On free infinite divisibility for classical Meixner distributions

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Abstract

We prove that symmetric Meixner distributions, whose probability densities are proportional to $|\Gamma(t+ix)|^2$, are freely infinitely divisible for $0 < t \leq \frac{1}{2}$. The case $t = \frac{1}{2}$ corresponds to the law of Lévy's stochastic area whose probability density is $\frac{1}{\cosh(\pi x)}$. A logistic distribution, whose probability density is proportional to $\frac{1}{\cosh^2(\pi x)}$, is freely infinitely divisible too.

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1 Introduction

The free convolution $\mu \boxplus \nu$ of probability measures μ and ν on $\mathbb{R}$ is the distribution of $X + Y$, where X and Y are free self-adjoint random variables respectively following the distributions μ and ν. A probability measure ν on $\mathbb{R}$ is said to be *freely infinitely divisible* if, for any $n \in \{1, 2, 3, \cdots\}$, there exists ν_n such that

$$\nu = \underbrace{\nu_n \boxplus \cdots \boxplus \nu_n}_{n \text{ times}}.$$

This concept was introduced in [V86] and its basic characterization was established in [BV93]. The most important freely infinitely divisible distributions are Wigner's semicircle law and the free Poisson law.

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Recent work has increased examples of probability measures which are infinitely divisible in both senses, classical and free: the Gaussian distribution [BBLS11], chi-square distribution $\frac{1}{\sqrt{\pi x}}e^{-x}1_{[0,\infty)}(x)\,dx$ [AHS], positive Boolean stable law with stability index $\alpha \in (0,\frac{1}{2}]$ [AHb] and Student distribution $\frac{1}{B(\frac{1}{2},n-\frac{1}{2})}\frac{1}{(1+x^2)^n}\,1_{\mathbb{R}}(x)\,dx$ for $n = 1,2,3,\cdots$ [H]. It is not yet clear whether a general theory of the intersection of free and classical infinite divisibility exists. We will add two more examples, Meixner distributions and the logistic distribution, which may contribute to a solution.

We will prove that symmetric *Meixner distributions*

$$\rho_t(dx) := \frac{4^t}{2\pi\Gamma(2t)}|\Gamma(t+ix)|^2\,dx, \quad x \in \mathbb{R}$$

are freely infinitely divisible for $0 < t \le \frac{1}{2}$, where $\Gamma(z)$ is the gamma function defined by:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \quad z > 0.$$

The gamma function satisfies the functional relation $\Gamma(z+1) = z\Gamma(z)$, which extends Γ to a meromorphic function in $\mathbb{C}$ with poles at $z = 0,-1,-2,-3,\dots$ [AS70, Chapter 6]. The measures ρ_t are probability distributions of a Lévy process, called a Meixner process [ST98], since the characteristic function of ρ_t is given by

$$\widehat{\rho}_t(z) = \left(\frac{1}{\cosh(\frac{z}{2})}\right)^{2t}. \tag{1.1}$$

Hence ρ_t is classically infinitely divisible for any $t > 0$. The measure ρ_t orthogonalizes Meixner-Pollaczek polynomials $\{P_n^{(t)}(x)\}_{n=0}^\infty$ which satisfy the recurrence relation [KLS10]

$$xP_n^{(t)}(x) = P_{n+1}^{(t)}(x) + \frac{n(n+2t-1)}{4}P_{n-1}^{(t)}(x), \quad n \ge 1,$$

with initial conditions $P_0^{(t)}(x) = 1,\ P_1^{(t)}(x) = x$.

If $t = \frac{1}{2}$, the measure $\rho_{1/2}$ coincides with

$$\mu_1(dx) = \frac{1}{\cosh(\pi x)}\,dx, \quad x \in \mathbb{R},$$

which is the law of *Lévy's stochastic area*[1]

$$\frac{1}{2}\int_0^1 (B_t^1\,dB_t^2 - B_t^2\,dB_t^1),$$

where (B_t^1, B_t^2) is a standard two-dimensional Brownian motion [L51]. The moments m_n of the rescaled measure $\dfrac{1}{2\cosh(\pi x/2)}\,dx$ are *Euler numbers* (with positive signs):

$$(m_0, m_2, m_4, m_6, m_8, \cdots) = (1,1,5,61,1385,50521,\cdots), \quad m_{2n+1} = 0,\ n \ge 0.$$

[1]This measure is also called the *hyperbolic secant distribution*.

See [AS70, Chapter 23] for Euler numbers.

