

# Directed and multi-directed animals on the square lattice with next nearest neighbour edges

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December 11, 2013

## 1 Introduction

An *animal* on a lattice is a finite and connected set of vertices. The enumeration of animals (up to a translation) is a longstanding problem in statistical physics and combinatorics. The problem, however, is extremely difficult, and little progress has been made [13, 11]. A more realistic goal, therefore, is to enumerate natural subclasses of animals.

The class of *directed animals* is one of the most classical of these subclasses. Let  $\mathcal{L}$  be a directed lattice and let  $s$  be a vertex of  $\mathcal{L}$ . A finite set  $A$  of vertices of  $\mathcal{L}$  is a directed animal of source  $s$  if, for every vertex  $t$  in  $A$ , there exists a directed path from  $s$  to  $t$  visiting only vertices in  $A$ . The vertices in  $A$  are called the *sites* of  $A$ ; the *area* of  $A$  is its number of sites. Directed animals have been enumerated in a variety of lattices; let us cite, non-exhaustively, the square and triangular lattices [8, 10, 2], the lattices  $\mathcal{L}_n$  for  $n \geq 1$  [4, 7], and the “strange” or  $n$ -decorated lattices [6, 3] (Figure 1). Unsolved lattices include, most notably, the honeycomb lattice [12].

The class of *multi-directed animals* is a superclass of directed animals introduced by Bousquet-Mélou and Rechnitzer [5] on the directed square and triangular lattices; this was based on initial work by Klarner [14]. However, their definition is not as nice as that of the directed animals.

The goal of this paper is to enumerate directed and multi-directed on a new lattice. We denote by  $\mathcal{N}$  the directed lattice with vertices  $\mathbb{Z}^2$  and arcs  $\{\leftarrow, \nearrow, \uparrow, \searrow, \rightarrow\}$  (Figure 1, below right). In physicists’ terms, this is the square lattice with added next nearest neighbor bonds. It can be seen as a generalization of Bousquet-Mélou and Conway’s lattice  $\mathcal{L}_3$  (which has arcs  $\{\searrow, \uparrow, \nearrow\}$ ) [4].

Several techniques have been used to enumerate directed animals on the various lattices. Among them are direct bijections with other combinatorial objects [10], comparison with gas models [8, 3, 15, 1] and the use of Viennot’s theory of heaps of pieces [17, 2, 7, 5, 18]. In this paper, we use the last method and define heaps of polymers, which we show to be in bijection with directed animals on the lattice  $\mathcal{N}$ .

The paper is organised as follows. In Section 2, we define heaps of polymers and show their links with directed and multi-directed animals on the lattice  $\mathcal{N}$ . In Section 3, we use these objects to enumerate directed animals and derive asymptotic results. Finally, in Section 4, we enumerate multi-directed animals;

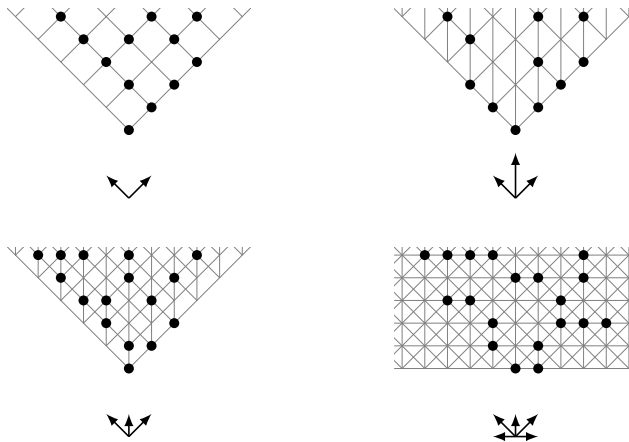


Figure 1: Directed animals on a selection of lattices. Above left: the square lattice. Above right: the triangular lattice. Below left: the lattice  $\mathcal{L}_3$ . Below right: the lattice  $\mathcal{N}$ .

we also give asymptotic enumeration and show that the generating function of multi-directed animals is not D-finite.

## 2 Animals on the lattice $\mathcal{N}$ and heaps of polymers

*Remark.* In the introduction, we defined directed animals as a set of sites of a lattice. In this section, for legibility's sake, we choose to represent directed animals as polyominoes, that is, as sets of cells of the dual square lattice (Figures 3 and 4).

### 2.1 Definitions

Let  $A$  be an animal on the lattice  $\mathcal{N}$  defined in the introduction. We call *segment* of  $A$  a maximal set of horizontally consecutive sites. The segments of the animal  $A$  are the strongly connected components of  $A$  in the lattice  $\mathcal{N}$ . As such, they play a prominent role in our study.

We now define heaps of polymers, which play the same role in the lattice  $\mathcal{N}$  as heaps of dimers do in the square and triangular lattices [17, 2].

**Definition 1.** We call *polymer* a closed real interval of the form  $[i, j]$ , where  $i$  and  $j$  are integers such that  $j > i$ . The integer  $j - i$  is the *length* of the polymer. Two polymers are called *concurrent* if they intersect, even by a point.

A *heap of polymers* is a finite sequence of polymers, considered up to commutation of non-concurrent polymers. The *total length* of a heap is the sum of the lengths of the polymers composing it.

