A search for primes p such that Euler number E_{p-3} is divisible by p

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Abstract

Let $p > 3$ be a prime. Euler numbers E_{p-3} first appeared in H. S. Vandiver's work (1940) in connection with the first case of Fermat Last Theorem. Vandiver proved that $x^p + y^p = z^p$ has no solution for integers x, y, z with $\gcd(xyz, p) = 1$ if $E_{p-3} \equiv 0 \pmod{p}$. Numerous combinatorial congruences recently obtained by Z.-W. Sun and by Z.-H. Sun involve the Euler numbers E_{p-3}. This gives a new significance to the primes p for which $E_{p-3} \equiv 0 \pmod{p}$.

For the computation of residues of Euler numbers E_{p-3} modulo a prime p, we use the congruence which runs significantly faster than other known congruences involving E_{p-3}. Applying this congruence, a computation via `Mathematica 8` shows that only three primes less than 10^7 satisfy the condition $E_{p-3} \equiv 0 \pmod{p}$ (such primes are 149, 241 and 2946901, and they are given as a Sloane's sequence A198245). By using related computational results and statistical considerations similar to those on search for Wieferich and Fibonacci-Wieferich and Wolstenholme primes, we conjecture that there are infinitely many primes p such that $E_{p-3} \equiv 0 \pmod{p}$. Moreover, we propose a conjecture on the asymptotic estimate of number of primes p in an interval $[x, y]$ such that $E_{p-3} \equiv A \pmod{p}$ for some integer A with $|A| \in [K, L]$.

Keywords: Euler number, E_{p-3}, congruence modulo a prime, supercongruence, Fermat quotient

1. Introduction

Euler numbers E_n $(n = 0, 1, 2, \ldots)$ (e.g., see [13, pp. 202–203]) are integers defined recursively by

$$E_0 = 1, \quad \text{and} \quad \sum_{\substack{0 \le k \le n \\ k \ even}} \binom{n}{k} E_{n-k} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

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(it is well known that $E_{2n-1}=0$ for each $n=1,2,\ldots$). The first few Euler numbers are $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, E_{10}=-50521, E_{12}=2702765, E_{14}=-199360981, E_{16}=19391512145$. It is well known that Euler numbers can also be defined by the generating function

$$\frac{2}{e^x+e^{-x}}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}.$$

It is well known that $E_n=E_n(0)$ ($n=0,1,\ldots$), where $E_n(x)$ is the classical Euler polynomial (see e.g., [15, p. 61 *et seq.*]).

Recall that *Bernoulli numbers* B_n $(n=0,1,2,\ldots)$ are rational numbers defined by the formal identity

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}.$$

It is easy to see that $B_n=0$ for odd $n\geq 3$, and the first few nonzero terms of (B_n) are $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$ and $B_8=-1/30$. It is well known that $B_n=B_n(0)$, where $B_n(x)$ is the classical Bernoulli polynomial (see e.g., [15, p. 61 *et seq.*]).

A significance of Euler numbers, and especially of E_{p-3} with a prime p, is closely related to Fermat Last Theorem (see [13, Lecture X, Section 2]). In 1850 Kummer (see e.g., [13, Theorem (3A), p. 86 and Theorems (2A)–(2F), pp. 99–103] proved that Fermat Last Theorem holds for each *regular prime*, that is, for each prime p that does not divide the numerator of any Bernoulli number B_{2n} with $n=1,2,\ldots,(p-3)/2$. In 1940 H. S. Vandiver [24] likewise proved for Euler-regular primes. Paralleling the previous definition of a (irr)regular prime (with respect to the Bernoulli numbers) following Vandiver [24], a prime p is said to be *Euler-irregular primes* (shortly E*-irregular*) if and only if it divides at least one of the Euler numbers E_{2n} with $1\leq n\leq (p-3)/2$. Otherwise, that is if p does not divide $E_2,E_4,\ldots,E_{p-3}$, a prime p is called E*-regular*. The smallest E-irregular prime is $p=19$, which divides $E(10)=-50521$. The first few E-irregular primes are $19,31,43,47,61,67,71,79,101,137,139,149,193,223,241$ (with $p=241$ dividing both E_{210} and E_{238}, and hence having an E-irregularity index of 2) (see [4]). In 1954 L. Carlitz [1] proved that there are infinitely many E-irregular primes p, i.e., $p\mid E_2E_4\cdots E_{p-3}$. Using modular arithmetic to determine divisibility properties of the corresponding Euler numbers, the E-irregular primes less than 10000 were found in 1978 by R. Ernvall and T. Metsänkylä [4].