The *logistic distribution*

$$\mu_2(dx) = \frac{\pi}{2\cosh^2(\pi x)}\,dx, \quad x \in \mathbb{R},$$

is know to be classically infinitely divisible [B92], and we are going to prove that it is freely infinitely divisible too. This measure orthogonalizes *continuous Hahn polynomials* $\{P_n(x)\}_{n=0}^\infty$ which satisfy the recurrence relation [KLS10]

$$xP_n(x) = P_{n+1}(x) + \frac{n^4}{4(4n^2-1)}P_{n-1}(x), \quad n \geq 1,$$

with initial conditions $P_0(x) = 1,\ P_1(x) = x$.

The moments m'_n of the rescaled measure $\frac{\pi}{4\cosh^2(\pi x/2)}\,dx$ are

$$(m'_0, m'_2, m'_4, m'_6, m'_8, \cdots) = \left(1, \frac{1}{3}, \frac{7}{15}, \frac{31}{21}, \frac{127}{15}, \cdots\right), \quad m'_{2n+1} = 0,\ n \geq 0,$$

which can be written as $m'_n = |(2-2^n)B_n|$ in terms of *Bernoulli numbers* B_n [AS70].

2 Preliminaries

Let $\mathbb{C}^+$ and $\mathbb{C}^-$ be the upper half-plane and the lower half-plane respectively. Basic tools for proving free infinite divisibility of a probability measure μ are the Cauchy transform

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x}\,\mu(dx), \quad z \in \mathbb{C}^+$$

and its reciprocal $F_\mu(z) := \frac{1}{G_\mu(z)}$. Let $\Gamma_{\alpha,M}$ be a truncated cone

$$\Gamma_{\alpha,M} := \{z \in \mathbb{C}^+ : \operatorname{Im} z > M,\ |\operatorname{Re} z| < \alpha \operatorname{Im} z\}, \quad \alpha, M > 0.$$

The reciprocal Cauchy transform maps $\mathbb{C}^+$ to $\mathbb{C}^+$ analytically, and it satisfies $\operatorname{Im} F_\mu(z) \geq \operatorname{Im} z$ for $z \in \mathbb{C}^+$. For any $0 < \varepsilon < \alpha$ and μ, there exist $M > 0$ and a unique univalent inverse map F_μ^{-1} from $\Gamma_{\alpha-\varepsilon,(1+\varepsilon)M}$ into $\mathbb{C}^+$ such that $F_\mu(\Gamma_{\alpha,M}) \supset \Gamma_{\alpha-\varepsilon,(1+\varepsilon)M}$ and $F_\mu \circ F_\mu^{-1} = \mathrm{Id}$ in $\Gamma_{\alpha-\varepsilon,(1+\varepsilon)M}$ [BV93].

Free convolution and free infinite divisibility can be characterized by the *Voiculescu transform* of μ defined by

$$\phi_\mu(z) := F_\mu^{-1}(z) - z \tag{2.1}$$

in a domain of the form $\Gamma_{\beta,L}$.

Theorem 2.1 ([BV93]). (1) *The free convolution $\mu \boxplus \nu$ is a unique probability measure such that*

$$\phi_{\mu\boxplus\nu}(z) = \phi_\mu(z) + \phi_\nu(z)$$

in a common domain of the form $\Gamma_{\beta,L}$.

(2) *A probability measure μ on $\mathbb{R}$ is freely infinitely divisible if and only if $-\phi_\mu$ analytically extends to a Pick function, i.e. an analytic function which maps $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$.*

In terms of analytic properties of F_μ^{-1}, a useful subclass of freely infinitely divisible distributions is introduced.

Definition 2.2. A probability measure μ is said to be in the class $\mathcal{UI}$ if F_μ^{-1} defined in a domain of the form $\Gamma_{\beta,L}$ analytically extends to a univalent map in $\mathbb{C}^+$. Equivalently, $\mu \in \mathcal{UI}$ if and only if there exists a simply connected open set $\mathbb{C}^+ \subset \Omega \subset \mathbb{C}$ such that

(i) F_μ analytically extends to a univalent map in Ω,

(ii) $F_\mu(\Omega) \supset \mathbb{C}^+$.

This equivalence is proved just by applying Riemann mapping theorem.

Remark 2.3. In [AHa] we required F_μ to be univalent in $\mathbb{C}^+$ in the definition of $\mu \in \mathcal{UI}$, but this automatically follows. If F_μ^{-1} is analytic in $\mathbb{C}^+$, then $F_\mu^{-1} \circ F_\mu(z) = z$ for $z \in \mathbb{C}^+$ by Identity Theorem, so that F_μ is univalent in $\mathbb{C}^+$.

