FREE ASSOCIATIVE ALGEBRAS, NONCOMMUTATIVE GRÖBNER BASES, AND UNIVERSAL ASSOCIATIVE ENVELOPES FOR NONASSOCIATIVE STRUCTURES

MURRAY R. BREMNER

Abstract. These are the lecture notes from my short course of the same title at the CIMPA Research School on *Associative and Nonassociative Algebras and Dialgebras: Theory and Algorithms - In Honour of Jean-Louis Loday (1946–2012)*, held at CIMAT, Guanajuato, Mexico, February 17 to March 2, 2013. The underlying motivation is to apply the theory of noncommutative Gröbner bases in free associative algebras to the construction of universal associative envelopes for nonassociative structures defined by multilinear operations. Trilinear operations were classified by the author and Peresi in 2007. In her Ph.D. thesis of 2012, Elgendy studied the universal associative envelopes of nonassociative triple systems obtained by applying these trilinear operations to the 2-dimensional simple associative triple system. In these notes I use computer algebra to extend some aspects of her work to the 4-dimensional and 6-dimensional simple associative triple systems.

1. Introduction

The primary goal of these lecture notes is to apply the theory of noncommutative Gröbner bases in free associative algebras to the construction of universal associative envelopes for nonassociative structures defined by multilinear operations. Throughout I will take an algorithmic approach, developing just enough theory to motivate the computational methods. Some of the easier proofs and examples are left as exercises for the reader. Along the way, I will mention a number of open research problems. I begin by recalling the basic definitions of the most familiar examples of nonassociative structures: finite dimensional Lie and Jordan algebras and their universal associative enveloping algebras. Unless otherwise indicated, I will work over an arbitrary field F.

1.1. **Lie algebras.** Lie algebras are defined by the polynomial identities of degree ≤ 3 satisfied by the Lie bracket $[x, y] = xy - yx$ in every associative algebra, namely anticommutativity and the Jacobi identity:

$$[x, x] \equiv 0, \qquad [[x, y], z] + [[y, z], x] + [[z, x], y] \equiv 0.$$

Every polynomial identity satisfied by the Lie bracket in every associative algebra is a consequence of these two identities; see Corollary 7.2.

Definition 1.1. Let A be an associative algebra with product denoted xy. We write A^- for the Lie algebra which has the same underlying vector space as A, but the original associative operation is replaced by the Lie bracket $[x, y] = xy - yx$. Let L be a Lie algebra over F. If L is isomorphic to a subalgebra of A^- then we call A an **associative envelope** for L.

Example 1.2. Let $L = \mathfrak{sl}_n(F)$ be the special linear Lie algebra of all $n \times n$ matrices of trace 0 over F. Then clearly L is a subalgebra of A^- where $A = M_n(F)$ is the associative algebra of all $n \times n$ matrices.

Definition 1.3. The **universal associative envelope** $U(L)$ of the Lie algebra L is the unital associative algebra satisfying the following **universal property**, which implies that $U(L)$ is unique up to isomorphism:

- There is a morphism of Lie algebras $\alpha\colon L \to U(L)^-$ such that for any unital associative algebra A and any morphism of Lie algebras $\beta\colon L \to A^-$, there is a unique morphism of associative algebras $\gamma\colon U \to A$ satisfying $\beta = \gamma \circ \alpha$.

In the terminology of category theory, this says that the functor sending a Lie algebra L to its universal associative envelope $U(L)$ is the left adjoint of the functor sending an associative algebra A to the Lie algebra A^-.

Lemma 1.4. *The subset $\alpha(L)$ generates $U(L)$. If A is an associative envelope for L, and A is generated by the subset L, then A is isomorphic to a quotient of $U(L)$; that is, $A \approx U(L)/I$ for some ideal I.*

Proof. Exercise. □

We will see later that $U(L)$ is always infinite dimensional, and that the map α is always injective, so that L is isomorphic to a subalgebra of $U(L)^-$. These are corollaries of the PBW theorem (Theorem 7.1) that we will prove using the theory of noncommutative Gröbner bases.

Example 1.5. Let L be the n-dimensional Lie algebra with basis $\{x_1, \ldots, x_n\}$ and trivial commutation relations $[x_i, x_j] = 0$ for all i, j. Then $U(L) \approx F[x_1, \ldots, x_n]$, the algebra of commutative associative polynomials in n variables over F.

1.2. **Jordan algebras.** Assume that $\operatorname{char} F \neq 2$. Jordan algebras are defined by the polynomial identities of degree ≤ 4 satisfied by the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$ in every associative algebra, commutativity and the Jordan identity:

$$x \circ y \equiv y \circ x, \qquad ((x \circ x) \circ y) \circ x \equiv (x \circ x) \circ (y \circ x).$$

In contrast to Lie algebras, there exist further identities satisfied by the Jordan product in every associative algebra which are not consequences of these two identities. The simplest such identities were discovered almost 50 years ago; they have degree 8 and are called the Glennie identities [56].

Definition 1.6. Let A be an associative algebra with product denoted xy. We write A^+ for the Jordan algebra which has the same underlying vector space as A, but the original associative operation is replaced by the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. Let J be a Jordan algebra over F. If J is isomorphic to a subalgebra of A^+ then we call A an **associative envelope** for J.

Example 1.7. Let $S_n(F)$ be the Jordan algebra of symmetric $n \times n$ matrices with entries in F, and let $A = M_n(F)$ be the associative algebra of all $n \times n$ matrices.

Exercise 1.8. Modify Definition 1.3 to define universal associative envelopes for Jordan algebras. State and prove the analogue of Lemma 1.4 for Jordan algebras.

If J is finite dimensional, then so is its universal associative envelope $U(J)$. On the other hand, the natural map from J to $U(J)$ may not be injective; hence, strictly speaking, the universal associative envelope $U(J)$ may not be an associative envelope in the sense of Definition 1.6.

Example 1.9. Let J be the n-dimensional Jordan algebra with basis $\{x_1, \dots, x_n\}$ and trivial products $x_i \circ x_j = 0$ for all i, j. Then $U(J) \approx \Lambda(x_1, \dots, x_n)$, the exterior (Grassmann) algebra on n generators over F, and so $\dim U(J) = 2^n$.

We have the following definition, which has no analogue for Lie algebras.

Definition 1.10. If a Jordan algebra J has an associative envelope then we call J a **special** Jordan algebra. Otherwise, we call J an **exceptional** Jordan algebra.

Example 1.11. The vector space $H_3(\mathbb{O})$ of 3×3 Hermitian matrices over the 8-dimensional division algebra $\mathbb{O}$ of real octonions is closed under the Jordan product and is a 27-dimensional exceptional Jordan algebra.

2. Free Associative Algebras

These lecture notes on the theory of noncommutative Gröbner bases follow closely the exposition by de Graaf [43, §§6.1-6.2]. The most famous paper on this topic is by Bergman [9], but similar results were published a little earlier by Bokut [11]. Bokut's approach was based on Shirshov's work on Lie algebras [99]. (Shirshov's papers have appeared recently in English translation [100].) For further references, including current research directions, see Section 11.

Definition 2.1. Let $X = \{x_1, x_2, \dots, x_n, \dots\}$ be an **alphabet**: a set of indeterminates (sometimes called **letters**), finite or countably infinite. We impose a total order on X by setting $x_i \prec x_j$ if and only if $i < j$. We write X^* for the set of **words** (also called **monomials**) $w = x_{i_1} x_{i_2} \cdots x_{i_k}$ where $x_{i_1}, x_{i_2}, \dots, x_{i_k} \in X$ and $k \geq 0$. (If $k = 0$ then we have the **empty word** denoted $w = 1$.) The **degree** of a word $w = x_{i_1} x_{i_2} \cdots x_{i_k}$ is the number of letters it contains, counting repetitions: $\deg(w) = k$. We define **concatenation** on X^* by $(u, v) \mapsto uv$ for any $u, v \in X^*$; this associative operation makes X^* into the **free monoid** generated by X.

Example 2.2. If $X = \{a\}$ has only one element, then $X^* = \{\, a^k \mid k \geq 0 \,\}$ is the set of all non-negative powers of a. The multiplication on X^* is given by $a^i a^j = a^{i+j}$, so X^* is commutative. If X has two or more elements, then X^* is noncommutative. For example, if $X = \{a, b\}$ then there are 2^k distinct words of degree k for all $k \geq 0$:

$$
\begin{array}{ll}
k = 0\colon & 1 \\
k = 1\colon & a,\ b \\
k = 2\colon & a^2,\ ab,\ ba,\ b^2 \\
k = 3\colon & a^3,\ a^2b,\ aba,\ ab^2,\ ba^2,\ bab,\ b^2a,\ b^3 \\
k = 4\colon & a^4,\ a^3b,\ a^2ba,\ a^2b^2,\ aba^2,\ abab,\ ab^2a,\ ab^3, \\
 & ba^3,\ ba^2b,\ baba,\ bab^2,\ b^2a^2,\ b^2ab,\ b^3a,\ b^4
\end{array}
$$

Definition 2.3. A nonempty word $u \in X^*$ is a **subword** (also called a **factor** or a **divisor**) of $w \in X^*$ if $w = v_1 u v_2$ for some $v_1, v_2 \in X^*$. If $v_1 = 1$ then u is a **left** subword of w; if $v_2 = 1$ then u is a **right** subword of w. We say that u is a **proper** subword of w if $u \neq w$.

Definition 2.4. The total order on X extends to a total order on X^*, called the **deglex** (degree lexicographical) order, as follows: If $u, w \in X^*$ then $u \prec w$ (we say u **precedes** w) if and only if either

(i) $\deg(u) < \deg(w)$, or
(ii) $\deg(u) = \deg(w)$ where $u = v x_i u'$ and $w = v x_j w'$ for some $v, u', w' \in X^*$ and $x_i, x_j \in X$ with $x_i < x_j$.

In condition (ii) we find the common left subword v of highest degree, and then compare the next letters x_i and x_j using the total order on X. We write $u \preceq v$ when $u \prec v$ or $u = v$. We often write $v \succ u$ to mean $u \prec v$.

Example 2.5. Let $X = \{a, b\}$ with $a \prec b$. We list the words in X^* of degree ≤ 3 in deglex order; this is the same order as in Example 2.2:

$$1 \prec a \prec b \prec a^2 \prec ab \prec ba \prec b^2 \prec a^3 \prec a^2b \prec aba \prec ab^2 \prec ba^2 \prec bab \prec b^2a \prec b^3.$$

Exercise 2.6. Let $X = \{a, b, c\}$ with $a \prec b \prec c$. List the words in X^* of degree ≤ 3 in deglex order. Do the same with $c \prec b \prec a$.

Definition 2.7. A total order on X^* is **multiplicative** if for all $u, v, w \in X^*$ with $u \prec v$ we have $uw \prec vw$ and $wu \prec wv$. (More concisely, we could require the single condition that $w_1uw_2 \prec w_1vw_2$ for all $u, v, w_1, w_2 \in X^*$.)

Definition 2.8. A total order on X^* satisfies the **descending chain condition** (**DCC**) if whenever $w_1, w_2, \ldots, w_n, \cdots \in X^*$ with $w_1 \succeq w_2 \succeq \cdots \succeq w_n \succeq \cdots$ then for some n we have $w_n = w_{n+1} = \cdots$; that is, there do not exist infinite strictly decreasing sequences. Equivalently, for any $w \in X^*$ the set $\{v \in X^* \mid v \prec w\}$ is finite. The DCC allows us to use induction on X^* with respect to the total order.

Lemma 2.9. *The total order $\prec$ on X^* from Definition 2.4 is multiplicative and satisfies the descending chain condition.*

Proof. Exercise. □

Definition 2.10. We write $F\langle X\rangle$ for the vector space with basis X^* over F. Concatenation in X^* extends bilinearly to $F\langle X\rangle$:

$$\Big(\sum_i a_iu_i\Big)\Big(\sum_j b_jv_j\Big) = \sum_{i,j} a_ib_ju_iv_j \quad (a_i, b_j \in F;\, u_i, v_j \in X^*).$$

This multiplication makes $F\langle X\rangle$ into the **free associative algebra** generated by X over F. This is a **unital** algebra, since the empty word acts as the unit element. Elements of $F\langle X\rangle$ are linear combinations of monomials in X^*, and we refer to them as **noncommutative polynomials** in the variables X with coefficients in F. (Here *noncommutative* means *not necessarily commutative*.)

Example 2.11. If $X = \{a\}$ has only one element, then $F\langle X\rangle$ is the same as $F[a]$, the familiar algebra of commutative associative polynomials in one variable. If X has two or more elements, then $F\langle X\rangle$ and $F[X]$ do not coincide: $F[X]$ is commutative but $F\langle X\rangle$ is noncommutative.

Definition 2.12. Consider a nonzero element $f \in F\langle X\rangle$. We write

$$f = \sum_{i\in\mathcal{I}} a_iu_i \quad (a_i \in F;\, u_i \in X^*),$$

where $\mathcal{I}$ is a nonempty finite index set and $a_i \neq 0$ for all $i \in \mathcal{I}$. The **support** of f is the set of all monomials occurring in f:

$$\text{support}(f) = \{\, u_i \mid i \in \mathcal{I}\,\}.$$

(If $f = 0$ then by convention its support is the empty set $\emptyset$.) For nonzero $f \in F\langle X\rangle$, the support is a nonempty finite subset of X^*; the greatest element of support(f) with respect to the total order $\prec$ on X^* is the **leading monomial** of f, denoted

$LM(f)$. The coefficient of $LM(f)$ is the **leading coefficient** of f, denoted $lc(f)$. We say that f is **monic** if $lc(f) = 1$. For any subset $S \subseteq F\langle X\rangle$, we write

$$LM(S) = \{\, LM(f) \mid f \in S \,\}.$$

Example 2.13. For $X = \{a, b, c\}$ and $cab - bca + da - cb + a^2 \in F\langle X\rangle$ we have

$$\text{support}(f) = \{\, a^2,\, cb,\, da,\, bca,\, cab\}, \qquad LM(f) = cab, \qquad lc(f) = 1.$$

Definition 2.14. The **standard form** of a nonzero element $f \in F\langle X\rangle$ consists of f divided by $lc(f)$ with the monomials in reverse deglex order. Thus the standard form is monic and the leading monomial occurs in the first (leftmost) position. The polynomial f in the previous example is in standard form.

3. Universal Associative Envelopes of Lie and Jordan Algebras

We use the concepts of the previous section to construct the universal associative envelopes of Lie and Jordan algebras.

Definition 3.1. Every associative algebra A is isomorphic to a quotient $F\langle X\rangle/I$ for some set X and some ideal $I \subseteq F\langle X\rangle$. If I is generated by the subset $G \subset I$ then the pair (X, G) is a **presentation** of A by **generators** and **relations**.

3.1. **Lie algebras.** Let L be a Lie algebra of finite dimension d over F with basis $X = \{\, x_1, \dots, x_d \,\}$. The structure constants $c_{ij}^k \in F$ are given by the equations

$$[x_i, x_j] = \sum_{k=1}^{d} c_{ij}^k x_k \qquad (1 \le i, j \le d).$$

Let $F\langle X\rangle$ be the free associative algebra generated by X. (By a slight abuse of notation, we regard the basis elements of L as formal variables, but this should not cause confusion.) Let I be the ideal in $F\langle X\rangle$ generated by the $d(d{-}1)/2$ elements

$$x_i x_j - x_j x_i - \sum_{k=1}^{d} c_{ij}^k x_k \qquad (1 \le j < i \le d).$$

The quotient algebra $U(L) = F\langle X\rangle/I$ is the universal associative envelope of L.

Example 3.2. We consider the Lie algebra $\mathfrak{sl}_2(F)$ of 2×2 matrices of trace 0 over a field F of characteristic 0. We use the following notation for basis elements:

$$h = E_{11} - E_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The structure constants are given by these equations:

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

From these equations we obtain the following set of generators G for the ideal I:

$$he - eh - 2e, \qquad hf - fh + 2f, \qquad ef - fe - h.$$

The universal associative envelope of $\mathfrak{sl}_2(F)$ is the quotient $U(\mathfrak{sl}_2(F)) = F\langle h, e, f\rangle/I$.

3.2. **Jordan algebras.** If J is a Jordan algebra with structure constants

$$x_i \circ x_j = \sum_{k=1}^{d} c_{ij}^k x_k \qquad (1 \le i, j \le d),$$

then we consider the ideal I generated by the $d(d{+}1)/2$ elements

$$\tfrac{1}{2}(x_i x_j + x_j x_i) - \sum_{k=1}^{d} c_{ij}^k x_k \qquad (1 \le j \le i \le d),$$

and $U(J) = F\langle X\rangle/I$ is the universal associative envelope of J.

Example 3.3. We consider the Jordan algebra $S_2(F)$ of symmetric 2×2 matrices over a field F of characteristic 0. We use the following notation for basis elements:

$$a = E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad c = E_{12} + E_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The structure constants are given by these equations:

$$a \circ a = 2a, \quad a \circ b = 0, \quad a \circ c = c, \quad b \circ b = 2b, \quad b \circ c = c, \quad c \circ c = 2a + 2b.$$

From these equations we obtain the following set G of generators for the ideal I:

$$a^2 - a, \quad ba + ab, \quad ca + ac - c, \quad b^2 - b, \quad cb + bc - c, \quad c^2 - b - a.$$

The universal associative envelope of $S_2(F)$ is the quotient $U(S_2(F)) = F\langle a, b, c\rangle/I$.