More information on heaps of pieces in general can be found in [17]. We recall below some elementary definitions that we use in this paper.

Let  $H$  be a heap of polymers and  $\alpha$  be a polymer of  $H$ . The polymer  $\alpha$  is *minimal* if  $H$  can be written  $\alpha H'$ ; it is *maximal* if  $H$  can be written  $H'\alpha$ .

Graphically, a heap is built by dropping polymers in succession; a polymer either falls on the ground or on another polymer concurrent to it (Figure 2). The polymers that lie on the ground are exactly the minimal polymers of the heap.

Finally, the set of heaps of polymers is equipped with a product. Let  $H_1$  and  $H_2$  be two heaps. The product  $H_1 H_2$  is obtained by dropping  $H_2$  on top of  $H_1$ . Let  $H$  be a heap and  $\alpha$  a polymer of  $H$ ; there exists a unique factorisation  $H = H_1 H_2$ , where  $H_2$  is a pyramid with minimal polymer  $\alpha$ . We call it the factorisation obtained by *pushing*  $\alpha$ . Both operations are illustrated in Figure 2.

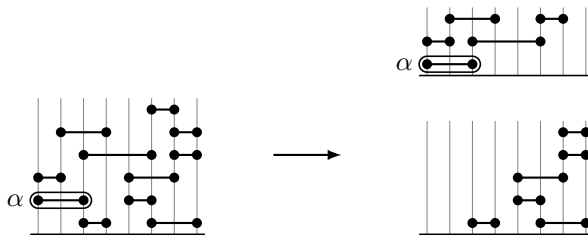


Figure 2: Left: a heap of polymers with a distinguished polymer  $\alpha$ . Right: the two heaps obtained by pushing the polymer  $\alpha$ . The product of these two heaps is equal to the heap on the left.

**Definition 2.** Let  $A$  be an animal on the lattice  $\mathcal{N}$ . If  $B$  is a segment of  $A$ , say  $B = \{(i, k), \dots, (j-1, k)\}$ , we call *projection* of the segment  $B$  the polymer  $[i, j]$  and *height* of  $B$  the integer  $k$ . We call *projection* of  $A$ , and we denote by  $\pi(A)$ , the heap built as the sequence of the projections of all segments of  $A$  in increasing height order.

Examples are shown in Figures 3 and 4.

As the segments of  $A$  are the maximal sets of adjacent sites, the above definition entails that the projections of two segments of  $A$  at the same height cannot be concurrent. The projections of segments at the same height thus commute, which ensures that the projection  $\pi(A)$  is well-defined. Moreover, the projection of a segment composed of  $\ell$  sites is a segment of length  $\ell$ . This means that the area of  $A$  is equal to the total length of the heap  $\pi(A)$ .

## 2.2 Directed animals and pyramids of polymers

Let  $A$  be an animal; we say that a site  $t$  of  $A$  is *connected* to another site  $s$  if there exists a directed path from  $s$  to  $t$  visiting only sites of  $A$ . In particular, all the sites of a given segment are connected to each other.

We say that an animal  $A$  is *directed* if it is directed of source  $s$ , for some site  $s$  of  $A$ . The source  $s$  is not unique; it may be any site of the bottommost segment of  $A$  (see Figure 3, left). By convention, we call *source* of  $A$  the leftmost site of the bottommost segment.

A heap of polymers is called a *pyramid* if it has a unique minimal polymer. Definition 2 implies that two segments that are connected in the lattice  $\mathcal{N}$  have

concurrent projections; this entails that the restriction of the projection  $\pi$  is a bijection between directed animals and pyramids of polymers (Figure 3). This bijection works identically to the classical bijection between directed animals on the square lattice and strict pyramids of dimers [17, 2]. We use this bijection to enumerate directed animals in Section 3.

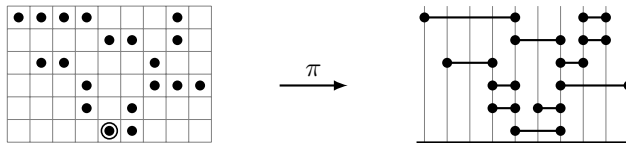


Figure 3: Left: a directed animal on the lattice  $\mathcal{N}$  with its source circled. Right: its projection, obtained by replacing each segment consisting of  $\ell$  sites by a polymer of length  $\ell$ . The projection has a unique minimal polymer, *i.e.* is a pyramid; moreover, replacing back every polymer by a segment recovers the animal.

### 2.3 Multi-directed animals and connected heaps of polymers

We now define another class of animals, called multi-directed animals, which generalize directed animals. This is inspired by the work of Bousquet-Mélou and Rechnitzer [5] on the directed square and triangular lattices, but our definition is slightly different. This is discussed further below.

Let  $A$  be an animal. For any abscissa  $i$ , we denote by  $b(i)$  the ordinate of the bottommost site of  $A$  at abscissa  $i$  (or  $b(i) = +\infty$  if there is no site of  $A$  at abscissa  $i$ ). We call *source* of  $A$  a site that realizes a local minimum of  $b$  and *keystone* of  $A$  a site that realizes a local maximum. By convention, in case several consecutive sites realize a minimum or maximum, the source or keystone is the leftmost one (Figure 4, left). Like the choice of the source of a directed animal, this is a purely conventional choice that does not alter the definition below.

