

A RECONSTRUCTION THEOREM FOR QUANTUM COHOMOLOGY OF FANO BUNDLES ON \mathbb{P}^n

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ABSTRACT. We present a reconstruction theorem for Fano vector bundles on projective space which recovers the small quantum cohomology for the projectivisation of the bundle from a small number of low-degree Gromov–Witten invariants. We provide an extended example in which we calculate the quantum cohomology of a certain Fano 9-fold and deduce from this, using the quantum Lefschetz theorem, the quantum period sequence for a Fano 3-fold of Picard rank 2 and degree 24. This example is new, and is important for the Fanosearch programme.

1. INTRODUCTION

We say that a vector bundle E is Fano if the total space of the projectivisation, $X := \mathbb{P}(E)$, is a Fano manifold. In this paper we study rank- r Fano vector bundles on n -dimensional projective space \mathbb{P}^n . It is well known that the ring $H^\bullet(X; \mathbb{Z})$ has two integer generating classes, p and ξ_E , given by the tautological bundle on \mathbb{P}^n and the relative tautological bundle on the projectivisation $\mathbb{P}(E)$, respectively. These are canonical up to a choice of normalisation of E .

The Mori cone of X has precisely two extremal rays $R_1, R_2 \subset \overline{NE}_X \subset H_2(X; \mathbb{R})$. We make the assumption on the bundle E (**Assumption 1**) that the primitive generators of the rays R_1, R_2 form an integer basis of $H_2(X; \mathbb{Z})$. We label this basis by A_1, A_2 .

Without loss of generality we may take A_2 to be the class of a line in the fibre. The representative for A_1 is less clear and indeed it is possible that it may not be represented by an algebraic curve (cf. [19, Question 3 and Example 4]). It is not clear whether for Fano bundles Assumption 1 always holds, although there exist counter examples for Fano varieties in general.

It follows from Assumption 1 that we can normalise the bundle such that the basis A_1, A_2 of $H_2(X; \mathbb{Z})$ is Poincaré dual to the basis $\widehat{p}, \widehat{\xi}_E$ of $H^2(X; \mathbb{Z})$. Here $\widehat{\alpha}$ denotes the dual of $\alpha \in H^\bullet(X)$ with respect to the pairing $(\alpha, \beta) = \int_X \alpha \cup \beta$ on cohomology. In the following we shall assume that E is normalised in this manner and denote the tautological class by ξ without reference to the corresponding bundle.

Given Assumption 1 we determine the Gromov–Witten invariants with target class represented by curves in the fibre of the bundle map. This

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generalises some early results of Qin and Ruan [24]. We then impose an additional assumption on our normalised bundle (**Assumption 2**): that $r + 1 + c_1 > 0$, where $c_1(E) = c_1p$. This assumption is essentially an assumption on the rank of the bundle, and holds automatically in the case $r > n$. Assumption 2 insures that quantum multiplication of two classes $\alpha, \beta \in H^\bullet(X; \mathbb{Z})$ with $\deg(\alpha) + \deg(\beta) \leq n$ has no quantum corrections determined by Gromov–Witten invariants with class $aA_1 + bA_2$, with a and b both non-zero.

We show (**Theorem 1**) that for a Fano bundle E satisfying Assumptions 1 & 2 the small quantum cohomology of X is determined by the Gromov–Witten invariants with class $A \in H_2(X)$ of degree $-K \cdot A \leq n + 1$. This gives us good control over the invariants required to determine the quantum cohomology. To place such a condition seems reasonable, as evidence suggests that low rank Fano bundles are split [2]. The projectivisation of a split bundle on \mathbb{P}^n is toric, so the quantum cohomology is determined by existing theory of Givental [12].

In the second part of the paper we provide an extended example, using Theorem 1 to calculate the quantum cohomology for the Fano 9-fold $\mathbb{P}(\wedge^2 \Omega_{\mathbb{P}^4})$, the projectivisation of second wedge of the cotangent bundle on \mathbb{P}^4 . From the quantum multiplication data we produce the small J -function, following Guest [16,17]. We observe in the J -function some tantalising traces of modularity, in the form of the Apéry numbers.

We apply the quantum Lefschetz theorem [10] to the result for the 9-fold to compute the quantum period for the rank-2 Fano 3-fold No. 17 in the Mori–Mukai list [22], which can be given as a complete intersection in the total space of the projectivised bundle. This result is important in the classification of Fano 3-folds as carried out in the Fanosearch programme [8].

While preparing the first draft of this paper the method we describe here was the only way to obtain the quantum cohomology of this Fano 3-fold. More recently Coates–Corti¹–Galkin–Kasprzyk have shown that the quantum period of this Fano 3-fold can be computed using Abelian/Non-Abelian correspondence [6]. Nonetheless the method presented here is less taxing than the alternative approach, and the methods presented in this paper also apply to more general Fano bundles.

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¹Being scooped by your own supervisors is an awkward experience.

2. NOTATION AND ASSUMPTIONS

Throughout we will use Fulton’s convention for projectivised bundles; we regard $\mathbb{P}(E)$ as the bundle of one dimensional linear subspaces in the fibres of E , not one dimensional quotients. Since we work exclusively with cohomology in even degree, we say that α has degree $\frac{n}{2}$ if $\alpha \in H^n$.

2.1. Fano Bundles and Extremal Rays. A vector bundle E is defined to be Fano if its projectivisation, $\mathbb{P}(E)$, is a Fano manifold. Let $E \rightarrow \mathbb{P}^n$ be a Fano bundle of rank r . Let $p = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ be the hyperplane class on \mathbb{P}^n , $\xi_E = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ the relative hyperplane class on $\mathbb{P}(E)$. We shall write $c_i(E) = c_i p^i$ for the Chern classes of E .

Let $X = \mathbb{P}(E)$, we have that the anti-canonical class is given by

$$-K_X = (n + 1)p + c_1(E) + r\xi_E$$

Note that while it might appear that $-K_X$ depends on the normalisation of E , one can easily check that for $E' = E \otimes \mathcal{O}(d)$, $c_1(E') = c_1(E) + dp$, while $\xi_{E'} = \xi_E - dp$.

We fix a basis for $H^\bullet(X; \mathbb{Z})$, $\phi_1, \dots, \phi_{r(n+1)}$, generated as a ring by $\mathbf{1}$, p and ξ_E . This basis is given lexicographic ordering where we take p before ξ_E . Let (\cdot, \cdot) denote the intersection pairing on cohomology, so that $(\alpha, \beta) = \int_M \alpha \cup \beta$. We denote by $\hat{\alpha}$ the dual to α defined by this pairing and denote by $\phi^i = \hat{\phi}_i$ the basis dual to ϕ_i . We will regularly denote the cup product of cohomology classes implicitly.

