# ADDENDUM TO UNIQUENESS OF CERTAIN POLYNOMIALS CONSTANT ON A LINE 

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#### Abstract

The computer calculations in [2] to classify sharp polynomials with nonegative coefficients constant on the line $x+y=1$ have been extended to degrees 19 and 21. In degree 19 a surprisingly large number of 13 sharp polynomials was found, while in degree 21 only the group invariant polynomial exists.


## 1. Introduction

Let $\mathcal{H}(2, d)$ be the space of polynomials $p(x, y)$ of two variables with nonnegative coefficients such that $p(x, y)=1$ whenever $x+y=1$. It is known [1] that the degree $d$ satisfies $d \leq 2 N-3$, where $N$ is the number of nonzero coefficients of $p$. Furthermore, for each odd degree $d$, there exists a group invariant polynomial with precisely $d=2 N-3$. We call polynomials satisfying equality the sharp polynomials. See [1,2] for more information, background, and motivation.

In [2] using computer code, we have have classified the sharp polynomials in $\mathcal{H}(2, d)$ up to degree 17. Due to the increase in the speed of computers over the last several years and improvements to the computer code to be described below, it was possible to run the computer code for degrees 19 and 21 . The computation with all the improvements takes approximately 5 days for degree 19 on a relatively recent 4 -core CPU . For degree 21, the computation took over 8 months.

The computer has found 13 sharp polynomials polynomials in degree 19 up to symmetry of the variables, several of which were unexpected. Two of the polynomials are symmetric. The number of sharp polynomials in degree 19 is stunning as there have been only 16 sharp polynomials in all odd degrees up to degree 17.

In degree 21, the computer code found that up to swapping of variables, there are no sharp polynomials besides the group invariant one, which was previously known.

We have therefore computed one new term in the sequence A143106 on OEIS [5], of degrees where the group invariant polynomial is the unique one up to swapping of variables. The sequence is now known to be:

$$
\begin{equation*}
1,3,5,9,17,21 \tag{1}
\end{equation*}
$$

We have also computed two new terms for the sequence A143107 on OEIS [6]. That is, a sequence whose $N$ th term is the number of sharp polynomials of degree $2 N-3$, or in other words, the number of polynomials in $\mathcal{H}(2,2 N-3)$ with $N$ nonzero coefficients. In this sequence symmetry is not taken into account and therefore there are 24 polynomials altogether for degree 19 and 2 polynomials for degree 21 . The sequence is now known to be:

$$
\begin{equation*}
0,1,1,2,4,2,4,8,4,2,24,2 . \tag{2}
\end{equation*}
$$

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## 2. The polynomials

Let us start with degree 21. In this degree only the group invariant polynomial exists. That is, up to swapping of variables the only sharp degree 21 polynomial is the group invariant one.

$$
\begin{array}{r}
x^{21}+21 x^{19} y^{1}+189 x^{17} y^{2}+952 x^{15} y^{3}+2940 x^{13} y^{4}+5733 x^{11} y^{5}+7007 x^{9} y^{6}+ \\
5148 x^{7} y^{7}+2079 x^{5} y^{8}+385 x^{3} y^{9}+21 x^{1} y^{10}+y^{21} \tag{3}
\end{array}
$$

In degree 19, the situation is dramatically different. Let us list the polynomials we found. First we have the group invariant polynomial.

$$
\begin{array}{r}
x^{19}+19 y x^{17}+152 y^{2} x^{15}+665 y^{3} x^{13}+1729 y^{4} x^{11}+2717 y^{5} x^{9}+2508 y^{6} x^{7}+ \\
1254 y^{7} x^{5}+285 y^{8} x^{3}+19 y^{9} x+y^{19} \tag{4}
\end{array}
$$

Then we have several polynomials with integer coefficients.

$$
\begin{align*}
& x^{19}+19 y x^{17}+152 y^{2} x^{15}+665 y^{3} x^{13}+1729 y^{4} x^{11}+2090 y^{5} x^{9}+627 y^{9} x^{5}+ \\
& 627 y^{5} x^{5}+285 y^{8} x^{3}+19 y^{9} x+y^{19}  \tag{5}\\
& x^{19}+285 y^{3} x^{13}+1425 y^{4} x^{11}+19 y^{9} x^{9}+2679 y^{5} x^{9}+19 y x^{9}+2508 y^{6} x^{7}+ \\
& 1254 y^{7} x^{5}+285 y^{8} x^{3}+19 y^{9} x+y^{19}  \tag{6}\\
& x^{19}+285 y^{3} x^{13}+1425 y^{4} x^{11}+19 y^{9} x^{9}+2052 y^{5} x^{9}+19 y x^{9}+627 y^{9} x^{5}+ \\
& 627 y^{5} x^{5}+285 y^{8} x^{3}+19 y^{9} x+y^{19} \tag{7}
\end{align*}
$$

