Quadrant marked mesh patterns in 132-avoiding permutations III

Sergey Kitaev
University of Strathclyde
Livingstone Tower, 26 Richmond Street
Glasgow G1 1XH, United Kingdom
sergey.kitaev@cis.strath.ac.uk

Jeffrey Remmel
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
jremmel@ucsd.edu

Mark Tiefenbruck
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
mtiefenb@math.ucsd.edu

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Abstract

Given a permutation $\sigma = \sigma_1 \dots \sigma_n$ in the symmetric group S_n, we say that σ_i matches the marked mesh pattern $MMP(a,b,c,d)$ in σ if there are at least a points to the right of σ_i in σ which are greater than σ_i, at least b points to the left of σ_i in σ which are greater than σ_i, at least c points to the left of σ_i in σ which are smaller than σ_i, and at least d points to the right of σ_i in σ which are smaller than σ_i.

This paper is continuation of the systematic study of the distribution of quadrant marked mesh patterns in 132-avoiding permutations started in [9] and [10] where we studied the distribution of the number of matches of $MMP(a,b,c,d)$ in 132-avoiding permutations where at most two elements of of a,b,c,d are greater than zero and the remaining elements are zero. In this paper, we study the distribution of the number of matches of $MMP(a,b,c,d)$ in 132-avoiding permutations where at least three of a,b,c,d are greater than zero. We provide explicit recurrence relations to enumerate our objects which can be used to give closed forms for the generating functions associated with such distributions. In many cases, we provide combinatorial explanations of the coefficients that appear in our generating functions.

Keywords: permutation statistics, quadrant marked mesh pattern, distribution

Contents

1 Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [2] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns. This notion was further studied in [1, 3, 5, 6, 9, 12].

Kitaev and Remmel [6] initiated the systematic study of distribution of quadrant marked mesh patterns on permutations. The study was extended to 132-avoiding permutations by Kitaev, Remmel and Tiefenbruck in [9, 10], and the present paper continues this line of research. Kitaev and Remmel also studied the distribution of quadrant marked mesh patterns in up-down and down-up permutations [7, 8].

Let $\sigma = \sigma_1 \dots \sigma_n$ be a permutation written in one-line notation. Then we will consider the graph of σ, $G(\sigma)$, to be the set of points (i, σ_i) for $i = 1, \dots, n$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1. Then if we draw a coordinate system centered at a point (i, σ_i), we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as pictured in Figure 1. For any $a, b, c, d \in \mathbb{N} = \{0, 1, 2, \dots\}$ and any $\sigma = \sigma_1 \dots \sigma_n \in S_n$, the set of all permutations of length n, we say that σ_i matches the quadrant marked mesh pattern $\mathrm{MMP}(a, b, c, d)$ in σ if, in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are at least a points in quadrant I, at least b points in quadrant II, at least c points in quadrant III, and at least d points in quadrant IV. For example, if $\sigma = 471569283$, the point $\sigma_4 = 5$ matches the marked mesh pattern $\mathrm{MMP}(2, 1, 2, 1)$ since, in $G(\sigma)$ relative to the coordinate system with the origin at $(4, 5)$, there are 3 points in quadrant I, 1 point in quadrant II, 2 points in quadrant III, and 2 points in quadrant IV. Note that if a coordinate in $\mathrm{MMP}(a, b, c, d)$ is 0, then there is no condition imposed on the points in the corresponding quadrant.

In addition, we considered patterns $\mathrm{MMP}(a, b, c, d)$ where $a, b, c, d \in \mathbb{N} \cup \{\emptyset\}$. Here when a coordinate of $\mathrm{MMP}(a, b, c, d)$ is the empty set, then for σ_i to match $\mathrm{MMP}(a, b, c, d)$ in $\sigma = \sigma_1 \dots \sigma_n \in S_n$, it must be the case that there are no points in $G(\sigma)$ relative to the coordinate system with the origin at (i, σ_i) in the corresponding quadrant. For example, if $\sigma = 471569283$, the point $\sigma_3 = 1$ matches the marked mesh pattern $\mathrm{MMP}(4, 2, \emptyset, \emptyset)$ since in $G(\sigma)$ relative to the coordinate system with the origin at $(3, 1)$, there are 6 points in quadrant I, 2 points in quadrant II, no points in quadrants III and IV. We let $\mathrm{mmp}^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches $\mathrm{MMP}(a,b,c,d)$ in σ.

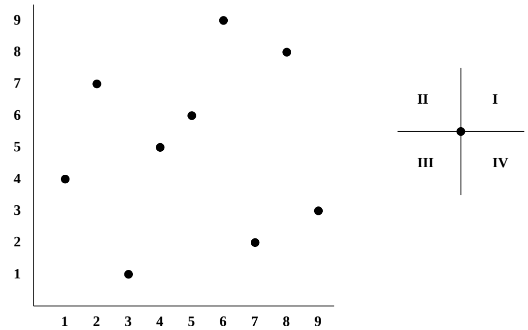
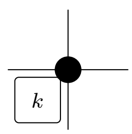
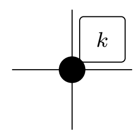


Figure 1: The graph of $\sigma = 471569283$.

Note how the (two-dimensional) notation of Úlfarsson [12] for *marked mesh patterns* corresponds to our (one-line) notation for quadrant marked mesh patterns. For example,

$\mathrm{MMP}(0,0,k,0) =$ 

, $\mathrm{MMP}(k,0,0,0) =$ 

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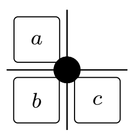
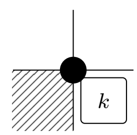
$\mathrm{MMP}(0,a,b,c) =$ 

and $\mathrm{MMP}(0,0,\emptyset,k) =$ 

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Given a sequence $w = w_1 \dots w_n$ of distinct integers, let $\mathrm{red}(w)$ be the permutation found by replacing the i-th largest integer that appears in σ by i. For example, if $\sigma = 2754$, then $\mathrm{red}(\sigma) = 1432$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group S_j, we say that the pattern τ *occurs* in $\sigma = \sigma_1 \dots \sigma_n \in S_n$ provided there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. Let $S_n(\tau)$ denote the set of permutations in S_n which avoid τ. In the theory of permutation patterns, τ is called a *classical pattern*. See [4] for a comprehensive introduction to patterns in permutations.

It has been a rather popular direction of research in the literature on permutation patterns to study permutations avoiding a 3-letter pattern subject to extra restrictions (see [4, Subsection 6.1.5]). In [9], we started the study of the generating functions

$$Q_{132}^{(a,b,c,d)}(t,x) = 1 + \sum_{n\geq 1} t^n Q_{n,132}^{(a,b,c,d)}(x)$$

where for any $a,b,c,d \in \{\emptyset\} \cup \mathbb{N}$,

$$Q_{n,132}^{(a,b,c,d)}(x) = \sum_{\sigma \in S_n(132)} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}.$$

For any a, b, c, d, we will write $Q_{n,132}^{(a,b,c,d)}(x)|_{x^k}$ for the coefficient of x^k in $Q_{n,132}^{(a,b,c,d)}(x)$.

There is one obvious symmetry for such generating functions which is induced by the fact that if $\sigma \in S_n(132)$, then $\sigma^{-1} \in S_n(132)$. That is, the following lemma was proved in [9].

Lemma 1. ([9]) *For any* $a, b, c, d \in \{\emptyset\} \cup \mathbb{N}$,

$$Q_{n,132}^{(a,b,c,d)}(x) = Q_{n,132}^{(a,d,c,b)}(x).$$

In [9], we studied the generating functions $Q_{132}^{(k,0,0,0)}(t,x)$, $Q_{132}^{(0,k,0,0)}(t,x) = Q_{132}^{(0,0,0,k)}(t,x)$, and $Q_{132}^{(0,0,k,0)}(t,x)$ where k can be either the empty set or a positive integer as well as the generating functions $Q_{132}^{(k,0,\emptyset,0)}(t,x)$ and $Q_{132}^{(\emptyset,0,k,0)}(t,x)$. In [10], we studied the generating functions $Q_{n,132}^{(k,0,\ell,0)}(t,x)$, $Q_{n,132}^{(k,0,0,\ell)}(t,x) = Q_{n,132}^{(k,\ell,0,0)}(t,x)$, $Q_{n,132}^{(0,k,\ell,0)}(t,x) = Q_{n,132}^{(0,0,\ell,k)}(t,x)$, and $Q_{n,132}^{(0,k,0,\ell)}(t,x)$, where $k, \ell \geq 1$. We also showed that sequences of the form $(Q_{n,132}^{(a,b,c,d)}(x)|_{x^r})_{n\geq s}$ count a variety of combinatorial objects that appear in the *On-line Encyclopedia of Integer Sequences* (OEIS) [11]. Thus, our results gave new combinatorial interpretations of certain classical sequences such as the Fine numbers and the Fibonacci numbers as well as provided certain sequences that appear in the OEIS with a combinatorial interpretation where none had existed before. Another particular result of our studies in [9] is enumeration of permutations avoiding simultaneously the patterns 132 and 1234, while in [10], we made a link to the *Pell numbers*.

The main goal of this paper is to continue the study of $Q_{132}^{(a,b,c,d)}(t,x)$ and combinatorial interpretations of sequences of the form $(Q_{n,132}^{(a,b,c,d)}(x)|_{x^r})_{n\geq s}$ in the case where $a, b, c, d \in \mathbb{N}$ and at least three of these parameters are non-zero.

Next we list the key results from [9] and [10] which we need in this paper.

Theorem 2. *([9, Theorem 4])*

$$Q_{132}^{(0,0,0,0)}(t,x) = C(xt) = \frac{1-\sqrt{1-4xt}}{2xt}$$

and, for $k \geq 1$,

$$Q_{132}^{(k,0,0,0)}(t,x) = \frac{1}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)}.$$

Hence

$$Q_{132}^{(1,0,0,0)}(t,0) = \frac{1}{1-t}$$

and, for $k \geq 2$,

$$Q_{132}^{(k,0,0,0)}(t,0) = \frac{1}{1 - tQ_{132}^{(k-1,0,0,0)}(t,0)}.$$

Theorem 3. *([9, Theorem 8]) For $k \geq 1$,*

$$Q_{132}^{(0,0,k,0)}(t,x) = \frac{1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j) - \sqrt{(1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j))^2 - 4tx}}{2tx}$$
$$= \frac{2}{1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j) + \sqrt{(1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j))^2 - 4tx}}$$

and

$$Q_{132}^{(0,0,k,0)}(t,0) = \frac{1}{1-t(C_0+C_1t+\cdots+C_{k-1}t^{k-1})}.$$

Theorem 4. *([10, Theorem 5]) For all $k,\ell \geq 1$,*

$$Q_{132}^{(k,0,\ell,0)}(t,x) = \frac{1}{1-tQ_{132}^{(k-1,0,\ell,0)}(t,x)}. \tag{1}$$

Theorem 5. *([10, Theorem 11]) For all $k,\ell \geq 1$,*

$$Q_{132}^{(k,0,0,\ell)}(t,x) = \\ \frac{C_\ell t^\ell + \sum_{j=0}^{\ell-1} C_j t^j (1-tQ_{132}^{(k-1,0,0,0)}(t,x) + t(Q_{132}^{(k-1,0,0,\ell-j)}(t,x) - \sum_{s=0}^{\ell-j-1} C_s t^s))}{1-tQ_{132}^{(k-1,0,0,0)}(t,x)}. \tag{2}$$

Theorem 6. *([10, Theorem 14]) For all $k,\ell \geq 1$,*

$$Q_{132}^{(0,k,\ell,0)}(t,x) = \\ \frac{C_{k-1}t^{k-1} + \sum_{j=0}^{k-2} C_j t^j \left(1-tQ_{132}^{(0,0,\ell,0)}(t,x) + t(Q_{132}^{(0,k-i-1,\ell,0)}(t,x) - \sum_{s=0}^{k-i-2} C_s t^s)\right)}{1-tQ_{132}^{(0,0,\ell,0)}(t,x)}. \tag{3}$$

Theorem 7. *([10, Theorem 17]) For all $k,\ell \geq 1$,*

$$Q_{132}^{(0,k,0,\ell)}(t,x) = \frac{\Phi_{k,\ell}(t,x)}{1-t} \tag{4}$$

where

$$\Phi_{k,\ell}(t,x) = \sum_{j=0}^{k+\ell-1} C_j t^j - \sum_{j=0}^{k+\ell-2} C_j t^{j+1} + t\left(\sum_{j=0}^{k-2} C_j t^j \left(Q_{132}^{(0,k,0,\ell-j-1)}(t,x) - \sum_{s=0}^{k-j-2} C_s t^s\right)\right) + \\ t\left(Q_{132}^{(0,k,0,0)}(t,x) - \sum_{u=0}^{k-1} C_u t^u\right)\left(Q_{132}^{(0,0,0,\ell)}(t,x) - \sum_{v=0}^{\ell-1} C_v t^v\right) + \\ t\left(\sum_{j=1}^{\ell-1} C_j t^j \left(Q_{132}^{(0,k,0,\ell-j)}(t,x) - \sum_{w=0}^{k+\ell-j-2} C_w t^w\right)\right).$$

As it was pointed out in [9], *avoidance* of a marked mesh pattern without quadrants containing the empty set can always be expressed in terms of multi-avoidance of (possibly many) classical patterns. Thus, among our results we will re-derive several known facts in permutation patterns theory. However, our main goals are more ambitious aimed at finding distributions in question.