In his book [13, p. 203] P. Ribenboim noticed that "it is not all surprizing that the connection, via Kummer's theorem, between the primes dividing certain Bernoulli numbers and the truth of Fermat's theorem, would suggest a similar theorem using the Euler numbers." Vandiver [24] proved that $x^p+y^p=z^p$ has no solution for integers x,y,z with $\gcd(xyz,p)=1$ if $E_{p-3}\equiv 0\,(\bmod\,p)$. The analogous result was proved by Cauchy (1847) and Genocchi (1852) (see [13, p. 29, Lecture II, Section 2]) with the Bernoulli number B_{p-3} instead of E_{p-3}. Further, in 1950 M. Gut [8] proved that the condition $E_{p-3}\equiv E_{p-5}\equiv E_{p-7}\equiv E_{p-9}\equiv E_{p-11}\equiv 0\,(\bmod\,p)$ is necessary for the Diophantine equation $x^{2p}+y^{2p}=z^{2p}$ to be solvable.

Furthermore, numerous combinatorial congruences recently obtained by Z.-W. Sun in [20]–[22] and by Z.-H. Sun in [17] involve Euler numbers E_{p-3} with a prime p. Many of these congruences become "supercongruences" if and only if $E_{p-3}\equiv 0\,(\bmod\,p)$ (A *supercongruence* is a congruence whose modulus is a prime power.) This gives a significance to primes p for which $E_{p-3}\equiv 0\,(\bmod\,p)$. The first two primes 149 and 241 have also been discoverded by Z.-W. Sun [20].

In this note, we focus our attention to the computational search for residues of Euler numbers E_{p-3} modulo a prime p. By the congruence obtained in 1938 by E. Lehmer [9, p. 359], for each prime $p \geq 5$

$$\sum_{k=1}^{[p/4]} \frac{1}{k^2} \equiv (-1)^{(p-1)/2} 4E_{p-3} \pmod{p}, \tag{1}$$

where $[a]$ denotes the integer part of a real number a. Usually (cf. [4]), if $E_{p-3} \equiv 0 \,(\bmod\, p)$ then we say that $(p, p-3)$ is an *E-irregular pair*. It was founded in [4] that in the range $p < 10^4$ $(p, p-3)$ is an E-irregular pair for $p = 149$ and $p = 241$.

For our computations presented in Section 3 we do not use Lehmer's congruence (1) including harmonic number of the second order. Our computation via `Mathematica 8` which uses the expression including the harmonic number (of the first order) is very much faster than those related to the congruence (1). Here we report that only three primes less than 10^7 satisfy the condition $E_{p-3} \equiv 0 \,(\bmod\, p)$. Using our computational results and statistical considerations similar to those in relation to a search for Wieferich and Fibonacci-Wieferich and Wolstenholme primes (cf. [2, p. 447] and [11]), we conjecture that there are infinitely many primes p such that $E_{p-3} \equiv 0 \,(\bmod\, p)$.

2. A congruence used in our computation

Here, as usually in the sequel, for integers m, n, rs with $n \neq 0$ and $s \neq 0$, and a prime power p^a we put $m/n \equiv r/s \,(\bmod\, p^e)$ if and only if $ms \equiv nr \,(\bmod\, p^e)$, and the residue class of m/n is the residue class of mn' where n' is the inverse of n modulo p^e.

In what follows p always denotes a prime. The Fermat Little Theorem states that if p is a prime and a is an integer not divisible by p, then $a^{p-1} \equiv 1 \,(\bmod\, p)$. This gives rise to the definition of the *Fermat quotient of p to base a*,

$$q_p(a) := \frac{a^{p-1} - 1}{p},$$

which is an integer. It is well known that divisibility of Fermat quotient $q_p(a)$ by p has numerous applications which include the Fermat Last Theorem and squarefreeness testing (see [5], [7] and [13]). If $q_p(2)$ is divisible by p, p is said to be *Wieferich prime*. Despite several intensive searches, only two Wieferich primes are known: $p = 1093$ and $p = 3511$ (see [2] and [3]). Another class of primes initially defined because of Fermat Last Theorem are *Fibonacci-Wieferich primes*, sometimes called *Wall-Sun-Sun primes*. A prime p is said to be *Fibonacci-Wieferich prime* if the Fibonacci number $F_{p-\left(\frac{p}{5}\right)}$ is divisible by p^2, where $\left(\frac{p}{5}\right)$ denotes the Legendre symbol (see [18]). A search in [11] and [3] shows that there are no Fibonacci-Wieferich primes less than 9.7×10^{14}.