As X is the projectivisation of a vector bundle bundle over projective space, the Mori cone of X , \overline{NE}_X , has exactly two extremal rays R_1, R_2 (pg. 25 in [7]). We may take primitive generators of the rays $A_1, A_2 \in H_2(X; \mathbb{Z})$ such that $R_1 = \mathbb{R}_+ A_1$ and $R_2 = \mathbb{R}_+ A_2$. Up to a choice of ordering A_2 is represented by a line in the fibre of $P : A_2 = \text{PD}(\widehat{\xi_E})$. The form that A_1 takes in general is less clear and, as we noted in the introduction, it may not be representable as an algebraic curve. We shall make the following assumption on the generators of the extremal rays of X :

Assumption 1. We assume the following equivalent conditions:

- $A_1, A_2 \in H_2(X; \mathbb{Z})$ form an integral basis.
- $p \cdot A_1 = 1$

Assumption 1 implies that $A_1 = \text{PD} [p^{n-1} \xi_E^{r-1} + (k + c_1) p^n \xi_E^{r-2}]$, for some integer k . A sufficient condition for Assumption 1 to hold is if the class A_1 is realised by rational curves corresponding to lines in the base \mathbb{P}^n . This occurs when A_1 is the class of a section of $P|_\ell$ for $\ell \subset \mathbb{P}^n$ a line. Note that ℓ need not be a generic line: unless the bundle is uniform it is reasonable to expect that any line realising A_1 will be some jumping line for the vector bundle E [1]. In greater generality one might expect that A_1 would be the class of a section of $P|_C$, for some curve C of degree > 1 i.e. corresponding to a jumping conic or higher degree rational curve. This

would contravene Assumption 1. At present we do not have examples of such a situation, though we have no general statement to rule it out.

Given Assumption 1 it trivially follows that:

$$\begin{aligned} p \cdot A_1 &= 1, & p \cdot A_2 &= 0 \\ \xi_E \cdot A_1 &= k, & \xi_E \cdot A_2 &= 1 \end{aligned}$$

By choosing the basis $p, (\xi - kp)$, this product becomes diagonal. This is equivalent to twisting E by $\mathcal{O}(k)$, which does not change $\mathbb{P}(E)$. From here on in we assume that E is normalised in this way so that the product is diagonalised, that is:

$$\begin{aligned} p \cdot A_1 &= 1, & p \cdot A_2 &= 0 \\ \xi_E \cdot A_1 &= 0, & \xi_E \cdot A_2 &= 1 \end{aligned}$$

Under these conditions we are therefore free to assume that E is normalised such that $A_1 = \text{PD}(\widehat{p})$ and $A_2 = \text{PD}(\widehat{\xi}_E)$, and we do so in what follows, dropping the subscript E on ξ . Note that this normalisation implies that ξ is nef while $\xi - p$ is not.

Since E is a Fano bundle, $-K_X = (n+1+c_1)p+r\xi$ is ample. We therefore have

$$(1) \quad -K_x \cdot A_1 = n+1+c_1 > 0.$$

With this normalisation fixed, we will later also make the assumption that:

Assumption 2. E satisfies $r+1+c_1 > 0$.

This assumption is essentially a condition on the rank of the bundle and is fulfilled if $r > n$ (combining this with (1), Assumption 2 follows easily). The purpose of this assumption will be made more transparent in the following section as it pertains to the degree of certain quantum corrections.

We shall denote by P and Ξ the extremal contractions given by the linear systems induced by p and ξ respectively.

$$\begin{array}{ccc} X & \xrightarrow{\Xi} & Y \\ P \downarrow & & \\ \mathbb{P}^n & & \end{array}$$

2.2. Gromov–Witten and Quantum Cohomology. We briefly review some aspects of Gromov–Witten theory to fix notation. In the following we use the fact that A_1, A_2 form an integer basis for $H_2(X; \mathbb{Z})$.

Let X_{0,n,aA_1+bA_2} denote the moduli space of n -pointed genus-zero stable maps to X of image class $aA_1 + bA_2$ from curves of genus 0 with n marked points [11, 20]. There are evaluation maps $\text{ev}_i : X_{0,n,aA_1+bA_2} \rightarrow X, 1 \leq i \leq n$, and, given cohomology classes $\alpha_1, \alpha_n \in H^\bullet(X)$, we define genus zero Gromov–Witten invariants as

$$\langle \alpha_1, \dots, \alpha_n \rangle_{0,n,aA_1+bA_2}^X := \int_{[X_{0,n,aA_1+bA_2}]^{\text{vir}}} \text{ev}_1^* \alpha_1 \wedge \dots \wedge \text{ev}_n^* \alpha_n$$

where $[X_{0,n,aA_1+bA_2}]^{\text{vir}}$ is the virtual fundamental class of X_{0,n,aA_1+bA_2} [3, 4, 21]. Recall that the virtual dimension of X_{0,n,aA_1+bA_2} is

$$\dim_{\text{vir}}(X_{0,n,aA_1+bA_2}) = \dim(X) - 3 + n - K \cdot (aA_1 + bA_2).$$

The small quantum product on a manifold M is a deformation of the usual (classical) product in cohomology. The deformation takes the form of quantum corrections, which are governed by genus-zero Gromov–Witten invariants.

Let $\Lambda := \mathbb{C}[H^2(X; \mathbb{Z})]$, an element of which is a finite sum $\sum_{d \in H^2(X; \mathbb{Z})} \lambda_d q^d$, with symbols multiplying as follows, $q^{d_1} q^{d_2} = q^{d_1+d_2}$. We let $q_1 = q^{A_1}$ and $q_2 = q^{A_2}$ be the elements of Λ associated to A_1 and A_2 corrections respectively.

We consider the small quantum product as the following operation:

$$\star : H^\bullet(X; \mathbb{C}) \times H^\bullet(X; \mathbb{C}) \rightarrow H^\bullet(X; \mathbb{C}) \otimes \Lambda$$

with

$$\alpha \star \beta = \sum_{i=1}^{(n+1)r} \sum_{a,b \geq 0} \langle \alpha, \beta, \phi_i \rangle_{0,3,aA_1+bA_2}^X \phi^i q_1^a q_2^b$$

We define $d_1 = \deg(q_1) = -K \cdot A_1 = n + 1 + c_1(E)$ and $d_2 = \deg(q_2) = -K \cdot A_2 = r$. With this definition, quantum product makes $H^\bullet(X, \Lambda)$ into a graded ring. This fact, which is a simple consequence of the virtual dimension of X_{0,n,aA_1+bA_2} , constrains the degree of image class for the Gromov–Witten invariants associated to quantum corrections based on the degree of the classes being multiplied. We will often refer to the degree preserving nature of quantum multiplication by little more than “for degree reasons”. To calculate small quantum multiplication by p and ξ we need only consider 2-point invariants, due to the divisor axiom [3].

Note that Assumption 2 implies that $d_1 + d_2 > n$, so the quantum multiplication of two classes $\alpha, \beta \in H^\bullet(X; \mathbb{Z})$ with $\deg(\alpha) + \deg(\beta) \leq n$ has no quantum corrections determined by Gromov–Witten invariants with class $aA_1 + bA_2$, with a and b both non-zero.

3. RECONSTRUCTION THEOREM FOR FANO BUNDLES

Lemma 1. *Let $X = \mathbb{P}(E)$ for $E \rightarrow Y$, a Fano bundle (not necessarily on \mathbb{P}^n) of rank r and let Y be smooth of dimension n . Let ξ be the relative hyperplane class on X , $\pi : X \rightarrow Y$ the induced bundle map, and $A = \text{PD}(\widehat{\xi})$ the extremal curve corresponding to lines in the fibre of π . The Gromov–Witten invariants of type $\langle \alpha, \beta \rangle_{0,2,kA}^X$ vanish for $k \geq 2$. Furthermore $\langle \alpha, \beta \rangle_{0,2,A}^X = \pi_* \alpha \cdot \pi_* \beta$*

Proof. This result generalises Lemmas 3.6 and 3.7 of [24].