Next, three polynomials with rational coefficients with denominator 25.

$$
\begin{array}{r}
x^{19}+19 y x^{17}+152 y^{2} x^{15}+\frac{15371 y^{3} x^{13}}{25}+\frac{6137 y^{4} x^{11}}{5}+\frac{4807 y^{5} x^{9}}{5}+ \\
\frac{1254 y^{13} x^{3}}{25}+\frac{4617 y^{8} x^{3}}{25}+\frac{1254 y^{3} x^{3}}{25}+19 y^{9} x+y^{19} \\
x^{19}+\frac{5871 y^{3} x^{13}}{25}+\frac{4617 y^{4} x^{11}}{5}+19 y^{9} x^{9}+\frac{4617 y^{5} x^{9}}{5}+19 y x^{9}+ \\
\frac{1254 y^{13} x^{3}}{25}+\frac{4617 y^{8} x^{3}}{25}+\frac{1254 y^{3} x^{3}}{25}+19 y^{9} x+y^{19} \\
x^{19}+\frac{5871 y^{3} x^{13}}{25}+\frac{4617 y^{4} x^{10}}{5}+19 y^{9} x^{9}+\frac{4617 y^{6} x^{9}}{5}+19 y x^{9}+ \\
\frac{1254 y^{13} x^{3}}{25}+\frac{4617 y^{8} x^{3}}{25}+\frac{1254 y^{3} x^{3}}{25}+19 y^{9} x+y^{19} \tag{10}
\end{array}
$$

Then we have two polynomials with denominator 56.

$$
\begin{align*}
x^{19}+\frac{855 y x^{17}}{56}+\frac{646 y^{2} x^{15}}{7}+\frac{1938 y^{3} x^{13}}{7}+\frac{2907 y^{4} x^{11}}{7}+ \\
\frac{3553 y^{5} x^{9}}{14}+\frac{323 y^{8} x^{3}}{7}+\frac{209 y^{17} x}{56}+\frac{323 y^{9} x}{28}+\frac{209 y x}{56}+y^{19} \tag{11}
\end{align*}
$$

$$
\begin{align*}
& x^{19}+\frac{855 y x^{17}}{56}+\frac{323 y^{2} x^{15}}{7}+\frac{323 y^{8} x^{9}}{7}+\frac{323 y^{5} x^{9}}{2}+ \\
& \frac{323 y^{2} x^{9}}{7}+\frac{323 y^{8} x^{3}}{7}+\frac{209 y^{17} x}{56}+\frac{323 y^{9} x}{28}+\frac{209 y x}{56}+y^{19} \tag{12}
\end{align*}
$$

Finally, rather surprisingly, there are 4 very similar polynomials with denominator 110, two of which are symmetric in $x$ and $y$.

$$
\begin{align*}
& x^{19}+\frac{19 y x^{17}}{2}+\frac{323 y^{2} x^{15}}{11}+\frac{323 y^{3} x^{13}}{11}+\frac{323 y x^{6}}{55}+ \\
& \frac{323 y^{13} x^{4}}{11}+\frac{323 y^{14} x^{2}}{11}+\frac{19 y^{17} x}{2}+\frac{323 y^{6} x}{55}+\frac{399 y x}{110}+y^{19}  \tag{13}\\
& x^{19}+\frac{19 y x^{17}}{2}+\frac{323 y^{2} x^{15}}{11}+\frac{323 y^{3} x^{13}}{11}+\frac{323 y x^{6}}{55}+ \\
& \frac{323 y^{13} x^{3}}{11}+\frac{323 y^{15} x^{2}}{11}+\frac{19 y^{17} x}{2}+\frac{323 y^{6} x}{55}+\frac{399 y x}{110}+y^{19}  \tag{14}\\
& x^{19}+\frac{19 y x^{17}}{2}+\frac{323 y^{2} x^{14}}{11}+\frac{323 y^{4} x^{13}}{11}+\frac{323 y x^{6}}{55}+ \\
& \frac{323 y^{13} x^{4}}{11}+\frac{323 y^{14} x^{2}}{11}+\frac{19 y^{17} x}{2}+\frac{323 y^{6} x}{55}+\frac{399 y x}{110}+y^{19}  \tag{15}\\
& x^{19}+\frac{19 y x^{17}}{2}+\frac{323 y^{2} x^{14}}{11}+\frac{323 y^{4} x^{13}}{11}+\frac{323 y x^{6}}{55}+ \\
& \frac{323 y^{13} x^{3}}{11}+\frac{323 y^{15} x^{2}}{11}+\frac{19 y^{17} x}{2}+\frac{323 y^{6} x}{55}+\frac{399 y x}{110}+y^{19} \tag{16}
\end{align*}
$$

