Extremal subgraphs of the d-dimensional grid graph

Geir Agnarsson [*] *Kshitij Lauria* [†]

Abstract

For each natural number n we determine, both asymptotically and exactly, the maximum number of edges an induced subgraph of order n of the d-dimension a grid graph $\mathbb{Z}^d$ can have. The asymptotic bound is obtained by using a theorem Bollobás and Thomason, and the exact bound is obtained by induction. This generalizes some earlier results for the case $d = 2$ on one hand, and for $n \leq 2^d$ on the other.

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1 Introduction

The purpose of this article is to determine the maximum number of edges an induced subgraph on n vertices of the d-dimensional rectangular grid graph $\mathbb{Z}^d$ can have. The very first non-trivial result in an exact manner for the case $d = 2$ appears in [6], where it is shown that this maximum number of edges is given by $\lfloor 2n - 2\sqrt{n} \rfloor$. Some other interesting and related exact results appear in [4], where the author Peter Braß studies $f(n, k)$, the maximum number of unit distances among n points in the plane, where the additional restriction is added that only those unit distances are counted that are among a fixed set of k directions. Here the maximum is taken over all sets of n points and all sets of k directions. The case $k = 1$ is trivial, whereas for the case $k = 2$ it suffices to consider subgraphs of $\mathbb{Z}^2$, and so it coincides with the mentioned result from [6], and so $f(n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor$. Other values of $f(n, k)$ have not yet been determined exactly.

In this paper we assume d to be fixed and n an unrestricted positive integer variable. Note that when $n \leq 2^d$, the problem reduces to determine the maximum number of edges of an induced subgraph on n vertices of the d-dimensional hypercube Q_d, a study already done in part in the 1970's as described in [3]. We will briefly revisit this case in the last Section 7. In [3] a recap and references of know results regarding the case $n \leq 2^d$ and induced subgraphs of the hypercube are presented.

The considerations in this article were in part initially inspired by the heuristic integer sequence $0, 1, 2, 4, 5, 7, 9, 12, 13, 15, 17, 20, 21, 23, 25, \ldots$ [2, A007818], describing the maximal number of edges joining $n = 1, 2, 3, \ldots$ vertices in the cubic rectangular grid $\mathbb{Z}^3$, for which no general formula nor procedure to compute it is given. – First we set forth our basic terminology and definitions.

[*]Department of Mathematical Sciences, George Mason University, MS 3F2, 4400 University Drive, Fairfax, VA – 22030, `geir@math.gmu.edu`

[†]Computer Science Department, Brown University, Box 1910, 115 Waterman St., 4th Floor Providence, RI – 02912, `klauria@cs.brown.edu`

Notation and terminology The set of integers will be denoted by $\mathbb{Z}$, the set of natural numbers $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$, and the set $\{1, 2, 3, \ldots, n\}$ by $[n]$. Unless otherwise stated, all graphs in this article will be finite, simple and undirected. For a graph G, its set of vertices will be denoted by $V(G)$ and its set of edges by $E(G)$. Clearly $E(G) \subseteq \binom{V(G)}{2}$, the set of all 2-element subsets of $V(G)$. We will sometimes denote an edge with endvertices u and v by uv instead of the actual 2-set $\{u, v\}$. The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. By an *induced subgraph* H of G we mean a subgraph H such that $V(H) \subseteq V(G)$ in the usual set theoretic sense, and such that if $u, v \in V(H)$ and $uv \in E(G)$, then $uv \in E(H)$. If $U \subseteq V(G)$ then the subgraph of G induced by U will be denoted by $G[U]$. For $d \in \mathbb{N}$, a *rectangular grid* $\mathbb{Z}^d$ in our context is a infinite graph with the point set $\mathbb{Z}^d$ as its vertices and where two points $\tilde{x} = (x_1, \ldots, x_d)$ and $\tilde{y} = (y_1, \ldots, y_d)$ are connected by an edge iff the *Manhattan distance* $D(\tilde{x}, \tilde{y}) = \sum_{i=1}^{d} |x_i - y_i| = 1$. We will talk about $\mathbb{Z}^d$ both as a point set and an infinite graph. Most of the times we will restrict ourselves to the subgraph $\mathbb{N}^d$ of $\mathbb{Z}^d$. For $I \subseteq [d]$ let $\pi_I : \mathbb{Z}^d \to \mathbb{Z}^{|I|}$ be the projection where all the coordinates of $\tilde{x}$ that are not in I are omitted. Many times we will omit the set-brackets and simply list the present coordinates; for example, $\pi_{\hat{i}}$ and $\pi_{1,\ldots,i-1,i+1,\ldots,d}$ will mean the projection $\pi_{\{1,\ldots,i-1,i+1,\ldots,d\}}$ onto $\mathbb{Z}^{d-1}$. Adopting this notation for sets we will denote the complement of $I \subseteq [d]$ by either $\overline{I}$ or $\widehat{I}$, when emphasizing the fact that we are omitting elements from the index set I.

Organization of article The rest of this article is organized as follows. In Section 2 we show that when considering the maximum number of edges a finite set $S \subseteq \mathbb{Z}^d$ of a given order can induce, we can assume that the slices of S perpendicular to each axis are nested, in the sorted "Tower of Hanoi" fashion, or more precisely, we can assume S to be "fully nested" in the sense of Definition 2.1 here below.

In Section 3 we use a result by Bollobás and Thomason [5] to obtain a tight asymptotic upper bound for the maximum number $E_d(n)$ of edges a set $S \subseteq \mathbb{Z}^d$ with $|S| = n$ can induce, for fixed d and n.

In Section 4 we derive an important recursive inequality for $E_d(n)$ as stated in Lemma 4.1.

In Section 5 we introduce a specific class of fully nested sets, *d-cubicles* $[\![n]\!]^d$ for each $n \in \mathbb{N}$, and start verifying that these d-cubicles in $\mathbb{N}^d$ are sets that have exactly the maximum $E_d(n)$ number of induced edges. This will be done by induction on $n + d$.

In Section 6 we prove some important properties of the d-cubicles $[\![n]\!]^d$ and complete the inductive argument from previous Section 5 and obtain an exact formula for $E_d(n)$, as stated in Theorem 6.4, the main theorem of this paper.

In the final Section 7 we state some corollaries we obtain from Theorem 6.4 when we consider some special cases, which have been derived and reported in the literature. Hence, we point out how Theorem 6.4 is a generalization of some celebrated known results.

2 Fully nested sets

Let $S \subseteq \mathbb{Z}^d$ of order n. We denote the number of edges of the induced subgraph $G[S]$ of $\mathbb{Z}^d$ by $E_d(S)$, and let

$$E_d(n) = \max\{E_d(S) : S \subseteq \mathbb{Z}^d,\ |S| = n\} \tag{1}$$

be the maximum number of edges $G[S]$ can have. The main objective of this article is to determine $E_d(n)$, both asymptotically and exactly. By translation, and without loss of generality, we may assume that $S \subseteq \mathbb{N}^d$. For $i \in [d]$ let g_i be the *gravity along i-th axis* that acts on S in the following way: For each $\tilde{y} \in \pi_{\hat{\imath}}(S)$ order the elements of $S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y})$ linearly by their i-coordinate, say $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots$, and then replace the i-th coordinate $\pi_i(\tilde{x}_h)$ in each $\tilde{x}_h$ by its placement h in this linear order, thereby obtaining the set $g_i(S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y}))$ that induces a path on $|S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y})|$ points in $\mathbb{N}^d$ parallel to the i-th axis. From the partition $S = \bigcup_{\tilde{y} \in \pi_{\hat{\imath}}(S)}(S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y}))$, we let

$$g_i(S) := \bigcup_{\tilde{y} \in \pi_{\hat{\imath}}(S)} g_i(S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y})).$$

One can imagine the points of S represented as cube-like blocks in zero-gravity d-space, with the edges of the blocks parallel to the axes, and $g_i(S)$ the location of the blocks after the gravity g_i pulls the set S of blocks down towards the hyperplane with i-th coordinate equal 1.

Clearly we have $g_i^2(S) = g_i(g_i(S)) = g_i(S)$. Note that in general $g_i(g_j(S)) \neq g_j(g_j(S))$, so the operators $g_1, \ldots, g_d$ do not commute. We let $g := g_1 g_2 \cdots g_{d-1} g_d$ be the *total gravity* acting on the set S.

Definition 2.1. *A set $S \subseteq \mathbb{N}^d$ is* i-nested *if*

$$\pi_{\hat{\imath}}(S \cap \pi_i^{-1}(1)) \supseteq \pi_{\hat{\imath}}(S \cap \pi_i^{-1}(2)) \supseteq \pi_{\hat{\imath}}(S \cap \pi_i^{-1}(3)) \supseteq \cdots .$$

The set S is fully nested *of it is i-nested for each $i \in [d]$.*

Note that $g_i(S) = S$ holds iff for each $\tilde{y} \in \pi_{\hat{\imath}}(S)$ we have $\pi_i(S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y})) = [k]$ where $k = |S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y})|$. From this, and the mere definition of the inverse image in general, we see that for a set $S \subseteq \mathbb{N}^d$ we then have the following.

Observation 2.2. *S is i-nested iff $g_i(S) = S$.*

By symmetry we note that if $p : \mathbb{N}^d \to \mathbb{N}^d$ is the linear map that permutes the coordinates and $S \subseteq \mathbb{N}^d$ is a set, then S is fully nested iff $p(S)$ is fully nested.