4. Normal Forms of Noncommutative Polynomials

To understand the structure of the quotient algebra $F\langle X\rangle/I$, we need to find a basis for $F\langle X\rangle/I$ and express the product of any two basis elements as a linear combination of basis elements. This can be achieved easily if we can construct a *Gröbner basis* for the ideal I: a set of generators (not a linear basis) for I with special properties which will be explained in detail in this section and the next.

4.1. **Normal forms modulo an ideal.** A basis for $F\langle X\rangle/I$ is a subset B of $F\langle X\rangle$ consisting of coset representatives: the elements $b + I$ for $b \in B$ are linearly independent in $F\langle X\rangle/I$ and span $F\langle X\rangle/I$. Equivalently, B is a basis for a complement $C(I)$ to I in $F\langle X\rangle$, meaning that $F\langle X\rangle = I \oplus C(I)$, the direct sum of subspaces.

Lemma 4.1. *Assume that I is an ideal in $F\langle X\rangle$, and that B is a subset of $F\langle X\rangle$. Then the set $\{\, b + I \mid b \in B \,\}$ is a basis of the quotient $F\langle X\rangle/I$ if and only if B is a basis for a complement of I in $F\langle X\rangle$; that is, the elements of B are linearly independent in $F\langle X\rangle$ and $F\langle X\rangle = I \oplus \operatorname{span}(B)$.*

Proof. Exercise. □

Definition 4.2. Let I be an ideal in $F\langle X\rangle$. The set $N(I)$ of **normal words** modulo I is the subset of X^* consisting of all monomials which are *not* leading monomials of elements of I:

$$N(I) = \{\, w \in X^* \mid w \notin LM(I) \,\}.$$

The **complement** to I in $F\langle X\rangle$ is the subspace $C(I) \subseteq F\langle X\rangle$ with basis $N(I)$.

Proposition 4.3. *We have $F\langle X\rangle = I \oplus C(I)$.*

Proof. We follow de Graaf [43, Proposition 6.1.1] but fill in some details. The proof consists for the most part of writing out the details in the division algorithm for noncommutative polynomials.

First, we prove that $I \cap C(I) = \{0\}$. Assume that $f \in I$ and $f \in C(I)$. If $f \neq 0$ then since $f \in I$, its leading monomial $LM(f)$ belongs to $LM(I)$; but since $f \in C(I)$, its leading monomial belongs to $N(I)$, and hence does not belong to $LM(I)$. This contradiction implies that $f = 0$.

Second, we prove that any $f \in F\langle X\rangle$ can be written as $f = g + h$ where $g \in I$ and $h \in C(I)$. This is clear for $f = 0$ (take $g = h = 0$), so we assume that $f \neq 0$. We use induction on leading monomials with respect to the total order $\prec$ on X^*.

For the basis of the induction, assume that $LM(f) = 1$ (the empty word). Then $f = \alpha \in F \setminus \{0\}$. If $I = F\langle X\rangle$ then $N(I) = \emptyset$ and $C(I) = \{0\}$; we have $f = \alpha + 0$ where $\alpha \in I$ and $0 \in C(I)$. If $I \neq F\langle X\rangle$ then $1 \notin LM(I)$ so $1 \in N(I)$; we have $f = 0 + \alpha$ where $0 \in I$ and $\alpha \in C(I)$.

Since X^* satisfies the DCC, we may now assume the claim for all $f_0 \in F\langle X\rangle$ with $LM(f_0) \prec LM(f)$. This is the inductive hypothesis, which depends on the fact that only finitely many elements of X^* precede $LM(f)$. We have $f = \alpha LM(f) + f_0$ where $\alpha = lc(f) \in F$, and either $f_0 = 0$ or $LM(f_0) \prec LM(f)$.

If $f_0 = 0$ then $f = \alpha LM(f)$; if $LM(f) \in I$ then $f = \alpha LM(f) + 0 \in I + C(I)$, and if $LM(f) \notin I$ then $LM(f) \in N(I)$ and $f = 0 + \alpha LM(f) \in I + C(I)$.

If $f_0 \neq 0$ then $LM(f_0) \prec LM(f)$, and by induction we have $f_0 = g_0 + h_0$ where $g_0 \in I$ and $h_0 \in C(I)$. We now have two cases: $LM(f) \in N(I)$ and $LM(f) \notin N(I)$. If $LM(f) \in N(I)$ then

$$f = \alpha LM(f) + (g_0 + h_0) = g_0 + \big(\alpha LM(f) + h_0\big) \in I + C(I).$$

If $LM(f) \notin N(I)$ then by definition of $N(I)$ we have $LM(f) = LM(k)$ for some $k \in I \setminus \{0\}$. (We cannot assume that $LM(f) \in I$. This raises an important issue: we are non-constructively choosing an element $k \in I$ which has the same leading monomial as the element f. Finding an algorithm to construct such an element k is one of the main goals of the theory of noncommutative Gröbner bases.)

Write $k = \beta LM(k) + k_0$ where $\beta = lc(k) \in F \setminus \{0\}$, and either $k_0 = 0$ or $LM(k_0) \prec LM(k) = LM(f)$. Then

$$\begin{aligned} f - \frac{\alpha}{\beta}k &= \Big(\alpha LM(f) + (g_0 + h_0)\Big) - \frac{\alpha}{\beta}\Big(\beta LM(k) + k_0\Big) \\ &= \alpha LM(f) + g_0 + h_0 - \alpha LM(k) - \frac{\alpha}{\beta}k_0 \\ &= g_0 + h_0 - \frac{\alpha}{\beta}k_0 \quad \text{since } LM(f) = LM(k). \end{aligned}$$

If $k_0 = 0$ then

$$f = \Big(\frac{\alpha}{\beta}k + g_0\Big) + h_0 \in I + C(I).$$

If $k_0 \neq 0$ then by induction $k_0 = \ell_0 + m_0$ where $\ell_0 \in I$ and $m_0 \in C(I)$. We have

$$\begin{aligned} f &= \frac{\alpha}{\beta}k + g_0 + h_0 - \frac{\alpha}{\beta}k_0 \\ &= \frac{\alpha}{\beta}k + g_0 + h_0 - \frac{\alpha}{\beta}(\ell_0 + m_0) \\ &= \Big(\frac{\alpha}{\beta}k + g_0 - \frac{\alpha}{\beta}\ell_0\Big) + \Big(h_0 - \frac{\alpha}{\beta}m_0\Big). \end{aligned}$$

The first three terms belong to I, and the last two terms belong to $C(I)$. $\square$

Corollary 4.4. *Let I be an ideal in $F\langle X\rangle$. Then every element $f \in F\langle X\rangle$ has a unique decomposition $f = g + h$ where $g \in I$ and $h \in C(I)$.*

Proof. This follows immediately from the definition of direct sum. $\square$

Definition 4.5. For any element $f \in F\langle X\rangle$ and any ideal $I \subseteq F\langle X\rangle$, the element $h \in C(I)$ which is uniquely determined by Corollary 4.4 is called the **normal form** of f modulo I, and is denoted $NF_I(f)$ or $NF(f)$ if I is understood.

Lemma 4.6. *Let $I \subseteq F\langle X\rangle$ be an ideal. Define a product $f \cdot g$ on $C(I)$ as follows: For any $f, g \in C(I)$ set $f \cdot g = NF_I(fg)$. Then the algebra consisting of the vector space $C(I)$ with the product $f \cdot g$ is isomorphic to the quotient algebra $F\langle X\rangle/I$.*

Proof. Exercise. $\square$

Lemma 4.6 shows how to find a basis and structure constants for $F\langle X\rangle/I$. But this depends on being able to determine the basis $N(I)$ of the complement $C(I)$, and to calculate the normal form $NF_I(f)$ for every element $f \in F\langle X\rangle$.

4.2. **Computing normal forms.** Our next task is to find an algorithm for which the input is an element $f \in F\langle X\rangle$ and an ideal $I \subseteq F\langle X\rangle$ given by a set G of generators, and the output is the normal form $NF_I(f)$. We present an algorithm for computing the normal form $NF(f, G)$ of f with respect to the set G. Unfortunately, the output of this algorithm depends on the set G; that is, if G_1 and G_2 are two generating sets for the same ideal I, then we may have $NF(f, G_1) \neq NF(f, G_2)$. Furthermore, even for one set G, the output may depend on the choice of reductions performed at each step of the algorithm; see Example 4.9 below. Therefore in general the output is not the normal form of f modulo I. The important property of a Gröbner basis is that if G is a Gröbner basis for I then $NF(f, G) = NF_I(f)$.

Definition 4.7. Let f be an element of $F\langle X\rangle$ and let G be a finite subset of $F\langle X\rangle$. We say that f is in **normal form** with respect to G if the following condition holds:

- For every generator $g \in G$ and every monomial $w \in \text{support}(f)$, the leading monomial $LM(g)$ is not a subword of w.

We first give an informal description of the algorithm for computing the normal form of f with respect to G. This algorithm is similar to the calculation in the proof of Proposition 4.3; it is a division algorithm for noncommutative polynomials. We may assume without loss of generality that the elements of G are monic.

Consider the set $LM(G)$ of leading monomials of the elements of G. For each $v \in LM(G)$ and $w \in \text{support}(f)$ we can easily determine if v is a subword of w. If this never occurs, then f is in normal form with respect to G, and the algorithm terminates. Otherwise, $w = u_1 v u_2$ for some $u_1, u_2 \in X^*$, and f contains the term αw for some $\alpha \in F \setminus \{0\}$. There exists $g \in G$ with $LM(g) = v$; we replace f by

$$f_2 = f - \alpha u_1 g u_2.$$

This reduction step eliminates from f the term αw. Repeating this procedure, we obtain a sequence $f_1 = f, f_2, f_3, \ldots, f_n, \ldots$ of elements of $F\langle X\rangle$; this sequence converges since X^* satisfies the DCC. This algorithm is given in pseudocode in Figure 1.

NormalForm(f, G)

Input: An element $f \in F\langle X\rangle$ and a finite monic subset $G \subset F\langle X\rangle$.

Output: The normal form of f with respect to G.

(1) Set $n \leftarrow 0$, $f_0 \leftarrow 0$, $f_1 \leftarrow f$.
(2) While $f_n \neq f_{n+1}$ do:
  (a) Set $n \leftarrow n+1$.
  (b) If $w = u_1 v u_2$ for some $v \in LM(G)$ and $w \in \text{support}(f_n)$ then
$$\text{set } f_{n+1} \leftarrow f_n - \alpha u_1 g u_2 \text{ where } v = LM(g)$$
  else set $f_{n+1} \leftarrow f_n$.
(3) Return f_n.

FIGURE 1. Algorithm for a normal form of f with respect to G

Lemma 4.8. *For the algorithm of Figure 1, we have*
$$LM(f_1) \succeq LM(f_2) \succeq LM(f_3) \succeq \cdots \succeq LM(f_n) \succeq \cdots,$$
and so $LM(f_n) = LM(f_{n+1}) = \cdots$ for some $n \geq 1$. Hence the algorithm terminates, and its output f_n is a normal form of f with respect to G.

Furthermore, $f_n + I = f + I$ in $F\langle X\rangle/I$; that is, f_n is congruent to f modulo the ideal I generated by G.

Proof. Exercise. □

A normal form of f with respect to G is not uniquely determined by the algorithm of Figure 1: the output depends on the choices made of v and w in step (2)(b). In particular, it follows that the output of the algorithm does not necessarily equal $NF_I(f)$, which is uniquely determined by Corollary 4.4.

Example 4.9. Let $X = \{a, b, c\}$ and let $I \subset F\langle X\rangle$ be the ideal generated by
$$G = \{\ a^2 - a,\ \ ba + ab,\ \ b^2 - b,\ \ ca + ac - c,\ \ cb + bc - c,\ \ c^2 - b - a\ \}.$$
(We have seen this set before in Example 3.3.) For convenience, we write each generator in standard form, and the generators are sorted in deglex order of their leading monomials. We compute the normal form of $f_1 = c^2 b$ with respect to G in two different ways, and obtain two different answers. We will see in Example 6.9 that $NF_I(c^2 b) = b$, so neither of these two calculations produces the desired result.

(1) Starting with $g_6 = c^2 - b - a$ we obtain
$$f_2 = f_1 - g_6 b = c^2 b - (c^2 b - b^2 - ab) = b^2 + ab.$$
Next using $g_3 = b^2 - b$ we obtain
$$f_3 = f_2 - g_3 = b^2 + ab - (b^2 - b) = ab + b.$$
No further reductions are possible; the algorithm terminates with output $ab + b$.

(2) Starting with $g_5 = cb + bc - c$ we obtain
$$f_2 = f_1 - c g_5 = c^2 b - (c^2 b + cbc - c^2) = -cbc + c^2.$$
Next using g_5 again we obtain
$$f_3 = f_2 + g_5 c = -cbc + c^2 + (cbc + bc^2 - c^2) = bc^2.$$

Using $g_6 = c^2 - b - a$ gives

$$f_4 = f_3 - bg_6 = bc^2 - (bc^2 - b^2 - ba) = b^2 + ba.$$

Using $g_3 = b^2 - b$ gives

$$f_5 = f_4 - g_3 = b^2 + ba - (b^2 - b) = ba + b.$$

Finally, using $g_2 = ba + ab$ we obtain

$$f_6 = f_5 - g_2 = ba + b - (ba + ab) = -ab + b.$$

No further reductions are possible; the algorithm terminates with output $-ab + b$.

5. Gröbner Bases for Ideals in $F\langle X\rangle$

If the set G of generators of the ideal I has a certain special property, stated in the next definition, then the output of the algorithm of Figure 1 is uniquely determined, and equals the normal form of f modulo I.

Definition 5.1. Let X be a finite set and let G be a set of generators for the ideal I in the free associative algebra $F\langle X\rangle$. We say that G is a **Gröbner basis** for I if the following condition holds:

- For every nonzero element $f \in I$ there is a generator $g \in G$ such that $LM(g)$ is a subword of $LM(f)$.

In other words, the leading monomial of every nonzero element of the ideal contains a subword equal to the leading monomial of some generator of the ideal.

Remark 5.2. A Gröbner basis is not a basis in the sense of linear algebra: it is not a basis for I as a vector space over F, but rather a set of generators for I. In this context, *basis* means *set of generators*. Unfortunately, this misleading terminology is so well-established that we have no choice but to accept it.

The next theorem shows why Gröbner bases are so important. Recall that the set $N(I)$ of all normal words modulo I is the complement of $LM(I)$ in X^*: the set of all words which are *not* leading monomials of elements of I. If we have a Gröbner basis for I, then we can easily compute $N(I)$ using part (a) of the next theorem, and we can easily compute $NF_I(f)$ for all $f \in F\langle X\rangle$ using part (b):

Theorem 5.3. *If G is a Gröbner basis for the ideal $I \subseteq F\langle X\rangle$ then:*

(a) $N(I) = \{\, w \in X^* \mid \text{for all } g \in G,\ LM(g) \text{ is not a subword of } w \,\}$.
(b) *For all $f \in F\langle X\rangle$ we have $NF_I(f) = NF(f,G)$: the normal form of f modulo I equals the normal form of f with respect to G.*

Proof. Part (a) follows immediately from Definitions 4.2 and 5.1. For part (b), consider $f \in F\langle X\rangle$ and let $h = NF(f,G)$ be the normal form of f with respect to G computed by the algorithm of Figure 1. For any $w \in \text{support}(h)$, since $h \in I$ and G is a Gröbner basis for I, we know by Definition 4.7 that for all $g \in G$, $LM(g)$ is not a subword of w. Part (a) of the theorem now shows that $w \in N(I)$; since this holds for all $w \in \text{support}(h)$, we have $h \in C(I)$. By the last statement of Lemma 4.8 we know that $f - h \in I$. Clearly $f = (f - h) + h \in I \oplus C(I)$, and hence the uniqueness of the decomposition in Corollary 4.4 implies that $h = NF_I(f)$. □

Theorem 5.3 is a beautiful result, but we still have the following problem:

- Find an algorithm for which the input is a set G of generators for the ideal $I \subseteq F\langle X\rangle$, and for which the output is a Gröbner basis of I.

This requires defining overlaps and compositions for two generators $g_1, g_2 \in G$ (Definition 5.10), and proving the Composition (Diamond) Lemma (Lemma 6.2).

Definition 5.4. Let X be a finite set and let G be a finite subset of $F\langle X\rangle$. We say that G is **self-reduced** if the following two conditions hold:

(1) Every $g \in G$ is in normal form with respect to $G \setminus \{g\}$.
(2) Every $g \in G$ is in standard form; in particular, $lc(g) = 1$.