Lemma 2.4 ([AHa])**.** (1) *If $\mu \in \mathcal{UI}$, then μ is freely infinitely divisible.*

(2) *The class $\mathcal{UI}$ is closed with respect to the weak convergence.*

(3) *The class $\mathcal{UI}$ is not closed under free convolution, i.e. $\mu, \nu \in \mathcal{UI}$ does not imply $\mu \boxplus \nu \in \mathcal{UI}$.*

This class was essentially introduced in [BBLS11] to show that the normal law is freely infinitely divisible, and this class has been successfully applied to several probability measures [ABBL10, AB, AHa, AHb, H]. Examples are presented below, mostly taken from the aforementioned references.

Example 2.5. The following probability measures belong to $\mathcal{UI}$.

(1) Wigner's semicircle law

$$\mathbf{w}(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,1_{[-2,2]}(x)\,dx, \quad F_{\mathbf{w}}^{-1}(z) = z + \frac{1}{z}.$$

(2) The free Poisson law (or Marchenko-Pastur law)

$$\mathbf{m}(dx) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,1_{(0,4]}(x)\,dx, \quad F_{\mathbf{m}}^{-1}(z) = z + \frac{z}{z-1}.$$

(3) The Cauchy distribution

$$\mathbf{c}(dx) = \frac{1}{\pi(1+x^2)}\,1_{\mathbb{R}}(x)\,dx, \quad F_{\mathbf{c}}^{-1}(z) = z - i.$$

(4) [AHa] The beta distribution

$$\boldsymbol{\beta}_a(dx) = \frac{\sin(\pi a)}{\pi a}\left(\frac{1-x}{x}\right)^a 1_{(0,1)}(x)\,dx, \quad F_{\boldsymbol{\beta}_a}^{-1}(z) = \frac{1}{1-(1-\frac{a}{z})^{\frac{1}{a}}}$$

for $\frac{1}{2} \le |a| < 1$. $\boldsymbol{\beta}_{\frac{1}{2}}$ is equal to $\mathbf{m}$ up to scaling.

(5) [BBLS11] The Gaussian distribution

$$\mathbf{g}(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} 1_{\mathbb{R}}(x)\,dx.$$

(6) [ABBL10] The q-Gaussian distribution

$$\mathbf{g}_q(dx) = \frac{\sqrt{1-q}}{\pi}\sin\theta(x)\prod_{n=1}^{\infty}(1-q^n)|1-q^n e^{2i\theta(x)}|^2\, 1_{\left[-\frac{2}{\sqrt{1-q}},\frac{2}{\sqrt{1-q}}\right]}(x)\,dx$$

for $q \in [0,1)$, where $\theta(x)$ is the solution of $x = \frac{2}{\sqrt{1-q}}\cos\theta$, $\theta \in [0,\pi]$. When $q \to 1$, $\mathbf{g}_q$ converges weakly to $\mathbf{g}$, and $\mathbf{g}_0$ coincides with $\mathbf{w}$. For $q \in (0,1)$, the density function of $\mathbf{g}_q$ can be written as [LM95]

$$\frac{1}{2\pi} q^{-\frac{1}{8}}(1-q)^{\frac{1}{2}}\Theta_1\left(\frac{\theta(x)}{\pi}, \frac{1}{2\pi i}\log q\right),$$

where $\Theta_1(z,\tau) := 2\sum_{n=0}^{\infty}(-1)^n (e^{i\pi\tau})^{(n+\frac{1}{2})^2}\sin(2n+1)\pi z$ is a Jacobi theta function.

(7) [AB] The ultraspherical distribution

$$\mathbf{u}_n(dx) = \frac{1}{16^n B(n+\frac{1}{2}, n+\frac{1}{2})}(4-x^2)^{n-\frac{1}{2}} 1_{[-2,2]}(x)\,dx, \quad n = 1,2,3,4,\cdots,$$

where $B(p,q)$ is the beta function. The semicircle law $\mathbf{w}$ appears in the case $n=1$ and the normal law $\mathbf{g}$ in the limit $n \to \infty$ if $\mathbf{u}_n$ are suitably scaled.

(8) [H] The Student distribution

$$\mathbf{t}_n(dx) = \frac{1}{B(\frac{1}{2}, n-\frac{1}{2})}\frac{1}{(1+x^2)^n} 1_{\mathbb{R}}(x)\,dx, \quad n = 1,2,3,\cdots.$$

$\mathbf{t}_1$ coincides with $\mathbf{c}$, and if suitably scaled, $\mathbf{t}_n$ weakly converge to $\mathbf{g}$ as $n \to \infty$.