Our next objective is to prove the following theorem, the statement of which is seemingly obvious for dimensions $d \leq 3$.

Theorem 2.3. *For $S \subseteq \mathbb{N}^d$ we have that $g(S) = S$ iff $g_i(S) = S$ for each $i \in [d]$, that is S is i-nested for each i.*

Clearly, if S is i-nested for each i, then $g(S) = S$. To verify the other implication, we need a couple of lemmas.

Claim 2.4. *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two strings of real numbers with $a_i \geq b_i$ for each i. If $a'_1 \geq a'_2 \geq \cdots \geq a'_n$ and $b'_1 \geq b'_2 \geq \cdots \geq b'_n$ are sortings of these strings in descending order, then for each $i = 1, \ldots, n$ we have $a'_i \geq b'_i$.*

Proof. We may assume that the a_is are already ordered $a_1 \geq a_2 \geq \cdots \geq a_n$. By assumption we then have $a_i \geq b_i, a_{i+1}, \ldots, a_n$ and hence $a_i \geq b_i, b_{i+1}, \ldots, b_n$. Since the b_i's are sorted descendingly $b'_1 \geq b'_2 \geq \cdots \geq b'_n$, we have in particular $a_i \geq b'_i, b'_{i+1}, \ldots, b'_n$. ☐

A direct consequence of Claim 2.4 is the following:

Corollary 2.5. *Let $d \geq 1$ and $a : [N_1] \times \cdots \times [N_d] \to \mathbb{R}$ a function. Let $k < d$ and assume the values $a(\tilde{x})$ are in a descending order w.r.t. the first k coordinates, i.e.*

$$\begin{aligned} a(x_1, \ldots, x_{\ell-1}, 1, x_{\ell+1}, \ldots, x_d) &\geq a(x_1, \ldots, x_{\ell-1}, 2, x_{\ell+1}, \ldots, x_d) \\ &\vdots \\ &\geq a(x_1, \ldots, x_{\ell-1}, N_\ell, x_{\ell+1}, \ldots, x_d), \end{aligned}$$

for each $\ell \in \{1, \ldots, k\}$. If now $a' : [N_1] \times \cdots \times [N_d] \to \mathbb{R}$ is the function obtained from a by sorting the $N_1 \cdots N_k N_{k+2} \cdots N_d$ strings $(a(\tilde{x}) : x_{k+1} \in \{1, \ldots, N_{k+1}\})$ w.r.t. the $(k+1)$-th coordinate, for each fixed $\pi_{\widehat{k+1}}(\tilde{x})$, then $a'(\tilde{x})$ are in a descending order w.r.t. the first $k+1$ coordinates.

By induction we then have by Corollary 2.5 the following.

Corollary 2.6. *Let $d \geq 1$ and $a : [N_1] \times \cdots \times [N_d] \to \mathbb{R}$ a function. Let $a' : [N_1] \times \cdots \times [N_d] \to \mathbb{R}$ be the function obtain from a by first sorting the $N_1 \cdots N_{d-1}$ strings $(a(\tilde{x}) : x_d \in \{1, \ldots, N_d\})$ w.r.t. the d-th coordinate in a descending order, then w.r.t. the $(d-1)$-th coordinate etc., finally sorting w.r.t. the first coordinate. In this case the values $a'(\tilde{x})$ are in a descending order w.r.t. each of the d coordinates.*

Consider a set $S \subseteq \mathbb{N}^d$ of order n. As a finite set, there are $N_1, \ldots, N_d \in \mathbb{N}$ such that $S \in [N_1] \times \cdots \times [N_d]$ and we have the indicator function $\mathbf{1}_S : [N_1] \times \cdots \times [N_d] \to \{0, 1\}$ of S, so $\mathbf{1}_S(\tilde{x}) = 1$ iff $\tilde{x} \in S$. Note that, by definition, the set $g_i(S)$ is the set whose indicator function $\mathbf{1}'_S$ is obtained by sorting the strings $\mathbf{1}_S(\tilde{x})$ w.r.t. the i-th coordinate. Hence, gravity along i corresponds to sorting the indicator function w.r.t. the i-coordinate. By Corollary 2.6 we have that

$$g_i g_1 \cdots g_d = g_1 \cdots g_d, \tag{2}$$

for any $i \in \{1, \ldots, d\}$. If $g(S) = S$, then by (2) $g_i(S) = S$ for each i, and hence we have Theorem 2.3.

Our final objective in this section is to show that $E_d(S)$ is at maximum when S is fully nested.

Lemma 2.7. *If $S \subseteq \mathbb{N}^d$ is a finite set, then $E_d(S) \leq E_d(g_i(S))$.*

Proof. The edges of $G[S] \subseteq \mathbb{N}^d$ are either parallel to the i-th axis, or not. Each edge parallel to the i-th axis is mapped by $\pi_{\hat{\imath}}$ to a single point $\tilde{x} \in \mathbb{N}^{d-1}$. Each edge that is not parallel to i-th axis is mapped by $\pi_{\hat{\imath}}$ to an edge $\{\tilde{x}, \tilde{y}\} \in \binom{\mathbb{N}^{d-1}}{2}$.

For $\tilde{x} \in \mathbb{N}^{d-1}$ there are at most $|S \cap \pi_{\hat{\imath}}^{-1}(\tilde{x})| - 1$ edges of $G[S]$ parallel to i-th axis that are mapped to $\tilde{x}$ under $\pi_{\hat{\imath}}$, and there are precisely $|g_i(S) \cap \pi_{\hat{\imath}}^{-1}(\tilde{x})| - 1 = |S \cap \pi_{\hat{\imath}}^{-1}(\tilde{x})| - 1$ edges of $G[g_i(S)]$ parallel to i-th axis that are mapped to $\tilde{x}$, since they form a connected path.

Each edge $\{\tilde{x}, \tilde{y}\} \in \binom{\mathbb{N}^{d-1}}{2}$ yields a matching between points/vertices of $G[S]$ in $S \cap \pi_{\hat{\imath}}^{-1}(\tilde{x})$ and $S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y})$. In particular, if k edges in $G[S]$ are mapped to $\{\tilde{x}, \tilde{y}\}$, then both $S \cap \pi_{\hat{\imath}}^{-1}(\tilde{x})$ and $S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y})$ have cardinality of k or greater. Therefore the number of edges in $G[g_i(S)]$ is given by

$$\min(|g_i(S) \cap \pi_{\hat{\imath}}^{-1}(\tilde{x})|, |g_i(S) \cap \pi_{\hat{\imath}}^{-1}(\tilde{y})|) = \min(|S \cap \pi_{\hat{\imath}}^{-1}(\tilde{x})|, |S \cap \pi_{\hat{\imath}}^{-1}(\tilde{y})|) \geq k,$$

and so $G[g_i(S)]$ has at least k edges mapped to $\{\tilde{x}, \tilde{y}\}$. $\square$

Corollary 2.8. *Among all finite sets $S \subseteq \mathbb{N}^d$ then $E_d(S)$ is at maximum when S is fully nested.*

3 Tight asymptotic bounds

The objective in this section is to derive an asymptotically tight upper bound for $E_d(n)$, which by Corollary 2.8, equals $E_d(S)$ for some finite fully nested set $S \subseteq \mathbb{N}^d$ of order n.

Let $S \subseteq \mathbb{N}^d$ be a fully nested set of order n. The edges of $G[S]$ are partitioned into i parts, the i-th part consisting of all edges parallel to the i-th axis. As noted in the proof of Lemma 2.7, each point $\tilde{x} \in \pi_{\hat{i}}(S)$ corresponds to a connected path of $G[S]$ since S is fully nested. Also, since there are n vertices of $G[S]$, and exactly $n_{\hat{i}} = n_{\hat{i}}(S) := |\pi_{\hat{i}}(S)|$ disjoint paths of $G[S]$ parallel to i-axis, the number of edges parallel to i-th axis is $n - n_{\hat{i}}$. From this we have the following exact count on the number of edges of $G[S]$.

Observation 3.1. *If $S \subseteq \mathbb{N}^d$ is fully nested set with $|S| = n$, then $E_d(S) = dn - (n_{\hat{1}} + \cdots + n_{\hat{d}})$.*

From the above Observation 3.1 we see that if we can compute the exact minimum value of $\sum_i n_{\hat{i}}$ for all fully nested sets $S \subseteq \mathbb{N}^d$ of order n, then we can determine $E_d(n)$ by subtracting that minimum value from dn.

By a theorem of Bollobás and Thomason [5, Thm 2, p.418] we have

$$n^{d-1} \leq \prod_{i=1}^{d} n_{\hat{i}}. \tag{3}$$

Note that equality holds in (3) for any set S of the form $S = S_1 \times \cdots \times S_d$. By Observation 3.1, the inequality of arithmetic and geometric mean, and (3) we obtain

$$E_d(S) = dn - \left(\sum_{i=1}^{d} n_{\hat{i}}\right) \leq d\left(n - \sqrt[d]{\prod_{i=1}^{d} n_{\hat{i}}}\right) \leq d(n - \sqrt[d]{n^{d-1}}) = dn(1 - n^{-1/d}) \tag{4}$$

and therefore for each $n \in \mathbb{N}$ we have

$$E_d(n) \leq dn(1 - n^{-1/d}). \tag{5}$$

Since equality holds in the inequality of arithmetic and geometric mean iff all the parameters are equal, we have that equality holds in (4) for a fully nested set S iff $S = [m]^d$ is a d-dimensional hypercube (or a *d-cube* for short) with m^d vertices. Hence for $n = m^d$ we have equality in (5). For each fixed d both the functions $E_d(n)$ and the upper bound on the right of (5) are clearly increasing functions of n. Also, since $E_d(n)$ is always an integer we have the following.