We demonstrate that for $k \geq 2$ there do not exist any rational curves of class kA which satisfy the intersection properties given by α and β . Connected curves of the class kA are restricted to live in a single fibre of π . Since we require that the curve intersects $\text{PD}(\alpha)$ and $\text{PD}(\beta)$, such a curve may only exist if $\text{PD}(\alpha)$ and $\text{PD}(\beta)$ both intersect a common fibre of π . We provide a dimension counting argument to show that no such common fibre exists.

The virtual dimension of $X_{0,2,kA}$ is

$$\dim_{\text{vir}}(X_{0,2,kA}) = \dim(X) - 3 + 2 + k \deg(q^A) = n + r - 2 + kr$$

If $k \geq 2$ then, for degree reasons, we only get non-zero invariants if $\deg(\alpha) + \deg(\beta) = n + r - 2 + kr \geq n + r - 2 + 2r$.

Letting s and t be the dimensions of $\text{PD}(\alpha)$ and $\text{PD}(\beta)$ respectively, we have that $s + t \leq n + r - 2r < n$. This implies that $\text{PD}(\alpha)$ and $\text{PD}(\beta)$ do not intersect in a common fibre of π , in particular $\pi_* \text{PD}(\alpha)$ and $\pi_* \text{PD}(\beta)$ do not intersect generically in Y . Since there are no rational curves with the correct intersection properties, the associated Gromov–Witten invariants vanishes.

The Gromov–Witten invariants $\langle \alpha, \beta \rangle_{0,2,A}^X$ count genuine lines in the fibres of π which intersect sufficiently generic cycles representing the classes $\text{PD}(\alpha)$ and $\text{PD}(\beta)$. The fibres are projective space and it is well known that the only non-zero, two-point invariants for projective space encode the fact that there is exactly one line between two points, see e.g. [18]. So we can rephrase the question: we wish to count fibres of P which contain a point of the cycle representing $\text{PD}(\alpha)$ and a point of the cycle representing $\text{PD}(\beta)$. This can be calculated via ordinary intersection product after pushdown by P , as stated.

□

Note that if α, β are elements of the basis $\{\phi_i\}$

$$\langle \alpha, \beta \rangle_{0,2,A_2} = \begin{cases} 1 & \text{if } \alpha = p^{n-l}\xi^{r-1}, \beta = p^n\xi^{r-1}, 1 \leq l \leq n \\ 0 & \text{otherwise} \end{cases}$$

We prove two reconstruction lemmas, which together allow us to produce the full quantum cohomology for X from a small number of input Gromov–Witten invariants. Note that throughout we are heavily reliant upon the assumption that the product between divisor classes and extremal curve classes is diagonal. The ‘ ξ -Lemma’ tells us that if we have the quantum multiplication of a class by p then we know for free the multiplication of the same class by ξ . The ‘ p -Lemma’ says that if we know $\xi \star p^k \xi^l$, then we get $p \star p^{k-1} \xi^{l+1}$. Note that once we have determined quantum multiplication by divisor classes we have determined the entirety of quantum cohomology, so we restrict our interest to quantum multiplication by divisors. Since quantum multiplication is distributive, we only need consider multiplication of basis elements.

Lemma 2 (ξ -Lemma). *Assume that for some i, k , with $i \geq k$, $p \star p^k \xi^{i-k}$ is known. We can then calculate $\xi \star p^k \xi^{i-k}$ with no geometric (i.e. moduli space) calculation. Put differently we already have sufficient Gromov–Witten invariants to calculate the quantum corrections in $\xi \star p^k \xi^{i-k}$.*

Proof. For degree reasons the quantum corrections in $p \star p^k \xi^{d-k}$ are determined by (and hence determine) all Gromov–Witten invariants of the form;

$$\langle \alpha, \beta \rangle_{0,2,aA_1+bA_2}^X$$

with $ad_1 + bd_2 \leq i + 1$ and $a \geq 1$.

Likewise, the quantum corrections for $\xi \star p^k \xi^{i-k}$ are determined by invariants of the form

$$\langle \alpha, \beta \rangle_{0,2,aA_1+bA_2}^X$$

with $ad_1 + bd_2 \leq i + 1$ and $b \geq 1$.

It is clear that the only Gromov–Witten invariants required to determine the quantum corrections in $\xi \star p^k \xi^{i-k}$, which are not already determined by $p \star p^k \xi^{i-k}$, are $\langle \alpha, \beta \rangle_{0,2,A_2}^X$, which are given by Lemma 1 □

Lemma 3 (p -Lemma). *Assume all quantum multiplication by divisor classes of classes of degree $\leq i - 1$ is known. Further, assume that $\xi \star p^{k+1} \xi^{i-k-1}$, $i - k - 1 \geq 1$ is known. Then we can calculate $p \star p^k \xi^{i-k}$ with no geometric calculation. Note that for $i \geq n + 1$ and $k = n$, $\xi \star p^{n+1} \xi^{i-n-1} = 0$, since $p^{n+1} = 0$*

Proof. Since quantum cohomology is both associative and commutative, we can use the knowledge of multiplication in lower degree to make the following manipulations:

$$\begin{aligned} p \star p^k \xi^{i-k} &= p \star (\xi \star p^k \xi^{i-k-1} - f_2 q_2) \\ &= \xi \star p^{k+1} \xi^{i-k-1} + \xi \star f_1 q_1 - p \star f_2 q_2 \\ &= p^{k+1} \xi^{i-k} + \text{known quantum corrections} \end{aligned}$$

where f_1 is the quantum correction from $p \star p^k \xi^{i-k-1}$ and f_2 the correction from $\xi \star p^k \xi^{i-k-1}$. Note that these are of degree $i - d_1$ and $i - d_2$ respectively so their multiplication by divisors is, by assumption, known. At each step the quantum corrections are all known and governed by invariants we already have, since they are necessarily of lower degree. □

For degree reasons quantum multiplication, by divisor classes, of classes of degree $i \leq n$ is completely determined by the Gromov–Witten invariants of the form $\langle \alpha, \beta \rangle_{0,2,A_2}^X$ and $\langle \alpha, \beta \rangle_{0,2,kA_1}^X$ for $kd_1 \leq n + 1$. Note in particular that there can be no corrections coming from $\langle \alpha, \beta \rangle_{0,2,A_1+A_2}^X$ invariants. Every basis element of degree $i \geq n + 1$ is divisible by ξ , so we can apply the p -Lemma, the proof of which requires that we can divide out a factor of ξ .

In the process of producing an algorithm from Lemmas 2 and 3 to carry out the reconstruction process, we consider cohomology classes as vectors in the lexicographical basis ϕ_i . Quantum multiplication by the basis classes p and ξ can be considered as left multiplication of cohomology vectors by $r(n+1) \times r(n+1)$ matrices, M_p and M_ξ respectively.

With this view point in mind, we see that the preceding lemmas can be reinterpreted.