(9) [AHb] The Boolean stable law

$$\frac{d\mathbf{b}_\alpha^\rho}{dx} = \begin{cases} \dfrac{\sin(\pi\rho\alpha)}{\pi}\dfrac{x^{\alpha-1}}{x^{2\alpha}+2x^\alpha\cos(\pi\rho\alpha)+1}, & x > 0, \\ \dfrac{\sin(\pi(1-\rho)\alpha)}{\pi}\dfrac{|x|^{\alpha-1}}{|x|^{2\alpha}+2|x|^\alpha\cos(\pi(1-\rho)\alpha)+1}, & x < 0, \end{cases}$$

for $0 < \alpha \le \frac{1}{2}$, $\rho \in [0,1]$.

If $\frac{1}{2} \le \alpha \le \frac{2}{3}$ and $2 - \frac{1}{\alpha} \le \rho \le \frac{1}{\alpha} - 1$, the Boolean stable law $\mathbf{b}_\alpha^\rho$ (defined as above too) is still freely infinitely divisible, but not in the class $\mathcal{UI}$ [AHb]. However, most of the known freely infinitely divisible distributions belong to $\mathcal{UI}$ as presented above.

In order to prove $\mu \in \mathcal{UI}$, the following sufficient condition is useful.

Proposition 2.6. *A probability measure μ on $\mathbb{R}$ is in $\mathcal{UI}$ if there exists a simple, continuous curve $\gamma = (\gamma(t))_{t\in\mathbb{R}} \subset \overline{\mathbb{C}^-}$ with the following properties:*

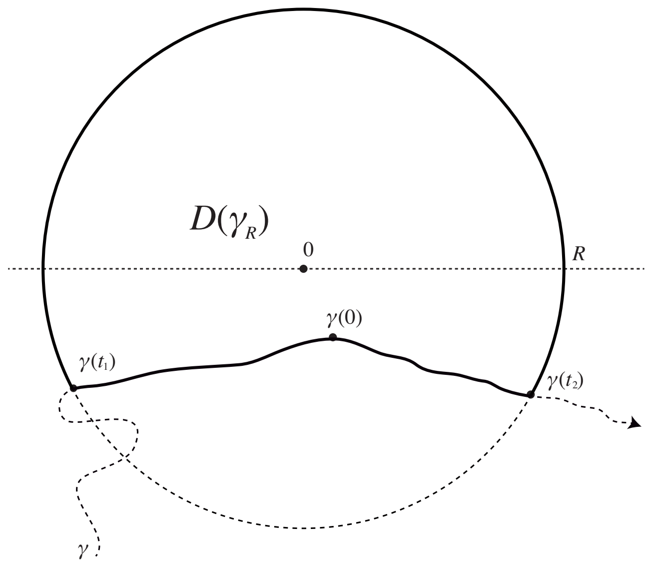
(A) $\lim_{t\to\infty} |\gamma(t)| = \lim_{t\to-\infty} |\gamma(t)| = \infty$*;*

(B) $F_\mu(\gamma) \subset \overline{\mathbb{C}^-}$*;*

(C) F_μ *extends to an analytic function in* $D(\gamma)$ *and to a continuous function on* $\overline{D(\gamma)}$*, where* $D(\gamma)$ *denotes the simply connected open set containing* $\mathbb{C}^+$ *with boundary* γ*;*

(D) $F_\mu(z) = z + o(z)$ *uniformly as* $z \to \infty,\ z \in D(\gamma)$.

Proof. For $R > |\gamma(0)|$, let $t_1 := \sup\{t < 0 : |\gamma(t)| \ge R\} \in (-\infty, 0)$ and $t_2 := \inf\{t > 0 : |\gamma(t)| \ge R\} \in (0,\infty)$. The circle $\{z \in \mathbb{C} : |z| = R\}$ is divided into two arcs by $\gamma(t_1), \gamma(t_2)$, and let A be the arc which contains $\{z \in \mathbb{C}^+ : |z| = R\}$. Consider a simple closed curve γ_R consisting of the arcs $(\gamma(t))_{t\in[t_1,t_2]}$ and A. From (D), we can take $R > 0$ large enough so that $|F_\mu(z) - z| \le \frac{1}{2}|z|$ for $z \in D(\gamma),\ |z| > R$. From the assumption (B), F_μ maps the simple closed curve γ_R to a curve surrounding each point of $\{z \in \mathbb{C}^+ : |z| < \frac{1}{2}R\}$ exactly once, and so the univalent map F_μ^{-1} can be defined in $\{z \in \mathbb{C}^+ : |z| < \frac{1}{2}R\}$ as the left inverse map of $F_\mu|_{D(\gamma_R)}$ which maps numbers with large positive imaginary parts to numbers with large positive imaginary parts. Here $D(\gamma_R)$ is the bounded Jordan domain surrounded by γ_R. Letting $R \to \infty$, we conclude by analytic continuation that F_μ^{-1} exists in $\mathbb{C}^+$ as a univalent map. □