**Definition 3.** Let  $A$  be an animal. The animal  $A$  is said *multi-directed* if it satisfies the two conditions:

- for every site  $t$  of  $A$ , there exists a source  $s$  such that  $t$  is connected to  $s$ ;
- for every keystone  $t$  of  $A$ , there exists two sources  $s_\ell$  and  $s_r$ , to the left and to the right of  $t$  respectively, such that  $t$  is connected to both  $s_\ell$  and  $s_r$ . Moreover, the directed paths connecting  $t$  to  $s_\ell$  and  $s_r$  do not go through a keystone at the same height as  $t$ .

As a directed animal has only one source and no keystone, every directed animal is multi-directed. A multi-directed animal is shown in Figure 4 (left).

We now describe a bijection between multi-directed animals and certain heaps of polymers. We say that a heap is *connected* if its projection to the  $x$ -axis is a real interval, like the one in Figure 4 (right).

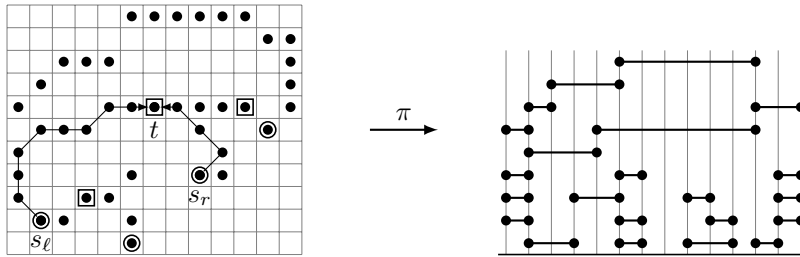


Figure 4: Left: a multi-directed animal with four sources (circled) and three keystones (boxed). The paths connecting one keystone, denoted by  $t$ , to the sources  $s_\ell$  and  $s_r$  are shown. Right: the projection of the animal, which is a connected heap of polymers with four minimal pieces.

**Proposition 4.** *Let  $n \geq 1$ . The projection  $\pi$  induces a bijection between multi-directed animals of area  $n$  and connected heaps of polymers of total length  $n$ , both taken up to a translation.*

*Proof.* Let  $A$  be a multi-directed animal. Since  $A$  is an animal, the projection  $\pi(A)$  is a connected heap (Figure 4). We therefore need to prove that for any connected heap  $H$ , there exists a unique multi-directed animal  $A$  such that  $\pi(A) = H$ .

Let us call *pre-animal* a finite set of sites with a connected projection. We define the sources and keystones of a pre-animal in the same manner as for an animal. A pre-animal is called *multi-directed* if it satisfies the conditions of Definition 3.

We prove by induction the following two statements: for every connected heap  $H$ , there exists a unique (up to a vertical translation) multi-directed pre-animal  $A$  such that  $\pi(A) = H$ ; moreover, the pre-animal  $A$  is an animal.

Let  $H$  be a connected heap. If  $H$  is reduced to a single polymer, the result is obvious. Otherwise, let  $\beta$  be a maximal polymer of  $H$ . Write  $H = H'\beta$  and let  $H_1, \dots, H_k$  be the connected components of the heap  $H'$ , from left to right. Assume that  $A$  is a multi-directed pre-animal such that  $\pi(A) = H$ . The pre-animal  $A$  is thus composed of a segment  $B$  such that  $\pi(B) = \beta$  and pre-animals  $A_1, \dots, A_k$ , the respective projections of which are  $H_1, \dots, H_k$ .

As the polymer  $\beta$  is maximal, no directed path visiting a site of  $B$  can reach a site not in  $B$ . Moreover, all the sources and keystones of the pre-animals  $A_1, \dots, A_k$  are also sources and keystones of the pre-animal  $A$ . As  $A$  is multi-directed, this forces all the  $A_i$ 's to be multi-directed. By the induction hypothesis, all the  $A_i$ 's are thus uniquely determined up to a vertical translation and are multi-directed animals. We now distinguish two cases.

- We have  $k = 1$ . By the first condition of Definition 3, the segment  $B$  is connected to a source of  $A_1$ . This forces  $A_1$  to touch  $B$  and uniquely determines the pre-animal  $A$ ; moreover, this means that  $A$  is an animal.
- We have  $k \geq 2$ . In this case, the segment  $B$  contains  $k - 1$  keystones, located between the animals  $A_i$  and  $A_{i+1}$  for  $i = 1, \dots, k - 1$ . By the second condition of Definition 3, every animal  $A_i$ , for  $i = 1, \dots, k$ , contains a source  $s_i$  such that  $B$  is connected to  $s_i$ . This forces the animals  $A_i$  to

touch the segment  $B$ . The pre-animal  $A$  is thus uniquely determined and is an animal.