Lemma 4 (*p*-lemma as linear algebra). *Given the i^{th} column of M_p we can determine the i^{th} column of M_ξ .*

Lemma 5 (*ξ* -lemma as linear algebra). *Assume the first i columns of M_p and M_q have been determined. We can calculate the $i+1^{\text{st}}$ column of M_p using linear algebra.*

Theorem 1. *Let $X = \mathbb{P}(E)$ with $E \rightarrow \mathbb{P}^n$ a Fano bundle of rank r , such that $r+1+c_1(E) > 0$. The quantum cohomology of X can be reconstructed from the Gromov–Witten invariants which map to target class A such that $-K \cdot A \leq n+1$. In particular in our fixed basis that is invariants of the form $\langle \alpha, \beta \rangle_{0,2,A_2}^X$ (which are determined by Lemma 1) and $\langle \alpha, \beta \rangle_{0,2,kA_1}^X$ for $kd_1 \leq n+1$.*

Proof. The proof proceeds by the construction of an algorithm. We alternately apply Lemma 3 followed by Lemma 2 to calculate the multiplication for all basis elements following the lex ordering. We produce this algorithm in pseudo-code below.

We first describe in words, roughly how the algorithm proceeds. Consider the first basis element of degree $n+1$ (under lexicographical ordering), i.e. $p^n \xi$. Since we are in the special case of Lemma 3 ($p^{n+1} = 0$) we obtain $p \star p^n \xi$. We are now in a position to apply Lemma 2, to obtain $\xi \star p^n \xi$. One can easily check that we can now apply both lemmas to the next basis element in degree $n+1$ and so on. After obtaining the multiplication data for classes of degree $n+1$ we repeat the process for degree $n+2$ (again obtaining the first multiplication for free due to the vanishing of p^{n+1}). We repeat until we obtain the full quantum multiplication data.

The input for the reconstruction process is a pair of $r(n+1) \times r(n+1)$ matrices M_p and M_ξ . The columns corresponding to multiplication of degrees $\leq n$ are known² and we initialise the unknown entries as zero. By convention we label arrays with the first entry given index 1. We may think of the reconstruction process as giving an algorithm to fill in the rest of the matrices M_p and M_ξ .

The result of Lemma 3 can usefully be written as

$$p \star p^k \xi^{i-k} = \xi \star p \star p^k \xi^{i-k-1} - p \star (\xi \star p^k \xi^{i-k-1} - p^k \xi^{i-k})$$

We use this form in the algorithm as it lends itself easily to calculation by matrix multiplication.

²in the case $r > n$ this is the first $\frac{1}{2}(n+2)(n+1)$ columns

We define the following functions for use in our pseudo-code: Let $I(d, k)$ be the position that $p^k \xi^{d-k}$ appears in the basis of $H^\bullet(X)$, when given lexicographical ordering. In the case that $r > n$, this is given by

$$\begin{cases} \frac{1}{2}(d+2)(d+1) - k, & \text{for } d \leq n \\ \frac{1}{2}(n+1)(2d-n+2) - k, & \text{for } n < d < r \\ \frac{1}{2}((n+1)(2r-n) + (d-r+1)(2n-d+r)) \\ -k + d + 1 - r, & \text{for } r \geq d \end{cases}$$

Next, $V_{(d,k)}$ is the vector corresponding to the monomial $p^k \xi^{d-k}$

$$V_{(d,k)}[j] = \begin{cases} 1 & \text{for } j = I(d, k) \\ 0 & \text{otherwise} \end{cases}$$

Finally we define the Term function, which extracts from a polynomial the term specified along with the corresponding coefficient. It is given by the expression $\text{Term}(\text{polynomial}, \text{monomial})$. e.g. $\text{Term}(5x^2 + 3xy + 2x + 1, x) = 2x$

Let C be an r -vector with $C[i] := -c_i(E)$, note that we set the entries to $-c_i(E)$ to determine the form ξ^r takes.

We produce the following algorithm. Note that we aim for clarity of exposition and as such the algorithm is not minimised in terms of calculational expense.

```

## d loops over all the degrees to calculate
for  $d = n + 1 \rightarrow r + n$  do
    ## We loop over all classes in degree d
    ## Note that the expression below in  $I(.,.)$  determines the number of
    ## basis elements of the given degree
    ## together d and k determine which column of the matrix
    ## we're filling in
    for  $k = 0 \rightarrow (I(d+1, n) + I(d, n) - 1)$  do
        ## looping over j goes through each row of the selected column
        for  $j = 1 \rightarrow (n+1)r$  do
            ## applying the p-lemma
            ## note that we use the special form described above
             $M_p[j, I(d, n-k)] := (M_\xi \cdot M_p \cdot V_{(d-1, n-k)} - M_p \cdot (M_\xi \cdot V_{(d-1, n-k)} -$ 
 $V_{(d, n-k)})) [j]$ 
            ## applying the  $\xi$ -lemma
            ## we loop over the possible powers of  $q_1$ 
            for  $s = 1 \rightarrow (n+1)r$  do
                ## Extracting the quantum corrections from the relevant
                ##  $p$  multiplication. We just pull out the data
                ## from the corresponding column in  $M_p$ 
                ## Note the important factor  $\frac{1}{s}$ 
                ## from the divisor axiom
                 $M_\xi[j, I(d, n-k)] := M_\xi[j, I(d, n-k)] + \frac{1}{s} \text{Term}(M_p[j, I(d, n-k)], q_1^s q_2)$ 
            end for
        end for
    end for

```

```

end for
## Testing for special case where class =  $p^a \xi^{r-1}$ 
## i.e the case where we need to insert chern numbers for  $\xi^r$ .
if  $k = n + r - d - 1$  and  $d \geq r - 1$  then
  ## Insert additional quantum correction in special case
  ## This Gromov–Witten invariants is determined by Lemma 1
   $M_\xi[\mathbb{I}(d - r + 1, n - k), \mathbb{I}(d, n - k)] := q_2$ 
  ## Insert classical multiplication in special case,
  ## i.e  $\xi^r = -c_1 p \xi^{r-1} - c_2 p^2 \xi^{r-2} - \dots$ 
  ## We shift the entries as we insert them to account for the
  ## powers of  $p$  that the class we’re multiplying includes.
  for  $s = 0 \rightarrow k - 1$  do
     $M_\xi[\mathbb{I}(d + 1, n - s), \mathbb{I}(d, n - k)] := C[k - s]$ 
  end for
else
  ## Insert classical multiplication in generic case
  ## i.e. dealing with all cases where we don’t get  $\xi^r$ 
  ## note that by Lemma 1 there are no quantum corrections
  ## in this case
   $M_\xi[\mathbb{I}(d + 1, n - k), \mathbb{I}(d, n - k)] := 1$ 
end if
end for
end for

```

□

Remark 1. The reconstruction algorithm proposed in the proof to Theorem 1 makes use of the fact that, as a special case of Lemma 3, $p \star p^n \xi^{i-n}$ is known for $i > n$. The algorithm takes as input the columns of the matrices for quantum multiplication by p and ξ which correspond to classes of degree $i \leq n$.

The combination of Lemmas 2 and 3 in fact shows additionally that in any degree $i \leq n$, given the input $p \star p^i$ (and the Gromov–Witten invariants determined by Lemma 1), we can determine the quantum multiplication of all other classes of degree i without explicit calculation of any other Gromov–Witten invariants. This remark shows that the quantum cohomology for X can be recovered from $p \star p^k$, for $1 < k \leq n - 1$ alone.