Remark 2.7. Note that the map $F_\mu|_{D(\gamma_R)}$ may not be univalent in whole of $D(\gamma_R)$. The fact that each point of $\{z \in \mathbb{C}^+ : |z| < \frac{1}{2}R\}$ has rotation number 1 implies that there exists a subset S_R (which is in fact open and simply connected) of $D(\gamma_R)$ such that F_μ is univalent in S_R and that $F_\mu(S_R) = \{z \in \mathbb{C}^+ : |z| < \frac{1}{2}R\}$.

3 Proof for Meixner distributions

We present some properties of Meixner distributions.

(1) ρ_t is a probability measure for $t > 0$ because

$$\begin{aligned}\int_{\mathbb{R}} |\Gamma(t+ix)|^2\, dx &= \int_{\mathbb{R}} \left| \int_0^\infty s^{t+ix-1} e^{-s}\, ds \right|^2 dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{tu-e^u} e^{ixu}\, du \right|^2 dx \\ &= 2\pi \int_{\mathbb{R}} e^{2tu-2e^u}\, du = 2\pi \int_0^\infty \left(\frac{s}{2}\right)^{2t} e^{-s} \frac{ds}{s} = \frac{2\pi\Gamma(2t)}{4^t},\end{aligned}$$

where Plancherel's theorem was used in the third equality.

(2) $\rho_{1/2}$ coincides with μ_1 thanks to the formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$.

(3) By the residue theorem, $G_t := G_{\rho_t}$ has the series expansion

$$G_t(z) = \frac{4^t}{\Gamma(2t)} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(n+2t)}{n!} \cdot \frac{1}{z + i(t+n)},$$

which is convergent for $0 < t \le 1/2$.

(4) For any compact set $I \subset \mathbb{R}$, there is $M > 0$ such that

$$|\Gamma(t+zi)\Gamma(t-zi)| \le M e^{-\pi|x|} |x|^{2t-1}, \quad z = x+yi,\ |x| \ge 1,\ t, y \in I.$$

This estimate follows from Stirling's formula.

(5) The density function of ρ_t is symmetric, and moreover strictly decreasing on $[0,\infty)$ as the following calculation shows. We have $\frac{d}{dx}|\Gamma(t+xi)|^2 = -2|\Gamma(t+xi)|^2 \mathrm{Im}\, \psi(t+xi)$ by using the digamma function $\psi(z) = \frac{d}{dz}\log\Gamma(z)$. It is known that $\psi(z) = -\gamma - \sum_{n=0}^\infty \left(\frac{1}{z+n} - \frac{1}{n+1}\right)$, where γ is Euler's constant, and so $\mathrm{Im}\, \psi(t+xi) = \sum_{n=0}^\infty \frac{x}{(t+n)^2+x^2} > 0$ for $x > 0$.

We do not use the series expansion of $G_t(z)$; instead the following recursive relation is useful.

Proposition 3.1. *It holds that*

$$G_t(z - ti) = \frac{1}{z} + \frac{it}{z} G_{t+\frac{1}{2}}\left(z + \left(\frac{1}{2} - t\right) i\right), \quad \mathrm{Im}\, z > t, \quad t > 0. \tag{3.1}$$

Iterative use of this relation extends G_t to a meromorphic function in $\mathbb{C}$ with poles at $-(t+n)i$, $n = 0, 1, 2, \cdots$.

Proof. Assume $t > \frac{1}{2}$. Because $\Gamma(t+iz)\Gamma(t-iz)$ does not have a pole in $\{z \in \mathbb{C} : -\frac{1}{2} \le \operatorname{Im} z \le 0\}$ and vanishes rapidly as $\operatorname{Re} z \to \infty$ (see the above property (4)),

$$\begin{aligned} G_t\left(z - \frac{i}{2}\right) &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{1}{z - (x + \frac{i}{2})} \Gamma(t+ix)\Gamma(t-ix)\, dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{1}{z-x} \Gamma\left(t + \frac{1}{2} + ix\right) \Gamma\left(t - \frac{1}{2} - ix\right) dx, \quad \operatorname{Im} z > \frac{1}{2}. \end{aligned}$$

