 3$ and $R_{\frac{k+1}{k}}(m) \equiv j \pmod{k+1}$, $3 \leq j \leq k$, then, generally speaking, (30) is not an equality. Evidently, $N_k(m) \geq N_k(m-1)$ and it is interesting that the equality is attainable (see below sequences (37)-(40)).

Example 12. Let $k = 3$, $m = 2$. Then $v = \frac{4}{3}$ and, by (14), $R_{\frac{4}{3}}(2) = 29 \equiv 1 \pmod{4}$. Therefore, by (31), $N_3(2) = \frac{29+3}{4} = 8$. Indeed, interval $(3 \cdot 7, \ 4 \cdot 7)$ already contains only prime 23.

Example 13. Let $k = 3$, $m = 3$. Then, by (14), $R_{\frac{4}{3}}(3) = 59 \equiv 3 \pmod{4}$. Here $N_3(3) = 11$ which is essentially less than $\left\lceil R_{\frac{4}{3}}(3)/4 \right\rceil = 15$. Indeed, each interval

$$(3 \cdot 15, \ 4 \cdot 15), \ (3 \cdot 14, \ 4 \cdot 14), \ (3 \cdot 13, \ 4 \cdot 13), \ (3 \cdot 12, \ 4 \cdot 12), \ (3 \cdot 11, \ 4 \cdot 11)$$

contains more than 2 primes and only interval $(3 \cdot 10, \ 4 \cdot 10)$ contains only 2 primes.

In any case, Proposition 9 allows to calculate terms of sequence $\{N_k(m)\}$ for every considered values of k. So, we obtain the following few terms of $\{N_k(m)\}$:
for $k = 2$,

$$(36) \qquad 2, 5, 13, 14, 23, 25, 33, 43, 46, 58, 60, 61, 71, 77, 80, 88, 103, 104, \dots ;$$

for $k = 3$,

$$(37) \qquad 2, 8, 11, 17, 26, 38, 40, 41, 48, 57, 68, 68, 70, 87, 96, 100, 108, 109, \dots ;$$

for $k = 5$,

$$(38) \qquad 2, 7, 17, 24, 25, 38, 41, 58, 59, 64, 65, 73, 95, 97, 103, 106, 107, 108, \dots ;$$

for $k = 9$,

$$(39) \qquad 2, 14, 23, 23, 34, 36, 57, 58, 60, 60, 77, 86, 100, 100, 102, 123, 149, \dots ;$$

for $k = 14$,

$$(40) \quad 2, 11, 24, 37, 38, 39, 50, 96, 96, 96, 96, 97, 97, 125, 125, 132, 178, 178, \dots .$$

Remark 14. If, as in [1], [6], instead of intervals $(kn,\ (k+1)n)$, to consider intervals $[kn,\ (k+1)n]$, then sequences (5), (36)-(38) would begin with 1.

6. METHOD OF SMALL INTERVALS

If we know a theorem of the type: for $x \geq x_0(\Delta)$, the interval $(x,\ (1+\frac{1}{\Delta})x]$ contains a prime, then we can calculate a bounded number of the first terms of sequences (5) and (36)-(40). Indeed, put $x_1 = kn$, such that $n \geq \frac{x_0}{k}$. Then $(k+1)n = \frac{k+1}{k}x_1$ and, if $1 + \frac{1}{\Delta} < \frac{k+1}{k}$, i.e., $\Delta > k$, then

$$\left(x_1,\ (1+\frac{1}{\Delta})x_1\right] \subset (kn, (k+1)n).$$

Thus, if $n \geq \frac{x_0}{k}$, then the interval $(kn, (k+1)n)$ contains a prime, and, using method of finite descent, we can find $N_k(1)$. Further, put $x_2 = (1 + \frac{1}{\Delta})x_1$. Then interval $(x_2,\ (1+\frac{1}{\Delta})x_2]$ also contains a prime. Thus the union

$$\left(x_1,\ (1+\frac{1}{\Delta})x_1\right] \cup \left(x_2,\ (1+\frac{1}{\Delta})x_2\right] = \left(x_1, (1+\frac{1}{\Delta})^2 x_1\right]$$

contains at least two primes. This means that if $(1+\frac{1}{\Delta})^2 x_1 < (k+1)n$ or $(1+\frac{1}{\Delta})^2 < 1 + \frac{1}{k}$, then

$$\left(x_1,\ (1+\frac{1}{\Delta})^2 x_1\right] \subset (kn, (k+1)n)$$

and the interval $(kn, (k+1)n)$ contains at least two primes; again, using method of finite descent, we can find $N_k(2)$, etc., if $(1+\frac{1}{\Delta})^m < 1 + \frac{1}{k}$, then

$$\left(x_1,\ (1+\frac{1}{\Delta})^m x_1\right] \subset (kn, (k+1)n)$$

and the interval $(kn, (k+1)n)$ contains at least m primes and we can find $N_k(m)$. In this way, we can find $N_k(m)$ for $m < \frac{\ln(1+\frac{1}{k})}{\ln(1+\frac{1}{\Delta})}$. In 2002, Ramaré and Saouter [9] proved that interval $(x(1 - 28314000^{-1}),\ x)$ always contains a prime if $x > 10726905041$, or, equivalently, interval $(x,\ (1 + 28313999^{-1})x)$ contains a prime if $x > 10726905419$. This means that, e.g., we can find $N_{14}(m)$ for $m \leq 1954471$. Unfortunately, this method cannot give the exact estimates and formulas for $N_k(m)$ as (30)-(32).