Proposition 3.2. *For all $d, n \in \mathbb{N}$ we have*

$$E_d(n) \leq \lfloor dn(1 - n^{-1/d}) \rfloor. \tag{6}$$

This bound is asymptotically tight as $n \to \infty$ and equality holds for all d-th powers $n = m^d$.

Remark: Note that for $d \in \{1, 2\}$ then we have equality in (6) as shown in [6].

4 A recursive inequality

In this section we will derive a general recursive upper bound of $E_d(n)$, that is tight in the sense that it can be realized in some specific cases. By Corollary 2.8 it suffices to consider fully nested $S \subseteq \mathbb{N}^d$ of order n, and by Proposition 3.2 we may, when necessary, assume that n is not a d-th power of an integer.

CONVENTION: Just like a d-cube had $2d$ sides, a fully nested set $S \subseteq \mathbb{N}^d$ will, in our context, have $2d$ sides as well, namely $S \cap H$ where $H : x_i = k$, $i \in [d]$ and $k \in \{1, |\pi_i(S)|\}$ is one of the $2d$ supporting hyperplanes of S.

For any fully nested $S \subseteq \mathbb{N}^d$ not contained in a hyperplane, and any hyperplane $H_k : x_d = k$, where $2 \leq k \leq |\pi_d(S)|$, we obtain a partition or *cut* by

$$\begin{aligned} S_1 &= \{\tilde{x} \in S : \pi_d(\tilde{x}) < k\}, \\ S_2 &= \{\tilde{x} \in S : \pi_d(\tilde{x}) \geq k\}. \end{aligned}$$

Assume that $|S| = n$ and that S is optimal, so $E_d(S) = E_d(n)$. Since S is fully nested the number of edges parallel to the x_d axis that cut through the hyperplane H_k, in the sense that one endvertex is in S_2 and the other is in S_1, is given by $n_{\hat{d}}(S_2) = |\pi_{\hat{d}}(S_2)|$. From this we see that for our set S we then have

$$E_d(n) = E_d(S) = E_d(S_1) + E_d(S_2) + n_{\hat{d}}(S_2). \tag{7}$$

NOTE: There is no significance to the last coordinate x_d here. This can also be obtained by any cut perpendicular to any of the d coordinate axes.

If $h = |\pi_d(S)|$, then we have a partition $S_2 = S_{2;k} \cup \cdots \cup S_{2;h}$ where each $S_{2;i} = \{\tilde{x} \in S : \pi_d(\tilde{x}) = i\}$. Since S is fully nested we have

$$\pi_{\hat{d}}(S_{2;k}) \supseteq \cdots \supseteq \pi_{\hat{d}}(S_{2;h})$$

and $n_{\hat{d}}(S_2) = |\pi_{\hat{d}}(S_{2;k})| = n_{\hat{d}}(S_{2;k})$ and therefore

$$E_d(S_2) + n_{\hat{d}}(S_2) = \sum_{i=k}^{h} \left(n_{\hat{d}}(S_{2;i}) + E_{d-1}(\pi_{\hat{d}}(S_{2;i})) \right).$$

Since

$$\sum_{i=k}^{h} n_{\hat{d}}(S_{2;i}) = \sum_{i=k}^{h} |S_{2;i}| = |S_2|,$$

we obtain

$$E_d(S_2) + n_{\hat{d}}(S_2) = |S_2| + \sum_{i=k}^{h} E_{d-1}(\pi_{\hat{d}}(S_{2;i}) \tag{8}$$

By (1), the definition of E_{d-1} as a function $\mathbb{N} \to \mathbb{N}$, we clearly have $\sum_{i=k}^{h} E_{d-1}(\pi_{\hat{d}}(S_{2;i})) \leq E_{d-1}(|S_2|)$ and hence from (8) we then get

$$E_d(S_2) + n_{\hat{d}}(S_2) \leq |S_2| + E_{d-1}(|S_2|)$$

and hence from (7) we get the inequality

$$E_d(n) = E_d(S) \leq E_d(|S_1|) + E_{d-1}(|S_2|) + |S_2|. \tag{9}$$

We summarize in the following.

Lemma 4.1. *Let $n \in \mathbb{N}$ and $S \subseteq \mathbb{N}^d$ fully nested and optimal with $|S| = n$. Then for any cut that partitions S into two proper sets S_1 and S_2 of order n_1 and n_2 respectively, and so $n_1 + n_2 = n$, we have*

$$E_d(n) \leq E_d(n_1) + E_{d-1}(n_2) + n_2. \tag{10}$$

Note that (10) does not hold for any partition $n = n_1 + n_2$; only for the mentioned particular partitions. The main thing to notice in Lemma 4.1 is that there *exists* a proper partition of n that yields the desired inequality.

Let $S \subseteq \mathbb{N}^d$ be a fully nested and optimal with $|S| = n$. Of particular interest is the special cut when $k = h = |\pi_d(S)|$, so S_2 is a $(d-1)$-dimensional side of S. In this case $S_2 = S_{2;h}$ and so by (7) and (8) we obtain

$$E_d(n) = E_d(S) = E_d(S_1) + E_{d-1}(\pi_{\hat{d}}(S_2)) + |S_2|. \tag{11}$$

Note that if in addition S_1 and S_2 are also optimal, then (11) will yield an equality in (10). We will see that such an equality can be obtained in (10). With this in mind, it is our next objective to show that for each $n \in \mathbb{N}$ there is always a fully nested and optimal set S of order n and a cut with a partition $S = S_1 \cup S_2$ where S_2 is a side of S such that equality holds in (10). To do that we need results in the next section.

5 Pseudo cubes, pseudo cubics and their properties

In this section we define some specific representations for integers, their corresponding sets in $\mathbb{N}^d$, and prove some properties that will demonstrate that we can always assume that an optimal $S \subseteq \mathbb{N}^d$ is one of these corresponding sets.

Definition 5.1. *For $d \in \mathbb{N}$, call a number $n \in \mathbb{N}$ a* pseudo d-cubic *if $n = (m+1)^\ell m^{d-\ell} := [m, \ell]^d$ for some $m \in \mathbb{N}$ and $\ell \in \{0, 1, \ldots, d-1\}$. A* pseudo cubic *is then a pseudo d-cubic for some d.*

Any pseudo d-cubic n yields a corresponding pseudo d-cube $[\![m, \ell]\!]^d := [m+1]^\ell \times [m]^{d-\ell} \subseteq \mathbb{N}^d$.

Remarks: (i) Although the word "cubic" is an adjective, we will use it both as such and also as a noun. (ii) We do reserve the right to interpret $[m, \ell]^d$ when $\ell = d$ by the defining algebraic expression in Definition 5.1, so $[m, d]^d = [m+1, 0]^d$. (iii) A pseudo d-cube $[\![m, \ell]\!]^d$ has $2d$ sides; 2ℓ of which are copies of $[\![m, \ell-1]\!]^{d-1}$, and $2(d-\ell)$ of which are copies of $[\![m, \ell]\!]^{d-1}$, both types of sides are pseudo $(d-1)$-cubes.

With the above remark in mind, then clearly for d fixed, every $n \in \mathbb{N}$ is between two pseudo d-cubics: $[m, \ell]^d \leq n < [m, \ell+1]^d$ for some m and $\ell \in \{0, 1, \ldots, d-1\}$. Since $(m+1)^{\ell+1} m^{d-\ell-1} - (m+1)^\ell m^{d-\ell} = (m+1)^\ell m^{d-\ell-1}$, then the difference between two consecutive pseudo d-cubics is a pseudo $(d-1)$-cubic.

Recall the lexicographical ordering on $\mathbb{Z}^d$:

$$\tilde{x} < \tilde{y} \Leftrightarrow x_i = y_i \text{ for } 1 \leq i \leq j-1 \text{ and } x_j < y_j.$$

The lexicographical ordering is a total/linear ordering of the elements of $\mathbb{Z}^d$.

With the above convention we have similarly to the Pascal's Rule for binomial coefficient the following.

Claim 5.2. *For a fixed m and $\ell \in \{0, 1, \ldots, d-1\}$ we have for pseudo cubics that*

$$[m, \ell+1]^d = [m, \ell]^d + [m, \ell]^{d-1}.$$

Also, for a fixed d we have $[m, \ell]^d \leq [m', \ell']^d$ iff $(m, \ell) \leq (m', \ell')$ lexicographically.

Note that although the partition in Claim 5.2 could be defined for all integer values of ℓ, including negative ℓ, it is an integer partition only for $\ell \in \{0, 1, \ldots, d-1\}$.

In a similar fashion to the unique binomial representation of an integer [9, p. 55] and [7, Lemma 7.1], Claim 5.2 yields the following.

Proposition 5.3. *Every $n \in \mathbb{N}$ has a unique* pseudo d-cubic representation (d-PCR) *as*

$$n = [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c$$

where $c, m_c \geq 1$ and $(m_d, \ell_d) > (m_{d-1}, \ell_{d-1}) > \cdots > (m_c, \ell_c)$ lexicographically.