2 $Q_{n,132}^{(k,0,m,\ell)}(x) = Q_{n,132}^{(k,\ell,m,0)}(x)$ where $k, \ell, m \geq 1$

By Lemma 1, we know that $Q_{n,132}^{(k,0,m,\ell)}(x) = Q_{n,132}^{(k,\ell,m,0)}(x)$. Thus, we will only consider $Q_{n,132}^{(k,\ell,m,0)}(x)$ in this section.

Throughout this paper, we shall classify the 132-avoiding permutations $\sigma = \sigma_1 \ldots \sigma_n$ by the position of n in σ. That is, let $S_n^{(i)}(132)$ denote the set of $\sigma \in S_n(132)$ such that $\sigma_i = n$. Clearly each $\sigma \in S_n^{(i)}(132)$ has the structure pictured in Figure 2. That is, in the graph of σ, the elements to the left of n, $A_i(\sigma)$, have the structure of a 132-avoiding permutation, the elements to the right of n, $B_i(\sigma)$, have the structure of a 132-avoiding permutation, and all the elements in $A_i(\sigma)$ lie above all the elements in $B_i(\sigma)$. It is well-known that the number of 132-avoiding permutations in S_n is the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ and the generating function for the C_n's is given by

$$C(t) = \sum_{n \geq 0} C_n t^n = \frac{1 - \sqrt{1-4t}}{2t} = \frac{2}{1+\sqrt{1-4t}}.$$

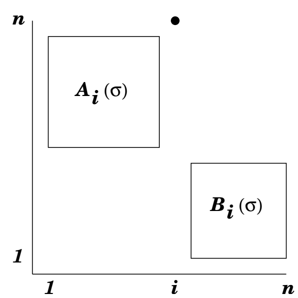

Figure 2: The structure of 132-avoiding permutations.

Suppose that $n \geq \ell$. It is clear that n can never match the pattern $\mathrm{MMP}(k, \ell, m, 0)$ for $k, m \geq 1$ in any $\sigma \in S_n(132)$. For $1 \leq i \leq n$, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(k-1,\ell,m,0)}(x)$ to $Q_{n,132}^{(k,\ell,m,0)}(x)$. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(k,\ell-i,m,0)}(x)$ to $Q_{n,132}^{(k,\ell,m,0)}(x)$ if $i < \ell$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s, σ_s) for any $s > i$. However if $i \geq \ell$, then our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(k,0,m,0)}(x)$ to $Q_{n,132}^{(k,\ell,m,0)}(x)$. It follows that for $n \geq \ell$,

$$Q_{n,132}^{(k,\ell,m,0)}(x) = \sum_{i=1}^{\ell-1} Q_{i-1,132}^{(k-1,\ell,m,0)}(x) Q_{n-i,132}^{(k,\ell-i,m,0)}(x) + \sum_{i=\ell}^{n} Q_{i-1,132}^{(k-1,\ell,m,0)}(x) Q_{n-i,132}^{(k,0,m,0)}(x).$$

Note that for $i < \ell$, $Q_{i-1,132}^{(k-1,\ell,m,0)}(x) = C_{i-1}$. Thus, for $n \geq \ell$,

$$Q_{n,132}^{(k,\ell,m,0)}(x) = \sum_{i=1}^{\ell-1} C_{i-1} Q_{n-i,132}^{(k,\ell-i,m,0)}(x) + \sum_{i=\ell}^{n} Q_{i-1,132}^{(k-1,\ell,m,0)}(x) Q_{n-i,132}^{(k,0,m,0)}(x). \tag{5}$$

Multiplying both sides of (5) by t^n and summing for $n \geq \ell$, we see that for $k, \ell \geq 1$,

$$\begin{aligned}
Q_{132}^{(k,\ell,m,0)}(t,x) &= \sum_{j=0}^{\ell-1} C_j t^j + \sum_{i=1}^{\ell-1} C_{i-1} t^i \sum_{u \geq \ell-i} Q_{u,132}^{(k,\ell-i,m,0)}(x) t^u + \\
&\quad t \sum_{n \geq \ell} \sum_{i=1}^{n} Q_{i-1,132}^{(k-1,\ell,m,0)}(x) t^{i-1} Q_{n-i,132}^{(k,0,m,0)}(x) t^{n-i} \\
&= \sum_{j=0}^{\ell-1} C_j t^j + \sum_{i=1}^{\ell-1} C_{i-1} t^i \left(Q_{132}^{(k,\ell-i,m,0)}(t,x) - \sum_{j=0}^{\ell-i-1} C_j t^j \right) + \\
&\quad t Q_{132}^{(k,0,m,0)}(t,x) \left(Q_{132}^{(k-1,\ell,m,0)}(t,x) - \sum_{s=0}^{\ell-2} C_s t^s \right) \\
&= C_{\ell-1} t^{\ell-1} + t Q_{132}^{(k,0,m,0)}(t,x) Q_{132}^{(k-1,\ell,m,0)}(t,x) + \\
&\quad \sum_{s=0}^{\ell-2} C_s t^s \left(1 + t Q_{132}^{(k,\ell-1-s,m,0)}(t,x) - t Q_{132}^{(k,0,m,0)}(t,x) - t \sum_{j=0}^{\ell-2-s} C_j t^j \right).
\end{aligned}$$

Thus, we have the following theorem.

Theorem 8.

$$\begin{aligned}
Q_{132}^{(k,\ell,m,0)}(t,x) = C_{\ell-1} t^{\ell-1} + t Q_{132}^{(k,0,m,0)}(t,x) Q_{132}^{(k-1,\ell,m,0)}(t,x) + \\
\sum_{s=0}^{\ell-2} C_s t^s \left(1 + t Q_{132}^{(k,\ell-1-s,m,0)}(t,x) - t Q_{132}^{(k,0,m,0)}(t,x) - t \sum_{j=0}^{\ell-2-s} C_j t^j \right).
\end{aligned} \tag{6}$$

Note that since we can compute $Q_{132}^{(k,0,m,0)}(t,x)$ by Theorem 4 and $Q_{132}^{(0,\ell,m,0)}(t,x)$ by Theorem 6, we can use (6) to compute $Q_{132}^{(k,\ell,m,0)}(t,x)$ for any $k, \ell, m \geq 1$.

2.1 Explicit formulas for $Q_{n,132}^{(k,\ell,m,0)}(x)|_{x^r}$

It follows from Theorem 8 that

$$Q_{132}^{(k,1,m,0)}(t,x) = 1 + t Q_{132}^{(k,0,m,0)}(t,x) Q_{132}^{(k-1,1,m,0)}(t,x) \tag{7}$$

and

$$Q_{132}^{(k,2,m,0)}(t,x) = 1 + t Q_{132}^{(k,0,m,0)}(t,x) (Q_{132}^{(k-1,2,m,0)}(t,x) - 1) + t Q_{132}^{(k,1,m,0)}(t,x).$$

Note that it follows from Theorems 4 and 6 that

$$\begin{aligned}
Q_{132}^{(1,1,1,0)}(t,0) &= 1 + t Q_{132}^{(1,0,1,0)}(t,0) Q_{132}^{(0,1,1,0)}(t,0) \\
&= 1 + t \frac{1-t}{1-2t} \frac{1-t}{1-2t} = \frac{1-3t+2t^2+t^3}{(1-2t)^2}.
\end{aligned}$$

Thus, the generating function of the sequence $(Q_{n,132}^{(1,1,1,0)}(0))_{n\geq 1}$ is $\left(\frac{1-t}{1-2t}\right)^2$ which is the generating function of the sequence A045623 in the OEIS. The n-th term a_n of this sequence has many combinatorial interpretations including the number of 1s in all partitions of $n+1$ and the number of 132-avoiding permutations of S_{n+2} which contain exactly one occurrence of the pattern 213. We note that for a permutation σ to avoid the pattern MMP$(1,1,1,0)$, it must simultaneously avoid the patterns 3124, 4123, 1324, and 1423. Thus, the number of permutations $\sigma \in S_n(132)$ which avoid MMP$(1,1,1,0)$ is the number of permutations in S_n that simultaneously avoid the patterns 132, 3124, and 4123.

Problem 1. Find simple bijections between the set of permutations $\sigma \in S_n(132)$ which avoid MMP$(1,1,1,0)$ and the other combinatorial interpretations of the sequence A045623 in the OEIS.

Note that it follows from Theorem 4 and our previous results that

$$\begin{aligned}
Q_{132}^{(2,1,1,0)}(t,0) &= 1+tQ_{132}^{(2,0,1,0)}(t,0)Q_{132}^{(1,1,1,0)}(t,0) \\
&= 1+t\frac{1-2t}{1-3t+t^2}\left(\frac{1-3t+2t^2+t^3}{(1-2t)^2}\right) \\
&= \frac{1-4t+4t^2+t^4}{1-5t+7t^2-2t^3}.
\end{aligned}$$

The sequence $(Q_{n,132}^{(2,1,1,0)}(0))_{n\geq 1}$ is the sequence A142586 in the OIES which has the generating function $\frac{1-3t+2t^2+t^3}{(1-3t+t^2)(1-2t)}$. That is, $\frac{1-4t+4t^2+t^4}{1-5t+7t^2-2t^3} - 1 = \frac{t(1-3t+2t^2+t^3)}{(1-3t+t^2)(1-2t)}$. This sequence has no listed combinatorial interpretation so that we have found a combinatorial interpretation of this sequence.