The proof is illustrated in Figure 5. □

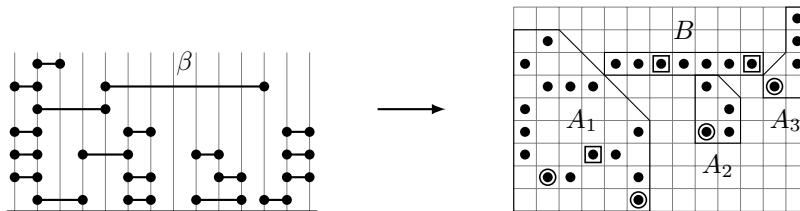


Figure 5: Left: a connected heap of polymers  $H$  with one distinguished maximal polymer  $\beta$ . Right: the multi-directed animal  $A$  such that  $\pi(A) = H$ . It is built by recursively building the animals  $A_1, A_2$  and  $A_3$  and vertically translate them so that they touch the segment  $B$ .

The definition of multi-directed animals can be used in the directed square and triangular lattices; moreover, the proof of Proposition 4 also works on connected heaps of dimers. Since Bousquet-Mélou and Rechnitzer's multi-directed animals are also in bijection with connected heaps [5], this means that they have the same enumeration. Our definition of multi-directed animals has the advantage of being more intrinsic; moreover, it has a vertical symmetry, which was not the case with Bousquet-Mélou and Rechnitzer's.

### 3 Enumeration of directed animals

#### 3.1 Exact enumeration

**Definition 5.** Let  $A$  be a directed animal. We say that  $A$  has *left half-width*  $i$ , and we write  $\text{lw}(A) = i$ , if the leftmost sites of  $A$  are at the abscissa  $-i$ .

A directed animal of left half-width 0 is called a *half-animal*. A pyramid of polymers is called a *half-pyramid* if it has no polymers to the left of its minimal polymer.

This definition is based on the convention that the leftmost site of the bottom segment of  $A$  has abscissa 0 (see above).

**Theorem 6.** *The generating function of half-animals satisfies the equation:*

$$S(t) = t(1 + 2S(t))(1 + S(t)). \quad (1)$$

*The generating function of directed animals taking into account the left half-width is:*

$$D(t, u) = S(t) + \frac{uS(t)^2}{1 - uR(t)}, \quad (2)$$

where the series  $R(t)$  is:

$$R(t) = S(t) + t(1 + S(t)). \quad (3)$$

In particular, the generating functions of half-animals and animals without regard for the left half-width are:

$$S(t) = \frac{1 - 3t - \sqrt{1 - 6t + t^2}}{4t}; \quad (4)$$

$$D(t, 1) = \frac{1}{4} \left( \frac{1+t}{\sqrt{1-6t+t^2}} - 1 \right). \quad (5)$$

Interestingly, the generating function  $S(t)$  is already known in combinatorics. Its coefficients are the little Schröder numbers (**A001003** in the OEIS [16]). This phenomenon is remindful of the triangular lattice, where the half-animals are enumerated by the Catalan numbers [2].

*Proof.* To prove the theorem, we use techniques similar to those of Bétréma and Penaud [2] to decompose half-pyramids and pyramids of polymers.

We decompose half-pyramids as shown in Figure 6. Let  $P$  be a half-pyramid and  $\alpha$  its minimal polymer. Let  $\beta$  be the lowest polymer in the column  $[0, 1]$  (i.e. such that  $[0, 1] \subseteq \beta$ ), apart from  $\alpha$ , if such a polymer exists. By pushing the polymer  $\beta$ , we get the factorisation  $P = \alpha P_1 P_2$ , where  $P_2$  is a pyramid of minimal polymer  $\beta$  (or the empty heap if  $\beta$  does not exist). By construction,  $P_2$  is either empty or a half-pyramid; moreover, the heap  $P_1$  has no polymers in the column  $[0, 1]$ .

We distinguish two cases.

1. The polymer  $\alpha$  has length 1. In this case, the heap  $P_1$  is a half-pyramid.
2. The polymer  $\alpha$  has length greater than 1. Write:

$$\alpha = [0, 1] \cup \alpha'$$

and let  $P'_1 = \alpha' P_1$ . The heap  $P'_1$  is a half-pyramid.

This decomposition yields the equation:

$$S(t) = t(1 + S(t))^2 + tS(t)(1 + S(t)),$$

which is equivalent to (1).

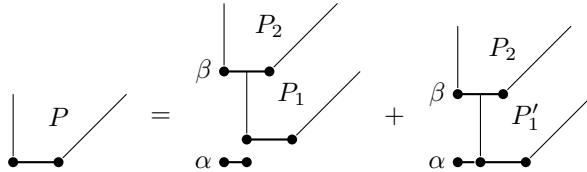


Figure 6: The two cases in the decomposition of half-pyramids. The generating function of the possible heaps  $P_1$  and  $P_2$  is  $1 + S(t)$ , while the generating function of the possible heaps  $P'_1$  is  $S(t)$ .

We decompose the general pyramids in a similar manner, shown in Figure 7. Let  $P$  be a pyramid with left half-width  $i$ . Let  $\alpha$  be the lowest polymer of  $P$  in the column  $[-i, -i + 1]$ .