Proof. We proceed in a greedy fashion; for a given n we choose the unique (m_d, ℓ_d) such that $[m_d, \ell_d]^d$ is the largest pseudo d-cubic less than or equal to n. We continue by letting $[m_{d-1}, \ell_{d-1}]^{d-1}$ be the largest pseudo $(d-1)$-cube less than or equal to $n - [m_d, \ell_d]^d$. By Claim 5.2 we have then have $(m_d, \ell_d) > (m_{d-1}, \ell_{d-1})$. The rest follows by induction on n. $\square$

When either d is fixed or irrelevant, we will just write PCR for d-PCR of a natural number n.

Remarks: (i) Note that for the d-PCR of n we have $m_d = \lfloor \sqrt[d]{n} \rfloor$. (ii) Also, since $[m_d, \ell_d]^d \leq n < [m_d, \ell_d + 1]^d$ then

$$\ell_d = \left\lfloor \frac{\log(n/m_d^d)}{\log(1 + 1/m_d)} \right\rfloor.$$

This can be used to obtain a quick recursive method to obtain the d-PCR of n as indicated in Observation 5.5 here below. (ii) By letting $m_1 = m_2 = \cdots = m_{c-1} = 0$ and $\ell_1 = \ell_2 = \cdots = \ell_{c-1} = 0$, and noting that $[0, 0]^d = 0$, we can, when needed, assume each d-PCR to have exactly d terms.

Observation 5.4. *If $n, n' \in \mathbb{N}$ have d-PCR given by*

$$\begin{aligned} n &= [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_1, \ell_1]^1, \\ n' &= [m'_d, \ell'_d]^d + [m'_{d-1}, \ell'_{d-1}]^{d-1} + \cdots + [m'_1, \ell'_1]^1, \end{aligned}$$

then $n \leq n'$ iff $(m_d, \ell_d, m_{d-1}, \ell_{d-1}, \ldots, m_1, \ell_1) \leq (m'_d, \ell'_d, m'_{d-1}, \ell'_{d-1}, \ldots, m'_1, \ell'_1)$ lexicographically.

The following observation is convenient when working recursively.

Observation 5.5. *If $n \in \mathbb{N}$ has a d-PCR given by*

$$n = [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c,$$

then the $(d-1)$-PCR of $n - [m_d, \ell_d]^d$ is given by

$$n - [m_d, \ell_d]^d = [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c.$$

The PCR of an integer gives rise to a special fully nested configuration. In order to describe this we need the following definition.

Definition 5.6. *For a set $J \subseteq [d]$ with $|J| = j$ and a fixed point $\tilde{a} \in \mathbb{N}^j$ we have a lifting map*

$$\lambda_{J;\tilde{a}} : \mathbb{Z}^{d-j} \to \mathbb{Z}^d,$$

such that $\pi_J \circ \lambda_{J;\tilde{a}} : \mathbb{Z}^{d-j} \to \mathbb{Z}^j$ is the constant map taking each element to $\tilde{a}$, and $\pi_{[d]\setminus J} \circ \lambda_{J;\tilde{a}} : \mathbb{Z}^{d-j} \to \mathbb{Z}^{d-j}$ is the identity map.

If the set J is given explicitly $J = \{h_1, \dots, h_j\}$, then we usually write $\lambda_{J;\tilde{a}}$ as $\lambda_{h_1,\dots,h_j;\tilde{a}}$.

From the d-PCR $n = [m_d, \ell_d]^d + \cdots + [m_c, \ell_c]^c$ where $c, m_c \geq 1$, we obtain the set $[\![n]\!]^d \subseteq \mathbb{N}^d$ of order n recursively by setting $[\![0]\!]^i := \emptyset$ for each i and

$$[\![n]\!]^d := [\![m_d, \ell_d]\!]^d \cup \lambda_{\ell_d+1;m_d+1}([\![n - [m_d, \ell_d]^d]\!]^{d-1}). \tag{12}$$

We list some properties of these sets $[\![n]\!]^d \subseteq \mathbb{N}^d$ that are immediate. From the d-PCR $n = [m_d, \ell_d]^d + \cdots + [m_c, \ell_c]^c$ we have the following.

Proposition 5.7. *For $n \in \mathbb{N}$ we have*

1. $|[\![n]\!]^d| = n$.
2. $[\![n]\!]^d \subseteq [\![n']\!]^d$ *iff* $n \leq n'$.
3. $[\![n]\!]^d$ *is fully nested.*

Proof. Since the defining union in the recursion (12) is disjoint, the first assertion follows by induction on n.

The second assertion follows from Proposition 5.3.

For the third assertion we see that since $[\![m_d, \ell_d]\!]^d$ is fully nested, then for each $i \neq \ell_m + 1$, $[\![n]\!]^d$ is i-nested iff $\lambda_{\ell_d+1;m_d+1}([\![n - [m_d, \ell_d]^d]\!]^{d-1})$ is i-nested, which by induction on n is i-nested. Since $[\![m_d, \ell_d]\!]^{d-1}$ contains $[\![n - [m_d, \ell_d]^d]\!]^{d-1}$ we also have $(\ell_m + 1)$-nestedness. $\square$

Definition 5.8. *For $n \in \mathbb{N}$, a set $[\![n]\!]^d \subseteq \mathbb{N}^d$ as given above, is called a d-*cubicle*, or simply a* cubicle *if d is irrelevant.*

As with cubics, we can also talk about a *side* of a cubicle as the intersection of the cubicle with one of its supporting hyperplanes, most notably the planes $x_i = 1$ for various i. These d sides of $[\![n]\!]^d$ are given by $\pi_{\hat{\imath}}([\![n]\!]^d)$ for each i. By the recursive definition (12) we also have the following.

Claim 5.9. *For each $n \in \mathbb{N}$ we have that $\pi_{\hat{\imath}}([\![n]\!]^d)$ is a $(d-1)$-cubicle. Hence, each projection of $[\![n]\!]^d$ is also a cubicle.*

Our next lemma will be useful in the next section.

Lemma 5.10. *For $n \in \mathbb{N}$ with d-PCR $n = [m_d, \ell_d]^d + \cdots + [m_c, \ell_c]^c$, then $[\![n]\!]^d \subseteq \mathbb{N}^d$ always has a side S with $|S| \geq \frac{n}{m_d+1}$. If further $\ell_d < d-1$ then every side $S = \pi_{\hat{\imath}}([\![n]\!]^d)$ where $i \in \{\ell_d + 2, \dots, d\}$ satisfies $|S| \geq \frac{n}{m_d}$.*

Proof. Since $[\![n]\!]^d \subseteq [\![m_d+1,0]\!]^d$ and $[\![n]\!]^d$ is fully nested we clearly have $|\pi_{\hat{\imath}}([\![n]\!]^d)| \geq \frac{n}{m_d+1}$.

If further $\ell_d < d-1$, then for each $i \in \{\ell_d+2,\dots,d\} \neq \emptyset$ we have $|\pi_{\hat{\imath}}([\![n]\!]^d)| = m_d$ and since $[\![n]\!]^d$ is fully nested we therefore have $|\pi_{\hat{\imath}}([\![n]\!]^d)| \geq \frac{n}{m_d}$. □

Our ultimate goal is to show that the d-cubicles are sets achieving the most edges among induced graphs in $\mathbb{N}^d$.

Definition 5.11. *For $n \in \mathbb{N}$ let $F_d(n)$ be the number of edges that the d-cubicle $[\![n]\!]^d$ induces in $\mathbb{N}^d$, that is $F_d(n) := E_d([\![n]\!]^d)$.*

Our goal is therefore to show that $E_d(n) = F_d(n)$, although $[\![n]\!]^d$ is by no means the unique fully nested configuration in $\mathbb{N}^d$ yielding the maximum number $E_d(n)$ of edges. The rest of this current section and the following next section will be devoted to obtain this goal.

In the same fashion as we derived (11) we get by Claim 5.2 the following recursion for each $\ell \in \{0,1,\dots,d\}$.

$$F_d([m,\ell+1]^d) = F_d([m,\ell]^d) + F_{d-1}([m,\ell]^{d-1}) + [m,\ell]^{d-1}. \tag{13}$$

Likewise, for $n \in \mathbb{N}$ with d-PCR $n = [m_d,\ell_d]^d + [m_{d-1},\ell_{d-1}]^{d-1} + \cdots + [m_c,\ell_c]^c$, we get by the recursive definition of $[\![n]\!]^d$ and Definition 5.11 that

$$F_d(n) = F_d([m_d,\ell_d]^d) + F_{d-1}(n - [m_d,\ell_d]^d) + n - [m_d,\ell_d]^d. \tag{14}$$

By Observation 5.5 and (14) we obtain recursively an expression for $F_d(n)$, namely

$$F_d(n) = \sum_{i=c}^{d} \left(F_i([m_i,\ell_i]^i) + (d-i)[m_i,\ell_i]^i\right), \tag{15}$$

where for each pseudo cube we again obtain recursively by (13) that

$$F_d([m,\ell]^d) = d[m,\ell]^d - \ell[m,\ell-1]^{d-1} - (d-\ell)[m,\ell]^{d-1}, \tag{16}$$

for $\ell \in \{0,1,\dots,d-1\}$. Note that for $\ell \in \{0,d-1\}$ then (16) yields a valid formula (one with m and the other with $m+1$), which by itself can be verified by induction using (13) as well. Hence, (16) yields an explicit formula for $F_d(n)$ for every pseudo d-cubic n.

Remarks: (i) The formula (16) for $F_d([m,\ell]^d)$ can also be obtained from Observation 3.1. (ii) Note that (16) for $\ell \in \{0,d-1\}$ matches the upper bound given in (6) for $n = [m,0]^d$, which shows that $F_d(n) = E_d(n)$ for every d-power of an integer n, something already stated clearly in Proposition 3.2.