Remark 5.5. Condition (1) in Definition 5.4 is stronger than the condition given by de Graaf [43, Definition 6.1.5], which requires only that for all $g \in G$ and for all $h \in G \setminus \{g\}$, $LM(h)$ is not a subword of $LM(g)$. The definition of de Graaf is analogous to the row-echelon form of a matrix, whereas our definition is analogous to the reduced row-echelon form (and is therefore somewhat more canonical).

Exercise 5.6. Referring to Remark 5.5, explain the analogy between row-echelon forms of matrices and self-reduced sets of noncommutative polynomials in $F\langle X\rangle$. (Consider finite sets of homogeneous polynomials of degree 1.)

By calling the algorithm of Figure 1 repeatedly, we can create an algorithm for which the input is a finite subset $G \subset F\langle X\rangle$ generating an ideal $I \subseteq F\langle X\rangle$ and the output is a self-reduced set which generates the same ideal. A naive approach would compute the set $\{\, NF(\,g, G \setminus \{g\}\,) \mid g \in G\,\}$. However, this set may not generate the same ideal, and it may not be self-reduced; so we have to be careful.

Example 5.7. Let $X = \{a, b, c\}$ with $a \prec b \prec c$, and let $G = \{c - a, c - b\}$. Then G is not self-reduced; computing the normal form of each element with respect to the other gives $c - a - (c - b) = b - a$ and $c - b - (c - a) = -b + a$ (with standard form $b - a$). Clearly the set $\{b - a\}$ does not generate the same ideal as G.

Example 5.8. Let $X = \{a, b, c, d\}$ with $a \prec b \prec c \prec d$, and consider the set

$$G = \{\, d - a,\ d - b,\ d - c\,\},$$

which is not self-reduced. One way to compute the normal form of each element with respect to the others is as follows, replacing each result by its standard form:

$$d - a - (d - b) = b - a, \quad d - b - (d - c) = c - b, \quad d - c - (d - a) = a - c \to c - a.$$

Clearly the set $\{\, b - a,\ c - b,\ c - a\,\}$ is not self-reduced.

Exercise 5.9. Using the algorithm of Figure 1, compose an algorithm whose input is a finite subset $G \subset F\langle X\rangle$ generating an ideal $I \subseteq F\langle X\rangle$ and whose output is a self-reduced set generating the same ideal. Hint: Avoid the problems illustrated by the last two examples by sorting G using deglex order of leading monomials.

Definition 5.10. Consider two nonzero elements $g_1, g_2 \in F\langle X\rangle$ in standard form; we allow $g_1 = g_2$. Set $w_1 = LM(g_1)$ and $w_2 = LM(g_2)$. Assume that

(1) w_1 is not a proper subword of w_2, and w_2 is not a proper subword of w_1 (we say "proper" because we allow $g_1 = g_2$).

(Condition (1) is satisfied if g_1, g_2 belong to a self-reduced set.) Assume also that

(2) for some words $u_1, u_2, v \in X^*$ with $v \neq 1$ we have $w_1 = u_1 v$ and $w_2 = v u_2$ (condition (1) implies that $u_1 \neq 1$ and $u_2 \neq 1$).

In this case, we call v an **overlap** between w_1 and w_2, and we have $w_1u_2 = u_1w_2$, where u_1 is a proper right subword of w_1, and u_2 is a proper left subword of w_2:

$$w_1u_2 = u_1vu_2 = u_1w_2.$$

The element $g_1u_2 - u_1g_2$ is called a **composition** of g_1 and g_2; the common term, a scalar multiple of u_1vu_2, cancels, since both g_1 and g_2 are monic. (In the theory of commutative Gröbner bases, compositions are often called S-polynomials.)

Example 5.11. Consider the following two words in X^* where $X = \{a, b, c\}$:

$$w_1 = a^2bcba, \qquad w_2 = bacba^2.$$

These words have the following overlaps:

- w_1 has a self-overlap: $w_1 = u_1v = vu_2$ for $u_1 = a^2bcb$, $v = a$, $u_2 = abcba$.
- w_1 and w_2 overlap: $w_1 = u_1v$, $w_2 = vu_2$ for $u_1 = a^2bc$, $v = ba$, $u_2 = cba^2$.
- w_2 and w_1 have overlaps of length 1 and length 2:
  - $w_2 = u_2v$, $w_1 = vu_1$ for $u_2 = bacba$, $v = a$, $u_1 = abcba$.
  - $w_2 = u_2v$, $w_1 = vu_1$ for $u_2 = bacb$, $v = a^2$, $u_1 = bcba$.

Example 5.12. Consider the last two generators from Example 3.3:

$$g_5 = cb + bc - c, \qquad g_6 = c^2 - b - a.$$

There is a composition of g_6 and g_5 corresponding to

$$w_6 = c^2, \quad w_5 = cb, \quad u_6 = c, \quad u_5 = b, \quad v = c.$$

We obtain

$$\begin{aligned} g_6u_5 - u_6g_5 &= (c^2 - b - a)b - c(cb + bc - c) = c^2b - b^2 - ab - c^2b - cbc + c^2 \\ &= -b^2 - ab - cbc + c^2 \xrightarrow{\text{sf}} cbc - c^2 + b^2 + ab, \end{aligned}$$

where the arrow denotes replacing the polynomial by its standard form.

Remark 5.13. The motivation for considering compositions is as follows. Suppose that $s = g_1u_2 - u_1g_2$ is a composition of g_1 and g_2, and that the normal form of s with respect to G is nonzero. Then $NF(s, G)$ is an element of the ideal I whose leading monomial is not divisible by any element of G. If we replace G by $G \cup \{NF(s, G)\}$, then we are one step closer to having a Gröbner basis for I.

6. The Composition (Diamond) Lemma

This lemma is fundamental to the theory of Gröbner bases, and leads to an algorithm for constructing a Gröbner basis for an ideal from a given set of generators for the ideal; the basic idea underlying this algorithm was given in Remark 5.13.

The origin of the name Diamond Lemma is roughly as follows; see also [9, 89]. We have an element $f \in F\langle X \rangle$, and we want to compute its normal form with respect to a finite subset $G \subset F\langle X \rangle$. At every step in the computation, there may be many different choices of reduction: many leading monomials of elements of G may occur as subwords of many monomials in f. We want to be sure that whatever sequence of reductions we perform, the final result will be the same. This condition is called the "resolution of ambiguities", and is illustrated by this "diamond":

$$\begin{array}{ccccc} & & g_0 = f = h_0 & & \\ & \swarrow & & \searrow & \\ g_i & & & & h_j \\ & \searrow & & \swarrow & \\ & & g_m = h_n & & \end{array}$$

Definition 6.1. Let $G = \{g_1, \dots, g_n\}$ be a set of generators for the ideal $I \subseteq F\langle X \rangle$. For any word $w \in X^*$ we define $I(G, w)$ to be the subspace of I spanned by the elements of the form ugv where $g \in G$, $u, v \in X^*$, and $LM(ugv) \prec w$:

$$I(G, w) = \Big\{ \sum_{i=1}^{n} \alpha_i u_i g_i v_i \,\Big|\, \alpha_i \in F;\ u_i, v_i \in X^*;\ LM(u_i g_i v_i) \prec w \Big\}.$$

Thus $I(G, w)$ is the subspace of I, relative to the set G of generators, consisting of the elements all of whose monomials precede w in the total order on X^*.

Lemma 6.2. Composition (Diamond) Lemma. *Let G be a monic self-reduced set generating the ideal $I \subseteq F\langle X \rangle$. Then these conditions are equivalent:*

1. *G is a Gröbner basis for I.*
2. *For every pair of generators $g, h \in G$, if $LM(g)u = vLM(h)$ for some $u, v \in X^*$, then $gu - vh \in I(G, t)$ where $t = LM(g)u = vLM(h)$.*

Remark 6.3. Condition (2) implies that every composition $gu - vh$ of the elements of G is a linear combination of elements of the form $u_i g_i v_i$ where $g_i \in G$ and $u_i, v_i \in X^*$ with $u_i LM(g_i) v_i \prec LM(g)u = vLM(h)$. The crucial point here is that we are only allowed to use elements of the form $u_i g_i v_i$.

Proof. (of Lemma 6.2) We follow closely the proof by de Graaf [43, Theorem 6.1.6].

(1) $\Longrightarrow$ (2): Assume that G is a Gröbner basis. For $g, h \in G$ let $f = gu - vh$ where $LM(g)u = vLM(h)$ for some $u, v \in X^*$. Clearly $f \in I$ and so $NF_I(f) = 0$. For $t = LM(g)u = vLM(h)$ we have $LM(f) \prec t$ since the leading terms of $LM(g)u$ and $vLM(h)$ cancel. When we apply the algorithm of Figure 1 to compute $NF(f, G)$, we repeatedly subtract terms of the form

$$\alpha u_1 k u_2 \quad (\alpha \in F;\ k \in G;\ u_1, u_2 \in X^*;\ LM(u_1 k u_2) \prec t).$$

Clearly all these terms belong to I and hence to $I(G, t)$. Since G is a Gröbner basis, we have $NF(f, G) = NF_I(f) = 0$. It follows that f is a sum of terms in $I(G, t)$, and hence $f \in I(G, t)$.

(2) $\Longrightarrow$ (1): We assume condition (2) and prove that G is a Gröbner basis for I. Let $f \in I$ be arbitrary; we have

$$f = \sum_{i=1}^{n} \alpha_i u_i g_i v_i \quad (\, \alpha_i \in F;\ u_i, v_i \in X^*;\ g_i \in G \,). \tag{1}$$

We need to show that $LM(g)$ is a subword of $LM(f)$ for some $g \in G$. We write

$$s_i = LM(u_i g_i v_i).$$

Renumbering the generators in G if necessary, we may assume that

$$s_1 = \cdots = s_\ell \succ s_{\ell+1} \succeq \cdots \succeq s_n. \tag{2}$$

Thus ℓ is the number of equal highest monomials in deglex order; the remaining monomials strictly precede these highest monomials; and we sort the remaining monomials in weak reverse deglex order.

If $\ell = 1$ then $s_1 \succ s_2$ and so $LM(f) = u_1 s_1 v_1 = u_1 LM(g_1) v_1$ as required.

We now assume $\ell \geq 2$. In this case we can rewrite equation (1) as follows:

$$f = \alpha_1 (u_1 g_1 v_1 - u_2 g_2 v_2) + (\alpha_1 + \alpha_2) u_2 g_2 v_2 + \sum_{i=3}^{n} \alpha_i u_i g_i v_i. \tag{3}$$

Since $\ell \geq 2$, we have

$$u_1 LM(g_1) v_1 = u_2 LM(g_2) v_2. \tag{4}$$

If $u_1 = u_2$ then $LM(g_1) v_1 = LM(g_2) v_2$. Hence either $LM(g_1)$ is a left subword of $LM(g_2)$, or $LM(g_2)$ is a left subword of $LM(g_1)$. But this contradicts the assumption that G is self-reduced. Hence $u_1 \neq u_2$, and so either u_1 is a proper left subword of u_2, or u_2 is a proper left subword of u_1.

Assume that u_1 is a proper left subword of u_2; a similar argument applies when u_2 is a proper left subword of u_1. We have $u_2 = u_1 u_2'$ where $u_2' \neq 1$. Then

$$u_1 LM(g_1) v_1 = u_1 u_2' LM(g_2) v_2 \quad \text{and so} \quad LM(g_1) v_1 = u_2' LM(g_2) v_2.$$

If v_1 is a right subword of v_2 then $LM(g_2)$ is a subword of $LM(g_1)$, again contradicting the assumption that G is self-reduced. Hence v_2 is a right subword of v_1, giving $v_1 = v_1' v_2$ where $v_1' \neq 1$. Then

$$LM(g_1) v_1' v_2 = u_2' LM(g_2) v_2 \quad \text{and so} \quad LM(g_1) v_1' = u_2' LM(g_2).$$

By the assumption that condition (2) holds, it follows that

$$g_1 v_1' - u_2' g_2 \in I(G, s) \text{ where } s = LM(g_1) v_1' = u_2' LM(g_2).$$

Therefore

$$u_1 (g_1 v_1' - u_2' g_2) v_2 = u_1 g_1 v_1' v_2 - u_1 u_2' g_2 v_2 = u_1 g_1 v_1 - u_2 g_2 v_2.$$

But $u_1 LM(g_1) v_1 = u_2 LM(g_2) v_2$ (since $\ell \geq 2$) and so cancellation gives

$$u_1 g_1 v_1 - u_2 g_2 v_2 \in I(G, t), \text{ where } t = u_1 LM(g_1) v_1.$$

It follows that we can rewrite equation (1) to obtain an expression of the same form where either

i) the new value of $LM(u_1 g_1 v_1)$ is lower in deglex order (this happens when $\ell = 2$ and $\alpha_1 + \alpha_2 = 0$), or

ii) the number ℓ, defined by the order relations (2), has decreased.

Since the total order on X^* satisfies the descending chain condition, after a finite number of steps we obtain an expression for f of the form (1) where $\ell = 1$, and then again $LM(f) = u_1 s_1 v_1 = u_1 LM(g_1) v_1$ as required. □

Lemma 6.4. *Consider two elements $g, h \in G$ in standard form, and let $s \in X^*$ be an arbitrary monomial. Set $u = sLM(h)$, $v = LM(g)s$ and $t = LM(g)sLM(h)$, so that $LM(g)u = vLM(h) = t$. Then we have $gu - vh \in I(G, t)$.*

Proof. Separate the leading monomials of g and h:

$$g = LM(g) + g_0, \qquad h = LM(h) + h_0,$$

where either $g_0 = 0$ or $LM(g_0) \prec LM(g)$, and either $h_0 = 0$ or $LM(h_0) \prec LM(h)$. We calculate as follows:

$$\begin{aligned} gu - vh &= \big(LM(g) + g_0\big)sLM(h) - LM(g)s\big(LM(h) + h_0\big) \\ &= g_0 sLM(h) - LM(g)sh_0 \\ &= g_0 s(h - h_0) - (g - g_0)sh_0 \\ &= g_0 sh - gsh_0. \end{aligned}$$

Then clearly $gu - vh = (g_0 s)h - g(sh_0) \in I(G,t)$ where $t = LM(g)sLM(h)$. □

Theorem 6.5. Main Theorem. *Suppose that G is a monic self-reduced set of generators for the ideal $I \subseteq F\langle X\rangle$. Then these two conditions are equivalent:*

1. *G is a Gröbner basis for I.*
2. *For every composition f of the generators in G, the normal form of f with respect to G is zero: $NF(f,G) = 0$.*

Proof. (1) $\Longrightarrow$ (2): Let G be a Gröbner basis for I, and let $f = g_1u_2 - u_1g_2$ be a composition of $g_1, g_2 \in G$ where $u_1, u_2 \in X^*$. Clearly $f \in I$, and hence by the definition of Gröbner basis, for some $g \in G$ the leading monomial $LM(g)$ is a subword of $LM(f)$; say $LM(f) = v_1LM(g)v_2$. If we define

$$f_1 = f - \alpha v_1 g v_2 \text{ where } \alpha = lc(f),$$

where the subtracted element belongs to I, then either $f_1 = 0$ or $LM(f_1) \prec LM(f)$. Repeating this argument, and using the DCC on X^*, we obtain $NF(f,G) = 0$ after a finite number of steps.

(2) $\Longrightarrow$ (1): Suppose that $f = g_1u_2 - u_1g_2$ is a composition of $g_1, g_2 \in G$ where $u_1, u_2 \in X^*$, and set $t = LM(g_1)u_2 = u_1LM(g_2)$. Assume that $NF(f,G) = 0$. Definition 5.10 implies that $u_2 \neq LM(g_2)$ and $u_1 \neq LM(g_1)$.

If u_2 is longer than $LM(g_2)$ then also u_1 is longer than $LM(g_1)$, and hence by Lemma 6.4 we have $f \in I(G,t)$.

If u_2 is shorter than $LM(g_2)$ then u_1 is shorter than $LM(g_1)$. Since $NF(f,G) = 0$ by assumption, the algorithm of Figure 1 outputs zero after a finite number of steps. But during each iteration of the loop in step (2) of that algorithm, we set

$$f_{n+1} \leftarrow f_n - \alpha u_1 g u_2,$$

where $LM(u_1gu_2) = LM(f_n) \preceq LM(f) \prec t$. Thus f is a linear combination of terms u_1gu_2 which strictly precede t in deglex order, showing that $f \in I(G,t)$.

In both cases we have $f \in I(G,t)$, and now Lemma 6.2 completes the proof. □

Remark 6.6. Theorem 6.5 suggests the Gröbner basis algorithm in Figure 2 for which the input is a set G generating the ideal $I \subseteq F\langle X\rangle$ and for which the output (assuming that the algorithm terminates) is a Gröbner basis for I.

Exercise 6.7. (a) Write a complete formal proof by induction (with basis and inductive hypothesis) of the statement "repeating this argument, and using the DCC on X^*, we obtain $NF(f,G) = 0$ after a finite number of steps" from part (1) $\Longrightarrow$ (2) of the proof of Theorem 6.5.

(b) Write a complete formal proof by induction (with basis and inductive hypothesis) of the statement "f is a linear combination of terms u_1gu_2 which strictly precede t in deglex order" from part (2) $\Longrightarrow$ (1) of the proof of Theorem 6.5.