Similarly,

$$\begin{aligned}
Q_{132}^{(3,1,1,0)}(t,0) &= 1+tQ_{132}^{(3,0,1,0)}(t,0)Q_{132}^{(2,1,1,0)}(t,0) \\
&= 1+t\frac{1-3t+t^2}{1-4t+3t^2}\,\frac{1-4t+4t^2+t^4}{1-5t+7t^2-2t^3} \\
&= \frac{1-5t+7t^2-2t^3+t^5}{1-6t+11t^2-6t^3}.
\end{aligned}$$

$$\begin{aligned}
Q_{132}^{(1,1,2,0)}(t,0) &= 1+tQ_{132}^{(1,0,2,0)}(t,0)Q_{132}^{(0,1,2,0)}(t,0) \\
&= 1+t\frac{1-t-t^2}{1-2t-t^2}\,\frac{1-t-t^2}{1-2t-t^2} \\
&= \frac{1-3t+3t^3+3t^4+t^5}{(1-2t-t^2)^2}.
\end{aligned}$$

$$\begin{aligned}
Q_{132}^{(2,1,2,0)}(t,0) &= 1+tQ_{132}^{(2,0,2,0)}(t,0)Q_{132}^{(1,1,2,0)}(t,0) \\
&= 1+t\frac{1-2t-t^2}{1-3t+t^3}\,\frac{1-3t+3t^3+3t^4+t^5}{(1-2t-t^2)^2} \\
&= \frac{1-4t+2t^2+4t^3+t^4+2t^5+t^6}{(1-2t-t^2)(1-3t+t^3)}.
\end{aligned}$$

Using (7) and Theorem 4, we have computed the following.

$$Q_{132}^{(1,1,1,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + (12+2x)t^4 + \left(28+12x+2x^2\right)t^5 + \left(64+48x+18x^2+2x^3\right)t^6 + \left(144+160x+97x^2+26x^3+2x^4\right)t^7 + \left(320+480x+408x^2+184x^3+36x^4+2x^5\right)t^8 + \left(704+1344x+1479x^2+958x^3+327x^4+48x^5+2x^6\right)t^9 + \cdots .$$

$$Q_{132}^{(1,1,2,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (38+4x)t^5 + \left(102+26x+4x^2\right)t^6 + \left(271+120x+34x^2+4x^3\right)t^7 + \left(714+470x+200x^2+42x^3+4x^4\right)t^8 + \left(1868+1672x+964x^2+304x^3+50x^4+4x^5\right)t^9 + \cdots .$$

$$Q_{132}^{(1,1,3,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (122+10x)t^6 + \left(351+68x+10x^2\right)t^7 + \left(1006+326x+88x^2+10x^3\right)t^8 + \left(2868+1364x+512x^2+108x^3+10x^4\right)t^9 + \cdots .$$

We can explain the highest and second highest coefficients of x in these series. That is, we have the following theorem.

Theorem 9.

(i) For all $m \geq 1$ and $n \geq 3+m$, the highest power of x that occurs in $Q_{n,132}^{(1,1,m,0)}(x)$ is x^{n-2-m} which appears with a coefficient of $2C_m$.

(ii) For $n \geq 5$, $Q_{n,132}^{(1,1,1,0)}(x)|_{x^{n-4}} = 6 + 2\binom{n-2}{2}$.

(iii) For $m \geq 2$ and $n \geq 4+m$ $Q_{n,132}^{(1,1,m,0)}(x)|_{x^{n-3-m}} = 2C_{m+1} + 8C_m + 4C_m(n-4)$.

Proof. It is easy to see that for the maximum number of MMP$(1,1,m,0)$-matches in a $\sigma \in S_n(132)$, the permutation must be of the form $(n-1)\tau(m+1)\ldots(n-2)n$ or $n\tau(m+1)\ldots(n-2)(n-1)$ where $\tau \in S_m(132)$. Thus, the highest power of x occurring in $Q_{n,132}^{(1,1,m,0)}(x)$ is x^{n-2-m} which occurs with a coefficient of $2C_m$.

For parts (ii) and (iii), we have the recursion that

$$Q_{n,132}^{(1,1,m,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(0,1,m,0)}(x) Q_{n-i,132}^{(1,0,m,0)}(x). \tag{8}$$

We proved in [10] that the highest power of x which occurs in either $Q_{n,132}^{(0,1,m,0)}(x)$ or $Q_{n,132}^{(1,0,m,0)}(x)$ is x^{n-1-m} and

$$Q_{n,132}^{(0,1,m,0)}(x)|_{x^{n-1-m}} = Q_{n,132}^{(1,0,m,0)}(x)|_{x^{n-1-m}} = C_m.$$

It is then easy to check that the highest coefficient of x in $Q^{(0,1,m,0)}_{i-1,132}(x)Q^{(1,0,m,0)}_{n-i,132}(x)$ is less than x^{n-3-m} for $i = 3,\dots,n-3$.

We also proved in [10] that

$$\begin{aligned} Q^{(1,0,1,0)}_{n,132}(x)|_{x^{n-3}} &= Q^{(0,1,1,0)}_{n,132}(x)|_{x^{n-3}} = 2 + \binom{n-1}{2} \text{ for } n \geq 4 \text{ and} \\ Q^{(1,0,m,0)}_{n,132}(x)|_{x^{n-m-2}} &= Q^{(0,1,m,0)}_{n,132}(x)|_{x^{n-m-2}} \\ &= C_{m+1} + C_m + 2C_m(n-2-m) \text{ for } n \geq 3+m \text{ and } m \geq 2. \end{aligned}$$

For $m = 1$, we are left with 4 cases to consider in the recursion (8). We start with the $m = 1$ case.

Case 1. $i = 1$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x)Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(1,0,1,0)}_{n-1,132}(x)|_{x^{n-4}}$ and

$$Q^{(1,0,1,0)}_{n-1,132}(x)|_{x^{n-4}} = 2 + \binom{n-2}{2} \text{ for } n \geq 5.$$

Case 2. $i = 2$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x)Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(1,0,1,0)}_{n-2,132}(x)|_{x^{n-4}}$ and

$$Q^{(1,0,1,0)}_{n-2,132}(x)|_{x^{n-4}} = 1 \text{ for } n \geq 5.$$

Case 3. $i = n-1$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x)Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(0,1,1,0)}_{n-2,132}(x)|_{x^{n-4}}$ and

$$Q^{(0,1,1,0)}_{n-2,132}(x)|_{x^{n-4}} = 1 \text{ for } n \geq 5.$$

Case 4. $i = n$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x)Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(0,1,1,0)}_{n-1,132}(x)|_{x^{n-4}}$ and

$$Q^{(0,1,1,0)}_{n-1,132}(x)|_{x^{n-4}} = 2 + \binom{n-2}{2} \text{ for } n \geq 5.$$

Thus, $Q^{(0,1,1,0)}_{n,132}(x)|_{x^{n-4}} = 6 + 2\binom{n-2}{2}$ for $n \geq 5$.

Next we consider the case when $m \geq 2$. Again we have 4 cases.

Case 1. $i = 1$. In this case, $Q^{(0,1,m,0)}_{i-1,132}(x)Q^{(1,0,m,0)}_{n-i,132}(x)|_{x^{n-3-m}} = Q^{(1,0,m,0)}_{n-1,132}(x)|_{x^{n-3-m}}$ and

$$Q^{(1,0,m,0)}_{n-1,132}(x)|_{x^{n-3-\ell}} = C_{m+1} + C_m + 2C_m(n-3-m) \text{ for } n \geq 4+m.$$

Case 2. $i = 2$. In this case, $Q^{(0,1,m,0)}_{i-1,132}(x)Q^{(1,0,m,0)}_{n-i,132}(x)|_{x^{n-3-m}} = Q^{(1,0,m,0)}_{n-2,132}(x)|_{x^{n-3-m}}$ and

$$Q^{(1,0,m,0)}_{n-2,132}(x)|_{x^{n-3-m}} = C_m \text{ for } n \geq 4+m.$$

Case 3. $i = n-1$. In this case, $Q^{(0,1,m,0)}_{i-1,132}(x)Q^{(1,0,m,0)}_{n-i,132}(x)|_{x^{n-3-m}} = Q^{(0,1,m,0)}_{n-2,132}(x)|_{x^{n-3-m}}$ and

$$Q^{(0,1,m,0)}_{n-2,132}(x)|_{x^{n-3-m}} = C_m \text{ for } n \geq 4+m.$$

Case 4. $i = n$. In this case, $Q^{(0,1,2,0)}_{i-1,132}(x) Q^{(1,0,m,0)}_{n-i,132}(x)|_{x^{n-3-m}} = Q^{(0,1,m,0)}_{n-1,132}(x)|_{x^{n-3-m}}$ and

$$Q^{(0,1,m,0)}_{n-1,132}(x)|_{x^{n-3-m}} = C_{m+1} + C_m + 2C_m(n-3-m) \text{ for } n \geq 4+m.$$

Thus, for $n \geq 4+m$,

$$\begin{aligned} Q^{(1,1,m,0)}_{n,132}(x)|_{x^{n-3-m}} &= 2C_{m+1} + 4C_m + 4C_m(n-3-m) \\ &= 2C_{m+1} + 8C_m + 4C_m(n-4-m). \end{aligned}$$

Thus, when $m = 2$, we obtain that

$$Q^{(0,1,2,0)}_{n,132}(x)|_{x^{n-5}} = 26 + 8(n-6) \text{ for } n \geq 6$$

and, for $m = 3$, we obtain that

$$Q^{(0,1,3,0)}_{n,132}(x)|_{x^{n-6}} = 68 + 20(n-7) \text{for } n \geq 7$$

which agrees with our computed series. $\square$

We also have computed that

$$\begin{aligned} &Q^{(2,1,1,0)}_{132}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (39+3x)t^5 + \left(107 + 22x + 3x^2\right)t^6 + \\ &\left(290 + 105x + 31x^2 + 3x^3\right)t^7 + \left(779 + 415x + 190x^2 + 43x^3 + 3x^4\right)t^8 + \\ &\left(2079 + 1477x + 909x^2 + 336x^3 + 58x^4 + 3x^5\right)t^9 + \\ &\left(5522 + 4922x + 3765x^2 + 1938x^3 + 570x^4 + 76x^5 + 3x^6\right)t^{10} + \cdots, \end{aligned}$$

$$\begin{aligned} &Q^{(2,1,2,0)}_{132}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (126+6x)t^6 + \left(376 + 47x + 6x^2\right)t^7 + \\ &\left(1115 + 250x + 59x^2 + 6x^3\right)t^8 + \left(3289 + 1110x + 386x^2 + 71x^3 + 6x^4\right)t^9 + \\ &\left(9660 + 4444x + 2045x^2 + 558x^3 + 83x^4 + 6x^5\right)t^{10} + \cdots, \text{ and} \end{aligned}$$

$$\begin{aligned} &Q^{(2,1,3,0)}_{132}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (414+15x)t^7 + \\ &\left(1293 + 122x + 15x^2\right)t^8 + \left(4025 + 670x + 152x^2 + 15x^3\right)t^9 + \\ &\left(12486 + 3124x + 989x^2 + 182x^3 + 15x^4\right)t^{10} + \cdots. \end{aligned}$$

Again one can easily explain the highest coefficient in $Q^{(2,1,m,0)}_{n,132}(x)$. That is, to have the maximum number of MMP$(2,1,m,0)$-matches in a $\sigma \in S_n(132)$, the permutation must be of the form

$$\begin{aligned} &(n-2)\tau(m+1)\ldots(n-3)(n-1)n, \\ &(n-1)\tau(m+1)\ldots(n-3)(n-2)n, \text{or} \\ &n\tau(m+1)\ldots(n-3)(n-2)(n-1) \end{aligned}$$

where $\tau \in S_m(132)$. Thus, the highest power of x occurring in $Q_{n,132}^{(2,1,m,0)}(x)$ is x^{n-3-m} which occurs with a coefficient of $3C_m$.