From (15) and (16) we then get the following explicit formula for $F_d(n)$.

Observation 5.12. *For $n \in \mathbb{N}$ with the above d-PCR we then have $F_d(n) = dn - \delta_d(n)$ where the discrepancy is given by*

$$\delta_d(n) = \sum_{i=c}^{d} \left(\ell_i[m_i,\ell_i-1]^{i-1} + (i-\ell_i)[m_i,\ell_i]^{i-1}\right).$$

We now prove some important properties of the function F_d. In order to do that we need to introduce some notation.

Notation: Let $n \in \mathbb{N}$ with d-PCR $n = [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c$. (i) Let $[n]^d_- := [m_d, \ell_d]^d$ denote the largest pseudo cubic $\leq n$, so $n = [n]^d_- + n'$ where $n' = [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c < [m_d, \ell_d]^{d-1}$, and (ii) let $[n]^d_+$ denote the smallest pseudo cubic $> n$, so for each $\ell_d \in \{0, 1, \ldots, d-1\}$

$$[n]^d_+ := [m_d, \ell_d + 1]^d.$$

Note that $[n]^d_-$ and $[n]^d_+$ are consecutive pseudo cubics and $[n]^d_- \leq n < [n]^d_+$. (iii) Let

$$[n]^{d-1}_\Delta := [n]^d_+ - [n]^d_-,$$

so $[n]^{d-1}_\Delta = [m_d, \ell_d]^{d-1}$ in terms of the d-PCR of n above. Since $m_d = \lfloor \sqrt[d]{n} \rfloor$ in terms of n and d alone, we obtain by partitioning $[\![n]\!]^d$ into "slices" of height one and order $[n]^{d-1}_\Delta$ along the (ℓ_d+1)-th coordinate that

$$F_d([n]^d_-) = \lfloor \sqrt[d]{n} \rfloor F_{d-1}([n]^{d-1}_\Delta) + (\lfloor \sqrt[d]{n} \rfloor - 1)[n]^{d-1}_\Delta, \tag{17}$$

and hence for each $i \in \{0, 1, \ldots, \lfloor \sqrt[d]{n} \rfloor\}$ we then obtain by (14) and (17) that

$$F_d(n) = F_{d-1}(n') + n' + i(F_{d-1}([n]^{d-1}_\Delta) + [n]^{d-1}_\Delta) + (\lfloor \sqrt[d]{n} \rfloor - i)F_{d-1}([n]^{d-1}_\Delta) + (\lfloor \sqrt[d]{n} \rfloor - i - 1)[n]^{d-1}_\Delta, \tag{18}$$

where $n' = n - [n]^d_- = n - [m_d, \ell_d]^d$. (iv) Needless to say we can recursively define

$$\begin{aligned} [n]^d_{1+} &:= [n]^d_+, \\ [n]^d_{(i+1)+} &:= [[n]^d_{i+}]^d_+, \text{ for } i \geq 1 \end{aligned}$$

thereby obtaining a strictly increasing sequence of consecutive pseudo d-cubics

$$[n]^d_+ < [n]^d_{2+} < \cdots < [n]^d_{i+} < \cdots,$$

the unique such sequence that contains every pseudo d-cubic strictly larger than n. Also, this can be done in the negative direction as well to obtain $[n]^d_- > [n]^d_{2-} > \cdots > [n]^d_{i-} > \cdots$, the unique sequence containing d-cubics less than or equal to n. However, here the recursion is slightly different, as $[n]^d_-$ is the largest pseudo cubic $\leq n$ as suppose to $< n$.

$$\begin{aligned} [n]^d_{1-} &:= [n]^d_-, \\ [n]^d_{(i+1)-} &:= [[n]^d_{i-} - 1]^d_-, \text{ for } i \geq 1. \end{aligned}$$

Let $d, n \in \mathbb{N}$ be fixed and consider the following statements.

$\mathbf{P}(d,n)$: $F_d(n_1) + F_d(n_2) \leq F_d([m,\ell]^d) + F_d(n')$, whenever $n_1 + n_2 = [m,\ell]^d + n' = n$ and $n_1, n_2 \leq [m,\ell]^d$.

$\mathbf{P}'(d,n)$: $F_d(n_1) + F_d(n_2) \leq F_d([n_1]^d_+) + F_d(n')$, whenever $n_1 + n_2 = [n_1]^d_+ + n' = n$ and $n_2 \leq n_1$.

For a fixed $d, n \in \mathbb{N}$ we clearly have the implication $\mathbf{P}(d,n) \Rightarrow \mathbf{P}'(d,n)$. We now briefly argue the reverse implication $\mathbf{P}'(d,n) \Rightarrow \mathbf{P}(d,n)$.

Assume $\mathbf{P}'(d,n)$ and let $n_1 + n_2 = [m,\ell]^d + n' = n$ where $n_1, n_2 \leq [m,\ell]^d$. With the notation above, there is an finite sequence

$$[n_1]^d_+ < \cdots < [n_1]^d_{j+} = [m,\ell]^d,$$

and with repeated use of $\mathbf{P}'(d,n)$ we obtain

$$F_d([n_1]^d_{i+}) + F_d(n^{(i)}) \le F_d([n_1]^d_{(i+1)+}) + F_d(n^{(i+1)})$$

for each i, where $[n_1]^d_{i+} + n^{(i)} = [n_1]^d_{(i+1)+} + n^{(i+1)} = n$, which yields $F_d(n_1) + F_d(n_2) \le F_d([m,\ell]^d) + F_d(n')$. This proves $\mathbf{P}'(d,n) \Rightarrow \mathbf{P}(d,n)$, and so we have $\mathbf{P}'(d,n) \Leftrightarrow \mathbf{P}(d,n)$.

For $d, n \in \mathbb{N}$ let our goal be phrased as the following statement.

$\mathbf{EF}(d,n) : E_d(n) = F_d(n)$.

To prove that $\mathbf{EF}(d,n)$ is valid for every $d, n \in \mathbb{N}$, we first note that both $\mathbf{EF}(d,n)$ and $\mathbf{P}(d,n)$ are trivially true whenever either $d = 1$ or $n = 1$. We then proceed to show that for any $N \in \mathbb{N}$

$$\mathbf{P}(d,n) \wedge \mathbf{EF}(d,n) \text{ for } d+n < N \Rightarrow \mathbf{P}'(d,n) \wedge \mathbf{EF}(d,n) \text{ for } d+n = N. \tag{19}$$

We conclude this section by proving the following implication

$$\mathbf{P}(d,n) \wedge \mathbf{EF}(d,n) \text{ for } d+n < N \Rightarrow \mathbf{P}'(d,n) \text{ for } d+n = N. \tag{20}$$

The remainder of (19) will be proved in the following section.

To prove (20) let $d, n, n_1, n_2, N \in \mathbb{N}$ and $n' \ge 0$ be such that $d + n = N$, $n_2 \le n_1$, and $n_1 + n_2 = [n_1]^d_+ + n' = n$. Write $n_1 = [n_1]^d_- + n'_1$, $n_2 = [n_2]^d_- + n'_2$, and $[n_1]^d_+ = [n_1]^d_- + [n_1]^{d-1}_\Delta$. By (13) and (14) we get

$$\begin{aligned} F_d([n_1]^d_+) &= F_d([n_1]^d_-) + F_d([n_1]^{d-1}_\Delta) + [n_1]^{d-1}_\Delta, \\ F_d(n_1) &= F_d([n_1]^d_-) + F_{d-1}(n'_1) + n'_1. \end{aligned}$$

Let $i \in \{0, 1, \ldots, \lfloor \sqrt[d]{n_2} \rfloor\}$ be such that

$$(i-1)[n_2]^{d-1}_\Delta + n'_2 + n'_1 < [n_1]^{d-1}_\Delta \le i[n_2]^{d-1}_\Delta + n'_2 + n'_1.$$

Such an i exists: since $n_2 \le n_1$ and $n_1 + n_2 = [n_1]^d_+ + n'$ where $n' \ge 0$, we get by subtracting $[n_1]^d_-$ from each side and rewriting n_2, that

$$\begin{aligned} n'_2 + \lfloor \sqrt[d]{n_2} \rfloor [n_2]^{d-1}_\Delta + n'_1 &= n_2 + n'_1 \\ &= n_2 + (n_1 - [n_1]^d_-) \\ &= n' + ([n_1]^d_+ - [n_1]^d_-) \\ &= n' + [n_1]^{d-1}_\Delta \\ &\ge [n_1]^{d-1}_\Delta. \end{aligned}$$