3.1. Special Cases. There are two special cases in which Theorem 1 can be significantly strengthened. The first is when the second extremal contraction realises X as the projectivisation of a vector bundle on some other space, the second when the contraction realises X as the blowup of some smooth space in a smooth locus. In both cases the entire quantum cohomology is determined by counting lines in the fibres of the two extremal maps P and Ξ . We first prove a lemma regarding the Gromov–Witten invariants of a blow-up of a smooth sub-variety in a smooth ambient space.

Lemma 6. *Let $X := \mathbb{P}(E)$ with E a Fano bundle satisfying Assumption 1. Let the extremal contraction associated to A_1 be given by $\Xi : X \rightarrow Y$ is the blow up of $Z \subset Y$, with both Y and Z smooth. Then the Gromov–Witten invariants of the form $\langle \alpha, \beta \rangle_{0,2,kA_1}^X$ vanish for $k \geq 2$.*

Furthermore, $\langle \alpha, \beta \rangle_{0,2,A_1}^X = \Xi|_ \iota^* \alpha \cdot \Xi|_* \iota^* \beta$, where $\Xi|$ is the restriction of Ξ to $D \subset X$, the exceptional divisor of the blow up, and $\iota : D \hookrightarrow X$ is the embedding of D in X .*

Proof. We summarise the geometry of the statement in the following diagram

$$\begin{array}{ccc} D & \xrightarrow{\iota} & X \\ \downarrow \Xi| & & \downarrow \Xi \\ Z & \hookrightarrow & Y \end{array}$$

Curves of class kA_1 are restricted to the fibres of Ξ . It is clear that they must live in the exceptional fibres of Ξ , which are precisely the fibres of $\Xi|$. The calculation may be carried out inside D : stable maps with target X and class kA_1 are in one to one correspondence with stable maps with target D and class ι^*kA_1 .

Since Y and Z are both smooth, the exceptional locus, D , is given by $\mathbb{P}(N_{Z/Y})$, when $N_{Z/Y}$ is the normal bundle of Z in Y . We are now in the case of Lemma 1. □

We have proved the following corollary to Theorem 1

Corollary 1. *Let $X = \mathbb{P}(E)$ with $E \rightarrow \mathbb{P}^n$ Fano bundle satisfying Assumptions 1 & 2 and with the second extremal contraction $\Xi : \mathbb{P}(E) \rightarrow Y$ given by either*

- (1) *the projectivisation of a bundle $E' \rightarrow Y$*
- (2) *the blow up of a smooth subvariety Z inside smooth Y*

then the quantum cohomology of X can be reconstructed from the Gromov–Witten invariants of the form $\langle \alpha, \beta \rangle_{0,2,A_1}$ and $\langle \alpha, \beta \rangle_{0,2,A_2}$. These Gromov–Witten invariants are determined by Lemmas 1 and 6.

4. THE GEOMETRY OF RANK-2 FANO 3-FOLD NO. 17

Our aim is to compute the quantum period sequence of M , the rank-2 Fano 3-fold No. 17 in Mori–Mukai [22]. M can be embedded as a complete intersection in $X = \mathbb{P}(E)$ for some Fano bundle E which we now describe.

Fix a vector space $V \simeq \mathbb{C}^5$ with $\mathbb{P}^4 := \mathbb{P}(V)$. Let $E := \Omega_{\mathbb{P}(V)}^2(2)$ be the second wedge of the bundle of holomorphic differentials on \mathbb{P}^4 , twisted by $\mathcal{O}(2)$. Let X be the total space $\mathbb{P}(E)$. We can naturally view X as the blow-up of the Plücker embedding of the Grassmannian $G(2, V^*) \subset \mathbb{P}(\wedge^2 V^*)$.

As above, let P be the canonical map induced by X 's bundle structure and Ξ the contraction induced by the second extremal ray. We illustrate this in the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{\Xi} & \mathbb{P}(\wedge^2 V^*) \\ P \downarrow & & \\ \mathbb{P}(V) & & \end{array}$$

4.1. Cohomology and Extremal Rays of X . The ordinary cohomology of X is determined by the Chern classes of E , which are easily obtained from the following exact sequence of bundles, taking the second wedge of the Euler sequence on $\mathbb{P}(V)$. By abuse of notation we refer by $V^* \otimes \mathcal{O}$ to the trivial V^* bundle on $\mathbb{P}(V)$.

$$(2) \quad 0 \rightarrow E \rightarrow \wedge^2 V^* \otimes \mathcal{O} \xrightarrow{v \lrcorner} V^* \otimes \mathcal{O}(1) \rightarrow \mathcal{O}(2) \rightarrow 0$$

Here $v \lrcorner$ denotes the contraction by the vector $v \in V$ representing the point in $\mathbb{P}(V)$. From (2) it is clear that X embeds into $\mathbb{P}(V) \times \mathbb{P}(\wedge^2 V^*) \cong \mathbb{P}^4 \times \mathbb{P}^9$. The total Chern class is $c(E) = 1 - 3p + 5p^2 - 5p^3$ and hence

$$(3) \quad H^\bullet(X) = \frac{\mathbb{C}[p, \xi]}{(p^5, \xi^6 - 3p\xi^5 + 5p^2\xi^4 - 5p^3\xi^3)}$$

We check that ξ is nef and prove that E is a Fano bundle.

Lemma 7. ξ is nef.

Proof. Since ξ is the relative hyperplane on $\mathbb{P}(E)$, it is a quotient of P^*E^* . E^* is generated by global sections (one sees this by dualising (2)) so ξ is also generated by global sections and is therefore nef. \square

Since $-K_X = 6\xi + 2p$, and both p and ξ are nef we see that $-K_X$ is ample; X is Fano.

In $H_2(X, \mathbb{Z})$ we fix classes A_1, A_2 . Recall that A_2 is the class of a line in the fibre $P : X \rightarrow \mathbb{P}(V)$. In this example A_1 is the class of a line in an exceptional fibre of $\Xi : X \rightarrow \mathbb{P}(\wedge^2 V^*)$ and is isomorphic, by P , to a generic line in \mathbb{P}^n . We have

$$p \cdot A_1 = 0, \xi \cdot A_1 = 1, p \cdot A_2 = 1, \xi \cdot A_2 = 0$$

from which follows $A_1 = \text{PD}(p^3\xi^5 - 3p^4\xi^4)$ and $A_2 = \text{PD}(p^4\xi^4)$.

Lemma 8. A_1, A_2 generate the extremal rays of the Mori cone.

Proof. Both p and ξ are nef, so, given the homology class $aA_1 + bA_2$ of any curve T , we have $a = \xi \cdot T \geq 0$ and $b = p \cdot T \geq 0$. On the other hand both A_1 and A_2 are contained in the Mori cone and therefore form the extremal rays. \square

The following discussion shows that X is given by the blow up of $\mathbb{P}(\wedge^2 V^*)$ along the Plücker embedding of $G(2, V^*)$.