GrobnerBasis(G)

Input: A finite subset $G \subset F\langle X\rangle$ generating an ideal $I \subseteq F\langle X\rangle$.

Output: If step (2) terminates, the output is a Gröbner basis of I.

(1) Set `newcompositions` $\leftarrow$ `true`.
(2) While `newcompositions` do:
  (a) Convert the elements of G to standard form.
  (b) Sort G by deglex order of leading monomials: $G = \{g_1, \dots, g_n\}$.
  (c) Convert G to a self-reduced set:
    • Set `selfreduced` $\leftarrow$ `false`.
    • While not `selfreduced` do:
      (i) Set `selfreduced` $\leftarrow$ `true`.
      (ii) Set $H \leftarrow \{\,\}$ (empty set).
      (iii) For $i = 1, \dots, n$ do:
        – Set $H \leftarrow H \cup \{\, NF(\, g_i,\ \{g_1, \dots, g_{i-1}\}\,)\,\}$.
      (iv) Convert the elements of H to standard form.
      (v) Sort H by deglex order of leading monomials.
      (vi) If $G \neq H$ then set `selfreduced` $\leftarrow$ `false`.
      (vii) Set $G \leftarrow H$.
  (d) Set `compositions` $\leftarrow \{\,\}$ (empty set).
  (e) Set `newcompositions` $\leftarrow$ `false`.
  (f) For $g \in G$ do for $h \in G$ do:
    • If $LM(g)$ and $LM(h)$ have an overlap w then:
      (i) Define u, v by $LM(g) = vw$ and $LM(h) = wu$.
      (ii) Set $s \leftarrow gu - vh$ (the composition of g and h).
      (iii) Replace s by its standard form.
      (iv) Set $t \leftarrow NF(s, G)$.
      (v) Replace t by its standard form.
      (vi) If $t \neq 0$ and $t \notin$ `compositions` then
        ∗ Set `newcompositions` $\leftarrow$ `true`.
        ∗ Set `compositions` $\leftarrow$ `compositions` $\cup \{t\}$.
(3) Return G.

FIGURE 2. Computing a Gröbner basis of the ideal I generated by G

Remark 6.8. A different approach to the Composition (Diamond) Lemma, emphasizing Shirshov's point of view which was developed by the Novosibirsk school of algebra, can be found in the works of Bokut and his co-authors. See in particular, Bokut [11], Bokut and Kukin [21, Chapter 1], Bokut and Shum [22], Bokut and Chen [13]. See also Mikhalev and Zolotykh [85].

Example 6.9. We compute a Gröbner basis for the ideal appearing in the construction of the universal associative envelope of the Jordan algebra $S_2(F)$ of symmetric 2×2 matrices. Let $X = \{a, b, c\}$ and let I be the ideal in $F\langle X\rangle$ generated by the self-reduced set G from Example 3.3:

$$(5) \qquad \left\{ \begin{array}{lll} g_1 = a^2 - a, & g_2 = ba + ab, & g_3 = b^2 - b, \\ g_4 = ca + ac - c, & g_5 = cb + bc - c, & g_6 = c^2 - b - a. \end{array} \right.$$

The first iteration of the algorithm produces 10 compositions (including 3 self-compositions); after putting them in standard form, denoted $p \xrightarrow{\text{sf}} q$, we obtain

$$
\begin{array}{ll}
g_1 a - a g_1 \xrightarrow{\text{sf}} 0, & g_2 a - b g_1 \xrightarrow{\text{sf}} s_1 = aba + ba, \\
g_3 a - b g_2 \xrightarrow{\text{sf}} s_2 = bab + ba, & g_3 b - b g_3 \xrightarrow{\text{sf}} 0, \\
g_4 a - c g_1 \xrightarrow{\text{sf}} s_3 = aca, & g_5 a - c g_2 \xrightarrow{\text{sf}} s_4 = cab - bca + ca, \\
g_5 b - c g_3 \xrightarrow{\text{sf}} s_5 = bcb, & g_6 a - c g_4 \xrightarrow{\text{sf}} s_6 = cac - c^2 + ba + a^2, \\
g_6 b - c g_5 \xrightarrow{\text{sf}} s_7 = cbc - c^2 + b^2 + ab, & g_6 c - c g_6 \xrightarrow{\text{sf}} s_8 = cb + ca - bc - ac.
\end{array}
$$

Computing normal forms of these compositions with respect to the set G using the algorithm of Figure 1, we obtain only two distinct nonzero results:

$$
\begin{aligned}
& s_1 - a g_2 + g_1 b - g_2 = -2ab \xrightarrow{\text{sf}} ab, \\
& s_2 - g_2 b + a g_3 - g_2 = -2ab \xrightarrow{\text{sf}} ab, \\
& s_3 - a g_4 + g_1 c = 0 \xrightarrow{\text{sf}} 0, \\
& s_4 - g_4 b + b g_4 - g_2 c + a g_5 - g_5 - g_4 = -2bc - 2ac + 2c \xrightarrow{\text{sf}} bc + ac - c, \\
& s_5 - b g_5 + g_3 c = 0 \xrightarrow{\text{sf}} 0, \\
& s_6 - g_4 c + a g_6 - g_2 = -2ab \xrightarrow{\text{sf}} ab, \\
& s_7 - g_5 c + b g_6 + g_2 = 2ab \xrightarrow{\text{sf}} ab, \\
& s_8 - g_5 - g_4 = -2bc - 2ac + 2c \xrightarrow{\text{sf}} bc + ac - c.
\end{aligned}
$$

So we define

$$
t_1 = ab, \qquad t_2 = bc + ac - c. \tag{6}
$$

We include the new generators (6) in the original set (5), obtaining a new set H of generators for the ideal I:

$$
\left\{
\begin{array}{lll}
g_1 = a^2 - a, & t_1 = ab, & g_2 = ba + ab, \\
g_3 = b^2 - b, & t_2 = bc + ac - c, & g_4 = ca + ac - c, \\
g_5 = cb + bc - c, & g_6 = c^2 - b - a. &
\end{array}
\right.
$$

For each element $h \in H$, we compute its normal form with respect to the elements which precede it in the total order on H (deglex order of leading monomials). In this simple example, all we need to do is to replace g_2 by $g_2 - t_1 = ba$:

$$
\left\{
\begin{array}{lll}
g_1 = a^2 - a, & t_1 = ab, & g_2' = ba, \\
g_3 = b^2 - b, & t_2 = bc + ac - c, & g_4 = ca + ac - c, \\
g_5 = cb + bc - c, & g_6 = c^2 - b - a. &
\end{array}
\right.
$$

We now verify that this set is a Gröbner basis: all compositions of these generators have normal form 0 with respect to this set.

Remark 6.10. Using the Gröbner basis of Example 6.9, it is easy to compute the normal form of any element of $F\langle X\rangle$ using Theorem 5.3(b). In particular, we can compute the normal form of the element $f = c^2 b$ from Example 4.9:

$$
f_1 - g_6 b - g_3 - t_1 = c^2 b - (c^2 - b - a)b - (b^2 - b) - ab = b.
$$

In this way, using Lemma 4.6, we can calculate the structure constants of the universal associative envelope $U(S_2(F))$.

Exercise 6.11. Let $J = S_2(F)$ be the Jordan algebra of symmetric 2×2 matrices. A basis for $U(J) = F\langle a, b, c\rangle / I$ consists of the cosets of the monomials which do not any leading monomial from the Gröbner basis (Example 6.9).

(a) Write down the (finite) set of basis monomials for $U(J)$.
(b) Using Lemma 4.6, compute the structure constants for $U(J)$: express products of basis monomials as linear combinations of basis monomials.
(c) Determine explicitly the structure of the associative algebra $U(J)$. (Use the algorithms in my survey paper [24] on the Wedderburn decomposition.)

Example 6.12. Here is an example, from de Graaf [43, page 226], of a generating set G for which the algorithm of Figure 2 never terminates. This also shows why we must consider self-compositions of generators. Let $X = \{a, b\}$ and define

$$G_0 = \{g_1 = aba - ba\}.$$

The first iteration of the algorithm produces one composition of g_1 with itself:

$$g_1 ba - abg_1 = (aba - ba)ba - ab(aba - ba) = -baba + ab^2a \xrightarrow{\text{sf}} baba - ab^2a.$$

Computing the normal form of this composition with respect to G_0 gives

$$(baba - ab^2a) - b(aba - ba) = -ab^2a + b^2a \xrightarrow{\text{sf}} ab^2a - b^2a.$$

Including this with g_1 gives a new generating set, which is already self-reduced:

$$G_1 = \{\, g_1 = aba - ba,\; g_2 = ab^2a - b^2a \,\}.$$

The second iteration produces three compositions:

$$\begin{aligned} g_1b^2a - abg_2 &= (aba - ba)b^2a - ab(ab^2a - b^2a) = -bab^2a + ab^3a \\ &\xrightarrow{\text{sf}} bab^2a - ab^3a, \\ g_2ba - ab^2g_1 &= (ab^2a - b^2a)ba - ab^2(aba - ba) = -b^2aba + ab^3a \\ &\xrightarrow{\text{sf}} b^2aba - ab^3a, \\ g_2b^2a - ab^2g_2 &= (ab^2a - b^2a)b^2a - ab^2(ab^2a - b^2a) = -b^2ab^2a + ab^4a \\ &\xrightarrow{\text{sf}} b^2ab^2a - ab^4a. \end{aligned}$$

Computing the normal forms of these compositions with respect to G_1 gives

$$(bab^2a - ab^3a) - b(ab^2a - b^2a) = -ab^3a + b^3a \xrightarrow{\text{sf}} ab^3a - b^3a,$$

$$(b^2aba - ab^3a) - b^2(aba - ba) = -ab^3a + b^3a \xrightarrow{\text{sf}} ab^3a - b^3a,$$

$$(b^2ab^2a - ab^4a) - b^2(ab^2a - b^2a) = -ab^4a + b^4a \xrightarrow{\text{sf}} ab^4a - b^4a.$$

Including these with g_1, g_2 gives a new generating set, which is already self-reduced:

$$G_2 = \{\, g_1 = aba - ba,\; g_2 = ab^2a - b^2a\, g_3 = ab^3a - b^3a,\; g_4 = ab^4a - b^4a \,\}.$$

It is now easy to verify that the algorithm never terminates; see Exercise 6.13.

Exercise 6.13. (a) Work out in detail the next iteration for Example 6.12.
(b) State and prove a conjecture for the elements of the set G_n obtained at the end of the n-th iteration of the Gröbner basis algorithm in Example 6.12.

Example 6.14. Here is another (much more complicated) example in which self-compositions play an essential role. We set $X = \{a, b\}$ and consider the following two elements of $F\langle a, b\rangle$ which clearly form a self-reduced set:

$$g_1 = aba - a^2b - a, \qquad g_2 = bab - ab^2 - b.$$

(1) The first iteration of the Gröbner basis algorithm produces three compositions:

$$\begin{aligned}
s_1 = g_1ba - abg_1 &= (aba - a^2b - a)ba - ab(aba - a^2b - a) \\
&= ababa - a^2b^2a - aba - ababa + aba^2b + aba \\
&= aba^2b - a^2b^2a, \\
s_2 = g_2a - bg_1 &= (bab - ab^2 - b)a - b(aba - a^2b - a) \\
&= baba - ab^2a - ba - baba + ba^2b + ba \\
&= ba^2b - ab^2a, \\
s_3 = g_2ab - bag_2 &= (bab - ab^2 - b)ab - ba(bab - ab^2 - b) \\
&= babab - ab^2ab - bab - babab + ba^2b^2 + bab \\
&= ba^2b^2 - ab^2ab.
\end{aligned}$$

We compute the normal form of each composition with respect to $\{g_1, g_2\}$:

$$\begin{aligned}
s_1 - g_1ab - a^2g_2 &= aba^2b - a^2b^2a - (aba - a^2b - a)ab - a^2(bab - ab^2 - b) \\
&= aba^2b - a^2b^2a - aba^2b + a^2bab + a^2b - a^2bab + a^3b^2 + a^2b \\
&= -a^2b^2a + a^3b^2 + 2a^2b \xrightarrow{\text{sf}} a^2b^2a - a^3b^2 - 2a^2b = h_1, \\
s_2 &= h_2, \\
s_3 + abg_2 + g_1b^2 &= ba^2b^2 - ab^2ab + ab(bab - ab^2 - b) + (aba - a^2b - a)b^2 \\
&= ba^2b^2 - ab^2ab + ab^2ab - abab^2 - ab^2 + abab^2 - a^2b^3 - ab^2 \\
&= ba^2b^2 - a^2b^3 - 2ab^2 = h_3.
\end{aligned}$$

We combine these compositions with the original generators and sort them:

$$\begin{aligned}
&g_1 = aba - a^2b - a, \qquad g_2 = bab - ab^2 - b, \qquad h_2 = ba^2b - ab^2a, \\
&h_1 = a^2b^2a - a^3b^2 - 2a^2b, \qquad h_3 = ba^2b^2 - a^2b^3 - 2ab^2.
\end{aligned}$$

Self-reducing this set eliminates h_3 since $h_3 - h_2b - abg_2 - g_1b^2 = 0$.
(2) The second iteration produces five compositions with these normal forms:

$$\begin{aligned}
&h_4 = ba^3b - ab^2a^2 + ba^2, \quad h_5 = ba^3b^2 - a^2b^3a, \quad h_6 = a^3b^3a - a^4b^3 - 3a^3b^2, \\
&h_7 = ba^4b^2 - ab^2a^3b + 2ba^3b, \quad h_8 = a^4b^4a - a^5b^4 + 2a^3b^3a - 6a^4b^3 - 6a^3b^2.
\end{aligned}$$

Combining these compositions with g_1, g_2, h_2, h_1 and self-reducing the resulting set eliminates h_5 and replaces h_7 and h_8 with these elements:

$$h_7' = ba^4b^2 - a^2b^3a^2 + 2ab^2a^2 - 2ba^2, \qquad h_8' = a^4b^4a - a^5b^4 - 4a^4b^3.$$

(3) The third iteration of the algorithm produces 18 compositions:

$$\begin{aligned}
&ba^4b - ab^2a^3 + 2ba^3, \\
&ba^5b^2 - a^2b^3a^3 + 2ba^4b + 2ab^2a^3 - 2ba^3, \\
&ba^5b^2 - ab^2a^4b + 3ba^4b,
\end{aligned}$$

$$
\begin{aligned}
& ba^6b^2 - ab^2a^5b + 4ba^5b, \\
& ba^5b^3 - a^3b^4a^2 + 3a^2b^3a^2 - 6ab^2a^2 + 6ba^2, \\
& ba^6b^3 - a^2b^3a^4b + 4ba^5b^2 + 2ab^2a^4b, \\
& ba^6b^3 - ab^2a^5b^2 + 4ba^5b^2, \\
& ba^5b^4 - a^4b^5a, \\
& a^5b^5a - a^6b^5 - 5a^5b^4, \\
& ba^7b^3 - a^2b^3a^5b + 6ba^6b^2 + 2ab^2a^5b + 4ba^5b, \\
& ba^7b^3 - ab^2a^6b^2 + 5ba^6b^2, \\
& a^5b^5a^2 - a^6b^5a - 5a^6b^4 - 20a^5b^3, \\
& a^6b^6a - a^7b^6 + 6a^5b^5a - 12a^6b^5 - 30a^5b^4, \\
& a^6b^6a - a^7b^6 + 8a^5b^5a - 14a^6b^5 - 40a^5b^4, \\
& ba^8b^4 - a^2b^3a^6b^2 + 8ba^7b^3 + 2ab^2a^6b^2 + 10ba^6b^2, \\
& a^6b^6a^2 - a^7b^6a + 4a^5b^5a^2 - 10a^6b^5a - 20a^6b^4 - 80a^5b^3, \\
& a^7b^7a - a^8b^7 + 12a^6b^6a - 19a^7b^6 + 36a^5b^5a - 108a^6b^5 - 180a^5b^4, \\
& a^8b^8a - a^9b^8 + 12a^7b^7a - 20a^8b^7 + 36a^6b^6a - 120a^7b^6 + 24a^5b^5a \\
& \qquad - 240a^6b^5 - 120a^5b^4.
\end{aligned}
$$

At this point it seems clear that the algorithm will never terminate!

Exercise 6.15. Referring to Example 6.14:

(a) Verify the statements about the second and third iterations.
(b) Prove that the algorithm does not terminate.
(c) Determine a closed form for the generators at the end of the n-th iteration.

Remark 6.16. A rich source of examples of the behavior of the Gröbner basis algorithm comes from the construction of universal associative envelopes for nonassociative triple systems obtained from the trilinear operations classified by the author and Peresi [28]. A detailed study of the simplest non-trivial examples of this construction appears in the Ph.D. thesis of Elgendy [45]; see also her forthcoming paper [46]. Similar examples are discussed in §10 of these lecture notes.

7. Application: The PBW Theorem

We now present the beautiful combinatorial proof of the Poincaré-Birkhoff-Witt (PBW) Theorem discovered by Bokut [11] and independently by Bergman [9]. We follow the exposition given by de Graaf [43, Theorem 6.2.1]. The assumption that the Lie algebra is finite dimensional is not essential.