We have computed that

$$Q_{132}^{(1,2,1,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (37+5x)t^5 + \left(94 + 33x + 5x^2\right)t^6 + \left(232 + 144x + 48x^2 + 5x^3\right)t^7 + \left(560 + 520x + 277x^2 + 68x^3 + 5x^4\right)t^8 + \left(1328 + 1680x + 1248x^2 + 508x^3 + 93x^4 + 5x^5\right)t^9 + \cdots .$$

$$Q_{132}^{(1,2,2,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (122+10x)t^6 + \left(348 + 71x + 10x^2\right)t^7 + \left(978 + 351x + 91x^2 + 10x^3\right)t^8 + \left(2715 + 1463x + 563x^2 + 111x^3 + 10x^4\right)t^9 + \cdots, \text{ and}$$

$$Q_{132}^{(1,2,3,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (404+25x)t^7 + \left(1220 + 185x + 25x^2\right)t^8 + \left(3655 + 947x + 235x^2 + 25x^3\right)t^9 + \cdots .$$

Again, one can easily explain the highest coefficient in $Q_{n,132}^{(1,2,m,0)}(x)$. That is, to have the maximum number of MMP$(1,2,m,0)$-matches in a $\sigma \in S_n(132)$, one must be of the form

$$\begin{array}{l} (n-2)(n-1)\tau(m+1)\ldots(n-3)n, \\ (n-1)(n-2)\tau(m+1)\ldots(n-3)n, \\ n(n-2)\tau(m+1)\ldots(n-3)(n-1), \\ n(n-1)\tau(m+1)\ldots(n-3)(n-2), \text{ or} \\ (n-1)n\tau(m+1)\ldots(n-3)(n-2) \end{array}$$

where $\tau \in S_m(132)$. Thus, the highest power of x occurring in $Q_{n,132}^{(1,2,m,0)}(x)$ is x^{n-3-m} which occurs with a coefficient of $5C_m$.

Finally, we have computed that

$$Q_{132}^{(2,2,1,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (123+9x)t^6 + \left(351 + 69x + 9x^2\right)t^7 + \left(982 + 343x + 96x^2 + 9x^3\right)t^8 + \left(2707 + 1405x + 609x^2 + 132x^3 + 9x^4\right)t^9 + \cdots,$$

$$Q_{132}^{(2,2,2,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (411+18x)t^7 + \left(1265 + 147x + 18x^2\right)t^8 + \left(3852 + 809x + 183x^2 + 18x^3\right)t^9 + \left(11626 + 3704x + 1229x^2 + 219x^3 + 18x^4\right)t^{10} + \cdots, \text{ and}$$

$$Q_{132}^{(2,2,3,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + 429t^7 + (1385+45x)t^8 + \left(4436 + 381x + 45x^2\right)t^9 + \left(14118 + 2162x + 471x^2 + 45x^3\right)t^{10} + \left(44670 + 10361x + 3149x^2 + 561x^3 + 45x^4\right)t^{11} + \cdots .$$

Again, one can easily explain the highest coefficient in $Q_{n,132}^{(2,2,m,0)}(x)$. That is, to have the maximum number of MMP$(2,2,m,0)$-matches in a $\sigma \in S_n(132)$, one must be of the form

$$\begin{aligned}
&n(n-1)\tau(m+1)\ldots(n-4)(n-3)(n-2),\\
&(n-1)n\tau(m+1)\ldots(n-4)(n-3)(n-2),\\
&n(n-2)\tau(m+1)\ldots(n-4)(n-3)(n-1),\\
&n(n-3)\tau(m+1)\ldots(n-4)(n-2)(n-1),\\
&(n-1)(n-2)\tau(m+1)\ldots(n-4)(n-3)n,\\
&(n-2)(n-1)\tau(m+1)\ldots(n-4)(n-3)n,\\
&(n-1)(n-3)\tau(m+1)\ldots(n-4)(n-2)n,\\
&(n-2)(n-3)\tau(m+1)\ldots(n-4)(n-1)n, \text{ or}\\
&(n-3)(n-2)\tau(m+1)\ldots(n-4)(n-1)n
\end{aligned}$$

where $\tau \in S_m(132)$. Thus, the highest power of x occurring in $Q_{n,132}^{(2,2,m,0)}(x)$ is x^{n-4-m} which occurs with a coefficient of $9C_m$.

3 $Q_{n,132}^{(0,k,\ell,m)}(x) = Q_{n,132}^{(0,m,\ell,k)}(x)$ where $k,\ell,m \geq 1$

By Lemma 1, we only need to consider $Q_{n,132}^{(0,k,\ell,m)}(x)$. Suppose that $k,\ell,m \geq 1$ and $n \geq k+m$. It is clear that n can never match the pattern MMP$(0,k,\ell,m)$ for $k,\ell,m \geq 1$ in any $\sigma \in S_n(132)$. If $\sigma = \sigma_1 \ldots \sigma_n \in S_n(132)$ and $\sigma_i = n$, then we have three cases, depending on the value of i.

Case 1. $i < k$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since none of the elements σ_j for $j \leq k$ can match MMP$(0,k,\ell,m)$ in σ. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(0,k-i,\ell,m)}(x)$ to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s,σ_s) for any $s > i$. Thus, the permutations in Case 1 will contribute

$$\sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(0,k-i,\ell,m)}(x)$$

to $Q_{n,132}^{(0,k,\ell,m)}(x)$.

Case 2. $k \leq i \leq n-m$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,k,\ell,0)}(x)$ to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since the elements in $B_i(\sigma)$ will all be in the fourth quadrant relative to a coordinate system centered at (r,σ_r) for $r \leq i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(0,0,\ell,m)}(x)$ to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s, σ_s) for any $s > i$. Thus, the permutations in Case 2 will contribute

$$\sum_{i=k}^{n-m} Q_{i-1,132}^{(0,k,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,m)}(x)$$

to $Q_{n,132}^{(0,k,\ell,m)}(x)$.

Case 3. $i \geq n-m+1$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,k,\ell,m-(n-i))}(x)$ to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since the elements in $B_i(\sigma)$ will all be in the fourth quadrant relative to a coordinate system centered at (r, σ_r) for $r \leq i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of C_{n-i} to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since the elements in $B_i(\sigma)$ do not have enough elements to the right to match $\mathrm{MMP}(0,k,\ell,m)$ in σ. Thus, the permutations in Case 3 will contribute

$$\sum_{i=n-m+1}^{n} Q_{i-1,132}^{(0,k,\ell,m-(n-i))}(x) C_{n-i}$$

to $Q_{n,132}^{(0,k,\ell,m)}(x)$. Hence, for $n \geq k+m$,

$$\begin{aligned} Q_{n,132}^{(0,k,\ell,m)}(x) &= \sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(0,k-i,\ell,m)}(x) + \sum_{i=k}^{n-m} Q_{i-1,132}^{(0,k,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,m)}(x) + \\ & \quad \sum_{i=n-m+1}^{n} Q_{i-1,132}^{(0,k,\ell,m-(n-i))}(x) C_{n-i}. \end{aligned} \tag{9}$$

Multiplying (9) by t^n and summing, it is easy to compute that

$$\begin{aligned} Q_{132}^{(0,k,\ell,m)}(t,x) &= \sum_{p=0}^{k+m-1} C_p t^p + \\ & \sum_{i=0}^{k-2} C_i t^i \left(t Q_{132}^{(0,k-1-i,\ell,m)}(t,x) - t \sum_{r=0}^{k-i+m-2} C_r t^r \right) + \\ & t \left(Q_{132}^{(0,k,\ell,0)}(t,x) - \sum_{a=0}^{k-2} C_a t^a \right) \left(Q_{132}^{(0,0,\ell,m)}(t,x) - \sum_{b=0}^{m-1} C_b t^b \right) + \\ & \sum_{j=0}^{m-1} C_j t^j \left(t Q_{132}^{(0,k,\ell,m-j)}(t,x) - t \sum_{s=0}^{k+m-j-2} C_s t^s \right). \end{aligned}$$

Note that the $j=0$ term in the last sum is $tQ_{132}^{(0,k,\ell,m)}(t,x) - t\sum_{s=0}^{k+m-2} C_s t^s$. Thus, taking the term $tQ_{132}^{(0,k,\ell,m)}(t,x)$ over to the other side and combining the sum $t\sum_{s=0}^{k+m-2} C_s t^s$ with the sum $\sum_{p=0}^{k+m-1} C_p t^p$ to obtain $C_{k+m-1}t^{k+m-1} + (1-t)\sum_{p=0}^{k+m-2} C_p t^p$ and then dividing both sides by $1-t$ will yield the following theorem.

Theorem 10.

$$\begin{aligned}
Q_{132}^{(0,k,\ell,m)}(t,x) &= \sum_{p=0}^{k+m-2} C_p t^p + \frac{C_{k+m-1}t^{k+m-1}}{1-t} + \\
&\quad \frac{t}{1-t}\sum_{i=0}^{k-2} C_i t^i \left(Q_{132}^{(0,k-1-i,\ell,m)}(t,x) - \sum_{r=0}^{k-i+m-2} C_r t^r \right) + \\
&\quad \frac{t}{1-t}\left(Q_{132}^{(0,k,\ell,0)}(t,x) - \sum_{a=0}^{k-2} C_a t^a \right)\left(Q_{132}^{(0,0,\ell,m)}(t,x) - \sum_{b=0}^{m-1} C_b t^b \right) + \\
&\quad \frac{t}{1-t}\sum_{j=1}^{m-1} C_j t^j \left(Q_{132}^{(0,k,\ell,m-j)}(t,x) - \sum_{s=0}^{k+m-j-2} C_s t^s \right).
\end{aligned}$$

Note that since we can compute $Q_{132}^{(0,k,\ell,0)}(t,x) = Q_{132}^{(0,0,\ell,k)}(t,x)$ by Theorem 6, we can compute $Q_{132}^{(0,k,\ell,m)}(t,x)$ for all $k,\ell,m \geq 1$.

3.1 Explicit formulas for $Q_{n,132}^{(0,k,\ell,m)}(x)|_{x^r}$

It follows from Theorem 10 that

$$\begin{aligned}
Q_{132}^{(0,1,\ell,1)}(t,x) &= 1 + \frac{t}{1-t} + \frac{t}{1-t} Q_{132}^{(0,1,\ell,0)}(t,x)(Q_{132}^{(0,0,\ell,1)}(t,x) - 1) \\
&= \frac{1}{1-t} + \frac{t}{1-t} Q_{132}^{(0,1,\ell,0)}(t,x)(Q_{132}^{(0,0,\ell,1)}(t,x) - 1),
\end{aligned}$$

$$\begin{aligned}
Q_{132}^{(0,1,\ell,2)}(t,x) &= 1 + t + \frac{2t^2}{1-t} + \frac{t}{1-t} Q_{132}^{(0,1,\ell,0)}(t,x)(Q_{132}^{(0,0,\ell,2)}(t,x) - (1+t)) + \\
&\quad \frac{t^2}{1-t}(Q_{132}^{(0,1,\ell,1)}(t,x) - 1),
\end{aligned}$$

and

$$\begin{aligned}
Q_{132}^{(0,2,\ell,2)}(t,x) &= 1 + t + 2t^2 + \frac{5t^3}{1-t} + \frac{t}{1-t}(Q_{132}^{(0,1,\ell,2)}(t,x) - (1+t+2t^2)) + \\
&\quad \frac{t}{1-t}(Q_{132}^{(0,2,\ell,0)}(t,x) - 1)(Q_{132}^{(0,0,\ell,2)}(t,x) - (1+t)) + \\
&\quad \frac{t^2}{1-t}(Q_{132}^{(0,2,\ell,1)}(t,x) - (1+t)).
\end{aligned}$$

We used these formulas to compute the following.