A point $[w] \in \mathbb{P}(\wedge^2 V^*)$ is represented by a 2-form w . The fibre $\Xi^{-1}([w])$ over this point consists of the subset of points of $\mathbb{P}(V)$ which are represented by a vector v annihilated by w . Considering w as an antisymmetric 5×5 matrix $A: V \rightarrow V^*$, it is clear that w is generically of rank 4, i.e. there is a one-dimensional space of vectors v annihilated by w . So, generically the fibre $\Xi^{-1}([w])$ is a point. When w drops ranks it must be rank 2, since we throw away the zero form upon projectivisation. Where w is of rank 2, the fibre is a \mathbb{P}^2 . The locus of rank-2 forms corresponds to those elements of $\wedge^2 V^*$ which are decomposable to $w_1 \wedge w_2$, where w_1, w_2 are 1-forms. This is exactly the Plücker embedding of $G(2, V^*) \hookrightarrow \mathbb{P}(\wedge^2 V^*)$ as a codimension-3 subvariety. Hence X is isomorphic to the blow up of $G(2, V^*) \subset \mathbb{P}(\wedge^2 V^*)$.

Lemma 9. *The rational map $\mathbb{P}(\wedge^2 V^*) \dashrightarrow \mathbb{P}(V)$ is given by the linear system of quadrics containing $G(2, V^*)$.*

Proof. The map sends a 2-form w , thought of as an antisymmetric 5×5 matrix $A: V \rightarrow V^*$, to its kernel. By a version of the Cramer rule, we can describe the map explicitly by sending A to the vector of 4×4 Pfaffians:

$$\text{pf}(A) = (\text{pf}_0(A), \dots, \text{pf}_4(A)).$$

The statement then reduces to the fact that these Pfaffians generate the ideal of $G(2, V) \subset \mathbb{P}(\wedge^2 V^*)$. \square

4.2. M as a complete intersection in $\mathbb{P}(E)$. We denote by M the Fano 3-fold No. 17 in the Mori–Mukai list of rank 2 Fano 3-folds [22]. According to Mori–Mukai, M is the blow-up of a 3-dimensional quadric $Q \subset \mathbb{P}^4$ with centre $\Gamma \subset Q$, a nonsingular curve of genus 1 and degree 5.

Lemma 10. *M is a complete intersection of type $p \cap \xi^5$ in $\mathbb{P}(E)$*

Proof. It is well-known that the Plucker embedding $G(2, V^*) \hookrightarrow \mathbb{P}(\wedge^2 V^*)$ is of degree five. Using adjunction, one can easily check that the curve given by the complete intersection of 5 general hyperplanes with the Grassmannian has trivial canonical bundle and hence is genus 1:

$$\Gamma = h_1 \cap \dots \cap h_5 \cap G(2, V^*) \subset h_1 \cap \dots \cap h_5 \cong \mathbb{P}^4,$$

all of this taking place in the natural ambient $\mathbb{P}(\wedge^2 V^*)$. Since $\Xi: X \rightarrow \mathbb{P}(\wedge^2 V^*)$ is the blow-up of $\mathbb{P}(\wedge^2 V^*)$ along the Plücker embedding of $G(2, V^*)$, the discussion makes it clear that M is the complete intersection

$$M = \Xi^*(h_1) \cap \dots \cap \Xi^*(h_5) \cap \tilde{Q} \subset X$$

where \tilde{Q} is the proper transform of a quadric containing $G(2, V^*)$, i.e., by Lemma 9, a section of p . \square

Corollary 2. $-K_X = (p + \xi)|_X$

5. THE QUANTUM COHOMOLOGY OF X AND M

In this section we use the reconstruction theorem (Theorem 1) to calculate the quantum cohomology of X . By passing to a certain generating function, the J -function, of X we use Quantum Lefschetz [10] to obtain information about the quantum cohomology of M .

Since $X = \mathbb{P}(E)$ is the projectivisation of a Fano bundle and the extremal contraction Ξ is the blow up of $G(2, 5) \subset \mathbb{P}^9$ the quantum cohomology follows from Corollary 1.

We make use of the Schubert calculus for $G(2, 5)$, following notational conventions from [15]. Let N be the normal bundle to the embedding of $G(2, V^*)$ in $\mathbb{P}(\wedge^2 V^*)$. The exceptional divisor D is given by the projectivisation of N . Let Q be the tautological quotient bundle on $G(2, V^*)$. The normal bundle to the embedding into $\mathbb{P}(\wedge^2 V^*)$ is given by $Q^*(2\sigma_1)$ though for the sake of convenient relations in cohomology we will work instead with Q^* (of course $\mathbb{P}(N)$ and $\mathbb{P}(Q^*)$ are isomorphic). The cohomology of this bundle is given by

$$H^\bullet(\mathbb{P}(Q^*)) = \frac{H^\bullet(G(2, V^*))[\eta]}{(\eta^3 + \sigma_1\eta^2 + \sigma_2\eta + \sigma_3)}$$

where η is the relative hyperplane class of $\mathbb{P}(Q^*)$.

The following diagram describes the geometry of the situation.

$$\begin{array}{ccc} \mathbb{P}(Q^*) & \xrightarrow{\iota} & Y \\ \downarrow \Xi| & & \downarrow \Xi \\ G(3, V) & \hookrightarrow & \mathbb{P}(\wedge^2 V^*) \end{array}$$

For the purpose of calculation, note that $\iota^*\xi = \sigma_1$ and $\iota^*p = \eta$.

Theorem 2.

$$QH^\bullet(X) = \frac{\mathbb{C}[p, \xi, q_1, q_2]}{(R_1, R_2)}$$

where

$$R_1 = p^{*5} - q_1^2 p + 2q_1^2 \xi + 2q_1 p^{*3} - 2q_1 p^{*2} \star \xi - q_1 p \star \xi^{*2} - q_1 \xi^{*3}$$

and

$$R_2 = \xi^{*6} - 3p \star \xi^{*5} + 5p^{*2} \star \xi^{*4} - 5p^{*3} \star \xi^{*3} - q_2 - 5q_1 p \star \xi^{*3} + 10q_1 \xi^{*4}$$

Proof. Quantum multiplication is determined by Corollary 1. From Theorem 2.2 in [25], all that then remains is to evaluate the relations of the classical cohomology (3), replacing classical multiplication with quantum multiplication, from which the statement follows. \square

5.1. Quantum differential structure. Our ultimate goal is to compute the J -function [13], a generating function for certain genus-zero Gromov–Witten invariants, of M . As M is a complete intersection in X we will use the Quantum Lefschetz theorem [10], which expresses certain genus-zero Gromov–Witten invariants of M in terms of invariants of the ambient space X . The input that we need for the Quantum Lefschetz theorem is the J -function of X as well as the direct sum of line bundles which describe M as a complete intersection. In this section we describe a method to obtain an arbitrary number of terms in the power series expansion of the J -function of X , by solving a system of differential equations, called the quantum differential equations for X . Here we follow closely the excellent papers of Guest [16, 17].

Recall from Section 2, ϕ_1, \dots, ϕ_{30} is the lexicographical basis of $H^\bullet(X; \mathbb{Z})$ in p and ξ , with ϕ^1, \dots, ϕ^{30} the dual basis given by the intersection pairing. The J -function of X is the function $H^2(X; \mathbb{C}) \rightarrow H^\bullet(X; \mathbb{C}) \otimes \mathbb{C}[[1/z]]$ defined by

$$J_X(t) = e^{t/z} \left(1 + \sum_{\epsilon=1}^{30} \sum_{a,b \geq 0} e^{d \cdot t} \left\langle \frac{\phi^\epsilon}{z(z-\psi)} \right\rangle_{0,1,aA_1+bA_2}^X \right)$$

where we expand $\left\langle \frac{\phi^\epsilon}{z(z-\psi)} \right\rangle_{0,1,aA_1+bA_2}$ as $\sum_{k \geq 0} \langle \phi^\epsilon \psi^k \rangle_{0,1,aA_1+bA_2} \frac{1}{z^{k+2}}$.