1. The polymer  $\alpha$  is the minimal polymer of  $P$ , *i.e.*,  $P$  is a half-pyramid.
2. The polymer  $\alpha$  is not the lowest polymer in the column  $[-i + 1, -i + 2]$ . Let  $P = P_1P_2$  be the factorisation obtained by pushing  $\alpha$ ; then  $P_1$  is a pyramid of left half-width  $i - 1$  and  $P_2$  is a half-pyramid.
3. The polymer  $\alpha$  is not the minimal polymer of  $P$  but is the lowest polymer in the column  $[-i + 1, -i + 2]$ . In this case, it is of the form:

$$\alpha = [-i, -i + 1] \cup \alpha'.$$

Let  $P'$  be the pyramid  $P$  in which the polymer  $\alpha$  is replaced by  $\alpha'$ ; let  $\beta$  be the lowest polymer of  $P'$  in the column  $[-i, -i + 1]$ , if such a polymer exists, and let  $P' = P'_1P'_2$  be the factorisation obtained by pushing the polymer  $\beta$  (if  $\beta$  does not exist,  $P'_2$  is the empty heap).

By construction, the pyramid  $P'_1$  has left half-width  $i - 1$  and  $\alpha'$  is its lowest polymer in the column  $[-i + 1, -i + 2]$ . Therefore, the pyramid  $P$  can be recovered from the pyramid  $P'_1$  and the possibly empty half-pyramid  $P'_2$ .

In Case 2 above, the heap  $P_1$  may be any pyramid. However, in Case 3, we must ensure that the polymer  $\alpha$  is not the minimal polymer of  $P$ , which means that  $\alpha'$  is not the minimal polymer of  $P'_1$ . In other terms,  $P'_1$  must not be a half-pyramid. This decomposition thus yields the equation:

$$D(t, u) = S(t) + uD(t, u)S(t) + u(D(t, u) - S(t))(1 + S(t)),$$

which boils down to (2).

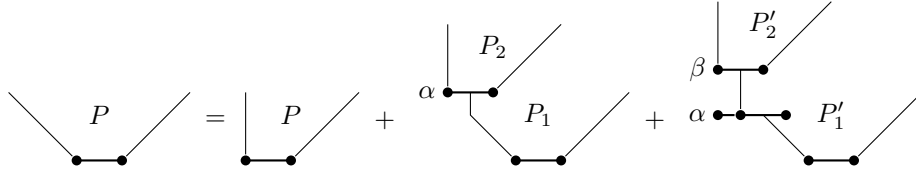


Figure 7: The three cases in the decomposition of pyramids. The generating function of the possible heaps  $P_1$  is  $D(t)$ , that of the possible heaps  $P_2$  is  $S(t)$ , that of the possible heaps  $P'_1$  is  $D(t) - S(t)$ , and that of the possible heaps  $P'_2$  is  $1 + S(t)$ .

Finally, solving the equations (1) and (2) yields the values (4) and (5).  $\square$

### 3.2 Asymptotics

**Proposition 7.** *Let  $n \geq 1$ . Let  $d(n)$  be the number of directed animals of area  $n$  and let  $\text{lw}(n)$  be the average left half-width of the same animals. As  $n$  tends to infinity, we have the following estimates:*

$$d(n) \sim 2^{-7/4} \frac{(3 + \sqrt{8})^n}{\sqrt{\pi n}};$$

$$\text{lw}(n) \sim 2^{-3/4} \sqrt{\pi n}.$$



In this result, we see that the numbers  $d(n)$  and  $\text{lw}(n)$  behave like their counterparts in the square and triangular lattices (in those cases, the number  $d(n)$  is asymptotically of the form  $\mu^n/\sqrt{n}$  and the number  $\text{lw}(n)$  is of the order of  $\sqrt{n}$ ; see [10]).

*Proof.* The exact values  $d(n)$  and  $\text{lw}(n)$  are obtained from the generating function  $D(t, u)$  as follows:

$$\begin{aligned} d(n) &= [t^n]D(t, 1); \\ \text{lw}(n) &= \frac{1}{d(n)}[t^n]\frac{\partial D}{\partial u}(t, 1). \end{aligned}$$

Both generating functions have radius of convergence  $\rho = 3 - \sqrt{8}$ . They have two singularities, at  $t = 3 \pm \sqrt{8}$ . They admit an analytic continuation in  $\mathbb{C} \setminus [3 - \sqrt{8}, 3 + \sqrt{8}]$ . As  $t$  tends to  $\rho$ , we have the following estimates:

$$\begin{aligned} D(t, 1) &= \frac{2^{-7/4}}{\sqrt{1-t/\rho}} + O(1); \\ \frac{\partial D}{\partial u}(t, 1) &= \frac{2^{-5/2}}{1-t/\rho} + O\left(\frac{1}{\sqrt{1-t/\rho}}\right). \end{aligned}$$

The results are then a consequence of classical singularity analysis [9, Theorem VI.4].  $\square$

## 4 Enumeration of multi-directed animals

We now turn our attention to the enumeration of multi-directed animals. By Proposition 4, this is equivalent to the enumeration of connected heaps of polymers.

### 4.1 Nordic decomposition

The Nordic decomposition was invented by Viennot to enumerate connected heaps of dimers, themselves linked to multi-directed animals in the triangular and square lattices [18]. We adapt this technique to connected heaps of polymers.

Let  $C$  be a connected heap of polymers that is not a pyramid. Let  $\alpha$  be the rightmost minimal polymer of  $C$ ; let  $C = C'P$  be the factorization obtained by pushing  $\alpha$ . Since  $C'$  might not be connected, let  $C' = C_1 \cdots C_n$  be the decomposition of  $C'$  in connected components, from left to right; let  $H$  be the heap  $C_2 \cdots C_n$ .