$Q_{132}^{(0,1,1,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + (13+x)t^4 + \left(33+8x+x^2\right)t^5 +$
$\left(81+39x+11x^2+x^3\right)t^6 + \left(193+150x+70x^2+15x^3+x^4\right)t^7 +$
$\left(449+501x+337x^2+122x^3+20x^4+x^5\right)t^8 +$
$\left(1025+1524x+1363x^2+719x^3+204x^4+26x^5+x^6\right)t^9 +$
$\left(2305+4339x+4891x^2+3450x^3+1450x^4+327x^5+33x^6+x^7\right)t^{10} + \cdots.$

$Q_{132}^{(0,1,2,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 2(20+x)t^5 + \left(113+17x+2x^2\right)t^6 +$
$\left(314+92x+21x^2+2x^3\right)t^7 + \left(859+404x+140x^2+25x^3+2x^4\right)t^8 +$
$\left(2319+1567x+745x^2+200x^3+29x^4+2x^5\right)t^9 +$
$\left(6192+5597x+3438x^2+1262x^3+272x^4+33x^5+2x^6\right)t^{10} + \cdots.$

$Q_{132}^{(0,1,3,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (127+5x)t^6 + \left(380+44x+5x^2\right)t^7 +$
$\left(1125+246x+54x^2+5x^3\right)t^8 + \left(3299+1135x+359x^2+64x^3+5x^4\right)t^9 +$
$\left(9592+4691x+1942x^2+492x^3+74x^4+5x^5\right)t^{10} + \cdots.$

Our next theorem will explain the coefficient of the highest and second highest powers of x that appear in $Q_{n,132}^{(0,1,\ell,1)}(x)$ in these series.

Theorem 11.

(i) For $n \geq 3+\ell$, the highest power of x that occurs in $Q_{n,132}^{(0,1,\ell,1)}(x)$ is $x^{n-2-\ell}$ which appears with a coefficient of C_ℓ.

(ii) For $n \geq 5$, $Q_{n,132}^{(0,1,1,1)}(x)|_{x^{n-4}} = 5 + \binom{n-2}{2}$.

(iii) For all $\ell \geq 2$ and $n \geq 4+\ell$, $Q_{n,132}^{(0,1,2,1)}(x)|_{x^{n-3-\ell}} = C_{\ell+1} + 6C_\ell + 2C_\ell(n-4-\ell)$.

Proof. It is easy to see that the maximum number of matches of MMP$(0,1,\ell,1)$ that are possible in a 132-avoiding permutation is a permutation of the form $n\ \alpha\ (n-1)\ \beta$ where α is a 132-avoiding permutation on the elements $n-\ell-1,\ldots,n-2$ and β is the decreasing permutation on the elements $1,\ldots,n-\ell-2$. Thus, the highest power in $Q_n^{(0,1,\ell,1)}$ is $x^{n-\ell-2}$ which has a coefficient of C_ℓ.

For parts (ii) and (iii), we note that it follows from (9) that

$$Q_{n,132}^{(0,1,\ell,1)}(x) = Q_{n-1,132}^{(0,1,\ell,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,1)}(x).$$

We proved in [10] that the highest power of x that appears in $Q_{n,132}^{(0,1,\ell,0)}(x) = Q_{n,132}^{(0,0,\ell,1)}(x)$ is $x^{n-\ell-1}$ which appears with a coefficient of C_ℓ for $n \geq \ell+2$. This implies that the highest power of x that appears in $Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,1)}(x)$ is less than $x^{n-\ell-3}$ for $i = 3,\ldots,n-2$.

Hence we have four cases to consider when we are computing $Q^{(0,1,1,1)}_{n,132}(x)|_{x^{n-4}}$.

Case 1. $i=1$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x)Q^{(0,0,1,1)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(0,0,1,1)}_{n-1,132}(x)|_{x^{n-4}}$ and we proved in [10] that

$$Q^{(0,0,1,1)}_{n-1,132}(x)|_{x^{n-4}} = Q^{(0,1,1,0)}_{n-1,132}(x)|_{x^{n-4}} = 2+\binom{n-2}{2} \text{ for } n \geq 5.$$

Case 2. $i=2$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x)Q^{(0,0,1,1)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(0,0,1,1)}_{n-2,132}(x)|_{x^{n-4}}$ and we proved in [10] that

$$Q^{(0,0,1,1)}_{n-2,132}(x)|_{x^{n-4}} = Q^{(0,1,1,0)}_{n-2,132}(x)|_{x^{n-4}} = 1 \text{ for } n \geq 5.$$

Case 3. $i=n-1$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x)Q^{(0,0,1,1)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(0,1,1,0)}_{n-2,132}(x)|_{x^{n-4}}$ and we proved in [10] that

$$Q^{(0,1,1,0)}_{n-2,132}(x)|_{x^{n-4}} = 1 \text{ for } n \geq 5.$$

Case 4. $Q^{(0,1,1,1)}_{n-1,132}(x)|_{x^{n-4}}$. By part (i), we know that $Q^{(0,1,1,1)}_{n-1,132}(x)|_{x^{n-4}} = 1$ for $n \geq 5$.

Thus, $Q^{(0,1,1,1)}_{n,132}(x)|_{x^{n-4}} = 5+\binom{n-2}{2}$ for $n \geq 5$.

Again there are four cases to consider when computing $Q^{(0,1,\ell,1)}_{n-1,132}(x)|_{x^{n-3-\ell}}$ for $\ell \geq 2$.

Case 1. $i=1$. In this case, $Q^{(0,1,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,1)}_{n-i,132}(x)|_{x^{n-3-\ell}} = Q^{(0,0,\ell,1)}_{n-1,132}(x)|_{x^{n-3-\ell}}$ and we proved in [10] that

$$\begin{aligned} Q^{(0,0,\ell,1)}_{n-1,132}(x)|_{x^{n-3-\ell}} &= Q^{(0,1,\ell,0)}_{n-1,132}(x)|_{x^{n-n-3-\ell}} \\ &= C_{\ell+1}+C_\ell+2C_\ell(n-3-\ell) \\ &= C_{\ell+1}+3C_\ell+2C_\ell(n-4-\ell) \text{ for } n \geq 4+\ell. \end{aligned}$$

Case 2. $i=2$. In this case, $Q^{(0,1,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,1)}_{n-i,132}(x)|_{x^{n-3-\ell}} = Q^{(0,0,\ell,1)}_{n-2,132}(x)|_{x^{n-3-\ell}}$ and we proved in [10] that

$$Q^{(0,0,\ell,1)}_{n-2,132}(x)|_{x^{n-3-\ell}} = Q^{(0,1,\ell,0)}_{n-2,132}(x)|_{x^{n-3-\ell}} = C_\ell \text{ for } n \geq 4+\ell.$$

Case 3. $i=n-1$. In this case, $Q^{(0,1,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,1)}_{n-i,132}(x)|_{x^{n-3-\ell}} = Q^{(0,1,\ell,0)}_{n-2,132}(x)|_{x^{n-3-\ell}}$ and we proved in [10] that

$$Q^{(0,1,\ell,0)}_{n-2,132}(x)|_{x^{n-3-\ell}} = C_\ell \text{ for } n \geq 4+\ell.$$

Case 4. $Q^{(0,1,\ell,1)}_{n-1,132}(x)|_{x^{n-3-\ell}}$. By part (i), we know that

$$Q^{(0,1,\ell,1)}_{n-1,132}(x)|_{x^{n-3-\ell}} = C_\ell \text{ for } n \geq 4+\ell.$$

Thus,

$$Q^{(0,1,\ell,1)}_{n,132}(x)|_{x^{n-3-\ell}} = C_{\ell+1}+6C_\ell+2C_\ell(n-4-\ell) \text{ for } n \geq 4+\ell.$$

For example, when $\ell = 2$, we have that

$$Q_{n,132}^{(0,1,2,1)}(x)|_{x^{n-5}} = 17 + 4(n-6) \text{ for } n \geq 6$$

and, for $\ell = 3$, we have that

$$Q_{n,132}^{(0,1,3,1)}(x)|_{x^{n-5}} = 44 + 10(n-7) \text{ for } n \geq 7$$

which agrees with the series we computed. □

The sequence $(Q_{n,132}^{(0,1,1,1)}(0))_{n\geq 1}$ starts out $1, 2, 5, 13, 33, 81, 193, 449, \ldots$. This is the sequence A005183 in OEIS. Using the fact that $Q_{132}^{(0,1,1,0)}(t,0) = Q_{132}^{(0,0,1,1)}(t,0) = \frac{1-t}{1-2t}$, one can show that

$$Q_{132}^{(0,1,1,1)}(t,x) = \frac{1-4t+5t^2-t^3}{(1-2t)^2(1-t)}$$

from which it is possible to show that $Q_{n,132}^{(0,1,1,1)}(0) = (n-1)2^{n-2} + 1$ for $n \geq 1$.

The sequence $(Q_{n,132}^{(0,1,1,1)}(x)|_x)_{n\geq 4}$ starts out $1, 8, 39, 150, 501, 1524 \ldots$. This seems to be the sequence A055281 in the OEIS. The n-th term of this sequence $(n^2-n+4)2^{n+1} - 7 - n$ and is the number of *directed column-convex polyominoes* of area $n+5$ having along the lower contour exactly 2 *reentrant corners*.

Problem 2. Verify that the sequence $(Q_{n,132}^{(0,1,1,1)}(x)|_x)_{n\geq 4}$ is counted by

$$(n^2 - 9n + 24)2^{n-3} - 3 - n$$

and if so, find a bijective correspondence with the polyominoes described in A055281 in the OEIS.

We have computed that

$$\begin{aligned}
&Q_{132}^{(0,1,1,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 2(20+x)t^5 + \left(111 + 19x + 2x^2\right)t^6 + \\
&\left(296 + 106x + 25x^2 + 2x^3\right)t^7 + \left(761 + 456x + 178x^2 + 33x^3 + 2x^4\right)t^8 + \\
&\left(1898 + 1677x + 947x^2 + 295x^3 + 43x^4 + 2x^5\right)t^9 + \\
&\left(4619 + 5553x + 4191x^2 + 1901x^3 + 475x^4 + 55x^5 + 2x^6\right)t^{10} + \cdots .
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(0,1,2,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 4(32+x)t^6 + \\
&\left(385 + 40x + 4x^2\right)t^7 + \left(1135 + 243x + 48x^2 + 4x^3\right)t^8 + \\
&\left(3281 + 1170x + 351x^2 + 56x^3 + 4x^4\right)t^9 + \\
&\left(9324 + 4905x + 2016x^2 + 483x^3 + 64x^4 + 4x^5\right)t^{10} + \cdots .
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(0,1,3,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (419+10x)t^7 + \\
&\left(1317 + 103x + 10x^2\right)t^8 + \left(4085 + 644x + 123x^2 + 10x^3\right)t^9 + \\
&\left(12514 + 3229x + 900x^2 + 143x^3 + 10x^4\right)t^{10} + \\
&\left(37913 + 14282x + 5222x^2 + 1196x^3 + 163x^4 + 10x^5\right)t^{11} + \cdots .
\end{aligned}$$

Again we can explain the coefficients of the highest and second highest coefficients in $Q_{n,132}^{(0,1,\ell,2)}(x)$ for large enough n.

Theorem 12.

(i) For $n \geq \ell+4$, the highest power of x that appears in $Q_{n,132}^{(0,1,\ell,2)}(x)$ is $x^{n-3-\ell}$ which occurs with a coefficient of $2C_\ell$.

(ii) For $n \geq 7$, $Q_{n,132}^{(0,1,1,2)}(x)|_{x^{n-5}} = 13 + \binom{n-2}{2}$.

(iii) For all $\ell \geq 2$ and $n \geq 5+\ell$, $Q_{n,132}^{(0,1,\ell,2)}(x)|_{x^{n-4-\ell}} = 2C_{\ell+1} + 15C_\ell + 4C_\ell(n-5-\ell)$.