Since $t \in H^2(X; \mathbb{C})$ is nilpotent as an element of $H^\bullet(X)$ the expression $e^{t/z}$ makes sense in $H^\bullet(X) \otimes \mathbb{C}[[1/z]]$.

By writing $t \in H^2(X; \mathbb{C})$ as $t = t_0 + t_1 p + t_2 \xi$, we regard the J -function as a function of $q_1 = e^{t_1}, q_2 = e^{t_2}$:

$$J_X(q) = q_1^{p/z} q_2^{\xi/z} \left(1 + \sum_{\epsilon=1}^{30} \sum_{a,b \geq 0} q_1^a q_2^b \left\langle \frac{\phi^\epsilon}{z(z-\psi)} \right\rangle_{0,1,aA_1+bA_2} \right)$$

Here $q_1^{p/z} = \exp(p \log q_1 / z) = \exp(tp/z)$

J_X satisfies a system of differential operators, called quantum differential operators [17]. Let $M_p(q), M_\xi(q)$ denote the matrices of quantum multiplication by p and ξ with respect to the basis ϕ_1, \dots, ϕ_{30} , which are easily obtained from the Corollary 1 and the discussion above. The reader may find $M_p(q), M_\xi(q)$ in Appendix A.

Consider the system of differential equations:

$$z q_1 \frac{\partial}{\partial q_1} s = M_p(q) s$$

$$z q_2 \frac{\partial}{\partial q_2} s = M_\xi(q) s$$

where s is a vector valued function of $t \in H^2(X; \mathbb{C})$ (or equivalently a multivalued vector function of q_1, q_2 .)

This system admits a fundamental solution matrix, the rows of which are given by vectors J_i

$$S = \begin{pmatrix} - & - & J_1 & - & - \\ - & - & J_2 & - & - \\ & & \vdots & & \\ - & - & J_{30} & - & - \end{pmatrix}$$

The row-vector J_{30} is the expansion of the J -function as a vector valued function in $H^\bullet(X; \mathbb{C}) \otimes \mathbb{C}[[1/z]]$ in terms of the basis ϕ_1, \dots, ϕ_{30} . The differential system is equivalent to 60 differential equations in J_1 through J_{30} . By solving for the rows J_1 through J_{29} in terms of J_{30} we are left with 31 differential equations in J_X . By applying Groebner basis techniques we find a generating set for the ideal formed by these equations in the Weyl algebra. These (non-unique) differential equations are *quantum differential equations* and define J_X up to scalar. Using the RosenfeldGroebner tool in the DifferentialAlgebra package of Maple 16 we obtain the following result.

Lemma 11. *The ideal of quantum differential operators for X is generated by*

$$(4) \quad \begin{aligned} \Delta_1 = & D_2^{10} - q_2 D_1^4 - 2q_2 D_1^3 D_2 - 4q_2 D_1^2 D_2^2 - 3q_2 D_1 D_2^3 - q_2 D_2^4 \\ & - 2q_1 q_2 D_1^2 - 2z q_2 D_1^3 - 2z q_1 q_2 D_1 D_2 - 8z q_2 D_1^2 D_2 - 3q_1 q_2 D_2^2 \\ & - 9z q_2 D_1 D_2^2 - 4z q_2 D_2^3 - q_1^2 q_2 - 4z q_1 q_2 D_1 - 4z^2 q_2 D_1^2 - 7z q_1 q_2 D_2 \\ & - 9z^2 q_2 D_1 D_2 - 6z^2 q_2 D_2^2 - 5z^2 q_1 q_2 - 3z^3 q_2 D_1 - 4z^3 q_2 D_2 - z^4 q_2 \end{aligned}$$

$$(5) \quad \begin{aligned} \Delta_2 = & D_1 D_2^7 - 2D_2^8 + 5q_2 D_1^2 + 5q_2 D_1 D_2 + 2q_2 D_2^2 + 5q_1 q_2 + 5z q_2 D_1 \\ & + 4z q_2 D_2 + 2z^2 q_2 \end{aligned}$$

$$(6) \quad \Delta_3 = 5D_1^3 D_2^3 - 5D_1^2 D_2^4 + 3D_1 D_2^5 - D_2^6 + 5q_1 D_1 D_2^3 - 10q_1 D_2^4 + q_2$$

$$(7) \quad \begin{aligned} \Delta_4 = & D_1^5 + 2q_1 D_1^3 - 2q_1 D_1^2 D_2 - q_1 D_1 D_2^2 - q_1 D_2^3 + q_1^2 D_1 + 2z q_1 D_1^2 \\ & - 2q_1^2 D_2 - 3z q_1 D_1 D_2 - 2z q_1 D_2^2 + z^2 q_1 D_1 - 2z^2 q_1 D_2 \end{aligned}$$

where $D_i = z q_i \frac{\partial}{\partial q_i}$

Corollary 3. *J satisfies $\Delta_1 J = \Delta_2 J = \Delta_3 J = \Delta_4 J = 0$*

The identity component of J , denoted J^0 , is a power series in q_1 and q_2 , i.e. $J^0 = (J, \phi^0) = \sum_{i,j \geq 0} c_{i,j} q_1^i q_2^j$. The differential system gives recursion relations for the coefficients in this power series. The coefficients are fixed by demanding that $c_{0,0} = 1$. We can find J^0 up to arbitrary order and by observation of finite terms try to find a general solution for the coefficients $c_{i,j}$ which solves the differential system. Such a solution, however, has not been forthcoming. We present the coefficients $c_{i,j}$ for $i \leq 7, j \leq 7$ in the the

following matrix $A = (a_{i,j})$. We have cleared the denominators by setting $a_{i+1,j+1} = i!^2 j!^6 c_{i,j}$.

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 20 & 51 & 104 & 185 & 300 & 455 \\ 0 & 4 & 73 & 447 & 1756 & 5320 & 13539 & 30373 \\ 0 & 0 & 90 & 1445 & 10904 & 55220 & 216110 & 703955 \\ 0 & 0 & 36 & 2148 & 33001 & 282085 & 1690515 & 7926751 \\ 0 & 0 & 0 & 1500 & 54500 & 819005 & 7606080 & 51405305 \\ 0 & 0 & 0 & 400 & 50350 & 1447150 & 21460825 & 211463875 \\ 0 & 0 & 0 & 0 & 24500 & 1590050 & 39750270 & 584307365 \end{bmatrix}$$

We remark that the coefficients are all zero below the leading ‘slant diagonal’. Additionally, in the form presented where we have cleared denominators, the leading diagonal $a_{i,i}$ is given by the Apéry numbers [23], so we expect that a closed formula may be hard to find. The occurrence of the Apéry numbers here may indicate hidden modular symmetries of J_0 (cf [5, 26]). Golyshev has observed a striking connection between the quantum differential equations for Fano 3-folds of Picard rank one and modular forms [14], and it is possible that this connection persists to the case of higher Picard rank.

