On the Hankel transform of C-fractions

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Abstract

We study the Hankel transforms of sequences whose generating function can be expressed as a C-fraction. In particular, we relate the index sequence of the non-zero terms of the Hankel transform to the powers appearing in the monomials defining the C-fraction. A closed formula for the Hankel transforms studied is given. As every power-series can be represented by a C-fraction, this gives in theory a closed form formula for the Hankel transform of any sequence. The notion of multiplicity is introduced to differentiate between Hankel transforms.

1 Introduction

Given a sequence a_n, we denote by h_n the general term of the sequence with $h_n = |a_{i+j}|_{0 \le i,j \le n}$. The sequence h_n is called the Hankel transform of a_n [7, 8, 10]. If the sequence a_n has generating function $g(x)$, then by an abuse of language we can also refer to h_n as the Hankel transform of $g(x)$.

A well known example of Hankel transform is that of the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$, where we find that $h_n = 1$ for all n. Hankel determinants occur naturally in many branches of mathematics, from combinatorics [1] to number theory [12] and to mathematical physics [17].

We shall be interested in characterizing the Hankel transform of sequences whose generating functions can be expressed as the following type of C-fraction:

$$g(x) = \cfrac{1}{1+\cfrac{a_1x^{q_1}}{1+\cfrac{a_2x^{q_2}}{1+\cfrac{a_3x^{q_3}}{1+\cdots}}}}, \qquad (1)$$

for appropriate values of coefficients $a_1, a_2, a_3, \ldots$ and exponents $q_1, q_2, q_3, \ldots$. The results will depend on making explicit the relationship between this type of C-fraction, and $h(1/x)$, where $h(x)$ is the following type of continued fraction:

$$h(x) = \cfrac{x^{p_0}}{b_1 x^{p_1} + \cfrac{1}{b_2 x^{p_2} + \cfrac{1}{b_3 x^{p_3} + \cfrac{1}{b_4 x^{p_4} + \cdots}}}}. \tag{2}$$

We will then be able to use classical results [5] to conclude our study and to examine interesting examples.

2 Review of known results

The first part of this section reviews the close link between power series and C-fractions. Note that the "C" comes from the word "corresponding".

We commence with a power series

$$f_0(x) = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \dots. \tag{3}$$

We form the family of power series $\{f_n(x)\}$ by the relations

$$f_{n+1}(x) = \frac{a_{n+1} x^{q_{n+1}}}{f_n(x) - 1}, \quad n = 0, 1, 2, \dots, \tag{4}$$

where the q_n are positive integers chosen together with complex numbers a_n in such a way that if $f_n(x) \neq 1$, $f_{n+1}(0) = 1$. If no $f_n(x) = 1$, this process yields an infinite sequence of power series $f_0(x), f_1(x), f_2(x), \dots$. If some $f_n(x) = 1$, the process terminates and yields a finite set of power series $f_0(x), f_1(x), \dots, f_n(x)$. The continued fraction

$$1 + \cfrac{a_1 x^{q_1}}{1 + \cfrac{a_2 x^{q_2}}{1 + \cfrac{a_3 x^{q_3}}{1 + \cfrac{a_4 x^{q_4}}{1 + \cdots}}}}, \tag{5}$$

formed with these a_n and q_n is said to *correspond* to the power series (3) [6, 11]. Conversely, if we begin with a continued fraction of the form (5), we can form the n-th *approximant* $\frac{A_n(x)}{B_n(x)}$ by means of the recurrence relations

$$\begin{aligned} A_0 &= 1, & B_0 &= 1, \\ A_1 &= 1 + a_1 x^{q_1}, & B_1 &= 1, \\ A_n &= A_{n-1} + a_n x^{q_n} A_{n-2}, & B_n &= B_{n-1} + a_n x^{q_n} B_{n-2}, \\ & & n &= 2, 3, \dots. \end{aligned}$$

We have

$$\frac{A_n(x)}{B_n(x)} - \frac{A_{n-1}(x)}{B_{n-1}(x)} = \frac{(-1)^{n-1}a_1a_2a_3\cdots a_nx^{s_n}}{B_{n-1}(x)B_{n-2}(x)}, \tag{6}$$

where

$$s_n = q_1 + q_2 + \cdots + q_n.$$

By equation (6) the Taylor development of the rational function $\frac{A_{n-1}(x)}{B_{n-1}(x)}$ about the origin agrees with the development of $\frac{A_n(x)}{B_n(x)}$ up to but not including the term in x^{s_n}. Hence if (5) is nonterminating, the C-fraction (5) determines uniquely a *corresponding* power series.