Proof. For (i), it is easy to see that the maximum number of matches of $\mathrm{MMP}(0,1,\ell,1)$ that are possible in a 132-avoiding permutation is a permutation of the form $n\ \alpha\ (n-1)\ \beta$ where α is a 132-avoiding permutation on the elements $n-\ell-1,\ldots,n-2$ and $\beta = (n-\ell-2)(n-\ell-3)\ldots 321$ or $\beta = (n-\ell-2)(n-\ell-3)\ldots 312$. Thus, the highest power in $Q_{n,132}^{(0,1,\ell,2)}(x)$ is $x^{n-\ell-3}$ which has a coefficient of $2C_\ell$.

For (ii) and (iii), we note that the recursion for $Q_{n,132}^{(0,1,\ell,2)}(x)$ is

$$Q_{n,132}^{(0,1,\ell,2)}(x) = Q_{n-1,132}^{(0,1,\ell,2)}(x) + Q_{n-2,132}^{(0,1,\ell,1)}(x) + \sum_{i=1}^{n-2} Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,2)}(x).$$

Since the highest power of x that occurs in $Q_{n,132}^{(0,1,\ell,0)}(x)$ is $x^{n-1-\ell}$ and the highest power of x that occurs in $Q_{n,132}^{(0,0,\ell,2)}(x)$ is $n-2-\ell$, it follows that the highest power of x that occurs in $Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,2)}(x)$ is less than $x^{n-4-\ell}$ for $i = 4,\ldots,n-3$.

Thus, we have to consider five cases when computing $Q_{n,132}^{(0,1,\ell,2)}(x)|_{x^{n-4-\ell}}$.

Case 1. $Q_{n-1,132}^{(0,1,\ell,2)}(x)|_{x^{n-4-\ell}}$. By part (i),

$$Q_{n-1,132}^{(0,1,\ell,2)}(x)|_{x^{n-4-\ell}} = 2C_\ell \text{ for } n \geq \ell+5.$$

Case 2. $Q_{n-2,132}^{(0,1,\ell,1)}(x)|_{x^{n-4-\ell}}$. We have shown earlier that

$$Q_{n-1,132}^{(0,1,\ell,1)}(x)|_{x^{n-4-\ell}} = C_\ell \text{ for } n \geq \ell+5.$$

Case 3. $i = n-2$. In this case, $Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,2)}(x)$ equals $2Q_{n-3,132}^{(0,1,\ell,0)}(x)$. We have shown in [10] that $Q_{n-3,132}^{(0,1,\ell,0)}(x)|_{x^{n-4-\ell}} = C_\ell$ for $n \geq \ell+5$ so that we get a contribution of $2C_\ell$ in this case.

Case 4. $i = 2$. In this case, $Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,2)}(x)$ equals $Q_{n-2,132}^{(0,0,\ell,2)}(x)$. We have shown in [10] that

$$Q_{n-3,132}^{(0,0,\ell,2)}(x)|_{x^{n-4-\ell}} = 2C_\ell \text{ for } n \geq \ell+5.$$

Case 5. $i = 1$. In this case, $Q^{(0,1,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,2)}_{n-i,132}(x)$ equals $Q^{(0,0,\ell,2)}_{n-1,132}(x)$. We have shown in [10] that for $n \geq \ell + 5$,

$$Q^{(0,0,\ell,2)}_{n-3,132}(x)|_{x^{n-4-\ell}} = \begin{cases} 6 + 2\binom{n-2}{2} & \text{if } \ell = 1, \\ 2C_{\ell+1} + 8C_\ell + 4C_\ell(n-5-\ell) & \text{if } \ell \geq 2. \end{cases}$$

Thus, for $\ell = 1$, we get

$$Q^{(0,1,1,2)}_{n-i,132}(x)|_{x^{n-5}} = 13 + 2\binom{n-2}{2} \text{ for } n \geq 6$$

and, for $\ell \geq 2$,

$$Q^{(0,1,\ell,2)}_{n-i,132}(x)|_{x^{n-4-\ell}} = 2C_{\ell+1} + 15C_\ell + 4C_\ell(n-5-\ell) \text{ for } n \geq 5+\ell.$$

For example, when $\ell = 2$, we get

$$Q^{(0,1,2,2)}_{n-i,132}(x)|_{x^{n-6}} = 40 + 8(n-7) \text{ for } n \geq 7$$

and, for $\ell = 3$, we get

$$Q^{(0,1,2,2)}_{n-i,132}(x)|_{x^{n-6}} = 103 + 20(n-8) \text{ for } n \geq 8$$

which agrees with the series that we computed. $\square$

$$\begin{aligned}&Q^{(0,2,1,2)}_{132}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 4t^6(32+x) + \left(380 + 45x + 4x^2\right)t^7 + \\ &\left(1083 + 286x + 57x^2 + 4x^3\right)t^8 + \left(2964 + 1368x + 453x^2 + 73x^3 + 4x^4\right)t^9 + \\ &\left(7831 + 5501x + 2650x^2 + 717x^3 + 93x^4 + 4x^5\right)t^{10} + \\ &\left(20092 + 19675x + 12749x^2 + 5035x^3 + 1114x^4 + 117x^5 + 4x^6\right)t^{11} + \cdots .\end{aligned}$$

$$\begin{aligned}&Q^{(0,2,2,2)}_{132}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (421 + 8x)t^7 + \\ &\left(1328 + 94x + 4x^2\right)t^8 + \left(4103 + 641x + 110x^2 + 8x^3\right)t^9 + \\ &\left(12401 + 3376x + 885x^2 + 126x^3 + 8x^4\right)t^{10} + \\ &\left(36740 + 15235x + 5484x^2 + 1177x^3 + 142x^4 + 8x^5\right)t^{11} + \\ &\left(106993 + 62012x + 28872x^2 + 8452x^3 + 1517x^4 + 158x^5 + 8x^6\right)t^{12} + \cdots .\end{aligned}$$

$$\begin{aligned}&Q^{(0,2,3,2)}_{132}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + 429t^7 + \\ &(1410 + 20x)t^8 + \left(4601 + 241x + 20x^2\right)t^9 + \left(14809 + 1686x + 281x^2 + 20x^3\right)t^{10} + \\ &\left(46990 + 9187x + 2268x^2 + 321x^3 + 20x^4\right)t^{11} + \\ &\left(147163 + 43394x + 14144x^2 + 2930x^3 + 361x^4 + 20x^5\right)t^{12} + \cdots .\end{aligned}$$

It is easy to explain the coefficient of the highest power in $Q^{(0,2,\ell,2)}_n(x)$. That is, the maximum number of matches of MMP$(0,2,\ell,2)$ that are possible in a 132-avoiding permutation is a permutation of the form $n(n-1)\ \alpha\ (n-2)\ \beta$ or $(n-1)n\ \alpha\ (n-2)\ \beta$ where α is a 132-avoiding permutation on the elements $n-\ell-2, \ldots, n-3$ and $\beta = (n-\ell-3)(n-\ell-4)\ldots 321$ or $\beta = (n-\ell-3)(n-\ell-4)\ldots 312$. Thus, the highest power in $Q^{(0,1,\ell,2)}_n$ is $x^{n-\ell-5}$ which has a coefficient of $4C_\ell$.

4 $Q_{n,132}^{(\ell,k,0,m)}(x) = Q_{n,132}^{(\ell,m,0,k)}(x)$ where $k,\ell,m \geq 1$

By Lemma 1, we need only consider $Q_{n,132}^{(\ell,k,0,m)}(x)$. Suppose that $k,\ell,m \geq 1$ and $n \geq k+m$. It is clear that n can never match MMP$(\ell,k,0,m)$ for $k,\ell,m \geq 1$ in any $\sigma \in S_n(132)$. If $\sigma = \sigma_1 \ldots \sigma_n \in S_n(132)$ and $\sigma_i = n$, then we have three cases, depending on the value of i.

Case 1. $i < k$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since the elements in $A_i(\sigma)$ do not have enough elements to the left to match MMP$(\ell,k,0,m)$ in σ. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(\ell,k-i,0,m)}(x)$ to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s,σ_s) for any $s > i$. Thus, the permutations in Case 1 will contribute

$$\sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(\ell,k-i,0,m)}(x)$$

to $Q_{n,132}^{(\ell,k,0,m)}(x)$.

Case 2. $k \leq i < n-m$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(\ell-1,k,0,0)}(x)$ to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since the elements in $B_i(\sigma)$ will all be in the fourth quadrant and $\sigma_i = n$ is in the first quadrant relative to a coordinate system centered at (r,σ_r) for $r \leq i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(\ell,0,0,m)}(x)$ to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s,σ_s) for any $s > i$. Thus, the permutations in Case 2 will contribute

$$\sum_{i=k}^{n-m} Q_{i-1,132}^{(\ell-1,k,0,0)}(x) Q_{n-i,132}^{(\ell,0,0,m)}(x)$$

to $Q_{n,132}^{(\ell,k,0,m)}(x)$.

Case 3. $i \geq n-m+1$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(\ell-1,k,0,m-(n-i))}(x)$ to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since $\sigma_i = n$ will be in the first quadrant and the elements in $B_i(\sigma)$ will all be in the fourth quadrant relative to a coordinate system centered at (r,σ_r) for $r \leq i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of C_{n-i} to $Q_{n,132}^{(\ell,0,0,m)}(x)$ since σ_j where $j > i$ does not have enough elements to its right to match MMP$(\ell,k,0,m)$ in σ. Thus, the permutations in Case 3 will contribute

$$\sum_{i=n-m+1}^{n} Q_{i-1,132}^{(\ell-1,k,0,m-(n-i))}(x) C_{n-i}$$

to $Q_{n,132}^{(\ell,k,0,m)}(x)$. Thus, we have the following. For $n \geq k+m$,

$$\begin{aligned} Q_{n,132}^{(\ell,k,0,m)}(x) &= \sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(\ell,k-i,0,m)}(x) + \sum_{i=k}^{n-m} Q_{i-1,132}^{(\ell-1,k,0,0)}(x) Q_{n-i,132}^{(\ell,0,0,m)}(x) + \\ & \quad \sum_{i=n-m+1}^{n} Q_{i-1,132}^{(\ell-1,k,0,m-(n-i))}(x) C_{n-i}. \end{aligned} \tag{10}$$

Multiplying (10) by t^n and summing over n will yield the following theorem.

Theorem 13. *For all $\ell, k, m \geq 1$,*

$$\begin{aligned} Q_{132}^{(\ell,k,0,m)}(t,x) &= \sum_{p=0}^{k+m-1} C_p t^p + \\ & t \sum_{i=0}^{k-2} C_i t^i \left(Q_{132}^{(\ell,k-1-i,0,m)}(t,x) - \sum_{r=0}^{k-i+m-2} C_r t^r \right) + \\ & t \left(Q_{132}^{(\ell-1,k,0,0)}(t,x) - \sum_{a=0}^{k-2} C_a t^a \right) \left(Q_{132}^{(\ell,0,0,m)}(t,x) - \sum_{b=0}^{m-1} C_b t^b \right) + \\ & t \sum_{j=0}^{m-1} C_j t^j \left(Q_{132}^{(\ell-1,k,0,m-j)}(t,x) - \sum_{s=0}^{k+m-j-2} C_s t^s \right). \end{aligned} \tag{11}$$

Note that we can compute $Q_{132}^{(\ell,k,0,0)}(t,x) = Q_{132}^{(\ell,0,0,k)}(t,x)$ by Theorem 6 so that (11) allows us to compute $Q_{132}^{(\ell,k,0,m)}(t,x)$ for any $k, \ell, m \geq 0$.