5.2. The Regularised Quantum Period Sequence for M . As we noted in the preceding section, given a complete intersection $M \subset X$, Quantum Lefschetz [10] allows us to calculate part of the J_M from J_X and the Chern classes of the line bundles defining the intersection. In our example we are only interested in the identity component of the J -function, Quantum Lefschetz provides the entirety of J_M^0 from J_X^0 . Our aim is not to explain how Quantum Lefschetz works, but how one uses it in practical examples.

With this in mind we briefly outline the process of applying Quantum Lefschetz. The following statements hold for any ambient space X and complete intersection M .

Let $\mathcal{E} = \oplus L_i$ be the direct sum of line bundles corresponding to M and $\rho_i = c_1(L_i)$ the first Chern class of the line bundle summands. Given the J -function of X , $J_X(t, z) = \sum_{d \in H_2(X)} J_d(t, z) q^d$, one forms the hypergeometric modification

$$I_{X,M}(t, z) = \sum_d J_d(t, z) q^d \prod_i \prod_{k=1}^{\rho_i \cdot d} (\rho_i + kz)$$

We also consider a formal function (with same domain and target) $J_{X,M}(t, z)$, defined in [10], which has the following property

$$e(\mathcal{E}) J_{X,M}(j^* u, z) = j_* J_M(u, z)$$

where $e(\mathcal{E}) = \prod_i \rho_i$ is the Euler class of the bundle \mathcal{E} . The relation between J_M and J_X is indirectly realised by the mirror map, which relates $I_{X,M}$ to $J_{X,M}$. The mirror map is determined by comparing the asymptotics of the expressions. Considered as a power series in z^{-1} , $J_{X,M}$ is the unique function with the form $J_{X,M} = z + t + O(z^{-1})$. We may write $I_{X,M}$ in the form $F(t)z + G^0(q, t)\phi_0 + \sum_{i=1}^r G^i(t)\phi_i + O(z^{-1})$. By homogeneity considerations writing $I_{X,M}$ in this form is practicable, even in the case that J_X is only known up to finite order in q , as in the case in point. The mirror map is given by the following;

$$(8) \quad \frac{I_{M,X}}{F(t)} = J_{X,M} \left(\frac{G^0(q, t)}{F(t)}\phi_0 + \sum_{i=1}^r \frac{G^i(t)}{F(t)}\phi_i, z \right) = e^{\frac{G^0(q,t)}{F(t)}\phi_0} J_{X,M}(\tau, q)$$

where $\tau = \sum_{i=1}^r \frac{G^i(t)}{F(t)}\phi_i$. The second equality follows from the string equation and the definition of J .

The procedure may summarised as follows:

- (1) Calculate J_X
- (2) Produce the hypergeometric modification $I_{X,M}$
- (3) Calculate the mirror map from the asymptotics of $I_{X,M}$
- (4) Produce $J_{X,M}$ from $I_{X,M}$ using the mirror map

Note that we do not obtain the entirety of J_M by comparison with $J_{X,M}$: some information is lost in the pushforward. However, we can recover the full identity component J_M^0 by following the described method with J_X^0 , since pushforward of the identity is cup with the Euler class.

We now return to our particular example and previous definitions for $M \subset X = \mathbb{P}(\Omega^2(2))$ and, proceeding as above, produce J_X . As described in Section 4.2, M is a complete intersection in X given by the intersection of one hyperplane of class $PD(p)$ by 5 of $PD(\xi)$. The hypergeometric modification of $J_X = \sum_{a,b} J_{a,b}(t, z)q_1^a q_2^b$ is given by

$$I_{X,M}(t, z) = \sum_{a,b} J_{a,b}(t, z)q_1^a q_2^b \prod_{k=1}^a (p + kz) \prod_{k=1}^b (\xi + kz)^5$$

Restircting our attention to the identity component $J_X^0 = \sum_{i,j} c_{i,j} q_1^i q_2^j$, we have $I_{X,M}^0 = (I_{X,M}, \phi^0) = \sum_{i,j} d_{i,j} q_1^i q_2^j$, with $d_{i,j} = c_{i,j} i!(j!)^5$

If we set the degree of $deg(z) = 1$ and $deg(t^i) = 1 - deg(\phi_i)$, J_X is known to be homogeneous of degree 1. One can see that in our example the only possible contributions to the mirror map come from the identity component, J_X^0 , and in particular we need only consider $c_{0,0}, c_{1,0}, c_{0,1}$. We find that $F(t) = 1$, $G^0(q, t) = 1 + q_1$, $G^1(t) = G^2(t) = 1$.

Lemma 12. *The mirror map is given by;*

$$J_{M,X}(\tau_0, t_1, t_2, z) = I_{M,X}(t_0, t_1, t_2, z)$$

where $\tau_0 = t_0 + q_1$. We can more conveniently write this as

$$J_{M,X}(t_0, t_1, t_2, z) = e^{-q_1} I_{M,X}(t_0, t_1, t_2, z)$$

To produce the quantum period sequence from J_M^0 we restrict t to the anti-canonical direction in $H^2(M; \mathbb{C})$. As previously stated $-K_M = p + \xi$ and so restricting to the anti-canonical direction has the effect of setting $q_1 = q_2 = q = e^t$. The effect on J_M^0 is to collapse the sum to a power series in one variable with coefficients $d_i = \sum_{j+k=i} c_{j,k}$. The first ten terms in the period sequence are: 1, 0, 10, 42, 414, 3300, 29890, 275940, 2608270, 25305000

Since it is known that J_M^0 satisfies quantum differential equations, the period sequence also does. Given sufficient entries in the power series of the period sequence, we may find the differential operator which annihilates the sequence.

The Picard–Fuchs operator for the regularised period sequence of M is given by

$$\begin{aligned} (9) \quad & -17727940t^9 D^4 - 47452732t^8 D^4 - 177279400t^9 D^3 - 51239477t^7 D^4 \\ & - 400876912t^8 D^3 - 620477900t^9 D^2 - 28719434t^6 D^4 - 363088702t^7 D^3 \\ & - 1218943172t^8 D^2 - 886397000t^9 D - 8782543t^5 D^4 - 169273876t^6 D^3 \\ & - 958664473t^7 D^2 - 1562482112t^8 D - 425470560t^9 - 1322684t^4 D^4 \\ & - 42555106t^5 D^3 - 384463114t^6 D^2 - 1102964660t^7 D - 696963120t^8 \\ & - 37187t^3 D^4 - 5281118t^4 D^3 - 80112855t^5 D^2 - 392394560t^6 D - 456149412t^7 \\ & + 13026t^2 D^4 - 238966t^3 D^3 - 7132816t^4 D^2 - 69331328t^5 D - 148485888t^6 \\ & + 995t D^4 - 11442t^2 D^3 - 3879t^3 D^2 - 4318688t^4 D - 22881836t^5 - 24D^4 \\ & - 2278t D^3 + 16030t^2 D^2 + 146332t^3 D - 928456t^4 + 24D^3 + 35t D^2 + 9600t^2 D \\ & \qquad \qquad \qquad + 76072t^3 + 1920t^2 \end{aligned}$$

where $D = t \frac{d}{dt}$

This matches the Picard–Fuchs operator predicted by the mirror polytope [9].

APPENDIX A. QUANTUM MULTIPLICATION MATRICES

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