As the heap  $C$  is known up to a translation, we assume that the rightmost column of the heap  $C_1$  is  $[-2, -1]$ . This assumption fixes the polymer  $\alpha$ , say  $\alpha = [k, j]$ . We call *Nordic decomposition* of the heap  $C$  the quadruple  $(C_1, k, H, P)$ . This decomposition is illustrated in Figure 8.

**Proposition 8.** *The Nordic decomposition is a bijection between non-pyramid connected heaps and quadruples of the form  $(C_1, k, H, P)$  such that:*

- $C_1$  is a connected heap, taken up to a translation;

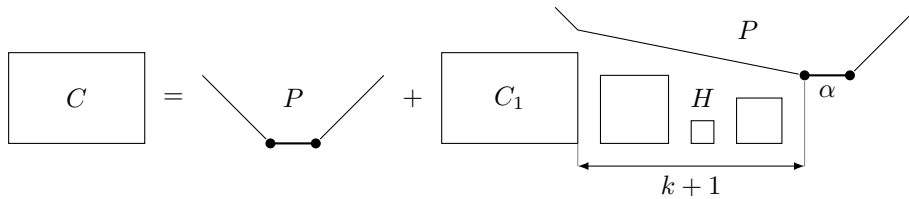


Figure 8: The Nordic decomposition of a non-pyramid connected heap: pushing the rightmost minimal polymer  $\alpha$  yields the pyramid  $P$ . The heap  $C_1$  is the leftmost connected component of the remaining heap. The other components compose the heap  $H$ , which lives in the gap of width  $k + 1$  between the heap  $C_1$  and the polymer  $\alpha$ .

- $k$  is a non-negative integer;
- $H$  is a heap the segments of which are included in  $[0, k - 1]$ ;
- $P$  is a pyramid with left half-width greater than  $k$ , taken up to a translation.

*Proof.* First, let  $C$  be a non-pyramid connected heap and let  $\mathcal{N} = (C_1, k, H, P)$  be its Nordic decomposition. We first show that the quadruple  $\mathcal{N}$  satisfies the conditions of the lemma. The heap  $C_1$  is connected by definition. Moreover, as the component  $C_1$  is not concurrent to  $\alpha$ , we have  $k \geq 0$ . Furthermore, as the components  $C_2, \dots, C_n$  are concurrent neither to  $C_1$  nor to  $\alpha$ , all the segments of the heap  $H$  are included in  $[0, k - 1]$ . Finally, for the heap  $C$  to be connected, one polymer of the pyramid  $P$  must be in the column  $[-1, 0]$ ; this shows that  $P$  has left half-width at least  $k + 1$ .

To conclude, we show that it is possible to recover the heap  $C$  from its Nordic decomposition  $\mathcal{N}$ . To do that, we first translate the heap  $C_1$  so that its rightmost column is  $[-2, -1]$ . Then, we translate the pyramid  $P$  so that its minimal polymer is of the form  $[k, j]$ . The heap  $C$  is then equal to the product  $C_1HP$ .  $\square$

## 4.2 Enumeration

We are now ready to enumerate multi-directed animals.

**Theorem 9.** *Let  $M = M(t)$  be the generating function of multi-directed animals. Let  $S = S(t)$ ,  $D = D(t, 1)$  and  $R = R(t)$  be the power series defined by (1), (2) and (3), respectively. Moreover, let  $Q = Q(t)$  be the power series defined by:*

$$Q(t) = (2 - 2t)S(t) - t. \quad (6)$$

*The generating function  $M$  is given by:*

$$M = \frac{D}{1 - \sum_{k \geq 0} S(1 + S)^k \frac{QR^k}{1 - QR^k}}. \quad (7)$$

This theorem is proved further below. First, we need to establish the following lemma, which enumerates the objects appearing in the Nordic decomposition.

**Lemma 10.** *Let  $k \geq 0$ ; let  $H_k(t)$  be the generating function of heaps of polymers included in  $[0, k - 1]$  and let  $D_{>k}(t)$  be the generating function of pyramids of polymers of left half-width more than  $k$ . We have the following identity:*

$$H_k(t)D_{>k}(t) = S(1 + S)^k \frac{QR^k}{1 - QR^k}.$$

*Proof.* Our proof is inspired by a similar result by Viennot on heaps of dimers [18]. Let  $\mathcal{A}_k$  be the set of heaps of polymers  $A$  satisfying the conditions:

- every polymer of  $A$  is of the form  $[i, j]$  with  $i \geq 0$ ;
- the rightmost minimal polymer of  $A$  is of the form  $[k, j]$ .

Let  $A_k(t)$  be the generating function of the set  $\mathcal{A}_k$ . Let  $A$  be a heap of  $\mathcal{A}_k$ . We decompose the heap  $A$  in two ways.