For the computation of residues of Euler numbers E_{p-3} modulo a prime p, it is suitable to use the following congruence which runs significantly faster than Lehmer's congruence (1).

Theorem ([17, Theorem 4.1(iii)]). *Let $p \geq 5$ be a prime. Then*

$$\sum_{k=1}^{[p/4]} \frac{1}{k} + 3q_p(2) - \frac{3p}{2} q_p(2)^2 \equiv (-1)^{(p+1)/2} p E_{p-3} \pmod{p^2}, \tag{2}$$

where $[a]$ denotes the integer part of a real number a.

Proof. Quite recently, Z.-W. Sun [20, Proof of Theorem 1.1, the congruence after (2.3)] noticed that by a result of Z.-H. Sun [17, Corollary 3.3],

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^{k-1}}{k}\equiv q_p(2)-\frac{p}{2}q_p(2)^2-(-1)^{(p+1)/2}pE_{p-3}\pmod{p^2}. \tag{3}$$

On the other hand, we have

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^{k-1}}{k}=\sum_{k=1}^{(p-1)/2}\frac{1}{k}-2\sum_{\substack{1\le k\le (p-1)/2\\ 2\mid k}}\frac{1}{k}=\sum_{k=1}^{(p-1)/2}\frac{1}{k}-\frac{1}{2}\sum_{j=1}^{[p/4]}\frac{1}{j}. \tag{4}$$

By the classical congruence proved in 1938 by E. Lehmer [9, the congruence (45), p. 358], for each prime $p\ge 5$

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k}\equiv -2q_p(2)+pq_p(2)^2\pmod{p^2}. \tag{5}$$

Substituting the congruence (5) into (4), we obtain

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^{k-1}}{k}\equiv -2q_p(2)+pq_p(2)^2\pmod{p^2}-\frac{1}{2}\sum_{j=1}^{[p/4]}\frac{1}{j}\pmod{p^2}. \tag{6}$$

Finally, substituting (6) into (3), we immediately obtain (2). □

3. The computation

Using the congruence (2), a computation via `Mathematica 8` shows that only three primes less than 10^7 satisfy the condition $E_{p-3}\equiv 0\,(\mathrm{mod}\,p)$ (such primes are 149, 241 and 2946901, and they are given as a sequence A198245 in [14]). Notice also that in 2011 [12, p. 3, Remarks], the author of this article reported that these three primes are only primes less than 3×10^6.

Recall that investigations of such primes have been recently suggested by Z.-W. Sun in [20]; namely, in [20, Remark 1.1] Sun found the first and the second such primes, 149 and 241, and used them to discover curious supercongruences (1.2)–(1.5) from Theorem 1.1 in [20] involving E_{p-3}.

Motivated by search for Wieferich and Fibonacci-Wieferich primes given in [2] and [3] and search for Wolstenholme primes given in [11], here we use similar computational considerations for Euler numbers E_{p-3} where p is a prime. Our computational results presented below suggest two conjectures on numbers E_{p-3} that are analogous to those on Wieferich ([2], [3]) and Wolstenholme primes [11]. Accordingly, we search primes p in the range $[10^5, 5\times 10^6]$ such that $E_{p-3}\equiv A\,(\mathrm{mod}\,p)$ with $|A|\le 100$ and/or $10^4\cdot |A/p|\le 1$. Our search employed the congruence (2) which runs significantly faster than Lehmer's congruence (1) and than the code

```
Print[{Prime[n]},Mod[EulerE[Prime[n]-3],Prime[n]]]
```

Here `EulerE[k]` gives E_k and `Mod[a,m]` gives $a(\mathrm{mod}\,m)$.
in `Mathematica 8`, as well as than some other known congruences involving Euler number E_{p-3}.