By using the basic relation $z\Gamma(z) = \Gamma(z+1)$, we obtain

$$\begin{aligned} G_t\left(z - \frac{i}{2}\right) &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{\Gamma\left(t+\frac{1}{2}+ix\right)\Gamma\left(t+\frac{1}{2}-ix\right)}{(z-x)(t-\frac{1}{2}-ix)}\, dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{1}{z+(t-\frac{1}{2})i} \left(\frac{1}{t-\frac{1}{2}-ix} - \frac{1}{iz-ix}\right) \left|\Gamma\left(t+\frac{1}{2}+ix\right)\right|^2 dx \\ &= \frac{ti}{z+(t-\frac{1}{2})i} \cdot \frac{4^{t+\frac{1}{2}}}{2\pi\Gamma(2t+1)} \int_{\mathbb{R}} \frac{1}{z-x} \left|\Gamma\left(t+\frac{1}{2}+ix\right)\right|^2 dx \\ &\quad + \frac{1}{(z+(t-\frac{1}{2})i)} \cdot \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{|\Gamma\left(t+\frac{1}{2}+ix\right)|^2}{t-\frac{1}{2}-ix}\, dx. \end{aligned}$$

In the last integral, we can again apply the formula $z\Gamma(z) = \Gamma(z+1)$, and moreover we deform the contour $\mathbb{R}$ to $\mathbb{R} + \frac{i}{2}$:

$$\begin{aligned} \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{|\Gamma\left(t+\frac{1}{2}+ix\right)|^2}{t-\frac{1}{2}-ix}\, dx &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \Gamma\left(t+\frac{1}{2}+ix\right)\Gamma\left(t-\frac{1}{2}-ix\right) dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \Gamma(t+ix)\Gamma(t-ix)\, dx \\ &= 1. \end{aligned}$$

The above calculations amount to $G_t\left(z - \frac{i}{2}\right) = \frac{1}{z+(t-\frac{1}{2})i} + \frac{it}{z+(t-\frac{1}{2})i} G_{t+\frac{1}{2}}(z)$, which holds for any $t > 0$ since $G_t(z)$ depends on $t > 0$ real analytically. The desired relation (3.1) follows from the simple replacement of z by $z + (\frac{1}{2} - t)i$. The right hand side of (3.1) is meromorphic in $\{z \in \mathbb{C} : \operatorname{Im} z > t - \frac{1}{2}\}$ with pole at 0, so that G_t extends to a meromorphic function in $\{z \in \mathbb{C} : \operatorname{Im} z > t - \frac{1}{2}\}$. Next we can write $G_{t+\frac{1}{2}}$ in terms of G_{t+1}, and so iteratively G_t can be written in terms of $G_{t+\frac{n}{2}}$ for any $n \in \mathbb{N}$. This procedure extends G_t to a meromorphic function in $\mathbb{C}$ with poles at $-(t+n)i$, $n = 0, 1, 2, \cdots$. ☐

Lemma 3.2. *If a probability measure μ has a density $p(x)$ such that $p(x) = p(-x)$, $p'(x) \le 0$ for a.e. $x > 0$ and $\lim_{x\to\infty} p(x) \log x = 0$, then it holds that $\operatorname{Re} G_\mu(x+yi) > 0$ for $x, y > 0$.*

Proof. The claim follows from the computation

$$\begin{aligned} \operatorname{Re} G_\mu(x+yi) &= \int_{\mathbb{R}} \frac{x-u}{(x-u)^2+y^2} p(u)\, du = -\frac{1}{2} \int_{\mathbb{R}} \left(\frac{\partial}{\partial u} \log\left((x-u)^2+y^2\right)\right) p(u)\, du \\ &= \frac{1}{2} \int_{\mathbb{R}} \log\left((x-u)^2+y^2\right) p'(u)\, du \\ &= \frac{1}{2} \int_0^\infty \log\left(\frac{(x+u)^2+y^2}{(x-u)^2+y^2}\right) (-p'(u))\, du > 0, \quad x, y > 0. \end{aligned}$$

The property $p'(-u) = -p'(u)$ was used at the final equality. $\square$

Theorem 3.3. *The Meixner distribution ρ_t is in $\mathcal{UI}$ for $0 < t \le \frac{1}{2}$.*