First, let  $\alpha$  be the lowest polymer in the column  $[0, 1]$ , if such a polymer exists. Pushing the polymer  $\alpha$  yields a possibly empty half-pyramid. We repeat the process in the columns  $[1, 2], \dots, [k - 1, k]$ ; in total, we get  $k$  possibly empty half-pyramids. Finally, since the existence of a minimal polymer of the form  $[k, j]$  is guaranteed, what remains is a half-pyramid. This decomposition yields the identity:

$$A_k(t) = S(t)(1 + S(t))^k.$$

Second, let  $\alpha$  be the rightmost minimal polymer of  $A$ . Pushing  $\alpha$  yields a pyramid with left half-width at most  $k$ ; moreover, what remains is a heap with all its polymers included in  $[0, k - 1]$ . We deduce this second identity:

$$A_k(t) = H_k(t)(D(t) - D_{>k}(t)).$$

Equating the two above expressions for  $A_k(t)$ , we find the identity:

$$H_k(t)D_{>k}(t) = S(t)(1 + S(t))^k \frac{D_{>k}(t)}{D(t) - D_{>k}(t)}.$$

Finally, the identity (2) shows that the generating function of pyramids with left half-width  $j$  is  $D_j(t) = S(t)^2 R(t)^{j-1}$  as soon as  $j \geq 1$ . Therefore, we have:

$$D_{>k}(t) = S(t)^2 \frac{R(t)^k}{1 - R(t)}.$$

An elementary computation from the definitions of  $D$ ,  $R$ , and  $Q$  (Equations (2), (3) and (6)) lets us rewrite this into:

$$D_{>k}(t) = D(t)Q(t)R(t)^k.$$

Together with the above identity, this establishes the lemma.  $\square$

This lemma is illustrated in Figure 9.

*Proof of Theorem 9.* With the notations of Lemma 10, we use Proposition 8 to write the functional equation:

$$M(t) = D(t) + \sum_{k \geq 0} M(t)H_k(t)D_{>k}(t),$$

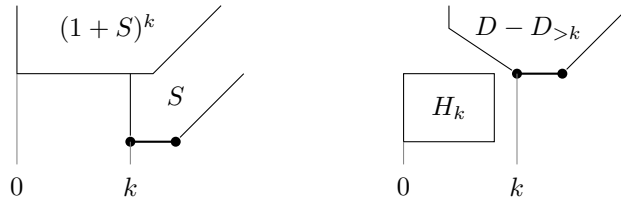


Figure 9: The two decompositions of the objects of  $\mathcal{A}_k$  involved in the proof of Lemma 12.

which is equivalent to

$$M(t) = \frac{D(t)}{1 - \sum_{k \geq 0} H_k(t) D_{>k}(t)}.$$

We conclude using Lemma 10.  $\square$

### 4.3 Nature of the series and asymptotics

We now derive asymptotic results from the formula of Theorem 9; we also prove that the generating function  $M$  of multi-directed animals is not D-finite. The last property is shared by its counterparts on the square and triangular lattices [5].

Let  $\rho = 3 - \sqrt{8}$  be the radius of convergence of the generating functions  $S$ ,  $Q$ ,  $R$  found in Section 3.2 and let  $\bar{\rho} = 3 + \sqrt{8}$ . Let  $\mathcal{B}$  be the curve in the complex plane consisting of the unit circle and the segment  $[\rho, \bar{\rho}]$ . Let  $\mathcal{D}$  be the domain interior to the curve  $\mathcal{B}$ .

**Theorem 11.** *The series  $M(t)$  admits a meromorphic continuation in the domain  $\mathcal{D}$ , with a unique dominant simple pole at  $\rho_M = 0.154\dots$ . Moreover, the series  $M(t)$  is not D-finite.*

To prove the theorem, we need the following two lemmas that deal with analytic properties of the generating functions occurring in  $M(t)$ .

**Lemma 12.** *For  $z$  in the domain  $\mathcal{D}$ , the values  $Q(z)$  and  $R(z)$  have modulus less than 1.*

*Proof.* We start by computing the values  $z$  such that  $|Q(z)| = 1$  or  $|R(z)| = 1$ . From the definitions of  $Q(t)$  and  $R(t)$ , we find the equations satisfied by the two power series:

$$\begin{aligned} 1 - \frac{(1-t)^2}{2t} R(t) + R(t)^2 &= 0; \\ 1 - \frac{1-4t+t^2}{t} Q(t) + Q(t)^2 &= 0. \end{aligned}$$

that we rewrite into:

$$\begin{aligned} R(t) + R(t)^{-1} &= \frac{t+t^{-1}}{2} - 1; \\ Q(t) + Q(t)^{-1} &= t + t^{-1} - 4. \end{aligned}$$

For the value  $R(z)$  (resp.  $Q(z)$ ), to be equal to  $e^{i\theta}$ , the quantity  $(z + z^{-1})/2 - 1$  (resp.  $z + z^{-1} - 4$ ), which is  $2 \cos \theta$ , must be a real in the segment  $[-2, 2]$ . The fact that  $z + z^{-1}$  is real implies that  $z$  is either real or has modulus 1. Using elementary real analysis, we find that  $|Q(z)| = 1$  if and only if  $z$  is in  $[\rho, \bar{\rho}]$  and that  $|R(z)| = 1$  if and only if  $z$  is in the curve  $\mathcal{B}$ .