Namely, in order to obtain data of Table 1 below concerning primes p with $10^5 < p < 5\times 10^6$ we used the code:

```
Do[If[Max[Min[Mod[(Mod[Numerator[HarmonicNumber[Floor[Prime[n]/4]]],
Prime[n]^2]PowerMod[Denominator[HarmonicNumber [Floor[Prime[n]/4]]],
-1,Prime[n]^2+3*(2^(Prime[n]-1)-1)/Prime[n]-PowerMod[2,-1,Prime[n]^2]
*(3*Prime[n])*((2^(Prime[n]-1)/Prime[n]^2)/((-1)^((Prime[n]+1)/2)
*Prime[n]),Prime[n]],Prime[n]-Mod[(Mod[Numerator[HarmonicNumber
[Floor[Prime[n]/4]]],Prime[n]^]*PowerMod[Denominator[HarmonicNumber
[Floor[Prime[n]/4]]],-1,Prime[n]^2+3*(2^(Prime[n]-1)-1)/Prime[n]
-PowerMod[2,-1,Prime[n]^2]*(3*Prime[n])*((2^(Prime[n]-1)-1) /Prime[n])^2)
/((-1)^((Prime[n]+1)/2)*Prime[n]),Prime[n]]]]==1000, Print[{n},
{Prime[n]},{Mod[(Mod[Numerator[HarmonicNumber[Floor[Prime[n]/4]]],
Prime[n]^2]*PowerMod[Denominator[HarmonicNumber[Floor[Prime[n]/4]]],-1,
Prime[n]^2+3*(2^(Prime[n]-1)-1)/Prime[n]-PowerMod[2,-1,Prime[n]^2]
*(3*Prime[n])*((2^(Prime[n]-1)/Prime[n]^2)/((-1)^((Prime[n]+1)/2)
*Prime[n]),Prime[n]]},{Prime[n]-Mod[(Mod[Numerator[HarmonicNumber
[Floor[Prime[n]/4]]],Prime[n]^2]*PowerMod[Denominator[HarmonicNumber
[Floor[Prime[n]/4]]],-1,Prime[n]^2+3*(2^(Prime[n]-1)-1)/Prime[n]
-PowerMod[2,-1,Prime[n]^2]*(3*Prime[n])*((2^(Prime[n]-1)/Prime[n]^2)
/((-1)^((Prime[n]+1)/2)*Prime[n]),Prime[n]]}]],{n,i,j}]
```

Here `Mod[a,m]` gives $a (\bmod m)$, `PowerMod[a,b,m]` gives $a^b (\bmod m)$ (and is faster than `Mod[a^b,m]`.

Further, in order to verify that there are no primes p between 5×10^6 and 10^7 such that $E_{p-3} \equiv 0 \,(\bmod\, p)$, we used the following code which is very much faster the previous code:

```
Do[Print[{n},{Prime[n]},Mod[Numerator[2*HarmonicNumber[Floor[
Prime[n]/4]]+6*(2^(Prime[n]-1)-1)/Prime[n]-3*(2^(Prime[n]-1)-1)^2
/Prime[n]],Prime[n]^2]],{n,i,j}]
```

Certainly $A = A(p)$ can take any of p values $(\bmod\, p)$. Assuming that A takes these values these values randomly, the "probability" that A takes any particular value (say 0) is $1/p$. From this, in accordance to the heuristic given in [2] related to the Wieferich primes, we might argue that the number of primes p in an interval $[x, y]$ such that $E_{p-3} \equiv 0 \,(\bmod\, p)$ is expected to be

$$\sum_{x\le p\le y} \frac{1}{p} \approx \log \frac{\log y}{\log x}. \tag{7}$$

If this is the case, we would be only expect to find about $0.998529 (\approx 1)$, such primes in the interval $[10^7, 10^{19}]$. On the other hand, since 9999991 is the greatest prime less than 10^7 and is is actually 664589th prime, by above estimate, we find that in the interval $[2, 10^7]$ we can expect about $\sum_{2\le p\le 10^7} 1/p = \sum_{k=1}^{664589} 1/p_k \approx 3.04145$ primes p such that $E_{p-3} \equiv 0 \,(\bmod\, p)$ (p_k is a kth prime); as noticed previously, our computation shows that all these primes are 149, 241 and 2946901.