## 3. No adjacent terms

One new optimization used to compute degree 21 requires a proof. For terminology see [1-3]. Previously the code has avoided polynomials with both terms $x^{j+1} y^{k}$ and $x^{j} y^{k+1}$. If $p(x, y)$ is sharp and contains both terms, one can obtain via an undoing a sharp polynomial with one of the terms missing. If we could undo both, then the polynomial could not have been sharp to begin with. However, using recent work, [3], we can prove a stronger assertion, to improve efficiency of the code.

Proposition 3.1. If $p \in \mathcal{H}(2, d), d>1$ and odd, is sharp, then $p$ contains no adjacent terms. That is, given any $j$ and $k$, at least two of the monomials of the form $x^{j+1} y^{k}, x^{j} y^{k+1}$, and $x^{j} y^{k}$ do not appear in $p$.
Proof. In [2] it was proved that $p$ must contain terms $x^{d}, y^{d}$. The Newton diagram (see [2]) of $q(x, y)=\frac{p(x, y)-1}{x+y-1}$ must therefore contain all $P \mathrm{~s}$ on the sides leading to pure monomials.

The top row (degree $d-1$ in $q$ ) must contain all terms. In [2] we have proved that the top row must be alternating $N$ s and $P s$. This fact follows from the observation that the only degree $d$ terms in $p$ are $x^{d}$ and $y^{d}$.

For $d>1$ then by the above we see that any adjacent terms would have to occur somewhere in the interior. We also assume that adjacent terms occur at the same degree. For example
if we got $x^{j+1} y^{k}$, and $x^{j} y^{k}$, we could multiply the $x^{j} y^{k}$ term by $(x+y)$ to obtain another sharp polynomial with two adjacent terms of same degree.

As $p$ is sharp, all terms in $p-1$ must correspond to sinks and sources. Using terminology from [3], the Newton diagram must be connected as $p-1$ has only one negative term. Furthermore, it was proved in [3] that the number of sources and sinks satisfies the same bound for connected diagrams as those arising from $\mathcal{H}(2, d)$. In other words, there can be at most $\frac{d+5}{2}$ sinks and sources. Therefore if we fill in any zeros with Ps or $N$ s or flip signs, without increasing the number of sinks and sources, we cannot in fact decrease the number of sinks and sources as it is already minimal. So we can never be in a situation where flipping $P \mathrm{~s}$ and $N_{\mathrm{s}}$ or setting zeros to $P \mathrm{~s}$ and $N$ s reduces the number of sinks or sources. Let us disqualify certain situations by showing we could reduce the number of sinks or sources.

Suppose we have two sinks next to each other in a configuration such as:


Here the rows correspond to a fixed degree of terms in $q$. For example, the middle row corresponds to the terms $x^{j-1} y^{k+1}, x^{j} y^{k}$, and $x^{j+1} y^{k-1}$.

We could flip the middle $P$ (term corresponding to $x^{j} y^{k}$ ) to an $N$ and decrease the number of sinks. There of course could also be zeros present. However, any time the top two sinks have both at least one $P$ or $N$ from a term that does not correspond to the $x^{j} y^{k}$, we could set the term corresponding to $x^{j} y^{k}$ to $N$ and remove two sinks. If both sinks have just zeros as in

simply switching the $P$ to a 0 will remove the two sinks. Therefore, we must have one of the sinks have simply zeros and we must have a $P$ at the $x^{j} y^{k}$ term (otherwise one of the sinks would not be there). In other words we have a situation such as


Now we cannot just flip the $P$ to an $N$. Doing so would kill one sink, convert one to a source and create a new sink, so it would not lower the number of sinks and sources. So let us start filling.