We have the following classical result [11]

Proposition 1. *If the continued fraction (5) corresponds to the power series (3), then the power series (3) corresponds to the continued fraction (5), and conversely.*

A division-free algorithm for the construction of the C-fraction (5) from the power series (3) is given by Frank [2, 3].

If we start with a power series $f(x) = \sum_{i=0}^{\infty} t_i x^i$, then by considering the sequence $1 + xf(x)$, which is in the form (3), we see that $f(x)$ corresponds to a C-fraction of the form

$$\cfrac{a_0x^{q_0}}{1+\cfrac{a_1x^{q_1}}{1+\cfrac{a_2x^{q_2}}{1+\cdots}}},$$

for appropriate values of $a_0, a_1, a_2, a_3, \ldots$ and $q_0, q_1, q_2, q_3, \ldots$.

We now recall known results concerning the Hankel transform of sequences whose generating functions are of the form $f(1/x)$ where $f(x)$ can be expressed as a continued fraction of the form

$$f(x) = \cfrac{b_0x^{p_0}}{b_1x^{p_1}+\cfrac{1}{b_2x^{p_2}+\cfrac{1}{b_3x^{p_3}+\cfrac{1}{b_4x^{p_4}+\cdots}}}}. \tag{7}$$

We have the following result [5].

Proposition 2. *Let h_n denote the Hankel transform of the sequence $[x^n]f(1/x)$ where $f(x)$ has the form (7) (give conditions on $b_0 = 1$ and $p_0 = 0$). Then h_n is zero for all n unless $n = p_1 + p_2 + \cdots + p_m$, for some m, in which case*

$$h_n = \prod_{i=1}^{m}(-1)^{\frac{p_i(p_i-1)}{2}} \cdot (-1)^{\sum_{i=0}^{m-1} ip_{i+1}} \prod_{i=1}^{m} \frac{1}{b_i^{p_i+2\sum_{j=i+1}^{m} p_j}}. \tag{8}$$

3 Main result

In order to obtain our main result, we need to relate C-fractions of the form

$$g(x) = \cfrac{1}{1 + \cfrac{a_1 x^{q_1}}{1 + \cfrac{a_2 x^{q_2}}{1 + \cfrac{a_3 x^{q_3}}{1 + \cdots}}}}$$

to continued fractions of the form

$$f(x) = \cfrac{x^{p_0}}{b_1 x^{p_1} + \cfrac{1}{b_2 x^{p_2} + \cfrac{1}{b_3 x^{p_3} + \cfrac{1}{b_4 x^{p_4} + \cdots}}}}.$$

We wish to find the conditions under which $f(1/x) = g(x)$. We look at the case of unit coefficients first. By equation (8), the corresponding Hankel transforms will then take on values from the set $\{-1, 0, 1\}$.

By successive divisions above and below the line, we can cast $f(x)$ in the form

$$f(x) = \cfrac{x^{p_0 - p_1}}{1 + \cfrac{x^{-p_1 - p_2}}{1 + \cfrac{x^{-p_2 - p_3}}{1 + \cdots}}},$$

and hence we have

$$f(1/x) = \cfrac{x^{-p_0 + p_1}}{1 + \cfrac{x^{p_1 + p_2}}{1 + \cfrac{x^{p_2 + p_3}}{1 + \cdots}}}.$$

Starting from $g(x)$ and proceeding to $f(x)$ is more problematic, since it is not clear what to choose as p_0. The Hankel transforms that we will be concerned with determine that we require the condition $-p_0 + p_1 = 0$, and hence that $p_1 = p_0$. We choose to set $p_0 = 1$. Then starting from the C-fraction

$$\cfrac{1}{1 + \cfrac{x^{q_1}}{1 + \cfrac{x^{q_2}}{1 + \cfrac{x^{q_3}}{1 + \cdots}}}}$$

we find the following continued fraction of type (2):

$$\cfrac{x^{p_0}}{x^{p_0}+\cfrac{1}{x^{q_1-p_0}+\cfrac{1}{x^{q_2-q_1+p_0}+\cfrac{1}{x^{q_3-q_2+q_1-p_0}+\cfrac{1}{x^{q_4-q_3+q_2-q_1+p_0}+\cdots}}}}}$$