Theorem 7.1. PBW Theorem. *If L is a finite dimensional Lie algebra over a field F with ordered basis $X = \{x_1, \dots, x_n\}$, then a basis of its universal associative envelope $U(L)$ consists of the monomials*

$$x_1^{e_1} \cdots x_n^{e_n} \quad (e_1, \dots, e_n \geq 0).$$

It follows immediately that:

(i) *$U(L)$ is infinite dimensional.*

(ii) *The natural map $L \to U(L)$ is injective.*
(iii) *L is isomorphic to a subalgebra of $U(L)^-$.*

Proof. The structure constants of L have the form

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k \quad (c_{ij}^k \in F),$$

where $c_{ji}^k = -c_{ij}^k$ and $c_{ii}^k = 0$. The universal associative envelope $U(L)$ is the quotient of the free associative algebra $F\langle X \rangle$ by the ideal I generated by the elements

$$g_{ij} = x_i x_j - x_j x_i - [x_i, x_j] = x_i x_j - x_j x_i - \sum_{k=1}^{n} c_{ij}^k x_k.$$

By anticommutativity of the Lie bracket, we may assume that $i > j$, and hence $x_i x_j$ is the leading monomial of g_{ij}. (If $i = j$ then $g_{ii} = 0$.) So we set

$$G = \{\, g_{ij} \mid 1 \le j < i \le n \,\}.$$

We will show that G is a Gröbner basis for the ideal I.

Consider the leading monomials of two distinct generators,

$$LM(g_{ij}) = x_i x_j \ (i > j), \qquad LM(g_{\ell k}) = x_\ell x_k \ (\ell > k).$$

The only possible compositions of these generators occur when either $j = \ell$ or $k = i$. It suffices to assume $j = \ell$, so we consider g_{ij} and g_{jk} where $i > j > k$. We have

$$LM(g_{ij})\, x_k = x_i x_j x_k = x_i\, LM(g_{jk}),$$

which produces the composition

$$\begin{aligned}
g_{ij} x_k - x_i g_{jk} &= \big(x_i x_j - x_j x_i - [x_i, x_j]\big) x_k - x_i \big(x_j x_k - x_k x_j - [x_j, x_k]\big) \\
&= x_i x_j x_k - x_j x_i x_k - [x_i, x_j] x_k - x_i x_j x_k + x_i x_k x_j + x_i [x_j, x_k] \\
&= -x_j x_i x_k - [x_i, x_j] x_k + x_i x_k x_j + x_i [x_j, x_k] \\
&= x_i x_k x_j - x_j x_i x_k - [x_i, x_j] x_k + x_i [x_j, x_k],
\end{aligned}$$

which is in standard form. (It is convenient to avoid explicit structure constants in this calculation; recall that $[x_i, x_j]$ is a homogeneous polynomial of degree 1.) To compute the normal form with respect to G, we first subtract $g_{ik} x_j$ and add $x_j g_{ik}$:

$$\begin{aligned}
& x_i x_k x_j - x_j x_i x_k - [x_i, x_j] x_k + x_i [x_j, x_k] \\
& \quad - \big(x_i x_k - x_k x_i - [x_i, x_k]\big) x_j + x_j \big(x_i x_k - x_k x_i - [x_i, x_k]\big) \\
&= x_i x_k x_j - x_j x_i x_k - [x_i, x_j] x_k + x_i [x_j, x_k] \\
& \quad - x_i x_k x_j + x_k x_i x_j + [x_i, x_k] x_j + x_j x_i x_k - x_j x_k x_i - x_j [x_i, x_k] \\
&= -[x_i, x_j] x_k + x_i [x_j, x_k] + x_k x_i x_j + [x_i, x_k] x_j - x_j x_k x_i - x_j [x_i, x_k] \\
&= -x_j x_k x_i + x_k x_i x_j - [x_i, x_j] x_k + x_i [x_j, x_k] + [x_i, x_k] x_j - x_j [x_i, x_k].
\end{aligned}$$

We next add $g_{jk} x_i$ and subtract $x_k g_{ij}$:

$$\begin{aligned}
& - x_j x_k x_i + x_k x_i x_j - [x_i, x_j] x_k + x_i [x_j, x_k] + [x_i, x_k] x_j - x_j [x_i, x_k] \\
& \quad + \big(x_j x_k - x_k x_j - [x_j, x_k]\big) x_i - x_k \big(x_i x_j - x_j x_i - [x_i, x_j]\big) \\
&= -x_j x_k x_i + x_k x_i x_j - [x_i, x_j] x_k + x_i [x_j, x_k] + [x_i, x_k] x_j - x_j [x_i, x_k] \\
& \quad + x_j x_k x_i - x_k x_j x_i - [x_j, x_k] x_i - x_k x_i x_j + x_k x_j x_i + x_k [x_i, x_j] \\
&= -[x_i, x_j] x_k + x_i [x_j, x_k] + [x_i, x_k] x_j - x_j [x_i, x_k] - [x_j, x_k] x_i + x_k [x_i, x_j]
\end{aligned}$$

$$= x_i[x_j, x_k] - [x_j, x_k]x_i + x_j[x_k, x_i] - [x_k, x_i]x_j + x_k[x_i, x_j] - [x_i, x_j]x_k.$$

We now observe that this last expression is equal to

$$[x_i, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]],$$

which is zero by the Jacobi identity. Thus every composition of the generators has normal form zero, and so G is a Gröbner basis.

The leading monomials of the elements of this Gröbner basis have the form $x_i x_j$ where $i > j$. A basis for $U(L)$ consists of all monomials w which do not have any of these leading monomials as a subword. That is, if w contains a subword $x_i x_j$ then $i \leq j$. It follows that the monomials in the statement of this Theorem form a basis for $U(L)$. In particular, the monomials $x_1, \dots, x_n$ of degree 1 are linearly independent in $U(L)$, and hence the natural map from L to $U(L)$ is injective. □

Corollary 7.2. *Every polynomial identity satisfied by the Lie bracket in every associative algebra is a consequence of anticommutativity and the Jacobi identity.*

Proof. Suppose that $p(a_1, \dots, a_n) \equiv 0$ is a polynomial identity which is not a consequence of anticommutativity and the Jacobi identity. Then the Lie polynomial $p(a_1, \dots, a_n)$ is a nonzero element of the free Lie algebra L generated by the variables $\{a_1, \dots, a_n\}$. Let A be any associative algebra, and let $\epsilon \colon L \to A^-$ be any morphism of Lie algebras. By definition of polynomial identity, we have $\epsilon(p) = 0$. Take $A = U(L)$ and let ϵ be the injective map $L \to U(L)^-$ obtained from the PBW theorem. Since $p \neq 0$ we have $\epsilon(p) \neq 0$, giving a contradiction. □

Remark 7.3. Lie algebras arose originally as tangent algebras of Lie groups. Weakening the requirement of associativity in the definition of Lie groups gives rise to various classes of nonassociative smooth loops, such as Moufang loops, Bol loops, and monoassociative loops. The corresponding tangent algebras are known respectively as Malcev algebras, Bol algebras, and BTQ algebras. Universal nonassociative envelopes for Malcev and Bol algebras have been constructed by Pérez-Izquierdo and Shestakov [91, 93]. This problem is still open for BTQ algebras, but see my recent paper with Madariaga [27]. All of these tangent algebras are special cases of Akivis and Sabinin algebras; for the universal nonassociative envelopes of these structures, see Shestakov and Umirbaev [97] and Pérez-Izquierdo [92].

The PBW Theorem shows that for every Lie algebra L, the original set of generators obtained from the structure constants is already a Gröbner basis. The original generators in I can be interpreted as rewriting rules in $U(L)$ as follows:

$$x_i x_j - x_j x_i - \sum_{k=1}^{n} c_{ij}^k x_k \in I \qquad \Longleftrightarrow \qquad x_i x_j = x_j x_i + \sum_{k=1}^{n} c_{ij}^k x_k \in U(L).$$

Repeated application of these rewriting rules allows us to work out explicit multiplication formulas for monomials in $U(L)$.

Exercise 7.4. Let L be the 2-dimensional solvable Lie algebra with basis $\{a, b\}$ where $[a, b] = b$; the other structure constants follow from anticommutativity. The basis of $U(L)$ obtained from the PBW theorem consists of the monomials $a^i b^j$ for $i, j \geq 0$. The ideal I is generated by $ab - ba - b$, and so in $U(L)$ we have the relation $ba = ab - b$. Use this and induction on the exponents to work out a formula for the product $(a^i b^j)(a^k b^\ell)$ as a linear combination of basis monomials.

Exercise 7.5. Let L be the 3-dimensional nilpotent Lie algebra with basis $\{a, b, c\}$ where $[a, b] = c$, $[a, c] = [b, c] = 0$. The PBW basis of $U(L)$ consists of the monomials $a^i b^j c^k$ for $i, j, k \geq 0$. In $U(L)$ we have $ba = ab - c$, $ac = ca$, $bc = cb$. State and prove a formula for $(a^i b^j c^k)(a^\ell b^m c^n)$ as a linear combination of basis monomials.

Exercise 7.6. Let L be the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(F)$ with basis $\{e, f, h\}$ where $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. The PBW basis of $U(L)$ consists of the monomials $f^i h^j e^k$ for $i, j, k \geq 0$. In $U(L)$ we have

$$eh = he - 2e, \qquad hf = fh - 2f, \qquad ef = fe + h.$$

State and prove a formula for $(f^i h^j e^k)(f^\ell h^m e^n)$ as a linear combination of basis monomials. (This exercise is harder than the previous two. Note that $\{h, e\}$ and $\{h, f\}$ span 2-dimensional solvable subalgebras. See also Example 3.2.)

8. Jordan Structures on 2×2 Matrices

In this section we study some examples of nonassociative structures whose universal associative envelopes are finite dimensional. The underlying vector space in all three examples is $M_2(F)$, the 2×2 matrices over a field F of characteristic $\neq 2$. We will use the following notation for the basis of matrix units:

$$a = E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \;\; b = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \;\; c = E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \;\; d = E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

8.1. **The Jordan algebra of 2×2 matrices.** We first make $M_2(F)$ into a Jordan algebra J using the Jordan product $x \circ y = xy + yx$. (For convenience we omit the scalar $\frac{1}{2}$.) The universal associative envelope $U(J)$ is isomorphic to $F\langle a, b, c, d\rangle / I$ where the ideal I is generated by the following set of 10 elements, obtained from the structure constants of J; this set is already self-reduced:

$$g_1 = a^2 - a, \qquad g_2 = ba + ab - b, \qquad g_3 = b^2, \qquad g_4 = ca + ac - c,$$
$$g_5 = cb + bc - d - a, \qquad g_6 = c^2, \qquad g_7 = da + ad, \qquad g_8 = db + bd - b,$$
$$g_9 = dc + cd - c, \qquad g_{10} = d^2 - d.$$

We obtain three distinct nonzero compositions from the pairs (g_5, g_2), (g_5, g_3), (g_6, g_5); computing their normal forms with respect to the set of generators gives:

$$s_1 = ad, \qquad s_2 = bd - ab, \qquad s_3 = cd - ac.$$

Combining these three compositions with the original ten generators gives a new set of 13 generators; self-reduction makes only minor changes

$$a^2 - a, \quad ad, \quad ba + ab - b, \quad b^2, \quad bd - ab, \quad ca + ac - c, \quad cb + bc - d - a,$$
$$c^2, \quad cd - ac, \quad da, \quad db + ab - b, \quad dc + ac - c, \quad d^2 - d.$$

Every composition of these 13 generators has normal form zero, and so this set is a Gröbner basis. There are only 9 monomials in $F\langle a, b, c, d\rangle$ which do not have the leading monomial of one of the Gröbner basis elements as a subword:

$$u_1 = 1, \;\; u_2 = a, \;\; u_3 = b, \;\; u_4 = c, \;\; u_5 = d, \;\; u_6 = ab, \;\; u_7 = ac, \;\; u_8 = bc, \;\; u_9 = abc.$$

The cosets of these monomials modulo I form a basis for $U(J)$. The multiplication table of $U(J)$ is displayed in Table 1, where u_i is denoted by i and dot indicates 0.

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	2	6	7	·	6	7	9	9
3	3	3−6	·	8	6	·	8−9	·	·
4	4	4−7	2+5−8	·	7	5−8+9	·	4	4−7
5	5	·	3−6	4−7	5	·	·	8−9	·
6	6	·	·	9	6	·	·	·	·
7	7	·	2−9	·	7	·	·	7	·
8	8	9	3	·	8−9	6	·	8	9
9	9	9	6	·	·	6	·	9	9

TABLE 1. Structure constants for $U(J)$ where $J = M_2(F)^+$

Exercise 8.1. (a) Verify the multiplication table for $U(J)$ by computing the normal form of each product of basis elements with respect to the Gröbner basis.

(b) Use the algorithms in my survey paper [24] to compute the structure of $U(J)$. Prove or disprove that $U(J) \approx F \oplus M_2(F) \oplus M_2(F)$. Compare your results with the known representation theory of Jordan algebras; see Jacobson [69].

8.2. **The 2×2 matrices as a Jordan triple system.** This subsection and the next introduce the topic of multilinear operations, which will be discussed systematically in Section 9. We consider the vector space $T = M_2(F)$ with a trilinear operation, the Jordan triple product $\langle x, y, z \rangle = xyz + zyx$. Working out the structure constants for this operation, we find that the ideal I appearing in the definition of $U(T)$ is generated by the following self-reduced set of 40 elements:

$$
\begin{gathered}
a^3 - a,\; aba,\; aca,\; ada,\; ba^2 + a^2b - b,\; bab,\; b^2a + ab^2,\; b^3,\; bca + acb - a, \\
bcb - b,\; bda + adb,\; bdb,\; ca^2 + a^2c - c,\; cab + bac - d,\; cac,\; cba + abc - a, \\
cb^2 + b^2c,\; cbc - c,\; c^2a + ac^2,\; c^2b + bc^2,\; c^3,\; cda + adc,\; cdb + bdc - a,\; cdc, \\
da^2 + a^2d,\; dab + bad,\; dac + cad,\; dad,\; dba + abd - b,\; db^2 + b^2d, \\
dbc + cbd - d,\; dbd,\; dca + acd - c,\; dcb + bcd - d,\; dc^2 + c^2d,\; dcd,\; d^2a + ad^2, \\
d^2b + bd^2 - b,\; d^2c + cd^2 - c,\; d^3 - d.
\end{gathered}
$$

These elements produce 36 distinct nonzero compositions:

$$
\begin{gathered}
ad,\; b^2,\; bd - ab,\; c^2,\; cd - ac,\; da,\; db - ba,\; dc - ca,\; d^2 - cb - bc + a^2,\; a^2d, \\
ab^2,\; abd - a^2b,\; acb + abc - a,\; ac^2,\; acd - a^2c,\; adb,\; adc,\; ad^2,\; bad,\; b^2c, \\
b^2d,\; bc^2,\; bcd - bac,\; bdc + acb - a,\; bdc - abc,\; bd^2 - b^2c - a^2b,\; bd^2 - abd, \\
bd^2 - a^2b,\; cad,\; cbd + bcd - a^2d - d,\; cbd + bcd - d,\; cbd + bac - d,\; c^2d, \\
cd^2 - acd,\; cd^2 + bc^2 - a^2c,\; cd^2 - a^2c.
\end{gathered}
$$

Taking the union of these two sets gives 76 generators, and self-reducing this set produces a set with only 22 elements:

$$
\begin{gathered}
ad,\; b^2,\; bd - ab,\; c^2,\; cd - ac,\; da,\; db - ba,\; dc - ca,\; d^2 - cb - bc + a^2, \\
a^3 - a,\; aba,\; aca,\; acb + abc - a,\; ba^2 + a^2b - b,\; bab,\; bca - abc,\; bcb - b,
\end{gathered}
$$

$$ca^2 + a^2c - c, \quad cab + bac - d, \quad cac, \quad cba + abc - a, \quad cbc - c.$$

All compositions of this new set have normal form zero, so we have a Gröbner basis. There are only 17 monomials in $F\langle a,b,c,d\rangle$ which do not have the leading monomial of one of these 22 generators as a subword, and the cosets of these monomials modulo I form a basis for the universal associative envelope $U(T)$:

$$1, \; a, \; b, \; c, \; d, \; a^2, \; ab, \; ac, \; ba, \; bc, \; ca, \; cb, \; a^2b, \; a^2c, \; abc, \; bac, \; a^2bc.$$

The multiplication table for $U(T)$ is an array of size 17×17.

Exercise 8.2. Use a computer algebra system to calculate the multiplication table for $U(T)$. Compute the Wedderburn decomposition of $U(T)$. Prove or disprove that $U(T) \approx F \oplus M_2(F) \oplus M_2(F) \oplus M_2(F) \oplus M_2(F)$. For the structure theory of Jordan triple systems, see Loos [80] and Meyberg [84].