4.1 Explicit formulas for $Q_{n,132}^{(\ell,k,0,m)}(x)|_{x^r}$

It follows from Theorem 13 that

$$Q_{132}^{(\ell,1,0,1)}(t,x) = 1+t+tQ_{132}^{(\ell-1,1,0,0)}(t,x) \left(Q_{132}^{(\ell,0,0,1)}(t,x) - 1 \right) + t \left(Q_{132}^{(\ell-1,1,0,1)}(t,x) - 1 \right), \tag{12}$$

$$\begin{aligned} Q_{132}^{(\ell,1,0,2)}(t,x) &= 1+t+2t^2 + tQ_{132}^{(\ell-1,1,0,0)}(t,x) \left(Q_{132}^{(\ell,0,0,2)}(t,x) - (1+t) \right) + \\ & t \left(Q_{132}^{(\ell-1,1,0,2)}(t,x) - (1+t) + t \left(Q_{132}^{(\ell-1,1,0,1)}(t,x) - 1 \right) \right), \end{aligned}$$

and

$$\begin{aligned} Q_{132}^{(\ell,2,0,2)}(t,x) &= 1+t+2t^2+5t^3 + t \left(Q_{132}^{(\ell,1,0,2)}(t,x) - (1+t+2t^2) \right) + \\ & t \left(Q_{132}^{(\ell-1,2,0,0)}(t,x) - 1 \right) \left(Q_{132}^{(\ell,0,0,2)}(t,x) - (1+t) \right) + \\ & t \left(Q_{132}^{(\ell-1,2,0,2)}(t,x) - (1+t+2t^2) + t \left(Q_{132}^{(\ell-1,2,0,1)}(t,x) - (1+t) \right) \right). \end{aligned}$$

One can use these formulas to compute the following.

$$\begin{aligned}&Q_{132}^{(1,1,0,1)}(t,x) = 1+t+2t^2+5t^3+2(5+2x)t^4+\left(17+17x+8x^2\right)t^5+\\&\left(26+44x+42x^2+20x^3\right)t^6+\left(37+90x+129x^2+117x^3+56x^4\right)t^7+\\&\left(50+160x+305x^2+397x^3+350x^4+168x^5\right)t^8+\\&\left(65+259x+615x^2+1029x^3+1268x^4+1098x^5+528x^6\right)t^9+\\&\left(82+392x+1113x^2+2259x^3+3503x^4+4167x^5+3564x^6+1716x^7\right)t^{10}+\cdots.\end{aligned}$$

It is easy to explain the highest coefficient of x in $Q_{n,132}^{(1,\ell,0,1)}(x)$.

Theorem 14. *For $n \geq 3+\ell$, the highest power of x that occurs in $Q_{n,132}^{(1,\ell,0,1)}(x)$ is $x^{n-2-\ell}$ which occurs with a coefficient of $4C_\ell C_{n-\ell-2}$.*

Proof. It is easy to see that the maximum number of MMP$(\ell,1,0,1)$ matches occurs in $\sigma \in S_n(132)$ when σ is of the form $n\ \tau\ \alpha\ (n-1)$, $n\ \tau\ (n-1)\ \alpha$, $(n-1)\ \tau\ \alpha\ n$, or $(n-1)\ \tau\ n\ \alpha$ where α is a 132-avoiding permutation on the elements $1,\ldots,\ell$ and τ is a 132-avoiding permutations of the elements $\ell+1,\ldots,n-2$. Thus, the highest power of x in $Q_{n,132}^{(\ell,1,0,1)}(x)$ for $n \geq \ell+3$ is $x^{n-2-\ell}$ which occurs with a coefficient of $4C_\ell C_{n-\ell-2}$. ☐

We can also explain the second highest coefficient in $Q_{n,132}^{(1,1,0,1)}(x)$.

Theorem 15. *For $n \geq 5$,*

$$Q_{n,132}^{(1,1,0,1)}(x)|_{x^{n-4}} = 8C_{n-3}+C_{n-4}.$$

Proof. In this case, the recursion for $Q_{n,132}^{(1,1,0,1)}(x)$ is

$$Q_{n,132}^{(1,1,0,1)}(x) = Q_{n-1,132}^{(0,1,0,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(0,1,0,0)}(x)Q_{n-i,132}^{(1,0,0,1)}(x).$$

It was proved in [9] that for $n \geq 1$, the highest power of x that occurs in $Q_{n,132}^{(0,1,0,0)}(x)$ is x^{n-1} which occurs with a coefficient of C_{n-1}. It was proved in [10] that for $n \geq 3$, the highest power of x that occurs in $Q_{n,132}^{(1,0,0,1)}(x)$ is x^{n-2} which occurs with a coefficient of $2C_{n-2}$. It follows that

$$\begin{aligned}Q_{n,132}^{(1,1,0,1)}(x)|_{x^{n-4}} &= Q_{n-1,132}^{(0,1,0,1)}(x)|_{x^{n-4}} + Q_{n-1,132}^{(1,0,0,1)}(x)|_{x^{n-4}} + Q_{n-2,132}^{(0,1,0,0)}(x)|_{x^{n-4}} +\\&\quad \sum_{i=2}^{n-2} Q_{i-1,132}^{(0,1,0,0)}(x)|_{x^{i-2}} Q_{n-i,132}^{(1,0,0,1)}(x)|_{x^{n-i-2}}.\end{aligned}$$

It was shown in [9] and [10] that

$$\begin{aligned}Q_{n-1,132}^{(0,1,0,1)}(x)|_{x^{n-4}} &= 2C_{n-3}+C_{n-4} \text{ for } n \geq 5,\\Q_{n-1,132}^{(1,0,0,1)}(x)|_{x^{n-4}} &= 3C_{n-3} \text{ for } n \geq 5, \text{ and}\\Q_{n-2,132}^{(0,1,0,0)}(x)|_{x^{n-4}} &= C_{n-3} \text{ for } n \geq 5.\end{aligned}$$

Thus, for $n \geq 5$,

$$\begin{aligned}
Q_{n,132}^{(1,1,0,1)}(x)|_{x^{n-4}} &= 2C_{n-3} + C_{n-4} + 3C_{n-3} + C_{n-3} + \sum_{i=2}^{n-2} C_{i-2} 2C_{n-i-2} \\
&= 6C_{n-3} + C_{n-4} + 2\sum_{i=2}^{n-2} C_{i-2}C_{n-i-2} \\
&= 6C_{n-3} + C_{n-4} + 2C_{n-3} = 8C_{n-3} + C_{n-4}.
\end{aligned}$$

$\square$

The sequence $(Q_{n,132}^{(1,1,0,1)}(0))_{n\geq 1}$ starts out $1, 2, 5, 10, 17, 26, 37, 50, 82, \ldots$ which is the sequence A002522 in the OEIS. The n-th element of the sequence has the formula $(n-1)^2+1$. This can be verified by computing the generating function $Q_{132}^{(1,1,0,1)}(t, 0)$. That is, we proved in [9] and [10] that

$$\begin{aligned}
Q_{132}^{(0,1,0,0)}(t,0) &= \frac{1}{1-t}, \\
Q_{132}^{(1,0,0,1)}(t,0) &= \frac{1-2t+2t^2}{(1-t)^3}, \text{ and} \\
Q_{132}^{(0,1,0,1)}(t,0) &= \frac{1}{1-t} + \frac{t^2}{(1-t)^2}.
\end{aligned}$$

Plugging these formulas into (12), one can compute that

$$Q_{132}^{(1,1,0,1)}(t,0) = \frac{1-3t+5t^2-2t^3+t^4}{(1-t)^3}.$$

Problem 3. Find a direct combinatorial proof of the fact that $Q_{n,132}^{(1,1,0,1)}(0) = (n-1)^2+1$ for $n \geq 1$.

$$\begin{aligned}
&Q_{132}^{(2,1,0,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (33+9x)t^5 + \left(71+43x+18x^2\right)t^6 + \\
&\left(146+137x+101x^2+45x^3\right)t^7 + \\
&\left(294+368x+367x^2+275x^3+126x^4\right)t^8 + \\
&\left(587+906x+1100x^2+1079x^3+812x^4+378x^5\right)t^9 + \\
&\left(1169+2125x+2973x^2+3463x^3+3352x^4+2526x^5+1188x^6\right)t^{10} + \cdots
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(3,1,0,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (116+16x)t^6 + \\
&\left(308+89x+32x^2\right)t^7 + \left(807+341x+202x^2+80x^3\right)t^8 + \\
&\left(2108+1140x+849x^2+541x^3+224x^4\right)t^9 + \\
&\left(5507+3583x+3046x^2+2406x^3+1582x^4+672x^5\right)t^{10} + \\
&\left(14397+10897x+10141x^2+9039x^3+7310x^4+4890x^5+2112x^6\right)t^{11} + \cdots.
\end{aligned}$$

It is not difficult to show that for $n \geq k+3$, the highest power or x that occurs in $Q_{n,132}^{(k,1,0,1)}(x)$ is x^{n-k-2} which appears with a coefficient of $(k+1)^2 C_{n-k-2}$. That is, the maximum number of occurrences of $MMP(k,1,0,1)$ for a $\sigma \in S_n(132)$ occurs when σ is of the form $x\tau\beta$ where $x \in \{n-k,\ldots,n\}$, β is a shuffle of 1 with the increasing sequence which results from $(n-k)(n-k+1)\ldots n$ by removing x, and τ is a 132-avoiding permutation on $2,\ldots,n-k-1$. Thus we have $k+1$ choices for x and, once x is chosen, we have $k+1$ choices for β, and C_{n-k-2} choices for τ.

$$\begin{aligned}
&Q_{132}^{(1,1,0,2)}(t,x) = 1+t+2t^2+5t^3+14t^4+(32+10x)t^5+\\
&\left(62+50x+20x^2\right)t^6+\left(107+149x+123x^2+50x^3\right)t^7+\\
&\left(170+345x+433x^2+342x^3+140x^4\right)t^8+\\
&\left(254+685x+1154x^2+1327x^3+1022x^4+420x^5\right)t^9+\\
&\left(362+1225x+2589x^2+3868x^3+4228x^4+3204x^5+1320x^6\right)t^{10}+\cdots.
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(2,1,0,2)}(t,x) = 1+t+2t^2+5t^3+14t^4+42t^5+(105+27x)t^6+\\
&\left(235+140x+54x^2\right)t^7+\left(494+470x+331x^2+135x^3\right)t^8+\\
&\left(1004+1301x+1275x^2+904x^3+378x^4\right)t^9+\\
&\left(2007+3248x+3960x^2+3773x^3+2674x^4+1134x^5\right)t^{10}+\cdots.
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(3,1,0,2)}(t,x) = 1+t+2t^2+5t^3+14t^4+42t^5+132t^6+(373+56x)t^7+\\
&\left(998+320x+112x^2\right)t^8+\left(2615+1233x+734x^2+280x^3\right)t^9+\\
&\left(6813+4092x+3131x^2+1976x^3+784x^4\right)t^{10}+\\
&\left(17749+12699x+11223x^2+8967x^3+5796x^4+2352x^5\right)t^{11}+\cdots.
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(1,2,0,2)}(t,x) = 1+t+2t^2+5t^3+14t^4+42t^5+(107+25x)t^6+\\
&\left(233+146x+50x^2\right)t^7+\left(450+498x+357x^2+125x^3\right)t^8+\\
&\left(794+1299x+1429x^2+990x^3+350x^4\right)t^9+\\
&\left(1307+2869x+4263x^2+4353x^3+2954x^4+1050x^5\right)t^{10}+\cdots.
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(2,2,0,2)}(t,x) = 1+t+2t^2+5t^3+14t^4+42t^5+132t^6+(348+81x)t^7+\\
&\left(811+457x+162x^2\right)t^8+\left(1747+1625x+1085x^2+405x^3\right)t^9+\\
&\left(3587+4663x+4443x^2+2969x^3+1134x^4\right)t^{10}+\\
&\left(7167+11864x+14360x^2+13201x^3+8792x^4+3402x^5\right)t^{11}+\cdots.
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(3,2,0,2)}(t,x) = 1+t+2t^2+5t^3+14t^4+42t^5+132t^6+429t^7+\\
&(1234+196x)t^8+2\left(1657+578x+196x^2\right)t^9+\\
&\left(8643+4497x+2676x^2+980x^3\right)t^{10}+\\
&\left(22345+14839x+11622x^2+7236x^3+2744x^4\right)t^{11}+\cdots.
\end{aligned}$$

Problem 4. In all the cases above, it seems that for $n \geq \ell + k + m + 1$, the highest power of x in $Q_{n,132}^{(\ell,k,0,m)}(x)$ is $x^{n-k-\ell}$ which appears with a coefficient of $a_{\ell,k,m} C_{n-\ell-k-m}$ for some constant $a_{\ell,k,m}$. Prove that this is the case and find a formula for $a_{\ell,k,m}$.