By Proposition (2), the position of the non-zero terms of the corresponding Hankel transform will be given by the indexing sequence $p_0, p_0+(q_1-p_0), p_0+(q_1-p_0)+(q_2-q_1+p_0), p_0+(q_1-p_0)+(q_2-q_1+p_0)+(q_3-q_2+q_1-p_0), \ldots$ or $p_0, q_1, q_2+p_0, q_3+q_1, q_4+q_2+p_0, \ldots$. This sequence can be realised by

$$\begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 1 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \vdots \end{pmatrix}.$$

The n-th term of this sequence m_n is given by

$$m_n = \sum_{k=0}^{n} \frac{1+(-1)^{n-k}}{2} \tilde{q}_k = \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{k-i} \tilde{q}_i = \sum_{k=0}^{n} p_k,$$

where $\tilde{q}_0 = p_0$, $\tilde{q}_n = q_n$ for $n > 0$, and $p_n = \sum_{k=0}^{n} (-1)^{n-k} \tilde{q}_k$. Note that since the above matrix is $\left(\frac{1}{1-x^2}, x\right)$ as a Riordan array, then if the g.f. of the sequence $q_1, q_2, q_3, \ldots$ is $G(x)$, then the g.f. of the index set is

$$\frac{1}{1-x^2}(1+xG(x)).$$

We next note that if

$$f(x) = \cfrac{b_0 x^{p_0}}{b_1 x^{p_1}+\cfrac{1}{b_2 x^{p_2}+\cfrac{1}{b_3 x^{p_3}+\cfrac{1}{b_4 x^{p_4}+\cdots}}}}$$

is to be such that $f(1/x)$ can be represented as

$$g(x) = \cfrac{a_0 x^{q_0}}{1+\cfrac{a_1 x^{q_1}}{1+\cfrac{a_2 x^{q_2}}{1+\cfrac{a_3 x^{q_3}}{1+\cdots}}}}$$

then we must have

$$a_k = \frac{1}{b_k b_{k+1}}. \tag{9}$$

Reversing this set of equations, beginning with $b_0 = 1$, we find that

$$b_{2n} = \frac{a_0 a_2 \cdots a_{2n-2}}{a_1 a_3 \cdots a_{2n-1}},$$

and

$$b_{2n+1} = \frac{a_1 a_3 \cdots a_{2n-1}}{a_0 a_2 \cdots a_{2n}}.$$

(See also [9], Theorem 3.6 and its corollaries). Substituting these values into Equation (8) and simplifying (where we take $a_0 = 1$, $p_0 = 1$), gives us the main result of this note.

Proposition 3. *The non-zero elements of the Hankel transform of the sequence with generating function given by the C-fraction*

$$\cfrac{1}{1+\cfrac{a_1x^{q_1}}{1+\cfrac{a_2x^{q_2}}{1+\cfrac{a_3x^{q_3}}{1+\cdots}}}}$$

are given by

$$h_n = \prod_{i=1}^{m}(-1)^{\frac{p_i(p_i+1)}{2}} \cdot (-1)^{1+\sum_{i=0}^{m-1} i p_{i+1}} \cdot \prod_{k=1}^{m} a_k^{\sum_{i=k}^{m} p_i},$$

where

$$p_i = \sum_{j=0}^{i}(-1)^{i-j}\tilde{q}_j \quad \textit{and} \quad n = \sum_{k=0}^{m} p_i,$$

and the sequence $\tilde{q}_n$ is given by $1, q_1, q_2, q_3, \ldots$.

Example 4. We consider the Fibonacci-inspired C-fraction

$$\cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x^2}{1+\cfrac{3x^3}{1+\cdots}}}}}$$

where $\tilde{q}_n = F_n + 0^n$ and $a_n = F_n$. Then we find that the non-zero terms of the Hankel transform are indexed by

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^{k-i}(F_i + 0^i) = F_{i+1}.$$

The non-zero terms, calculated as

$$\prod_{i=1}^{m}(-1)^{\frac{F_i(F_i+1)}{2}} \cdot (-1)^{1+\sum_{i=0}^{m-1} iF_{i+1}} \cdot \prod_{k=1}^{m} F_k^{\sum_{i=k}^{m} p_i},$$

begin

$$1, 1, 1, -2, 72, 1944000, \ldots.$$

We see emerging here an interesting phenomenon, which we will naturally term "multiplicity". In this case we note that $F_2 = F_3 = 1$, corresponding to the first two 1's of the non-zero Hankel elements above. The Hankel transform is thus given by

$$1, 1, -2, 0, 72, 0, 0, 1944000, 0, 0, 0, 0, 1547934105600000000, 0, 0, \ldots,$$

where the initial 1 has multiplicity 2.