8.3. **The 2×2 matrices with the Jordan tetrad.** We consider the vector space $Q = M_2(F)$ with a quadrilinear operation, the Jordan tetrad

$$\{w, x, y, z\} = wxyz + zyxw.$$

Working out the structure constants for this operation, we find that the ideal I is generated by the self-reduced set of 136 elements displayed in Table 2. Remarkably, there are 2769 distinct nontrivial compositions of these 136 generators. The most complicated normal form of these compositions is

$$bcbcdcd + bcbc^2d^2 + dcd^2 + c^2bd + cbdc + cbcd + bdc^2 - adcd - c.$$

Combining the original 136 generators with the 2769 compositions produces a new generating set of 2905 elements. After two iterations of self-reduction, this large set of generators collapses to the set 25 elements in Table 3 which form a Gröbner basis. There are only 25 monomials in $F\langle a,b,c,d\rangle$ which do not have the leading monomial of one of these 25 generators as a subword; the cosets of these monomials form a basis of the universal associative envelope $U(Q)$:

$$1, \quad a, \quad b, \quad c, \quad d, \quad a^2, \quad ab, \quad ac, \quad ba, \quad bc, \quad ca, \quad cb, \quad a^3, \quad a^2b, \quad a^2c,$$
$$abc, \quad ba^2, \quad bac, \quad ca^2, \quad cab, \quad a^3b, \quad a^3c, \quad a^2bc, \quad ba^2c, \quad a^3bc.$$

The multiplication table of $U(Q)$ is an array of size 25×25.

Exercise 8.3. Use a computer algebra system to calculate the multiplication table of $U(Q)$. Compute the Wedderburn decomposition of $U(Q)$. Prove or disprove that

$$U(T) \approx F \oplus M_2(F) \oplus M_2(F) \oplus M_2(F) \oplus M_2(F) \oplus M_2(F) \oplus M_2(F).$$

Remark 8.4. At present there is no general theory of the structures obtained from regarding the Jordan tetrad as a quadrilinear operation on an associative algebra. For the role played by tetrads in Jordan theory, see McCrimmon [83].

Exercise 8.5. Prove that if J is a finite dimensional Jordan algebra then its universal associative envelope is also finite dimensional.

Exercise 8.6. Prove that if J is an n-dimensional Jordan algebra with zero product, then $U(J)$ is the exterior algebra of an n-dimensional vector space.

Exercise 8.7. Prove that if J is the Jordan algebra of a symmetric bilinear form, then $U(J)$ is the corresponding Clifford algebra.

a^4-a, aba^2+a^2ba, ab^2a, aca^2+a^2ca, $acba+abca-a$, ac^2a, ada^2+a^2da, $adba+abda$, $adca+acda$, ad^2a, ba^3+a^3b-b, ba^2b, $baba+abab$, $baca+acab$, $bada+adab$, $b^2a^2+a^2b^2$, $b^2ab+bab^2$, b^3a+ab^3, b^4, $b^2ca+acb^2$, $b^2da+adb^2$, bca^2+a^2cb-a, $bcab+bacb-b$, $bcba+abcb-b$, bcb^2+b^2cb, bc^2a+ac^2b, bc^2b, $bcda+adcb$, bda^2+a^2db, $bdab+badb$, $bdba+abdb$, bdb^2+b^2db, $bdca+acdb-a$, $bdcb+bcdb-b$, bd^2a+ad^2b, bd^2b, ca^3+a^3c-c, ca^2b+ba^2c-d, ca^2c, $caba+abac$, cab^2+b^2ac, $caca+acac$, $cacb+bcac$, $cada+adac$, $cadb+bdac$, cba^2+a^2bc-a, $cbab+babc$, $cbac+cabc-c$, cb^2a+ab^2c, cb^3+b^3c, cb^2c, $cbca+acbc-c$, $cbcb+bcbc-d-a$, $cbda+adbc$, $cbdb+bdbc$, $c^2a^2+a^2c^2$, $c^2ab+bac^2$, $c^2ac+cac^2$, $c^2ba+abc^2$, $c^2b^2+b^2c^2$, $c^2bc+cbc^2$, c^3a+ac^3, c^3b+bc^3, c^4, $c^2da+adc^2$, $c^2db+bdc^2$, cda^2+a^2dc, $cdab+badc$, $cdac+cadc$, $cdba+abdc-a$, cdb^2+b^2dc, $cdbc+cbdc-c$, $cdca+acdc$, $cdcb+bcdc$, cdc^2+c^2dc, cd^2a+ad^2c, cd^2b+bd^2c-a, cd^2c, da^3+a^3d, da^2b+ba^2d, da^2c+ca^2d, da^2d, $daba+abad$, dab^2+b^2ad, $dabc+cbad$, $daca+acad$, $dacb+bcad$, dac^2+c^2ad, $dada+adad$, $dadb+bdad$, $dadc+cdad$, dba^2+a^2bd-b, $dbab+babd$, $dbac+cabd-d$, $dbad+dabd$, db^2a+ab^2d, db^3+b^3d, db^2c+cb^2d, db^2d, $dbca+acbd$, $dbcb+bcbd-b$, dbc^2+c^2bd, $dbda+adbd$, $dbdb+bdbd$, $dbdc+cdbd$, dca^2+a^2cd-c, $dcab+bacd-d$, $dcac+cacd$, $dcad+dacd$, $dcba+abcd$, dcb^2+b^2cd, $dcbc+cbcd-c$, $dcbd+dbcd-d$, dc^2a+ac^2d, dc^2b+bc^2d, dc^3+c^3d, dc^2d, $dcda+adcd$, $dcdb+bdcd$, $dcdc+cdcd$, $d^2a^2+a^2d^2$, $d^2ab+bad^2$, $d^2ac+cad^2$, $d^2ad+dad^2$, $d^2ba+abd^2-b$, $d^2b^2+b^2d^2$, $d^2bc+cbd^2-d$, $d^2bd+dbd^2$, $d^2ca+acd^2-c$, $d^2cb+bcd^2-d$, $d^2c^2+c^2d^2$, $d^2cd+dcd^2$, d^3a+ad^3, d^3b+bd^3-b, d^3c+cd^3-c, d^4-d.

Table 2. The 136 generators of the ideal I for the Jordan tetrad

ad, b^2, $bd-ab$, c^2, $cd-ac$, da, $db-ba$, $dc-ca$, $d^2-cb-bc+a^2$, aba, aca, $acb+abc-a^3$, bab, $bca-abc$, $bcb-ba^2-a^2b$, cac, $cba+abc-a^3$, $cbc-ca^2-a^2c$, a^4-a, ba^3+a^3b-b, ba^2b, ca^3+a^3c-c, ca^2b+ba^2c-d, ca^2c, $cabc+a^3c-c$.

Table 3. The Gröbner basis of the ideal I for the Jordan tetrad

9. Multilinear Operations

We now consider generalizations of the two basic nonassociative bilinear operations, the Lie bracket and the Jordan product, to n-linear operations for any integer $n \geq 2$. This discussion is based on my papers with Peresi [28, 29].

9.1. **Multilinear operations.** An n-linear operation $\omega(a_1,\ldots,a_n)$ over a field F is a linear combination of permutations of the monomial $a_1\cdots a_n$. We regard ω as a multilinear element of degree n in the free associative algebra on n generators:

$$\omega(a_1,\ldots,a_n) = \sum_{\sigma\in S_n} x_\sigma a_{\sigma(1)}\cdots a_{\sigma(n)} \qquad (x_\sigma \in F),$$

where the sum is over all permutations in the symmetric group S_n acting on $\{1,\dots,n\}$. We may also identify $\omega(a_1,\dots,a_n)$ with an element of FS_n, the group algebra of the symmetric group S_n:

$$\omega(a_1,\dots,a_n) = \sum_{\sigma \in S_n} x_\sigma \sigma \qquad (x_\sigma \in F).$$

The group S_n acts on FS_n by permuting the subscripts of the generators:

$$\sigma \cdot a_{\tau(1)} \cdots a_{\tau(n)} = a_{\sigma\tau(1)} \cdots a_{\sigma\tau(n)}.$$

Two n-linear operations are said to be equivalent if each is a linear combination of permutations of the other; that is, they generate the same left ideal in FS_n.

When discussing n-linear operations, we assume that the characteristic of F is either 0 or a prime $p > n$; this is a necessary and sufficient condition for the group algebra FS_n to be semisimple. In this case, FS_n is the direct sum of simple two-sided ideals, each isomorphic to a matrix algebra $M_d(F)$, and the projections of S_n to these matrix algebras define the irreducible representations of S_n.

9.2. **The case $n = 2$.** Every bilinear operation is equivalent to one of the following: the Lie bracket $[x,y] = xy - yx$, the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$, the original associative operation xy, and the zero operation. In other words, the only left ideals in the group algebra $FS_2 \approx F \oplus F$ are $\{0\} \oplus F$, $F \oplus \{0\}$, $F \oplus F$, and $\{0\} \oplus \{0\}$. The first summand F corresponds to the unit representation of S_2, and a basis for this summand is the idempotent $\frac{1}{2}(xy + yx)$. The second summand corresponds to the sign representation, and a basis for this summand is the idempotent $\frac{1}{2}(xy - yx)$. These two idempotents are orthogonal in the sense that their product is zero.

9.3. **The case $n = 3$.** Faulkner [49] classified the trilinear polynomial identities satisfied by a large class of nearly simple triple systems. Twenty years later, trilinear operations were classified up to equivalence in my work with Peresi [28]; we also determined the polynomial identities of degree 5 satisfied by these operations. The structure of the group algebra in this case is

$$FS_3 \approx F \oplus M_2(F) \oplus F.$$

The first and last summands correspond to the unit and sign representations respectively; bases for these summands are the following idempotents:

$$S = \tfrac{1}{6}(abc + acb + bac + bca + cab + cba),$$
$$A = \tfrac{1}{6}(abc - acb - bac + bca + cab - cba).$$

The middle summand $M_2(F)$ corresponds to the irreducible 2-dimensional representation of S_3. To find a basis for $M_2(F)$ corresponding to the matrix units E_{ij} $(i,j = 1,2)$ we use the representation theory of the symmetric group developed by Young [103] and simplified by Rutherford [96] and Clifton [40]. It follows that any trilinear operation can be represented as a triple of matrices:

$$\left[a, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, c \right].$$

As representatives of the equivalence classes we may take the triples in which each matrix is in row canonical form.

Using computer algebra [28], it can be shown that there are exactly 19 trilinear operations satisfying polynomial identities in degree 5 which do not follow from their identities in degree 3. Simplified forms of these operations were later discovered by the author [25] and Elgendy [46]. Together with these 19 operations, it is conventional to include the symmetric, alternating and cyclic sums, even though for these operations, every identity in degree 5 follows from those of degree 3; see my paper with Hentzel [26]. These 22 trilinear operations are given in Table 4. The first column gives the name of the operation; the second column gives the row canonical forms of the representation matrices of the corresponding element of the group algebra; the third column gives the the simplest representative of the equivalence class as a linear combination of permutations. (The parameter q represents the $(1,2)$ entry of the 2×2 matrix.)

9.4. **Associative n-ary algebras.** Simple associative triple systems were classified by Hestenes [64], Lister [75] and Loos [81]; their work was extended to simple associative n-ary systems by Carlsson [35]. The classification by Carlsson can be reformulated as follows. Let $(d_1, \dots, d_{n-1})$ be a sequence of $n{-}1$ positive integers; two such sequences are regarded as equivalent if they differ only by a cyclic permutation. For each $i = 1, \dots, n{-}1$, let V_i be a vector space of dimension d_i over F, and consider the direct sum $V = V_1 \oplus \cdots \oplus V_{n-1}$. Let A be the subspace of $\mathrm{End}_F(V)$ consisting of the linear operators $T\colon V \to V$ which satisfy the conditions

$$T(V_1) \subseteq V_2, \quad T(V_2) \subseteq V_3, \quad \dots, \quad T(V_{n-2}) \subseteq V_{n-1}, \quad T(V_{n-1}) \subseteq V_1.$$

Then A is a simple associative n-ary system, and every such system has this form. If we choose bases of the subspaces $V_1, \dots, V_{n-1}$ then we can represent the elements of A as $D \times D$ block matrices where $D = d_1 + \cdots + d_{n-1}$. The block in position (i,j) where $1 \le i,j \le n{-}1$ has size $d_i \times d_j$; nonzero entries may appear only in blocks $(2,1)$, …, $(n{-}1,n)$, $(n,1)$. To illustrate, for $n = 3,4,5$ we obtain the matrices of the following forms, where T_{ij} is an arbitrary block of size $d_i \times d_j$:

$$\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & T_{13} \\ T_{21} & 0 & 0 \\ 0 & T_{32} & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & T_{14} \\ T_{21} & 0 & 0 & 0 \\ 0 & T_{32} & 0 & 0 \\ 0 & 0 & T_{43} & 0 \end{bmatrix}.$$

9.5. **Special nonassociative n-ary systems.** If A is an associative n-ary system and $\omega(a_1, \dots, a_n)$ is an n-linear operation, then we obtain a nonassociative n-ary system A^ω by interpreting each monomial in ω as the corresponding product in A. Such a nonassociative n-ary system is called special (by analogy with special Jordan algebras) since it comes from a multilinear operation on an associative system.

In order to understand these nonassociative n-ary systems, we construct their universal associative envelopes using the theory of noncommutative Gröbner bases. The ultimate goal is to classify all the irreducible finite dimensional representations of these systems. This generalizes the familiar construction of the universal enveloping algebras of Lie and Jordan algebras, where a dichotomy arises: a finite dimensional simple Lie algebra has an infinite dimensional universal envelope and infinitely many isomorphism classes of irreducible finite dimensional representations, but a finite dimensional simple Jordan algebra has a finite dimensional universal envelope and only finitely many irreducible representations.

9.6. **Universal associative envelopes.** This subsection gives the precise definition of the universal associative envelope of a nonassociative n-ary system relative to an n-linear operation; we consider only the case of a special nonassociative n-ary

operation	$F \oplus M_2(F) \oplus F$	FS_3
symmetric sum	$\left[1, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 0\right]$	$abc + acb + bac + bca + cab + cba$
alternating sum	$\left[0, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 1\right]$	$abc - acb - bac + bca + cab - cba$
cyclic sum	$\left[1, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 1\right]$	$abc + bca + cab$
Lie $q = \infty$	$\left[0, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, 0\right]$	$abc - acb - bca + cba$
Lie $q = \frac{1}{2}$	$\left[0, \begin{bmatrix}1 & \frac{1}{2}\\0 & 0\end{bmatrix}, 0\right]$	$abc + acb - bca - cba$
Jordan $q = \infty$	$\left[1, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, 0\right]$	$abc + cba$
Jordan $q = 0$	$\left[1, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, 0\right]$	$abc + bac$
Jordan $q = 1$	$\left[1, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, 0\right]$	$abc + acb$
Jordan $q = \frac{1}{2}$	$\left[1, \begin{bmatrix}1 & \frac{1}{2}\\0 & 0\end{bmatrix}, 0\right]$	$abc + 2acb + 2cab + cba$
anti-Jordan $q = \infty$	$\left[0, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, 1\right]$	$abc - 2acb + 2cab - cba$
anti-Jordan $q = -1$	$\left[0, \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}, 1\right]$	$abc - acb$
anti-Jordan $q = \frac{1}{2}$	$\left[0, \begin{bmatrix}1 & \frac{1}{2}\\0 & 0\end{bmatrix}, 1\right]$	$abc - cba$
anti-Jordan $q = 2$	$\left[0, \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}, 1\right]$	$abc - bac$
fourth family $q = \infty$	$\left[1, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, 1\right]$	$abc - acb - bac$
fourth family $q = 0$	$\left[1, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, 1\right]$	$abc - acb + bca$
fourth family $q = 1$	$\left[1, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, 1\right]$	$abc - bac + cab$
fourth family $q = -1$	$\left[1, \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}, 1\right]$	$abc + bac + cab$
fourth family $q = 2$	$\left[1, \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}, 1\right]$	$abc + acb + bca$
fourth family $q = \frac{1}{2}$	$\left[1, \begin{bmatrix}1 & \frac{1}{2}\\0 & 0\end{bmatrix}, 1\right]$	$abc + acb + bac$
cyclic commutator	$\left[0, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, 0\right]$	$abc - bca$
weakly commutative	$\left[1, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, 0\right]$	$abc + acb + bac - cba$
weakly anticommutative	$\left[0, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, 1\right]$	$abc + acb - bca - cab$

TABLE 4. The twenty-two trilinear operations

system. The earliest discussion of this construction appears to be that of Birkhoff and Whitman [10, §2]; the presentation here follows my survey paper [25, §7.2].