Proof. We may assume that $0 < t < \frac{1}{2}$ since the set $\mathcal{UI}$ is closed with respect to the weak convergence. We will check conditions (A)–(D) for $F_t(z) := \frac{1}{G_t(z)}$ and $\gamma_t := \{x - ti : x \in \mathbb{R}\}$. (A) is trivial. To prove (B), we use Proposition 3.1:

$$\operatorname{Im} G_t\,(x - ti) = \frac{t}{x}\operatorname{Re} G_{t+\frac{1}{2}}\left(x + \left(\frac{1}{2} - t\right) i\right).$$

Since $\frac{d}{dx}|\Gamma(t + \frac{1}{2} + xi)|^2 < 0$ for $x > 0$, we can apply Lemma 3.2 to the measure $\rho_{t+\frac{1}{2}}$, to assert that $\operatorname{Re} G_{t+\frac{1}{2}}\left(x + \left(\frac{1}{2} - t\right) i\right) > 0$ for $x > 0$. Hence $\operatorname{Im} G_t\,(x - ti) > 0$ for $x > 0$ and also for $x < 0$ by symmetry. Hence condition (B) holds since $-ti$ is a pole of G_t.

From Proposition 3.1, G_t is a meromorphic function and so is F_t. If G_t had a zero in $\overline{D(\gamma_t)}$, there would be a point $z_0 \in \mathbb{C}^+ \cup \mathbb{R} \setminus \{0\}$ such that $G_t(z_0 - ti) = 0$. This implies $1 + tiG_{t+\frac{1}{2}}(z_0 + (\frac{1}{2} - t)i) = 0$ and so $G_{t+\frac{1}{2}}(z_0 + (\frac{1}{2} - t)i) = \frac{i}{t} \in \mathbb{C}^+$. This is a contradiction because $G_{t+\frac{1}{2}}$ maps $\mathbb{C}^+$ into $\mathbb{C}^-$. Thus condition (C) is proved.

Condition (D) can be checked as follows. Let $p_t(x)$ be the density function of ρ_t. In the integral $\int_{\mathbb{R}} \frac{1}{z-x}\,\rho_t(dx)$, one is allowed to replace the contour $\mathbb{R}$ by $C_t := \{x - \frac{3t}{2}i : -\infty < x < -\frac{3t}{2}\} \cup \{-\frac{3t}{2}i + \frac{3t}{2}e^{i\theta} : 0 \le \theta \le \pi\} \cup \{x - \frac{3t}{2}i : \frac{3t}{2} < x < \infty\}$:

$$\int_{\mathbb{R}} \frac{1}{z - x}\,\rho_t(dx) = \int_{C_t} \frac{1}{z - w} p_t(w)\,dw.$$

Clearly $1 = \int_{\mathbb{R}} p_t(x)\,dx = \int_{C_t} p_t(w)\,dw$, so we have $1 - zG_t(z) = \int_{C_t} \frac{1}{w-z}\,wp_t(w)\,dw$. If z tends to ∞ satisfying $z \in \overline{D(\gamma_t)}$, then $1 - zG_t(z)$ tends to 0 by Lebesgue convergence theorem. This implies $\left|\frac{F_t(z)-z}{z}\right| \to 0$, the conclusion. $\square$

Remark 3.4. The proof uses the inequality that $\operatorname{Re} G_{t+\frac{1}{2}}(x + yi) > 0$ for $x, y > 0$. If this inequality holds even for negative y, then we can prove the free infinite divisibility of ρ_t for $t > \frac{1}{2}$ too.

Remark 3.5. The free cumulant sequence $(r_n(\mu))_{n=1}^{\infty}$ of a probability measure μ with finite moments of all orders can be defined as the coefficients of series expansion of $F_\mu^{-1}(z) - z$:

$$F_\mu^{-1}(z) - z = \sum_{n=1}^{\infty} \frac{r_n(\mu)}{z^{n-1}},$$

see [NS06, Remark 16.18]. The free infinite divisibility of ρ_t $(0 < t \le \frac{1}{2})$ implies that the corresponding free cumulant sequence is conditionally nonnegative definite, i.e. the $N \times N$ matrix $(r_{m+n}(\rho_t))_{m,n=1}^{N}$ is nonnegative definite for any $N \ge 1$; see Theorem 13.16 of [NS06].[2] If $t = \frac{1}{2}$, the free cumulants up to the 10th order are given by

$$(r_2(\mu_2), r_4(\mu_1), r_6(\mu_1), \cdots) = (1, 3, 38, 947, 37394, \cdots), \quad r_{2n+1}(\mu_1) = 0,\ n \ge 0.$$

This sequence can be found in [OEIS].