Hence, we have $i[n_2]^{d-1}_\Delta + n'_2 + n'_1 = [n_1]^{d-1}_\Delta + n''$ where $0 \le n'' < [n_2]^{d-1}_\Delta$. Note that if $i = 0$ then $n'_1 + n'_2 \ge [n_1]^{d-1}_\Delta \ge [n_2]^{d-1}_\Delta$ and so $(i-1)[n_2]^{d-1}_\Delta + n'_2 + n'_1 \ge 0$. By (18) we then get

$$F_d(n_2) = i(F_{d-1}([n_2]^{d-1}_\Delta) + [n_2]^{d-1}_\Delta) + (\lfloor \sqrt[d]{n_2} \rfloor - i) F_{d-1}([n_2]^{d-1}_\Delta) + (\lfloor \sqrt[d]{n_2} \rfloor - i - 1)[n_2]^{d-1}_\Delta + F_{d-1}(n'_2) + n'_2$$

and we then obtain

$$\begin{aligned} & F_d(n_2) + F_d(n_1) \\ = \; & \Big\{ i(F_{d-1}([n_2]^{d-1}_\Delta) + [n_2]^{d-1}_\Delta) + (\lfloor \sqrt[d]{n_2} \rfloor - i) F_{d-1}([n_2]^{d-1}_\Delta) + (\lfloor \sqrt[d]{n_2} \rfloor - i - 1)[n_2]^{d-1}_\Delta + F_{d-1}(n'_2) + n'_2 \Big\} \\ & + F_d([n_1]^d_-) + F_{d-1}(n'_1) + n'_1 \\ = \; & \Big\{ iF_{d-1}([n_2]^{d-1}_\Delta) + F_{d-1}(n'_2) + F_{d-1}(n'_1) \Big\} + \Big\{ i[n_2]^{d-1}_\Delta + n'_2 + n'_1 \Big\} + F_d([n_1]^d_-) + G(n_2, i) \end{aligned} \tag{21}$$

where

$$G(n_2,i) := (\lfloor \sqrt[d]{n_2}\rfloor - i)F_{d-1}([n_2]_\Delta^{d-1}) + (\lfloor \sqrt[d]{n_2}\rfloor - i - 1)[n_2]_\Delta^{d-1}.$$

Note that $G(n_2,i)$ can be interpreted geometrically as the number of edges induced by a rectangular "box" in $\mathbb{N}^d$ of with base $[n_2]_\Delta^{d-1}$ and height $\lfloor \sqrt[d]{n_2}\rfloor - i$, that is, a box consisting of $\lfloor \sqrt[d]{n_2}\rfloor - i$ copies of $(d-1)$-cubicles of order $[n_2]_\Delta^{d-1}$ stacked one on top of the other.

By our induction hypothesis, then $F_{d-1} = E_{d-1}$ in (21), and recall that $i[n_2]_\Delta^{d-1} + n_2' + n_1' = [n_1]_\Delta^{d-1} + n''$, so we obtain

$$\begin{aligned} F_d(n_2) + F_d(n_1) &= \left\{ iE_{d-1}([n_2]_\Delta^{d-1}) + E_{d-1}(n_2') + E_{d-1}(n_1') \right\} \\ &\quad + \left\{ [n_1]_\Delta^{d-1} + n'' \right\} + F_d([n_1]_-^d) + G(n_2,i). \end{aligned} \tag{22}$$

If $i = 0$, we get by induction hypothesis that

$$iE_{d-1}([n_2]_\Delta^{d-1}) + E_{d-1}(n_2') + E_{d-1}(n_1') = E_{d-1}(n_2') + E_{d-1}(n_1') \le E_{d-1}([n_1]_\Delta^{d-1}) + E_{d-1}(n'').$$

If $i > 0$, we get since E_{d-1} is super-additive that

$$iE_{d-1}([n_2]_\Delta^{d-1}) + E_{d-1}(n_2') + E_{d-1}(n_1') \le E_{d-1}((i-1)[n_2]_\Delta^{d-1} + n_2' + n_1') + E_{d-1}([n_2]_\Delta^{d-1}),$$

which by inductive hypothesis is $\le E_{d-1}([n_1]_\Delta^{d-1}) + E_{d-1}(n'')$. With this in mind, and that $E_{d-1} = F_{d-1}$, we obtain from (22) that

$$F_d(n_1) + F_d(n_2) \le F_d([n_1]_-^d) + F_{d-1}([n_1]_\Delta^{d-1}) + [n_1]_\Delta^{d-1} + F_{d-1}(n'') + n'' + G(n_2,i). \tag{23}$$

Note that by definition of n'' we have $i[n_2]_\Delta^{d-1} + n_2' + n_1' = [n_1]_\Delta^{d-1} + n''$, and since $n_1 + n_2 = [n_1]_+^d + n'$, we have $n' = (\lfloor \sqrt[d]{n_2}\rfloor - i)[n_2]_\Delta^{d-1} + n''$. We now consider two cases.

First case $i = \lfloor \sqrt[d]{n_2}\rfloor$: In this case $G(n_2,i) = G(n_2, \lfloor \sqrt[d]{n_2}\rfloor) = -[n_2]_\Delta^{d-1}$, and (23) becomes

$$\begin{aligned} F_d(n_1) + F_d(n_2) &\le \left\{ F_d([n_1]_-^d) + F_{d-1}([n_1]_\Delta^{d-1}) + [n_1]_\Delta^{d-1} + \right\} + F_{d-1}(n'') + n'' - [n_2]_\Delta^{d-1} \\ &= F_d([n_1]_+^d) + F_{d-1}(n'') + n'' - [n_2]_\Delta^{d-1} \\ &\le F_d([n_1]_+^d) + F_{d-1}(n''), \end{aligned}$$

which is $\le F_d([n_1]_+^d) + F_d(n'')$, since $F_{d-1}(n'') = E_{d-1}(n'') \le E_d(n'') = F_d(n'')$ by induction hypothesis. This proves $\mathbf{P}'(d,n)$ in this case since $n' = n''$.

Second case $i < \lfloor \sqrt[d]{n_2}\rfloor$: Since $n'' < [n_2]_\Delta^{d-1}$, one can put n'' points on one $[n_2]_\Delta^{d-1}$-side of the $G(n_2,i)$-box mentioned here above, thereby obtaining $G(n_2,i) + F_{d-1}(n'') + n''$ edges. Hence we have

$$G(n_2,i) + F_{d-1}(n'') + n'' \le E_d((\lfloor \sqrt[d]{n_2}\rfloor - i)[n_2]_\Delta^{d-1} + n''),$$

and (23) yields

$$\begin{aligned} F_d(n_1) + F_d(n_2) &\le \left\{ F_d([n_1]_-^d) + F_{d-1}([n_1]_\Delta^{d-1}) + [n_1]_\Delta^{d-1} + \right\} + E_d((\lfloor \sqrt[d]{n_2}\rfloor - i)[n_2]_\Delta^{d-1} + n'') \\ &= F_d([n_1]_+^d) + F_d((\lfloor \sqrt[d]{n_2}\rfloor - i)[n_2]_\Delta^{d-1} + n'') \end{aligned}$$

which proves $\mathbf{P}'(d,n)$ in this case as well, since $n' = (\lfloor \sqrt[d]{n_2}\rfloor - i)[n_2]_\Delta^{d-1} + n''$ here in this case. This completes the proof of (20). We complete the proof of (19) in the following section.

6 The final steps in the proof of $E = F$

In this section we prove the following implication

$$\mathbf{P}(d,n) \wedge \mathbf{EF}(d,n) \text{ for } d+n < N \Rightarrow \mathbf{P}'(d,n) \wedge \mathbf{EF}(d,n) \text{ for } d+n = N.$$

From previous section we already have (20), so it suffices to verify the following implication

$$\mathbf{P}(d,n) \wedge \mathbf{EF}(d,n) \text{ for } d+n < N \Rightarrow \mathbf{EF}(d,n) \text{ for } d+n = N. \tag{24}$$

Before we delve into that, we need a property of fully nested sets in general.

Let $S \subseteq \mathbb{N}^d$ be a fully nested set with $|S| = n$. For each $i \in [d]$ let $h_i = |\pi_i(S)|$ be the *height* of S along i-th axis, and let $A_i = |\{\tilde{x} \in S : x_i = h_i\}|$ the *area* of the top layer of S along the i-th axis. Since S is fully nested we have $n = |S| \geq h_i A_i$ for each i and hence

$$\begin{aligned}
\frac{\min(A_1,\ldots,A_d)}{n} &= \min\left(\frac{A_1}{n},\ldots,\frac{A_d}{n}\right) \\
&\leq \min\left(\frac{A_1}{h_1A_1},\ldots,\frac{A_d}{h_dA_d}\right) \\
&= \min\left(\frac{1}{h_1},\ldots,\frac{1}{h_d}\right) \\
&= \frac{1}{\max(h_1,\ldots,h_d)}.
\end{aligned}$$

By Proposition 3.2 we can assume that n is not a d-th power of an integer. Hence, if n has d-PCR given by $n = [m_d,\ell_d]^d + [m_{d-1},\ell_{d-1}]^{d-1} + \cdots + [m_c,\ell_c]^c$, then either $c < d$, or $c = d$ and $1 \leq \ell_d \leq d-1$. In either case we have $m_d < \sqrt[d]{n} < m_d + 1$. By definition of h_i we have

$$m_d^d < n < h_1 \cdots h_d$$

and hence there is at least one i with $h_i \geq m_d + 1$, and so $\max(h_1,\ldots,h_d) \geq m_d + 1$. From above we then have

$$\frac{\min(A_1,\ldots,A_d)}{n} \leq \frac{1}{m_d+1}.$$

By symmetry we may assume $\min(A_1,\ldots,A_d) = A_d$ and so $A_d \leq \frac{n}{m_d+1}$. From this we get the following.