Suppose that B is a subspace, of an associative n-ary system A over the field F, which is closed under the n-linear operation

$$\omega(a_1, \dots, a_n) = \sum_{\sigma \in S_n} x_\sigma a_{\sigma(1)} \cdots a_{\sigma(n)} \qquad (x_\sigma \in F).$$

Set $d = \dim B$ and let $\{b_1, \dots, b_d\}$ be a basis of B over F; then we have the structure constants for the resulting nonassociative n-ary system B^ω:

$$\omega(b_{i_1}, \dots, b_{i_n}) = \sum_{j=1}^{d} c^j_{i_1 \cdots i_n} b_j \quad (1 \le i_1, \dots, i_n \le d).$$

Let $F\langle X \rangle$ be the free associative algebra generated by the symbols $X = \{b_1, \dots, b_d\}$ and consider the ideal $I \subseteq F\langle X \rangle$ generated by the following d^n elements:

$$\sum_{\sigma \in S_n} x_\sigma b_{i_{\sigma(1)}} \cdots b_{i_{\sigma(n)}} - \sum_{j=1}^{d} c^j_{i_1 \cdots i_n} b_j \quad (1 \le i_1, \dots, i_n \le d).$$

The quotient algebra $U(B^\omega) = F\langle X \rangle / I$ is the universal associative enveloping algebra of the nonassociative n-ary system B^ω. Since the n-ary structure on B^ω is special (that is, defined in terms of the associative structure on A), the natural map $B^\omega \to U(B^\omega)$ will necessarily be injective.

From this set of generators for the ideal I, we use the algorithm of Figure 2 to compute a Gröbner basis for I. We then use this Gröbner basis to obtain a monomial basis for the universal associative envelope $U(B^\omega)$. The multiplication table for $U(B^\omega)$ is then obtained by computing normal forms of products of basis monomials. The next section is devoted to examples of this procedure.

10. Special Nonassociative Triple Systems

In her Ph.D. thesis [45] and her forthcoming paper [46], Elgendy undertook a detailed study using noncommutative Gröbner bases of the universal associative envelopes of the nonassociative triple systems obtained by applying the trilinear operations of Table 4 to the 2-dimensional associative triple system A_1 of the form

$$\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix},$$

where $*$ represents an arbitrary scalar. She distinguished two classes of operations: those of Lie type, for which the universal envelopes are infinite dimensional; and those of Jordan type, for which the universal envelopes are finite dimensional. For the operations of Lie type, she discovered that the universal envelopes are closely related to the down-up algebras introduced by Benkart and Roby [8]. For the operations of Jordan type, she determined explicit Wedderburn decompositions of the universal envelopes and classified the irreducible representations; for these cases, she used the algorithms described in my survey paper [24].

In this section, I consider the same problem for the 4- and 6-dimensional associative triple systems A_2 and a_3 consisting of all matrices of the forms

$$\begin{bmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix}.$$

The resulting universal envelopes provide many examples of associative algebras, both finite dimensional and infinite dimensional, that deserve further study. It seems reasonable to expect that this will lead to generalizations of down-up algebras, and to nonassociative triple systems with many finite dimensional representations.

The computations are described in detail for A_2 and the results for A_1, A_2 and A_3 are summarized in Table 5. All calculations were done using Maple worksheets written by the author.

10.1. **Symmetric sum.** The original set of generators obtained from the structure constants consists of these 20 elements which form a Gröbner basis for the ideal:

$$\begin{gathered} a^3, \quad ba^2 + aba + a^2b, \quad b^2a + bab + ab^2, \quad b^3, \quad ca^2 + aca + a^2c - a, \\ cba + cab + bca + bac + acb + abc - b, \quad cb^2 + bcb + b^2c, \quad c^2a + cac + ac^2 - c, \\ c^2b + cbc + bc^2, \quad c^3, \quad da^2 + ada + a^2d, \quad dba + dab + bda + bad + adb + abd - a, \\ db^2 + bdb + b^2d - b, \quad dca + dac + cda + cad + adc + acd - d, \\ dcb + dbc + cdb + cbd + bdc + bcd - c, \quad dc^2 + cdc + c^2d, \\ d^2a + dad + ad^2, \quad d^2b + dbd + bd^2 - d, \quad d^2c + dcd + cd^2, \quad d^3. \end{gathered}$$

There are infinitely many monomials in $F\langle a, b, c, d\rangle$ which do not contain the leading monomial of one of these generators as a subword, and so the universal envelope is infinite dimensional. The first few dimensions of the homogeneous components of the associated graded algebra are as follows: 1, 4, 16, 44, 131, 344, 972, 2592, ...

10.2. **Alternating sum.** The original set of generators obtained from the structure constants consists of these 4 elements which form a Gröbner basis for the ideal:

$$\begin{gathered} cba - cab - bca + bac + acb - abc - b, \quad dba - dab - bda + bad + adb - abd + a, \\ dca - dac - cda + cad + adc - acd + d, \quad dcb - dbc - cdb + cbd + bdc - bcd - c. \end{gathered}$$

There are infinitely many monomials in $F\langle a, b, c, d\rangle$ which do not contain the leading monomial of one of these generators as a subword, and so in this case again, the universal associative envelope is infinite dimensional. The first few dimensions of the homogeneous components of the associated graded algebra are 1, 4, 16, 60, 225, 840, 3136, 11704, ... The *On-line Encyclopedia of Integer Sequences* [90], sequence A072335, suggests that the generating function for these dimensions is

$$\frac{1}{(1-x^2)(1-4x+x^2)}.$$

Since the generating function has such a simple form, it seems reasonable to expect that the universal envelope has an interesting structure, and that the original 4-dimensional alternating triple system has a large class of finite dimensional irreducible representations.

Open Problem 10.1. Prove the last claim about the generating function for the dimensions of the homogeneous components of the associated graded algebra.

Open Problem 10.2. Investigate the relationship between the universal envelopes for the symmetric and alternating sums and down-up algebras; see [8] and [46].

10.3. **Cyclic sum.** The original set of generators obtained from the structure constants consists of these 24 elements, forming a self-reduced set:

$$
\begin{gathered}
a^3, \quad ba^2+aba+a^2b, \quad b^2a+bab+ab^2, \quad b^3, \quad ca^2+aca+a^2c-a, \\
cab+bca+abc, \quad cba+bac+acb-b, \quad cb^2+bcb+b^2c, \quad c^2a+cac+ac^2-c, \\
c^2b+cbc+bc^2, \quad c^3, \quad da^2+ada+a^2d, \quad dab+bda+abd-a, \\
dac+cda+acd-d, \quad dba+bad+adb, \quad db^2+bdb+b^2d-b, \quad dbc+cdb+bcd, \\
dca+cad+adc, \quad dcb+cbd+bdc-c, \quad dc^2+cdc+c^2d, \quad d^2a+dad+ad^2, \\
d^2b+dbd+bd^2-d, \quad d^2c+dcd+cd^2, \quad d^3.
\end{gathered}
$$

This is not a Gröbner basis; there are 40 distinct nontrivial compositions of these generators, the most complicated of which has this normal form:

$$
\begin{aligned}
& bacda+bacad+acbda-acadb-abcad+abadc-2abacd+a^2cbd+a^2bdc \\
& -2a^2bcd-2bda-bad+adb-aca-abd-2a^2c+2a.
\end{aligned}
$$

Combining the original 24 generators with the 40 compositions gives a set of 64 elements; applying self-reduction to this set produces a new generating set of 59 elements. This new generating set produces 724 distinct nontrivial compositions. The combined set of 783 generators self-reduces to 62 elements, which form a Gröbner basis for the ideal:

$$
\begin{gathered}
a^3, \quad a^2b, \quad a^2d, \quad aba, \quad ab^2, \quad abc, \quad abd-a^2c, \quad aca+a^2c-a, \quad ada, \quad adb, \\
adc, \quad ad^2, \quad ba^2, \quad bab, \quad bac+acb-b, \quad bad, \quad b^2a, \quad b^3, \quad b^2c, \quad b^2d+acb-b, \\
bca, \quad bcb, \quad bc^2, \quad bcd, \quad bda+a^2c-a, \quad bdb-acb, \quad bdc-ac^2, \quad bd^2-acd, \\
ca^2, \quad cab, \quad cac+ac^2-c, \quad cad, \quad cba, \quad cb^2, \quad cbc, \quad cbd+ac^2-c, \quad c^2a, \quad c^2b, \\
c^3, \quad c^2d, \quad cda, \quad cdb, \quad cdc, \quad cd^2, \quad da^2, \quad dab, \quad dac+acd-d, \quad dad, \quad dba, \\
db^2, \quad dbc, \quad dbd+acd-d, \quad dca, \quad dcb, \quad dc^2, \quad dcd, \quad d^2a, \quad d^2b, \quad d^2c, \quad d^3, \\
a^2cb-ab, \quad a^2cd-ad.
\end{gathered}
$$

Only finitely many monomials in $F\langle a,b,c,d\rangle$ do not have a subword equal to the leading monomial of an element of this Gröbner basis. The universal associative envelope has a basis consisting of the cosets of these 26 monomials:

$$
\begin{gathered}
1, \quad a, \quad b, \quad c, \quad d, \quad a^2, \quad ab, \quad ac, \quad ad, \quad ba, \quad b^2, \quad bc, \quad bd, \quad ca, \quad cb, \quad c^2, \\
cd, \quad da, \quad db, \quad dc, \quad d^2, \quad a^2c, \quad acb, \quad ac^2, \quad acd, \quad a^2c^2.
\end{gathered}
$$

Exercise 10.3. Determine the radical of the universal envelope in this case, and the decomposition of the semisimple quotient into a direct sum of simple ideals.

10.4. **Lie $q = \infty$.** In this case we are studying a simple Lie triple system; see Lister [74]. The original set of generators obtained from the structure constants consists of 24 elements; after self-reduction, we are left with 20 elements:

$$
ba^2-2aba+a^2b, \quad b^2a-2bab+ab^2, \quad ca^2-2aca+a^2c+2a,
$$

$$cab - bca + bac - acb + b, \quad cba - bca - acb + abc + b, \quad cb^2 - 2bcb + b^2c,$$
$$c^2a - 2cac + ac^2 + 2c, \quad c^2b - 2cbc + bc^2, \quad da^2 - 2ada + a^2d,$$
$$dab - bda + bad - adb + a, \quad dac - cda + cad - adc - d,$$
$$dba - bda - adb + abd + a, \quad db^2 - 2bdb + b^2d + 2b, \quad dbc - cdb + cbd - bdc - c,$$
$$dca - cda - adc + acd, \quad dcb - cdb - bdc + bcd, \quad dc^2 - 2cdc + c^2d,$$
$$d^2a - 2dad + ad^2, \quad d^2b - 2dbd + bd^2 + 2d, \quad d^2c - 2dcd + cd^2.$$

There are 24 distinct compositions; the most complicated normal form is

$$bdcda - bdadc - bcdad + badcd + adcdb - adbdc - acdbd + abdcd + bd^2 + 3acd.$$

Combining the original 20 generators with the 24 compositions and applying self-reduction gives a new generating set of 16 elements, which is a Gröbner basis:

$$ba - ab, \quad dc - cd, \quad ca^2 - 2aca + a^2c + 2a, \quad cab - bca - acb + abc + b,$$
$$cb^2 - 2bcb + b^2c, \quad c^2a - 2cac + ac^2 + 2c, \quad c^2b - 2cbc + bc^2, \quad da^2 - 2ada + a^2d,$$
$$dab - bda - adb + abd + a, \quad dac - cda + cad - acd - d, \quad db^2 - 2bdb + b^2d + 2b,$$
$$dbc - cdb + cbd - bcd - c, \quad d^2a - 2dad + ad^2, \quad d^2b - 2dbd + bd^2 + 2d,$$
$$cbca - cacb - bcac + acbc + cb + bc, \quad dbda - dadb - bdad + adbd - da - ad.$$

There are infinitely many monomials in $F\langle a, b, c, d\rangle$ which do not contain the leading monomial of one of these generators as a subword, and so in this case, the universal associative envelope is infinite dimensional. The generating function for the dimensions seems to be as follows; see [90], sequence A038164:

$$\frac{1}{(1-x)^4(1-x^2)^4}.$$

The first few terms are 1, 4, 14, 36, 85, 176, 344, 624, 1086, 1800, 2892, 4488, ...

Open Problem 10.4. Investigate the universal enveloping algebras of Lie triple systems and their representation theory. Every finite dimensional Lie triple system can be embedded into a finite dimensional Lie algebra as the odd subspace of a 2-grading on the Lie algebra. For recent work, see Hodge and Parshall [65].

10.5. **Lie $q = \frac{1}{2}$.** In this case we are studying a simple anti-Lie triple system. The original set of generators obtained from the structure constants consists of 40 elements; after self-reduction, we are left with 20 elements:

$$ba^2 - a^2b, \quad b^2a - ab^2, \quad ca^2 - a^2c, \quad cab - bca - bac + acb - b,$$
$$cba + bca - acb - abc + b, \quad cb^2 - b^2c, \quad c^2a - ac^2, \quad c^2b - bc^2, \quad da^2 - a^2d,$$
$$dab - bda - bad + adb + a, \quad dac - cda - cad + adc - d,$$
$$dba + bda - adb - abd - a, \quad db^2 - b^2d, \quad dbc - cdb - cbd + bdc + c,$$
$$dca + cda - adc - acd, \quad dcb + cdb - bdc - bcd, \quad dc^2 - c^2d, \quad d^2a - ad^2,$$
$$d^2b - bd^2, \quad d^2c - cd^2.$$

There are 26 distinct nontrivial compositions of these generators, the most complicated of which has normal form

$$bdcda - bdadc - bcdad + badcd - adcdb + adbdc + acdbd - abdcd - bd^2 - acd.$$

Combining the original 20 generators with the 26 compositions and applying self-reduction gives a new generating set of 12 elements, which is a Gröbner basis:

$$\begin{aligned}
&a^2, \quad ba+ab, \quad b^2, \quad c^2, \quad dc+cd, \quad d^2, \quad cab-bca+acb+abc-b,\\
&dab-bda+adb+abd+a, \quad dac-cda-cad-acd-d,\\
&dbc-cdb-cbd-bcd+c, \quad cbca-cacb-bcac+acbc+cb-bc,\\
&dbda-dadb-bdad+adbd-da+ad.
\end{aligned}$$

There are infinitely many monomials in $F\langle a,b,c,d\rangle$ which do not contain the leading monomial of one of these generators as a subword, and so again in this case, the universal associative envelope is infinite dimensional. The first few dimensions of the homogeneous components in the associated graded algebra are 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, 560, 680, 816, 969, ... According to [90], sequence A000292, these are the tetrahedral numbers; the generating function is

$$\sum_{n=1}^{\infty} \binom{n+2}{3} x^n.$$

Open Problem 10.5. Investigate the universal enveloping algebras of anti-Lie triple systems. Every finite dimensional anti-Lie triple system can be embedded into a finite dimensional Lie superalgebra as the odd subspace. See the recent monograph by Musson [88] on Lie superalgebras and their enveloping algebras.

10.6. **Jordan $q=\infty$.** In this case we are studying a simple Jordan triple system. The original set of 40 generators is already self-reduced:

$$\begin{aligned}
&a^3, \quad aba, \quad aca-a, \quad ada, \quad ba^2+a^2b, \quad bab, \quad b^2a+ab^2, \quad b^3, \quad bca+acb-b,\\
&bcb, \quad bda+adb-a, \quad bdb-b, \quad ca^2+a^2c, \quad cab+bac, \quad cac-c, \quad cba+abc,\\
&cb^2+b^2c, \quad cbc, \quad c^2a+ac^2, \quad c^2b+bc^2, \quad c^3, \quad cda+adc, \quad cdb+bdc, \quad cdc,\\
&da^2+a^2d, \quad dab+bad, \quad dac+cad-d, \quad dad, \quad dba+abd, \quad db^2+b^2d,\\
&dbc+cbd-c, \quad dbd-d, \quad dca+acd, \quad dcb+bcd, \quad dc^2+c^2d, \quad dcd,\\
&d^2a+ad^2, \quad d^2b+bd^2, \quad d^2c+cd^2, \quad d^3.
\end{aligned}$$

There are 32 distinct nontrivial compositions; their normal forms are

$$\begin{aligned}
&a^2, \quad ab, \quad ba, \quad b^2, \quad c^2, \quad cd, \quad dc, \quad d^2, \quad a^2b, \quad a^2c, \quad a^2d, \quad ab^2, \quad abc,\\
&abd, \quad abd+a^2c, \quad ac^2, \quad acd, \quad adc, \quad ad^2, \quad bac, \quad bad, \quad b^2c, \quad b^2d,\\
&b^2d+bac, \quad bc^2, \quad bcd, \quad bdc+ac^2, \quad bdc, \quad bd^2+acd, \quad bd^2, \quad c^2d, \quad cd^2.
\end{aligned}$$

The combined set of 72 elements self-reduces to 20, forming a Gröbner basis:

$$\begin{aligned}
&a^2, \quad ab, \quad ba, \quad b^2, \quad c^2, \quad cd, \quad dc, \quad d^2, \quad aca-a, \quad ada, \quad bca+acb-b,\\
&bcb, \quad bda+adb-a, \quad bdb-b, \quad cac-c, \quad cbc, \quad dac+cad-d, \quad dad,\\
&dbc+cbd-c, \quad dbd-d.
\end{aligned}$$

There are only 19 monomials which do not contain the leading monomial of an element of the Gröbner basis, so the universal associative envelope is finite dimensional and has the cosets of the following monomials as a basis:

$$\begin{aligned}
&1, \quad a, \quad b, \quad c, \quad d, \quad ac, \quad ad, \quad bc, \quad bd, \quad ca, \quad cb, \quad da, \quad db,\\
&acb, \quad adb, \quad cad, \quad cbd, \quad acbd, \quad cadb.
\end{aligned}$$

Exercise 10.6. Compute the Wedderburn decomposition of the universal associative envelope of this Jordan triple system. In particular, prove or disprove that the envelope is isomorphic to $F \oplus M_3(F) \oplus M_3(F)$.