[2] If a measure μ has a compact support, the free infinite divisibility is equivalent to the conditional nonnegative definiteness of free cumulants. This equivalence can be extended to a measure with finite moments of all orders when the moment problem is determinate.

4 Proof for the logistic distribution

The free infinite divisibility of the logistic distribution μ_2 is proved with direct computation of the Cauchy transform. From residue theorem, it turns out that

$$\begin{aligned} G_{\mu_2}(z) &= \sum_{n=1}^{\infty} \frac{i}{(z+(n-\frac{1}{2})i)^2} \\ &= \sum_{n=1}^{\infty} \frac{2x(y+n-\frac{1}{2})}{[x^2+(y+n-\frac{1}{2})^2]^2} + i\sum_{n=1}^{\infty} \frac{x^2-(y+n-\frac{1}{2})^2}{[x^2+(y+n-\frac{1}{2})^2]^2}, \quad z=x+yi \in \mathbb{C}^+. \end{aligned} \tag{4.1}$$

Now we take $\gamma_{1/2} := \{x - \frac{i}{2} : x \in \mathbb{R}\}$. The imaginary part of G_{μ_2} on $\gamma_{1/2}$ can be written as

$$g(x) := \operatorname{Im} G_{\mu_2}\left(x - \frac{i}{2}\right) = \sum_{n=0}^{\infty} \frac{x^2-n^2}{(x^2+n^2)^2}.$$

Fortunately, g can be written by elementary functions.

Lemma 4.1. *The function g is given by* $g(x) = \frac{1}{2}\left(\frac{1}{x^2} + \left(\frac{\pi}{\sinh(\pi x)}\right)^2\right)$.

Proof. It is known that $\frac{1}{\sinh(\pi x)} = \frac{1}{\pi x} - \frac{\pi}{6}x + O(x^3)$ as $x \to 0$, and so $\left(\frac{\pi}{\sinh(\pi x)}\right)^2 = \frac{1}{x^2} + O(1)$, $x \to 0$. The poles of $\left(\frac{\pi}{\sinh(\pi x)}\right)^2$ are at $x = ni$ ($n \in \mathbb{Z}$) and the function $\left(\frac{\pi}{\sinh(\pi x)}\right)^2 - \sum_{n=-\infty}^{\infty} \frac{1}{(x-ni)^2}$ does not have a singular point. This function is bounded by a constant on $\mathbb{C}$ and so equal to a constant, which is actually zero as is known from the limit $x \to \infty$. Hence

$$\begin{aligned} \left(\frac{\pi}{\sinh(\pi x)}\right)^2 &= \sum_{n=-\infty}^{\infty} \frac{1}{(x-ni)^2} = \frac{1}{x^2} + \sum_{n=1}^{\infty}\left(\frac{1}{(x-ni)^2} + \frac{1}{(x+ni)^2}\right) \\ &= \frac{1}{x^2} + 2\sum_{n=1}^{\infty} \frac{x^2-n^2}{(x^2+n^2)^2}, \end{aligned}$$

leading to the conclusion. □

We easily find that $g(x) > 0$ for $x \neq 0$ thanks to Lemma 4.1, and the function F_{μ_2} vanishes at $-\frac{i}{2}$ since it is a pole of G_{μ_2}. Hence condition (B) is satisfied.

The following properties can be proved from (4.1):

(i) $\operatorname{Re} G_{\mu_2}(x+yi) > 0$ for $x > 0$ and $y \geq -\frac{1}{2}$;

(ii) $\operatorname{Im} G_{\mu_2}(yi) < 0$ for $y > -\frac{1}{2}$.

So G_{μ_2} does not have a zero in $\overline{D(\gamma_{1/2})}$ and so F_{μ_2} is analytic in $D(\gamma_{1/2})$, continuous on $\overline{D(\gamma_{1/2})}$. Consequently $\gamma_{1/2} = \{x - \frac{i}{2} : x \in \mathbb{R}\}$ satisfies condition (C).

Condition (D) is proved similarly to the case of ρ_t.

Open problems. The authors have not been able to solve the following questions.

(a) Free infinite divisibility for Meixner distributions ρ_t in the case $t > \frac{1}{2}$ and for non symmetric Meixner distributions.

(b) Free infinite divisibility for the measure with density $\frac{2\pi}{2^r B(\frac{r}{2},\frac{r}{2})}(\frac{1}{\cosh \pi x})^r$ for $r > 0$, $r \neq 1, 2$.

(c) Characterization of the class $\mathcal{UI}$ in terms of free Lévy measures.

(d) Combinatorial meaning of the free cumulant sequence of ρ_t, in particular of $\rho_{1/2}$.

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