Table 1. Primes p with $10^5 < p < 5 \times 10^6$ for which $E_{p-3} \equiv A \pmod{p}$ with $|A| \leq 100$ and/or with related values $|A/p| \leq 10^{-4}$ (given in multiples of 10^{-4})

p	A	$\|A/p\|$	p	A	$\|A/p\|$
105829	- 74	> 1	1355269	-60	0.442717
111733	45	> 1	1392323	-29	0.208285
127487	38	> 1	1462421	-78	0.533362
130489	-27	> 1	1546967	-43	0.277963
131617	9	0.683802	1743271	$107 (> 100)$	0.613789
162847	-85	> 1	1794049	$-131 (< -100)$	0.730192
165157	-46	> 1	1808497	$-121 (< -100)$	0.669109
171091	- 17	0.993623	1952131	$-153 (< -100)$	0.783759
171449	7	0.408285	1986539	$-157 (< -100)$	0.790319
191237	37	> 1	2053873	18	0.087639
192961	63	> 1	2114251	$211 (> 100)$	0.997989
200461	7	0.349195	2236349	4	0.017886
209393	27	> 1	2342381	$143 (> 100)$	0.610490
245471	39	> 1	2410627	$-219 (< -100)$	0.908477
246899	-54	> 1	2472731	$230 (> 100)$	0.930146
276371	-69	> 1	2583011	$159 (> 100)$	0.615561
290347	10	0.344415	2619847	$224 (> 100)$	0.855011
292183	53	> 1	2740421	225	0.821042
306739	-42	> 1	2890127	-34	0.117642
317263	-35	> 1	**2946901**	**0**	**0**
321509	84	> 1	3279833	$-111 (< -100)$	0.338432
342569	25	0.729780	3290689	$200 (> 100)$	0.607775
422789	-40	0.946098	3312653	$228 (> 100)$	0.688270
429397	-62	> 1	3340277	$226 (> 100)$	0.676591
440047	82	> 1	3355813	$116 (> 100)$	0.345669
479561	31	0.646425	3652613	$-290 (< -100)$	0.793952
501317	60	> 1	3818131	$-318 (< -100)$	0.832868
546631	92	> 1	3852677	75	0.194670
628301	73	> 1	3960377	-48	0.121201
636137	25	0.392997	4007747	$190 (> 100)$	0.474082
656147	-68	> 1	4121503	$-270 (< -100)$	0.655101
659171	-22	0.333753	4171229	$153 (> 100)$	0.366798
687403	-4	0.058190	4343659	$-252 (< -100)$	0.580156
717667	-42	0.585230	4392007	55	0.125227
719947	53	0.736165	4418497	70	0.158425
766261	-8	0.104403	4475707	$193 (> 100)$	0.431217
801709	53	0.661088	4541501	$120 (> 100)$	0.264230
920921	-82	0.890413	4551973	$-362 (-100)$	0.795260
924727	-8	0.086512	4564939	-63	0.138008
1064477	$106 (> 100)$	0.995794	4631399	$367 (> 100)$	0.792417
1080091	42	0.388856	4674347	$302 (> 100)$	0.646080
1159339	-38	0.327773	4706047	$220 (> 100)$	0.467484
1202843	21	0.174586	4751599	$-279 (< -100)$	0.587171
1228691	15	0.122081	4761677	$200 (> 100)$	0.420020
1285301	47	0.365673	4869517	-100	0.205359
1336469	-5	0.037412	4898099	$-236 (< -100)$	0.481820
1353281	78	0.576377	4928503	$-173 (< -100)$	0.351019

The second column of Table 1 shows that there are 61 primes between 10^5 and 5×10^6 for which $|A|\le 100$. Since the "probability" that $|A|\le 100$ for a prime $p\gg 200$ is equal to $201/p$, it follows that expected number of such primes between Mth prime p_M and Nth prime p_N with $N>M\gg 1000$ (that is, $p_N>p_M\gg 1000$) is equal to

$$Q(N,M,100)=201\sum_{p_M<p<p_N}\frac{1}{p}, \tag{8}$$

where the summation ranges over all primes p such that $p_M<p<p_N$. In particular, for the values $M=9593$ and $N=348513$ which correspond to the interval $[10^5,5\times 10^6]$ containing all primes from Table 1, we have

$$Q(348513,9593,100)=201\sum_{10^5<p<5\times 10^6}\frac{1}{p}\approx 201\cdot 0.292251=58.742451. \tag{9}$$

On the other hand, Table 1 shows that there are 61 primes between 10^5 and 5×10^6 for which $|A|\le 100$, which is $\approx 3.8431\%$ greater than related "expected number" 58.742451.

Because our program recorder all p with "small $|A|$", that is, with $|A|\le 100$, we compiled a large data set which can be used to give more rigorous (experimental) confirmation of both our Conjectures 1 and 2. Indeed, our program recorded 568 primes p in the interval $[10^5,5\times 10^6]$ for which $|A|\le 1000$. On the other hand, according to the formula (9), it follows that expected number of such primes is equal to

$$Q(348513,9593,1000)=2001\sum_{10^5<p<5\times 10^6}\frac{1}{p}\approx 2001\cdot 0.292251=584.794251 \tag{10}$$

which is $\approx 2.956\%$ greater than related "expected number" 568.