Example 5. The Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ provide an interesting example of the notion of multiplicity. They have generating function

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}},$$

and thus $a_n = -1$ for all $n > 0$ and $q_n = 1$ for all $n > 0$ (and hence $\tilde{q}_n = 1$ for all $n \geq 0$). We then have

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{n}\sum_{i=0}^{n}(-1)^{k-i} = \lfloor \frac{n+2}{2} \rfloor.$$

Thus the non-zero terms of the Hankel transform of C_n are indexed by

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$$

These terms are all calculated to equal 1, as is well-known. Thus we can write the Hankel transform of C_n as

$$1_2, 1_2, 1_2, \ldots,$$

where the sub-index 2 indicates that each 1 occurs with "multiplicity" 2. This is a shorthand way of saying that the index set is $1, 1, 2, 2, 3, 3, \ldots$.

Example 6. It is well known that the Hankel transform of the aerated Catalan numbers $C_{\frac{n}{2}}\frac{1+(-1)^n}{2}$ is also the all-1's sequence. This sequence has generating function

$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{1-\cdots}}},$$

where now $a_n = -1$ for all $n > 0$ and $q_n = 2$ for all $n > 0$ (and hence $\tilde{q}_n = 2 - 0^n$ for all $n \geq 0$). Now

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{n} \sum_{i=0}^{n} (-1)^{k-i}(2 - 0^i) = n + 1,$$

and hence the indexing set for this Hankel transform is $1, 2, 3, 4, 5, 6, \ldots$. That is, each 1 appears with multiplicity 1. Thus in a sense this is the original sequence with Hankel transform of all 1's.

Example 7. The generalized Rogers-Ramanujan continued fraction. We consider the continued fraction

$$\cfrac{1}{1 + \cfrac{\gamma x}{1 + \cfrac{\gamma x^2}{1 + \cfrac{\gamma x^3}{1 + \cfrac{\gamma x^4}{1 + \cdots}}}}}.$$

Here, $\tilde{q}_n = n + 0^n$, and $a_n = \gamma - \gamma 0^n = \gamma(1 - 0^n)$. We then have that p_n is the sequence

$$1, 0, 2, 1, 3, 2, 4, 3, 5, 4, 6, \ldots,$$

and $\sum_{k=0}^{n} p_k$ is the sequence that begins

$$1, 1, 3, 4, 7, 9, 13, 16, 21, 25, 31, \ldots.$$

The non-zero terms of the Hankel transform are, in order,

$$1, 1, -\gamma^6, \gamma^{12}, \gamma^{32}, \gamma^{52}, -\gamma^{94}, \gamma^{136}, \gamma^{208}, \gamma^{280}, -\gamma^{390}, \ldots.$$

The exponent sequence

$$0, 0, 6, 12, 32, 52, 94, \ldots$$

can be shown to have generating function

$$\frac{2x^2(x^3 + 3)}{(x + 1)^2(x - 1)^4}.$$

4 Conclusion

Since to each sequence a_n there corresponds the power series $\sum_{k=0}^{\infty} a_n x^n$, and to each power series there corresponds a C-fraction, the foregoing gives, in theory, a closed form formula for the Hankel transform of each sequence. Of course, this presupposes that the passage from generating function to C-fraction can be effected easily. The Q-D algorithm is one method for this.

We note that Heilermann's formula [7, 8] for the Hankel transform of a sequence with generating function of the form

$$\cfrac{1}{1-\alpha_1 x-\cfrac{\beta_1 x^2}{1-\alpha_2 x-\cfrac{\beta_2 x^2}{1-\cdots}}}$$

can be derived from the above result, due to the fact that $p_i = 1$ in this case, and the fact that although in this note Equation (8) has been expressed in the case of monomials $b_i x^{p_i}$, the result continues to be true for polynomials $Q_{p_i}(x) = b_i x^{p_i} + \cdots$ of degree p_i.

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