5 $Q_{n,132}^{(a,b,c,d)}(x) = Q_{n,132}^{(a,d,c,b)}(x)$ where $a, b, c, d \geq 1$

By Lemma 1, we only need to consider the case of $Q_{n,132}^{(a,b,c,d)}(x)$. Suppose that $a, b, c, d \geq 1$ and $n \geq b+d$. It is clear that n can never match the pattern MMP(a, b, c, d) for $a, b, c, d \geq 1$ in any $\sigma \in S_n(132)$. If $\sigma = \sigma_1 \dots \sigma_n \in S_n(132)$ and $\sigma_i = n$, then we have three cases, depending on the value of i.

Case 1. $i < b$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q_{n,132}^{(a,b,c,d)}(x)$ since the elements in $A_i(\sigma)$ do not have enough elements to the left to match MMP(a, b, c, d) in σ. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(a,b-i,c,d)}(x)$ to $Q_{n,132}^{(a,b,c,d)}(x)$ since $\sigma_1 \dots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s, σ_s) for any $s > i$. Thus, the permutations in Case 1 will contribute

$$\sum_{i=1}^{b-1} C_{i-1} Q_{n-i,132}^{(a,b-i,c,d)}(x)$$

to $Q_{n,132}^{(a,b,c,d)}(x)$.

Case 2. $b \leq i < n - d$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(a-1,b,c,0)}(x)$ to $Q_{n,132}^{(a,b,c,d)}(x)$ since the elements in $B_i(\sigma)$ will all be in the fourth quadrant and $\sigma_i = n$ is in the first quadrant relative to a coordinate system centered at (r, σ_r) for $r \leq i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(a,0,c,d)}(x)$ to $Q_{n,132}^{(a,b,c,d)}(x)$ since $\sigma_1 \dots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s, σ_s) for any $s > i$. Thus, the permutations in Case 2 will contribute

$$\sum_{i=b}^{n-d} Q_{i-1,132}^{(a-1,b,c,0)}(x) Q_{n-i,132}^{(a,0,c,d)}(x)$$

to $Q_{n,132}^{(a,b,c,d)}(x)$.

Case 3. $i \geq n - d + 1$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(a-1,b,c,d-(n-i))}(x)$ to $Q_{n,132}^{(a,b,c,d)}(x)$ since $\sigma_i = n$ will be in the first quadrant and the elements in $B_i(\sigma)$ will all be in the fourth quadrant relative to a coordinate system centered at (r, σ_r) for $r \leq i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of C_{n-i}

to $Q_{n,132}^{(a,b,c,d)}(x)$ since σ_j, where $j > i$, does not have enough elements to its right to match MMP(a,b,c,d) in σ. Thus, the permutations in Case 3 will contribute

$$\sum_{i=n-d+1}^{n} Q_{i-1,132}^{(a-1,b,c,d-(n-i))}(x) C_{n-i}$$

to $Q_{n,132}^{(a,b,c,d)}(x)$. Thus, we have the following. For $n \geq a+b+c+d+1$,

$$\begin{aligned} Q_{n,132}^{(a,b,c,d)}(x) &= \sum_{i=1}^{b-1} C_{i-1} Q_{n-i,132}^{(a,b-i,c,d)}(x) + \sum_{i=b}^{n-d} Q_{i-1,132}^{(a-1,b,c,0)}(x) Q_{n-i,132}^{(a,0,c.d)}(x) + \\ & \sum_{i=n-d+1}^{n} Q_{i-1,132}^{(a-1,b,c,d-(n-i))}(x) C_{n-i}. \end{aligned} \tag{13}$$

Multiplying (13) by t^n and summing, we obtain the following theorem.

Theorem 16. *For all $a,b,c,d \geq 1$,*

$$\begin{aligned} Q_{132}^{(a,b,c,d)}(t,x) &= \sum_{p=0}^{b+d-1} C_p t^p + \\ & t \sum_{i=0}^{b-2} C_i t^i \left(Q_{132}^{(a,b-1-i,c,d)}(t,x) - \sum_{r=0}^{b-i+d-2} C_r t^r \right) + \\ & t \left(Q_{132}^{(a-1,b,c,0)}(t,x) - \sum_{i=0}^{b-2} C_i t^i \right) \left(Q_{132}^{(a,0,c,d)}(t,x) - \sum_{j=0}^{d-1} C_j t^j \right) + \\ & t \sum_{j=0}^{d-1} C_j t^j \left(Q_{132}^{(a-1,b,c,d-j)}(t,x) - \sum_{s=0}^{b+d-j-2} C_s t^s \right). \end{aligned}$$

Thus, for example,

$$\begin{aligned} Q_{132}^{(1,1,1,1)}(t,x) &= 1 + t + t Q_{132}^{(0,1,1,0)}(t,x) \left(Q_{132}^{(1,0,1,1)}(t,x) - 1 \right) + \\ & t(Q_{132}^{(0,1,1,1)}(t,x) - 1). \end{aligned}$$

and, for $k \geq 2$,

$$\begin{aligned} Q_{132}^{(k,1,1,1)}(t,x) &= 1 + t + t Q_{132}^{(k-1,1,1,0)}(t,x) \left(Q_{132}^{(k,0,1,1)}(t,x) - 1 \right) + \\ & t(Q_{132}^{(k-1,1,1,1)}(t,x) - 1). \end{aligned}$$

$$\begin{aligned} & Q_{132}^{(1,1,1,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (38+4x)t^5 + \left(99 + 29x + 4x^2\right) t^6 + \\ & \left(249 + 135x + 41x^2 + 4x^3\right) t^7 + \left(609 + 510x + 250x^2 + 57x^3 + 4x^4\right) t^8 + \\ & \left(1457 + 1701x + 1177x^2 + 446x^3 + 77x^4 + 4x^5\right) t^9 + \\ & \left(3425 + 5220x + 4723x^2 + 2564x^3 + 759x^4 + 101x^5 + 4x^6\right) t^{10} + \cdots, \end{aligned}$$

$$Q_{132}^{(2,1,1,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (123+9x)t^6 + \left(350 + 70x + 9x^2\right) t^7 + \left(974 + 350x + 97x^2 + 9x^3\right) t^8 + \left(2667 + 1433x + 620x^2 + 133x^3 + 9x^4\right) t^9 + \left(7218 + 5235x + 3079x^2 + 1077x^3 + 178x^4 + 9x^5\right) t^{10} + \cdots,$$

and

$$Q_{132}^{(3,1,1,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (413+16x)t^7 + \left(1277 + 137x + 16x^2\right) t^8 + \left(3909 + 752x + 185x^2 + 16x^3\right) t^9 + \left(11881 + 3383x + 1267x^2 + 249x^3 + 16x^4\right) t^{10} + \cdots.$$

It is easy to explain the coefficient to the highest power that appears in $Q_{n,132}^{(k,1,1,1)}(x)$ for $k \geq 1$. That is, the maximum number of matches of MMP$(1,1,1,1)$ for $\sigma \in S_n(132)$ is when σ is of the form $x\ \alpha\ \beta$ where $x \in \{n-k, \ldots, n\}$, β is a shuffle of 1 with the sequence $(n-k)(n-k+1)\ldots n$ with x removed, and $\alpha = 23\ldots(n-k-1)$. Note that we have $k+1$ choices for x and, once we chosen x, we have $k+1$ choices for β. Thus, the highest power of x that occurs in $Q_{n,132}^{(k,1,1,1)}(x)$ is x^{n-k-3} which occurs with a coefficient of $(k+1)^2$ for $n \geq k+4$.

We also have

$$\begin{aligned} Q_{132}^{(0,1,1,0)}(t,0) &= \frac{1-t}{1-2t}, \\ Q_{132}^{(1,0,1,1)}(t,0) &= 1 + t\left(\frac{1-t}{1-2t}\right)^2, \text{ and} \\ Q_{132}^{(0,1,1,1)}(t,0) &= \frac{1-4t+5t^2-t^3}{(1-2t)^2(1-t)}, \end{aligned}$$

to compute that

$$Q_{132}^{(1,1,1,1)}(t,0) = \frac{1-6t+13t^2-11t^3+3t^4-2t^5+t^6}{(1-t)(1-2t)^3}.$$

Note that $Q_{132}^{(1,1,1,1)}(t,0)$ is the generating function of the permutations that avoid the patterns from the set $\{132, 52314, 52341, 42315, 42351\}$.

Finally, we can also determine the second highest coefficient of x in $Q_{n,132}^{(1,1,1,1)}(x)$.

Theorem 17. *For all $n \geq 6$,*

$$Q_{n,132}^{(1,1,1,1)}(x)|_{x^{n-5}} = 17 + 4\binom{n-3}{3}.$$

Proof. The recursion of $Q_{n,132}^{(1,1,1,1)}(x)$ is

$$Q_{n,132}^{(1,1,1,1)}(x) = Q_{n-1,132}^{(0,1,1,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(0,1,1,0)}(x) Q_{n-i,132}^{(1,0,1,1)}(x).$$

For $n \geq 3$, the highest power of x which occurs in $Q_{n,132}^{(0,1,1,0)}(x)$ is x^{n-2} and for $n \geq 4$, the highest power of x that occurs in $Q_{n,132}^{(1,0,1,1)}(x)$ is x^{n-3}. It follows that for $i = 2, \ldots, n-3$, the highest power of x that occurs in $Q_{i-1,132}^{(0,1,1,0)}(x) Q_{n-i,132}^{(1,0,1,1)}(x)$ is x^{n-6}. It follows that

$$\begin{aligned} Q_{n,132}^{(1,1,1,1)}(x)|_{x^{n-5}} &= Q_{n-1,132}^{(1,0,1,1)}(x)|_{x^{n-5}} + Q_{n-2,132}^{(1,0,1,1)}(x)|_{x^{n-5}} + 2Q_{n-3,132}^{(0,1,1,0)}(x)|_{x^{n-5}} + \\ & \quad Q_{n-2,132}^{(0,1,1,0)}(x)|_{x^{n-5}} + Q_{n-1,132}^{(0,1,1,1)}(x)|_{x^{n-5}}. \end{aligned}$$

But for $n \geq 6$, we have proved that

$$\begin{aligned} Q_{n-1,132}^{(1,0,1,1)}(x)|_{x^{n-5}} &= 6 + 2\binom{n-3}{2}, \\ Q_{n-2,132}^{(1,0,1,1)}(x)|_{x^{n-5}} &= 2, \\ 2Q_{n-3,132}^{(0,1,1,0)}(x)|_{x^{n-5}} &= 2C_1 = 2, \\ Q_{n-2,132}^{(0,1,1,0)}(x)|_{x^{n-5}} &= 2 + \binom{n-3}{2}, \text{ and} \\ Q_{n-1,132}^{(0,1,1,1)}(x)|_{x^{n-5}} &= 5 + \binom{n-3}{2}. \end{aligned}$$

Thus, $Q_{n,132}^{(1,1,1,1)}(x)|_{x^{n-5}} = 17 + 4\binom{n-3}{2}$. □

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