Instead, of selecting values based on $|A|\le 100$, we suggest to select them based on $A/p<q\times 10^{-4}$ (e.g., $q=1$) that would be consistent with the original selection criterion. In particular, in the third column of Table 1 there are 72 primes p contained in the interval $[10^5,5\times 10^6$ with related values $10^4\times A/p<1$.

Furthermore, since the "probability" that $|A/p|\le 10^{-4}$ for a prime $p\gg 10000$ is equal to

$$\frac{2\left[\frac{p}{10000}\right]+1}{p}\approx\frac{2}{10000},$$

it follows that expected number of such primes between Mth prime p_M and Nth prime p_N with $N>M\gg 1000$ (that is, $p_N>p_M\gg 10000$) is equal to

$$P(N,M)=\frac{2(N-M)}{10000}.$$

In particular, for the values $M=9593$ and $N=348513$ which correspond to the range $(10^5,5\times 10^6)$ of all primes from Table 1, we have

$$P(348513,9593)=\frac{677840}{10000}=67.7840$$

which is $\approx 5.855\%$ less than 72.

All the previous considerations and the well known fact that the series

$$\sum_{p\ \mathrm{prime}}\frac{1}{p}$$

diverges suggest the following conjecture.

Conjecture 1. There are infinitely many primes p such that $E_{p-3} \equiv 0 \pmod{p}$.

Since

$$\sum_{x \le p \le y} \frac{1}{p} \approx \log\log x - \log\log y,$$

in view of the previous comparison of our computational results with expected number of primes $p \in [10^5, 5 \times 10^6]$ for which $|A(p)| \le 100$ given by (9) (or primes $p \in [10^5, 5 \times 10^6]$ for which $|A(p)| \le 1000$ given by (10)), we can assume that expected number of primes p in an interval $[x, y]$ such that $K \le |A(p)| \le L$ is asymptotically equal to (cf. (7))

$$2(L - K) \cdot (\log\log b - \log\log a). \tag{11}$$

Using a larger data set which our program recorded, consisting of total 568 pairs $(p, A(p))$ such that $p \in [10^5, 5 \times 10^6]$ and $|A(p)| \le 1000$, we obtain experimental results presented in Table 2. In Table 2 the values in "column k" and in first and second row reflect the number of $p \in [10^5, 10^6]$ and $p \in [10^6, 5 \times 10^6]$, respectively, such that $A = A(p) \in [k \times 100, (k+1) \times 100]$ ($k = 0, 1, \ldots 9$). Expected numbers given in the last column of Table 2 are calculated by the formula (11).

Table 2.

	k										
Interval	0	1	2	3	4	5	6	7	8	9	Expected
$[10^5, 10^6]$	42	51	37	30	29	24	31	34	42	44	36.464
$[10^6, 5 \times 10^6]$	22	23	26	20	22	22	21	24	21	20	22.039

Table 2 presents a small snapshot of our experimental results. Notice that by the data of the last row, the relative error between the conjectured and experimental values for $k = 0, 1, \ldots, 9$ are respectively equal to 0.18%, 4.18%, 15.23%, 10.20%, 0.18%, 0.18%, 4.95%, 8.17%, 4.95%, 10.20%. Accordingly, we propose the following conjecture (cf. the same conjecture in [3, Conjecture 6.1] concerning the Wieferich primes; see also [2, Section 3]).

Conjecture 2. The number of primes $p \in [a, b]$ such that $|A| = |A(p)| \in [K, L]$ is *asymptotically*

$$2(L - K) \cdot (\log\log b - \log\log a).$$

Remarks. Recall that a prime p is said to be a *Wolstenholme prime* if it satisfies the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4},$$

or equivalently (cf. [10, Corollary on page 386]; also see [6]) that p divides the numerator of B_{p-3}. The only two known such primes are 16843 and 2124679, and by a result of R.J. McIntosh and E.L. Roettger from [11, pp. 2092–2093], these primes are the only two Wolstenholme primes less than 10^9. Nevertheless, using similar arguments to those given in Section 3 of this paper, McIntosh [10, page 387] conjectured that there are infinitely many Wolstenholme primes.

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