Observation 6.1. *Let $S \subseteq \mathbb{N}^d$ be a fully nested set with $|S| = n$. Then there is a suitable permutation of the coordinates of S and a partition $S = S_1 \cup S_2$ such that (11) holds and where $n_2 = |S_2| = A_d \leq \frac{n}{m_d+1}$.*

So if $S \subseteq \mathbb{N}^d$ is a fully nested set with $|S| = n$ and where the d-PCR is given by $n = [m_d,\ell_d]^d + [m_{d-1},\ell_{d-1}]^{d-1} + \cdots + [m_c,\ell_c]^c = [n]_-^d + n'$, then by 6.1 we can assume there is partition $S = S_1 \cup S_2$ with

$$n_1 = n - n_2 \geq \frac{m_d}{m_d+1} n \geq \frac{m_d}{m_d+1}[n]_-^d \geq [m_d,\ell_d - 1]^d.$$

Note, with our convention that $[m_d,0]^d = [m_d - 1,d]^d$ and keeping in mind that $m_d/(m_d+1) \geq (m_1-1)/m_d$, the above inequality is valid also for $\ell_d = 0$. We therefore have the following.

Observation 6.2. *Let $S \subseteq \mathbb{N}^d$ be a fully nested set with $|S| = n$, where the d-PCR is given by $n = [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c = [n]^d_- + n'$, then we can assume there is a partition $S = S_1 \cup S_2$ such that (11) holds and where $[n_1]^d_- = [n]^d_-$ or $[n_1]^d_- = [n]^d_{2-}$.*

Consider now an integer partition $n = n_1 + n_2$ where $n = [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c$ is the d-PCR of n and $n_2 \leq \frac{n}{m_d+1}$. We then have $n_1 \geq \frac{m_d}{m_d+1} n$ and we have the following.

Lemma 6.3. *If $n = n_1 + n_2$ where $n_1 \geq \frac{m_d}{m_d+1} n$ and $n_2 \leq \frac{n}{m_d+1}$, then $[\![n_1]\!]^d$ has a side that covers $[\![n_2]\!]^{d-1}$.*

Proof. Note that $n = n_1 + n_2$ where $n_2 \leq \frac{n}{m_d+1}$ is equivalent to $n = n_1 + n_2$ where $n_2 \leq n_1/m_d$. Let the d-PCR of n be given by $n = [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c$. By Observation 6.2 we have two cases to consider.

First case: $[n_1]^d_- = [n]^d_- = [m_d, \ell_d]^d$. Here $[\![n_1]\!]^d$ has a side $S = [\![m_d, \ell_d]\!]^{d-1}$ and since $n < [m_d, \ell_d + 1]^d$ we have $|S| \geq n/(m_d + 1)$. Since $n_2 \leq n/(m_d + 1) \leq |S|$, then S covers $[\![n_2]\!]^{d-1}$ by Proposition 5.7.

Second case: $[n_1]^d_- = [n]^d_{2-} = [m_d, \ell_d - 1]^d$. Here we need to consider two possibilities:

If $\ell_d \geq 1$, then $0 \leq \ell_d - 1 < d - 1$, so by Lemma 5.10 $[\![n_1]\!]^d$ has a side S with $|S| \geq n_1/m_d$. Since $n_2 \leq n_1/m_d$ then S covers $[\![n_2]\!]^{d-1}$ by Proposition 5.7.

If $\ell_d = 0$, then $[n_1]^d_- = [m_d - 1, d - 1]^d$, and $[\![n_1]\!]^d$ has a (bottom) side $S = [\![m_d - 1, d - 1]\!]^{d-1} = [\![m_d, 0]\!]^{d-1}$. Since $[m_d, 0]^d > n_1$ we have $|S| > n_1/m_d$, so as before S covers $[\![n_2]\!]^{d-1}$ by Proposition 5.7. $\square$

We now consider the two cases based on the above Observation 6.2.

First case: $[n_1]^d_- = [n]^d_- = [m_d, \ell_d]^d$. In this case we have for our partition $S = S_1 \cup S_2$ that $n_1 \geq [n]^d_-$ and hence $n_1 = [n]^d_- + n'_1$ where $n'_1 \geq 0$ and $n_2 \leq n - [n]^d_-$. Since $n_2 = n - n_1$, we have by (11) and induction hypothesis that

$$
\begin{aligned}
E_d(n) &= E_d(S_1) + E_{d-1}(S_2) + n_2 \\
&\leq F_d(n_1) + F_{d-1}(n_2) + n_2 \\
&= F_d([n]^d_- + n'_1) + F_{d-1}(n_2) + n_2
\end{aligned} \tag{25}
$$

Here the d-PCR of n_1 is obtained by adding $[n]^d_-$ to the d-PCR of n'_1. With this in mind we obtain by (14) that

$$F_d([n]^d_- + n'_1) = F_d([n]^d_-) + F_{d-1}(n'_1) + n'_1.$$

Substituting this expression into (25) we then obtain

$$E_d(n) \leq F_d([n]^d_-) + F_{d-1}(n'_1) + F_{d-1}(n_2) + n - [n]^d_-.$$

By induction hypothesis and the super-additivity of E_{d-1} we obtain

$$
\begin{aligned}
F_{d-1}(n'_1) + F_{d-1}(n_2) &= E_{d-1}(n'_1) + E_{d-1}(n_2) \\
&\leq E_{d-1}(n - [n]^d_-)) \\
&= F_{d-1}(n - [n]^d_-),
\end{aligned}
$$

and hence by (14) that

$$E_d(n) \leq F_d([n]^d_-) + F_{d-1}(n - [n]^d_-) + n - [n]^d_- = F_d(n),$$

which, by definition of $E_d(n)$, proves (24) in this case.

SECOND CASE: $[n_1]^d_- = [n]^d_{2-} = [m_d, \ell_d - 1]^d$ (Recall, if $\ell_d = 0$ then $[m_d, 0]^d = [m_d - 1, d]^d$ and hence $[n_1]^d_- = [m_d - 1, d-1]^d$.) In this case we have for our partition $S = S_1 \cup S_2$ that $[n]^d_{2-} \leq n_1 < [n]^d_- = [m_d, \ell_d]^d$ and hence $n_1 = [n_1]^d_- + n'_1$, where $0 \leq n'_1 < [n]^d_- - [n]^d_{2-} = [m_d, \ell_d - 1]^{d-1}$, a difference of two consecutive pseudo d-cubics, and hence itself a pseudo $(d-1)$-cubic. By (11) and by induction hypothesis we have, as in previous case, that

$$\begin{aligned} E_d(n) &= E_d(S_1) + E_{d-1}(S_2) + n_2 \\ &\leq F_d(n_1) + F_{d-1}(n_2) + n_2 \\ &= F_d([n_1]^d_- + n'_1) + F_{d-1}(n_2) + n_2. \end{aligned} \tag{26}$$

As in the previous case the d-PCR of n_1 is obtained by adding the d-PCR of n'_1 to $[n_1]^d_-$. With this in mind we obtain, as before, by (14) that

$$F_d([n_1]^d_- + n'_1) = F_d([n_1]^d_-) + F_{d-1}(n'_1) + n'_1.$$

Substituting this expression into (26) we then obtain

$$E_d(n) \leq F_d([n_1]^d_-) + F_{d-1}(n'_1) + F_{d-1}(n_2) + (n - [n]^d_{2-}). \tag{27}$$

By (14) and (13) we have

$$F_d(n) = F_d([n]^d_{2-}) + F_{d-1}([n]^d_- - [n]^d_{2-}) + F_{d-1}(n - [n]^d_-) + (n - [n]^d_{2-}). \tag{28}$$

By (27) and (28) we see that $E_d(n) \leq F_d(n)$ can be obtained from the following inequality

$$F_{d-1}(n'_1) + F_{d-1}(n_2) \leq F_{d-1}([n]^d_- - [n]^d_{2-}) + F_{d-1}(n - [n]^d_-). \tag{29}$$

We have $n'_1 < [n]^d_- - [n]^d_{2-}$. If also $n_2 < [n]^d_- - [n]^d_{2-}$, then (29) follows from our inductive hypothesis $\mathbf{P}(d-1, n'_1 + n_2)$ and so we have (24) in this case.

Otherwise we have $n_2 > [n]^d_- - [n]^d_{2-} = [m_d, \ell_d - 1]^{d-1}$. Writing $n_2 = [m_d, \ell_d - 1]^{d-1} + n''_2$ and noting that $n_2 \leq n/(m_d + 1)$ we obtain

$$n''_2 \leq \frac{n}{m_d + 1} - [m_d, \ell_d - 1]^{d-1} = \frac{n'}{m_d + 1} < \frac{[m_d, \ell_d]^{d-1}}{m_d + 1} = [m_d, \ell_d - 1]^{d-2},$$

and hence we have $[n_2]^{d-1}_- = [m_d, \ell_d - 1]^{d-1} = [n]^d_- - [n]^d_{2-}$. By (14) we then have

$$F_{d-1}(n_2) = F_{d-1}([n_2]^{d-1}_-) + F_{d-2}(n''_2) + n''_2,$$

and hence $E_d(n) \leq F_d(n)$, which can be obtained from (29), can therefore be obtained from

$$F_{d-1}(n'_1) + F_{d-2}(n''_2) + n''_2 \leq F_{d-1}(n').$$

By induction hypothesis we have $F_{d-1}(n') = E_{d-1}(n')$, and so (29) is valid if

$$F_{d-1}(n'_1) + F_{d-2}(n''_2) + n''_2 \leq E_{d-1}(n'). \tag{30}$$

Interpreting the quantity on the left of (30) as the number of edges of induced by a set $S' \subseteq \mathbb{N}^{d-1}$ with $|S'| = n' = n - [n]^d_-$, we see it can be realized if $[\![n'_1]\!]^{d-1}$ has a side that covers $[\![n''_2]\!]^{d-2}$. In that case we have

$$F_{d-1}(n'_1) + F_{d-2}(n''_2) + n''_2 = F_{d-1}(S') \leq E_{d-1}(n'),$$

where $S' \subseteq \mathbb{N}^{d-1}$ is obtained by attaching $[\![n_2'']\!]^{d-2}$ to one of the sides of $[\![n_1']\!]^{d-1}$. We will now verify this, and thereby completing the inductive step of (24).