Since  $R(0) = Q(0) = 0$ , we show by continuity that  $Q(z)$  and  $R(z)$  have modulus less than 1 for  $z$  in the domain  $\mathcal{D}$ .  $\square$

**Lemma 13.** *Let  $B = B(t)$  be the power series:*

$$B = B(t) = \sum_{k \geq 0} S(1+S)^k \frac{QR^k}{1 - QR^k}.$$

*Let  $\rho_B$  be the radius of convergence of  $B$ . The point  $\rho_B$  is a simple pole of  $B$  satisfying the equation:*

$$1 - 5\rho_B - 7\rho_B^2 + \rho_B^3 = 0. \quad (8)$$

*Moreover, the series  $B$  admits a meromorphic continuation in the domain  $\mathcal{D}$ . It is not  $D$ -finite.*

*Proof.* We start by rewriting the expression of  $B$  into:

$$B = \sum_{k \geq 0} \left( \sum_{j \geq 1} S(1+S)^k Q^j R^{jk} \right) = \sum_{j \geq 1} B_j,$$

with

$$B_j = \frac{SQ^j}{1 - (1+S)R^j}.$$

We use this form to locate the singularities of the power series  $B$  in the domain  $\mathcal{D}$ . Let  $z$  be in  $\mathcal{D}$ . By Lemma 12, we prove by continuity that both  $Q(z)$  and  $R(z)$  have modulus less than 1. Thus, the term  $B_j(z)$  and all its derivatives decrease exponentially. This shows that the infinite sum does not create any singularities; therefore, the only singularities of  $B$  in the domain  $\mathcal{D}$  are points such that  $(1+S)R^j = 1$  for some  $j \geq 1$ .

To find the radius of convergence of  $B$ , we use Pringsheim's Theorem [9, Theroem IV.6]. Since  $B$  has non-negative coefficients, its radius of convergence is its lowest real, positive singularity. Let  $j \geq 1$ . Consider the function  $f_j$  defined for  $x \in [0, \rho]$  by:

$$f_j(x) = (1 + S(x))R(x)^j.$$

Since  $S(0) = R(0) = 0$ ,  $R(\rho) = 1$  and  $S(\rho) = 1/\sqrt{2}$ , this function reaches 1 at a unique point that we denote by  $\rho_j$ . Moreover, since  $R(x) < 1$ , we have  $f_i(x) > f_j(x)$  for all  $0 < x < \rho$  and  $i < j$ . Therefore, the sequence  $(\rho_j)$  is strictly increasing. This proves that the radius of convergence of  $B$  is  $\rho_1$ , which is a simple pole. We thus have:

$$(1 + S(\rho_B))R(\rho_B) = 1.$$

The equation (8) is then obtained by performing elimination with the definitions (1) and (3) of  $S$  and  $R$ . Moreover, this proves that  $B$  is not  $D$ -finite since it has infinitely many singularities.

To show that  $B$  admits a meromorphic continuation in the domain  $\mathcal{D}$ , we show that its set of singularities has no point of accumulation in  $\mathcal{D}$ . Let  $z$  be such a point of accumulation. Thus, there exist a sequence  $(z_n)$  tending to  $z$  and a sequence  $(j_n)$  of integers such that:

$$(1 + S(z_n))R(z_n)^{j_n} = 1.$$

Since the function  $(1 + S)R^j - 1$  is holomorphic and has isolated zeroes, we may assume that  $j_n$  tends to infinity. By continuity, the term  $R(z_n)^{j_n}$  tends to  $1/(1 + S(z))$ , which is finite and nonzero. This forces that  $R(z) = 1$ , which is not possible by Lemma 12.  $\square$

*Proof of Theorem 11.* To prove the theorem, we rewrite the equation (7) into:

$$M(t) = \frac{D(t)}{1 - B(t)}.$$

We then use Lemma 13; since  $\rho_B$  is a simple pole of  $B$ , we have  $B(t) \rightarrow +\infty$  as  $t \rightarrow \rho_B$ . Therefore, the value  $B(t)$  reaches 1 at a unique point in  $[0, \rho_B]$ , which is a simple pole. Again, Pringsheim's Theorem shows that this point is the radius of convergence of  $M$ . Numerical estimates yield the announced value of  $\rho_M$ .

Since both  $D$  and  $B$  admit a meromorphic continuation in the domain  $\mathcal{D}$ , the series  $M$  also admits a meromorphic continuation. Moreover, since  $D$  is D-finite (since it is algebraic) and  $B$  is not,  $M$  is not D-finite.  $\square$

**Corollary 14.** *Let  $M_n$  be the number of multi-directed animals of area  $n$ . As  $n$  tends to infinity, this number satisfies:*

$$M_n \sim \lambda \mu^n,$$

with  $\mu = 1/\rho_M = 6.475\dots$ . Moreover, the average number of sources and the average width of the multi-directed animals of area  $n$  grow linearly with  $n$ .

*Proof.* The corollary stems from the fact that the class of multi-directed animals follows a supercritical sequence schema. In particular, [9, Theorem V.1] gives the estimate of  $M_n$  and shows that to decompose a connected heap of total length  $n$  all the way to a pyramid, a linear number of Nordic decompositions is needed on average. Since every decomposition adds at least one minimal polymer and one to the width of the heap, the average number of sources and the average width of the multi-animals are also linear.  $\square$

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