Take the smallest degree $k$ for which $q$ has a term missing. The row corresponding to degree $k$ and $k-1$ will therefore have a place that has a gap of zeros such as

$$
\begin{array}{lllllllll}
P & & 0 & & \cdots & & & & \\
& P & & N & \cdots & N & & P \tag{20}
\end{array}
$$

possibly with the $P$ s and $N_{\mathrm{s}}$ reversed. We know we always have such a situation, since the diagram is connected and all the sides are already filled.

If we fill the row of 0 s with alternating $P \mathrm{~s}$ and $N \mathrm{~s}$ the total number of sinks and sources cannot increase. We may have converted a sink to a source or vice versa, but we have not increased the total number.

We keep filling until we get to the row that is the middle row in the configuration

we notice that we could start filling that row with an $N$ on the right hand side and that would cancel one of the two adjacent sinks. And this would lead to a contradiction that the number of sinks plus sources was optimal.

## 4. The computer code

The code used is the C code described in detail in [2]. The new revision of the code that was used in this computation has been posted at 4]. The basic idea is to consider $p(x, 1-x)-1=0$, which provides a linear equation for the coefficients. We write this equation as a matrix $A$ that takes coefficients of degree $d-1$ or less to the degree $d$ coefficients. We know that the degree $d$ coefficients have the form $x^{d}+y^{d}$. We iterate over the list of possible monomials of degree $d-1$ or less, taking the corresponding columns of the matrix, we look for nonzero solutions. The idea of the algorithm is to find those submatrices that are not of full rank (have a nontrivial solution). Then we check if this solution has positive coefficients. There are several heuristics that are applied that can avoid doing row reduction at all. Already in [2], to improve speed of the row reduction, we first used mod $p$ arithmetic for reduction for a small prime, as most of the submatrices considered are full rank. This technique reduces the need to do row reduction in full integer arithmetic in vast majority of the cases. We have used $p=19$ in degree 17 or less, and in this calculation, we used $p=23$. The prime must not divide the degree, as most entries in the matrix are divisible by $d$.

The major improvements done in this revision are the following
(i) The row reduction is first done mod 2 before being done mode $p$. Mod 2 is much faster than $\bmod p$. The code actually does column reduction in $\bmod 2$ and considers columns as unsigned integers. The reduction removes one internal loop as adding columns together is simply an XOR operation. Unfortunately mod 2 arithmetic only eliminates $90 \%$ of the full rank cases, but it is approximately 4 times as fast on these $90 \%$. If the test fails with $\bmod 2$, we move to $\bmod p$ as before.
(ii) Just as the Mathematica code from [2], the C code now ignores polynomials that are "right side heavy." That is, polynomials where there are more terms with higher power of $y$ than of $x$. This optimization reduces the run time by approximately one third.
(iii) No adjacent terms can appear as was mentioned before. Skipping all these cases improved the runtime by another factor of one half.
(iv) Many other more minor optimizations were done, whose individual impact was harder to measure.
Overall, the new optimizations together with improvements in speed of computers since 2008, the current code runs approximately 25-50 times faster than it did in 2008.

## 5. Candidate sequence for uniqueness

In [2] we have stated that we tried the construction of section 8 up to degree 513 and listed the degrees not ruled out up to degree 149. The code was run to degree 1250 and it seems this is a proper place to record the results. Therefore, the construction of section 8 in [2] and hence an extension of the list in Proposition 7.2 is:
$1,3,5,9,17,21,33,41,45,53,69,77,81,93,105,113,117,125,129,141,149,153$, $161,165,177,185,201,213,221,225,249,261,269,273,285,297,305,309,333,341,345$, $357,365,369,381,405,413,417,429,437,441,453,465,473,489,501,521,525,537,549$, $581,585,597,609,617,621,633,645,653,665,689,693,701,705,725,729,741,753,765$, $773,777,789,809,825,833,837,845,861,881,885,897,905,909,921,933,953,957,969$, $981,993,1017,1029,1041,1049,1053,1061,1065,1085,1089,1097,1101,1113,1125,1137$, $1149,1157,1173,1185,1193,1197,1205,1229,1233$

## References

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