Recall the d-PCR $n = [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c = [n]_-^d + n'$. Since in this case we have $[n_1]_-^d = [n]_{2-}^d = [m_d, \ell_d - 1]^d$ and $[n_2]_-^{d-1} = [n]_-^d - [n]_{2-}^d = [m_d, \ell_d - 1]^{d-1}$ we have

$$n_1' \geq \frac{m_d}{m_d+1} n - [n]_{2-}^d = \frac{m_d}{m_d+1} n',$$

and since $n_1' + n_2'' = n'$ we also get

$$n_2'' = n' - n_1' \leq n' - \frac{m_d}{m_d+1} n' = \frac{n'}{m_d+1}.$$

Since $m_{d-1} \leq m_d$ and the $(d-1)$-PCR of n' is given by $n' = [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c$, that $[\![n_1']\!]^{d-1}$ has a side that covers $[\![n_2'']\!]^{d-2}$ now follows from Lemma 6.3. This completes the proof of (30) and hence (29), which then completes the proof of (24) in this case. – This completes the inductive proof of $\mathbf{EF}(d, n)$ for all $d, n \in \mathbb{N}$. By Observation 5.12 we have the following summarizing theorem, the main theorem of this article.

Theorem 6.4. *For $n \in \mathbb{N}$ with the d-PCR $n = [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c$ we have that the maximum number $E_d(n)$ of edges a set $S \subseteq \mathbb{N}^d$ with $|S| = n$ can induce is given by $E_d(n) = dn - \delta_d(n)$ where the discrepancy is given by*

$$\delta_d(n) = \sum_{i=c}^{d} \left(\ell_i [m_i, \ell_i - 1]^{i-1} + (i - \ell_i)[m_i, \ell_i]^{i-1} \right).$$

7 Some final observations and corollaries

An inequality on projections Let $S \subseteq \mathbb{N}^d$ be a fully nested set where $|S| = n = [m, \ell]^d$ is a pseudo d-cubic. In this case we have by Theorem 6.4 that

$$E_d(n) = E_d([m, \ell]^d) = dn - \ell[m, \ell - 1]^{d-1} + (d - \ell)[m, \ell]^{d-1}.$$

By Observation 3.1 we then have that

$$E_d(S) = dn - (n_{\hat{1}}(S) + \cdots + n_{\hat{d}}(S)) \leq dn - \ell[m, \ell - 1]^{d-1} + (d - \ell)[m, \ell]^{d-1},$$

and hence for any fully nested set $S \subseteq \mathbb{N}^d$ with $|S| = [m, \ell]^d$ we have

$$n_{\hat{1}}(S) + \cdots + n_{\hat{d}}(S) \geq \ell[m, \ell - 1]^{d-1} + (d - \ell)[m, \ell]^{d-1}. \tag{31}$$

Recall g_i, the gravity along i-th axis, from Section 2. For an arbitrary set $T \subseteq \mathbb{N}^2$ we have $|\pi_2(g_1(T))| = |\pi_2(T)|$ and

$$|\pi_1(g_1(T))| = \max\{|T \cap \pi_2^{-1}(x)| : x \in \mathbb{N}\} \leq |\pi_1(T)|.$$

Hence, if $S \subseteq \mathbb{N}^d$ is an arbitrary set (not necessarily fully nested) then $n_{\hat{i}}(S) = n_{\hat{i}}(g_i(S))$ and for $j \neq i$ we have a partition

$$S = \bigcup_{\tilde{x} \in \mathbb{N}^{d-2}} S_{\tilde{x}},$$

where $S_{\tilde{x}} = S \cap \pi_{\hat{i},\hat{j}}^{-1}(\tilde{x})$, and we then obtain

$$\begin{aligned} n_{\hat{i}}(S) &= \left| \pi_{\hat{i}} \left(\bigcup_{\tilde{x} \in \mathbb{N}^{d-2}} S_{\tilde{x}} \right) \right| \\ &= \sum_{\tilde{x} \in \mathbb{N}^{d-2}} |\pi_{\hat{i}}(S_{\tilde{x}})| \\ &\geq \sum_{\tilde{x} \in \mathbb{N}^{d-2}} |\pi_{\hat{i}}(g_j(S_{\tilde{x}}))| \\ &= \left| \pi_{\hat{i}} \left(g_j \left(\bigcup_{\tilde{x} \in \mathbb{N}^{d-2}} S_{\tilde{x}} \right) \right) \right| \\ &= |\pi_{\hat{i}}(g_j(S))|. \end{aligned}$$

Since $n_{\hat{i}}(S) \geq |\pi_{\hat{i}}(g_j(S))|$ for all $i, j \in \{1, \ldots, d\}$ we then obtain for the total gravity $g = g_1 g_2 \cdots g_d$ that $n_{\hat{i}}(S) = |\pi_{\hat{i}}(S)| \geq |\pi_{\hat{i}}(g(S))|$. From this and (31) we therefore we have the following corollary that relates the cardinality of a point set of $\mathbb{N}^d$ to that of its projections, in the spirit of Theorem 2 of Bollobás and Thomason [5, Thm 2, p.418].

Corollary 7.1. *For an arbitrary set $S \subseteq \mathbb{N}^d$ with $|S| = [m, \ell]^d$, we have*

$$n_{\hat{1}}(S) + \cdots + n_{\hat{d}}(S) \geq \ell[m, \ell - 1]^{d-1} + (d - \ell)[m, \ell]^{d-1}.$$

The above corollary can, of course, be generalized to an inequality for a general $n \in \mathbb{N}$ in terms of its d-PCR, although the formula will be more complicated.

The case of $d = 2$ Any $n \in \mathbb{N}$ has a 2-PCR given by $n = [m_2, \ell_2]^2 + [m_1, 0]^1$ where $\ell_2 \in \{0, 1\}$ and $m_1 = [m_1, 0]^1 < [m_2, \ell_2]^1$, and so $m_1 < [m_2, 1]^1 = m_2 + 1$ if $\ell_2 = 1$ and $m_1 < [m_2, 0]^1 = m_2$ if $\ell_2 = 0$. By Theorem 6.4 we have that $E_2(n) = 2n - \delta_2(n)$, where

$$\delta_2(n) = \ell_2[m_2, \ell_2 - 1]^1 + (2 - \ell_2)[m_2, \ell_2]^1 + [m_1, 0]^0 = \ell_2[m_2, \ell_2 - 1]^1 + (2 - \ell_2)[m_2, \ell_2]^1 + 1,$$

and hence $\delta_2(n) = 2m_2 + 1$ if $\ell_2 = 0$ and $\delta_2(n) = 2m_2 + 2$ if $\ell_2 = 1$. From this we see that $\delta_2(n) = \lceil 2\sqrt{n} \rceil$, which agrees with the formula $E_2(n) = \lfloor 2n - 2\sqrt{n} \rfloor$ given in [6].

The case $n < 2^d$ In this case the d-PCR of n has the form $n = [m_d, \ell_d]^d + [m_{d-1}, \ell_{d-1}]^{d-1} + \cdots + [m_c, \ell_c]^c$ where each $m_i = 1$ and $d - 1 \geq \ell_d > \ell_{d-1} > \cdots > \ell_c \geq 0$, which is exactly the usual binary representation of $n = 2^{\ell_d} + 2^{\ell_{d-1}} + \cdots + 2^{\ell_c}$. By Theorem 6.4 we have that $E_d(n) = dn - \delta_d(n)$, where

$$\delta_d(n) = \sum_{i=c}^{d} (\ell_i 2^{\ell_i - 1} + (i - \ell_i) 2^{\ell_i}) = \sum_{i=c}^{d} 2^{\ell_i - 1}(2i - \ell_i).$$

and hence

$$E_d(n) = \sum_{i=c}^{d} 2^{\ell_i - 1}(2d + \ell_i - 2i). \tag{32}$$

Now, since $n < 2^d$, the maximum number of edges a set $S \subseteq \mathbb{N}^d$ of order n can induce, is the same as the maximum number of edges a set S of order n in any rectangular grid can induce. So, $E_d(n) = f(n)$ where $f(n)$ is the total number of 1s in the binary representation of $1, \ldots, n-1$, as first proved in [8] and also stated in [3][Obs. 1.2]. The sequence $(f(n))_1^\infty = (0, 1, 2, 4, 5, 7, 9, 12, 13, 15, 17, 20, 22, 25, 28, 32, \ldots)$ is well known [1, A000788], and has appears naturally when analysing worst-case scenarios in sorting algorithms. It has been studied extensively in a variety of papers, as discussed in detail in [3], as it is one of the very few exact known solutions to a common divide-and-conquer recurrence relation [3][Obs. 1.2].

From (32) we have the following alternative explicit formula for $f(n)$ in terms of the binary representation of n.

Corollary 7.2. *For any $n \in \mathbb{N}$ with binary representation given by $n = 2^{\ell_d} + 2^{\ell_{d-1}} + \cdots + 2^{\ell_c}$, the total number of 1s appearing in the binary representations of $1, \ldots, n-1$ is given by*

$$f(n) = \sum_{i=c}^{d} 2^{\ell_i - 1}(2d + \ell_i - 2i).$$

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