We can also to consider a more general application of of this method. Consider a fixed infinite set P of primes which we call P-primes. Furthermore, consider the following generalization of v-Ramanujan numbers.

Definition 15. For $v > 1$, a (v, P)-Ramanujan number $(R_v^{(P)}(m))$, is the smallest integer such that if $x \geq R_v^P(m)$, then $\pi_P(x) - \pi_P(x/v) \geq m$, where $\pi_P(x)$ is the number of P-primes not exceeding x.

Note that every (v, P)-Ramanujan number is P-prime. If we know a theorem of the type: for $x \geq x_0(\Delta)$, the interval $\left(x, \ (1 + \frac{1}{\Delta})x\right]$ contains a P-prime, then, using the above described algorithm we can calculate a bounded number of the first (v, P) -Ramanujan numbers. For example, let P be the set of primes $p \equiv 1 \pmod 3$. From the result of Cullinan and Hajir [2] it follows, in particular, that for $x \geq 106706$, the interval $(x, \ 1.048x)$ contains a P-prime. Using the considered algorithm, we can calculate the first 14 $(2, P)$-Ramanujan numbers. They are

(41) $7, 31, 43, 67, 97, 103, 151, 163, 181, 223, 229, 271, 331, 337.$

Analogously, if P is the set of primes $p \equiv 2 \pmod 3$, then the sequence of $(2, P)$-Ramanujan numbers begins

(42) $11, 23, 47, 59, 83, 107, 131, 167, 227, 233, 239, 251, 263, 281, \dots;$

if P is the set of primes $p \equiv 1 \pmod 4$, then the sequence of $(2, P)$-Ramanujan numbers begins

(43) $13, 37, 41, 89, 97, 109, 149, 229, 233, 241, 257, 277, 281, 317, \dots;$

and, if P is the set of primes $p \equiv 3 \pmod 4$, then the sequence of $(2, P)$-Ramanujan numbers begins

(44) $7, 23, 47, 67, 71, 103, 127, 167, 179, 191, 223, 227, 263, 307, \dots.$

Denote by $N_k^{(P)}(m)$ the smallest number such that, for $n \geq N_k^{(P)}(m)$, the interval $(kn, (k+1)n)$ contains at least m P-primes. It is easy to see that formulas (30)-(32) hold for $N_k^{(P)}(m)$ and $R_{\frac{k+1}{k}}^{(P)}(m)$. In particular, in cases $k = 1, 2$ we have the formulas

(45) $N_1^{(P)}(m) = \dfrac{R_2^{(P)}(m) + 1}{2}, \quad N_2^{(P)}(m) = \left\lceil \dfrac{R_{\frac{3}{2}}^{(P)}(m)}{3} \right\rceil.$

Therefore, the following sequences for $N_1^{(P)}(m)$ for the considered cases of set P correspond to sequences (41)-(44) respectively:

(46) $4, 16, 22, 34, 49, 52, 76, 82, 91, 112, 115, 136, 166, 169, \dots;$

(47) $6, 12, 24, 30, 42, 54, 66, 84, 114, 117, 120, 126, 132, 141, \dots;$

(48) $7, 19, 21, 45, 49, 55, 75, 115, 117, 121, 129, 139, 141, 159, \dots;$

(49) $4, 12, 24, 34, 36, 52, 64, 84, 90, 96, 112, 114, 132, 154, \dots.$

7. Proof of Theorem 1

For $k \geq 1$, denote by $a(k)$ the least integer $n > 1$ for which the interval $(kn, \ (k+1)n)$ contains no prime; in the case, when such n does not exist, we put $a(k) = 0$. Taking into account (21), note that $a(k) = 0$ for $k = 1, 2, 3, 5, 9, 14, \ldots$. Consider sequence $\{a(k)\}$. Its first few terms are (A218831 in [13])

(50) $0, 0, 0, 2, 0, 4, 2, 3, 0, 2, 3, 2, 2, 0, 6, 2, 2, 3, 2, 6, 3, 2, 4, 2, 2, 7, 2, 2, 4, 3, \ldots$.

Calculations of $a(k)$ in the range $\{15, ..., 5 \times 10^7\}$ lead to values of $a(k)$ in the interval $[2, \ 16]$ which completes the proof. $\square$

8. Acknowledgment

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DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, BEER-SHEVA 84105, ISRAEL; E-MAIL: SHEVELEV@BGU.AC.IL

UNITED STATES; E-MAIL: CHARLES.GREATHOUSE@CASE.EDU

UNITED KINGDOM; E-MAIL: MOWS@MOPAR.FREESERVE.CO.UK