Exercise 10.7. Let T be a finite dimensional Jordan triple system. Prove that $U(T)$ is also finite dimensional.

10.7. **Jordan $q = 0$.** The original self-reduced set of generators has 40 elements. There are 20 distinct nontrivial compositions, and the combined set of 60 elements self-reduces to 27 elements. These 27 generators have 4 distinct nontrivial compositions, and the combined set of 31 elements self-reduces to 15 elements, which form a Gröbner basis for the ideal:

$$(7) \quad \left\{ \begin{array}{lllllllll} a^2, & ab, & ad, & ba, & b^2, & bc, & bd-ac, & c^2, & cd, \\ dc, & d^2, & aca-a, & acb-b, & cac-c, & dac-d. \end{array} \right.$$

The universal associative envelope has dimension 10 with basis consisting of the cosets of the elements 1, a, b, c, d, ac, ca, cb, da, db. See Exercise 10.8 below.

10.8. **Jordan $q = 1$.** The original self-reduced set of generators has 40 elements. There are 19 distinct nontrivial compositions, and the combined set of 59 elements self-reduces to 27 elements. These 27 generators have 6 distinct nontrivial compositions, and the combined set of 33 elements self-reduces to 15 elements, which form a Gröbner basis for the ideal. This Gröbner basis is the same as in the previous case, and so the universal envelopes are isomorphic.

10.9. **Jordan $q = \frac{1}{2}$.** The original self-reduced set of 40 generators has 94 distinct nontrivial compositions, and the combined set of 134 elements self-reduces to 15 elements, which form the Gröbner basis (7).

Exercise 10.8. Compute the Wedderburn decomposition of the universal envelope for the Gröbner basis (7). Prove or disprove that it is isomorphic to $F \oplus M_3(F)$.

10.10. **Anti-Jordan $q = \infty$.** The original self-reduced set of 24 generators has 76 distinct nontrivial compositions, and the combined set of 100 generators self-reduces to 15 elements, which is the Gröbner basis (7).

10.11. **Anti-Jordan $q = -1$.** The original self-reduced set of 24 generators has 37 distinct nontrivial compositions. The combined set of 61 elements self-reduces to 23 elements with 6 distinct nontrivial compositions. The combined set of 29 elements self-reduces to 15 elements, which is the Gröbner basis (7).

10.12. **Anti-Jordan $q = 2$.** The original self-reduced set of 24 generators has 37 distinct nontrivial compositions. The combined set of 61 elements self-reduces to 23 elements with 4 distinct nontrivial compositions. The combined set of 27 elements self-reduces to 15 elements, which is the Gröbner basis (7).

10.13. **Anti-Jordan $q = \frac{1}{2}$.** In this case we are studying a simple anti-Jordan triple system; see Faulkner and Ferrar [50] and the Ph.D. thesis of Bashir [6]. The original self-reduced set of 24 generators is as follows:

$$ba^2 - a^2b, \quad b^2a - ab^2, \quad bca - acb + b, \quad bda - adb - a, \quad ca^2 - a^2c, \quad cab - bac,$$
$$cba - abc, \quad cb^2 - b^2c, \quad c^2a - ac^2, \quad c^2b - bc^2, \quad cda - adc, \quad cdb - bdc,$$
$$da^2 - a^2d, \quad dab - bad, \quad dac - cad - d, \quad dba - abd, \quad db^2 - b^2d, \quad dbc - cbd + c,$$

$$dca - acd, \quad dcb - bcd, \quad dc^2 - c^2d, \quad d^2a - ad^2, \quad d^2b - bd^2, \quad d^2c - cd^2.$$

There are 32 distinct nontrivial compositions:

$$a^2, \quad ab, \quad ba, \quad b^2, \quad c^2, \quad cd, \quad dc, \quad d^2, \quad a^2b, \quad a^2c, \quad a^2d, \quad ab^2, \quad abc,$$
$$abd, \quad abd + a^2c, \quad ac^2, \quad acd, \quad adc, \quad ad^2, \quad bac, \quad bad, \quad b^2c, \quad b^2d,$$
$$b^2d + bac, \quad bc^2, \quad bcd, \quad bdc, \quad bdc + ac^2, \quad bd^2 + acd, \quad bd^2, \quad c^2d, \quad cd^2.$$

The combined set of 56 elements self-reduces to a Gröbner basis of 12 elements:

$$a^2, \quad ab, \quad ba, \quad b^2, \quad c^2, \quad cd, \quad dc, \quad d^2, \quad bca - acb + b, \quad bda - adb - a,$$
$$dac - cad - d, \quad dbc - cbd + c.$$

The universal associative envelope is infinite dimensional; the dimensions of the homogeneous components of the associated graded algebra appear to be

$$\tfrac{1}{2}(n+1)(n+3) \quad (n \text{ odd}), \qquad \tfrac{1}{2}(n+2)^2 \quad (n \text{ even}).$$

Open Problem 10.9. Prove that the universal envelope is infinite dimensional, and that the dimensions of the homogeneous components are as stated.

10.14. **Fourth family $q = \infty$.** The original set of 52 generators self-reduces to 44 elements, which have 140 distinct nontrivial compositions. Self-reducing the combined set of 184 generators produces the Gröbner basis (7).

10.15. **Fourth family $q = 0$.** The original set of 52 generators self-reduces to 44 elements, which have 88 distinct nontrivial compositions. Self-reducing the combined set of 132 generators produces the Gröbner basis (7).

10.16. **Fourth family $q = 1$.** The original set of 52 generators self-reduces to 44 elements, which have 76 distinct nontrivial compositions. Self-reducing the combined set of 120 generators produces the Gröbner basis (7).

10.17. **Fourth family $q = -1$.** The original set of 64 generators self-reduces to 44 elements, which have 209 distinct nontrivial compositions. Self-reducing the combined set of 253 generators produces the Gröbner basis (7).

10.18. **Fourth family $q = 2$.** The original set of 64 generators self-reduces to 44 elements, which have 227 distinct nontrivial compositions. Self-reducing the combined set of 271 generators produces the Gröbner basis (7).

10.19. **Fourth family $q = \frac{1}{2}$.** The original set of 64 generators self-reduces to 44 elements, which have 184 distinct nontrivial compositions. Self-reducing the combined set of 228 generators produces the Gröbner basis (7).

10.20. **Cyclic commutator.** The original set of 60 generators self-reduces to 40 elements, which have 86 distinct nontrivial compositions. Self-reducing the combined set of 126 generators produces the Gröbner basis (7).

10.21. **Weakly commutative operation.** The original set of 64 generators self-reduces to 60 elements, which have 15 distinct nontrivial compositions. Self-reducing the combined set of 75 generators produces the Gröbner basis (7).

10.22. **Weakly anticommutative operation.** The original set of 60 generators self-reduces to 44 elements, which have 41 distinct nontrivial compositions. Self-reducing the combined set of 85 generators produces the Gröbner basis (7).

operation	$U(A_1^\omega)$	$U(A_2^\omega)$	$U(A_3^\omega)$
Sym sum	4 1,2,4,**4,5**, . . .	20 1,4,16,44,131,344, . . .	56 1,6,36,160,750,3240, . . .
Alt sum	0 1,2,4,8,16,32, . . .	4 1,4,16,60,225,840, . . .	20 1,6,36,196,1071,5796, . . .
Cyc sum	4 1,2,4,**4,5**, . . .	24,40 $\mid$ 59,724 $\mid$ 62 26	unable to complete
Lie $q=\infty$	2 1,2,4,6,9,12, . . .	20,24 $\mid$ 16 1,4,14,36,85,176, . . .	70,140 $\mid$ 51 1,6,30,110,360,1026, . . .
Lie $q=\frac{1}{2}$	2 1,2,4,6,9,12, . . .	20,26 $\mid$ 12 1,4,10,20,35,56, . . .	70,147 $\mid$ 39 1,6,24,74,195,456, . . .
Jor $q=\infty$	6,4 $\mid$ 4 5	40,32 $\mid$ 20 19	126,107 $\mid$ 54 69
Jor $q=0$	6 9	40,20 $\mid$ 27,4 $\mid$ 15 10	126,97 $\mid$ 71,9 $\mid$ 32 17
Jor $q=1$	6 9	40,19 $\mid$ 27,6 $\mid$ 15 10	126,93 $\mid$ 71,18 $\mid$ 32 17
Jor $q=\frac{1}{2}$	6,4 $\mid$ 4 5	40,94 $\mid$ 15 10	126,542 $\mid$ 32 17
AJ $q=\infty$	2 1,2,4,6,9,12, . . .	24,76 $\mid$ 15 10	90,513 $\mid$ 32 17
AJ $q=-1$	2,2 $\mid$ 4,2 $\mid$ 4 5	24,37 $\mid$ 23,6 $\mid$ 15 10	90,135 $\mid$ 62,18 $\mid$ 32 17
AJ $q=\frac{1}{2}$	2 1,2,4,6,9,12, . . .	24,32 $\mid$ 12 1,4,8,12,18,24, . . .	90,107 $\mid$ 36 1,6,18,36,72,120, . . .
AJ $q=2$	2,2 $\mid$ 4,2 $\mid$ 4 5	24,37 $\mid$ 23,4 $\mid$ 15 10	90,137 $\mid$ 62,9 $\mid$ 32 17
4th $q=\infty$	6,4 $\mid$ 4 5	40,140 $\mid$ 15 10	146,1065 $\mid$ 32 17
4th $q=0$	6 9	44,88 $\mid$ 15 10	146,737 $\mid$ 32 17
4th $q=1$	6 9	44,76 $\mid$ 15 10	146,618 $\mid$ 32 17
4th $q=-1$	6,5 $\mid$ 4 5	44,209 $\mid$ 15 10	146,1432 $\mid$ 32 17
4th $q=2$	6,5 $\mid$ 4 5	44,227 $\mid$ 15 10	146,1601 $\mid$ 32 17
4th $q=\frac{1}{2}$	6,4 $\mid$ 4 5	44,184 $\mid$ 15 10	146,1347 $\mid$ 32 17
Cyc com	4,4 $\mid$ 4 5	40,86 $\mid$ 15 10	140,396 $\mid$ 32 17
Weak C	8,2 $\mid$ 4 5	60,15 $\mid$ 15 10	196,58 $\mid$ 32 17
Weak AC	4,4 $\mid$ 4 5	44,41 $\mid$ 15 10	160,124 $\mid$ 32 17

TABLE 5. Universal associative envelopes of nonassociative triple systems

10.23. **Summary.** Table 5 summarizes the results of Elgendy [45, 46] for $U(A_1^\omega)$, the results of this section for $U(A_2^\omega)$, and further computations for $U(A_3^\omega)$. Each entry in the table has the form

$$\begin{cases} \text{algorithm} \\ \text{dimension} \end{cases}$$

where "algorithm" describes the performance of the Gröbner basis algorithm, and "dimension" gives the dimension of the universal associative envelope. The "algorithm" data consists of a sequence of pairs x, y corresponding to the iterations of the algorithm; x is the size of the self-reduced set of generators at the start of the iteration, and y is the number of distinct nontrivial compositions in normal form at the end of the iteration (if $y = 0$ it is omitted). The "dimension" data consists either of a single number (in the case where the universal envelope is finite dimensional), or a sequence of numbers giving the first few dimensions of the homogeneous components of the associated graded algebra (in the case where the universal envelope is infinite dimensional). Dimensions in boldface indicate values that repeat indefinitely.

For example, consider the entry in row "Lie $q = \infty$" and column "$U(A_3^\omega)$", the Lie triple product on the 6-dimensional simple associative triple system:

$$\begin{cases} 70, 140 \,|\, 51 \\ 1, 6, 30, 110, 360, 1026, \dots \end{cases}$$

This means:

(a) The algorithm terminated after two iterations: the original self-reduced set of 70 generators produced 140 nontrivial compositions; the combined set of 210 generators self-reduced to a Gröbner basis of 51 elements.

(b) The universal associative envelope is infinite dimensional, and the generating function for the dimensions of the homogeneous components of the associated graded algebra begins with the terms

$$1 + 6z + 30z^2 + 110z^3 + 360z^4 + 1026z^5 + \cdots$$

In one case, $U(A_3^\omega)$ for the cyclic sum, the computations were so complicated that Maple 14 on my MacBook Pro was unable to complete them in a reasonable time. This may be related to the fact that the polynomial identities satisfied by this operation are extremely complicated; see my paper with Peresi [30].

10.24. **Conclusions.** The results of Table 5 suggest a slightly different classification of operations into "Lie type" and "Jordan type" from that of Elgendy [45, 46]. Two operations, the cyclic sum and the anti-Jordan $q = \infty$ operation, produce infinite dimensional envelopes for A_1^ω but finite dimensional envelopes for A_2^ω. It seems likely that A_1^ω is exceptional, owing to its small dimension, and that the universal associative envelopes will be finite dimensional when either of these operations is applied to a simple associative triple system of dimension > 2. If this is correct, then these two operations should be reclassified as having "Jordan type".

Four operations produced a non-semisimple envelope for A_1^ω: Jordan $q = 0, 1$ and fourth family $q = 0, 1$. In these cases, the 9-dimensional envelope has a 4-dimensional radical and a 5-dimensional semisimple quotient which is isomorphic to $F \oplus M_2(F)$. For these operations it seems very likely that $U(A_n^\omega)$ $(n = 2, 3)$ is semisimple and is isomorphic to $F \oplus M_{n+1}(F)$. The reason is that the dimension of the envelope (10 for $n = 2$ and 17 for $n = 3$) is the sum of the squares of the dimensions of the 1-dimensional trivial representation and the $(n+1)$-dimensional natural representation. This seems to hold for most of the operations: the universal envelopes are finite dimensional and the only irreducible representations are the trivial representation and the natural representation.

Conjecture 10.10. *Let $A_{p,q}$ $(p \le q)$ be the simple associative triple system consisting of $(p+q) \times (p+q)$ block matrices of the form*

$$\begin{bmatrix} 0 & p \times q \\ q \times p & 0 \end{bmatrix}.$$

Let ω be one of the following trilinear operations from Table 4: Jordan $(q = 0, 1, \frac{1}{2})$, anti-Jordan $(q = \infty, -1, 2)$, fourth family (all cases), cyclic commutator, weakly commutative, weakly anticommutative. Then, with finitely many exceptions, $U(A^\omega)$ is finite dimensional and semisimple and is isomorphic to $F \oplus M_{p+q}(F)$.

The operations not included in this conjecture are the first three operations (the symmetric, alternating, and cyclic sums), together with the four classical operations (the Lie, anti-Lie, Jordan, and anti-Jordan triple products). These seem likely to be the operations producing nonassociative triple systems with the most interesting representation theory. This is well-known for the four classical operations, owing to their close connection with Lie and Jordan algebras and superalgebras. On the other hand, very little is known about the representation theory of nonassociative triple systems arising from the first three operations.

Open Problem 10.11. Study the structure of the universal associative envelopes, and classify the finite dimensional irreducible representations, for the nonassociative triple systems $A^\omega_{p,q}$ where ω is the symmetric, alternating, and cyclic sum.

11. Bibliographical Remarks

The historical origins of the theory of Gröbner bases are complex, with similar ideas discovered in different contexts at different times by different people.

11.1. **The commutative case.** The most famous branch of the theory, owing to its close connections with algebraic geometry, is that of commutative Gröbner bases. Many of these ideas can be traced back to the work of Macaulay; his 1916 monograph on *The Algebraic Theory of Modular Systems* is available online [82]. The original work of Gröbner most often cited as the origin of the theory of commutative Gröbner bases is his 1939 paper on linear differential equations [61]; this has appeared in English translation [62] with commentary by the translator [1]. The modern form of the theory which emphasizes the algorithmic aspects originated in the 1965 Ph.D. thesis of Buchberger which has been translated into English [31] with commentary by the author [33]; see also his 1970 paper [32]. There are many textbooks on the theory of commutative Gröbner bases and their applications; see Adams and Loustaunau [2], Becker and Weispfennig [7], Cox et al. [42], Ene and Herzog [48], and Fröberg [51].

11.2. **The noncommutative case.** The theory of noncommutative Gröbner bases seems to have originated with the Russian school of nonassociative algebra; see the papers of Zhukov [104] and especially Shirshov [98, 99]. The first systematic statements of the Composition (Diamond) Lemma in the noncommutative case, and its application to the proof of the PBW theorem, were published almost simultaneously by Bokut [11] and Bergman [9]. The latter paper traces the origins of the theory to earlier work of Newman [89]. The computational complexity of algorithms for constructing noncommutative Gröbner bases has been studied by F. Mora [86]. The Ph.D. thesis of Keller [71, 72] on noncommutative Gröbner bases led to the software package Opal [59]. A more recent software package, with extensive online documentation, has been developed by Cohen and Gijsbers [41]. For some important papers on theory and algorithms for noncommutative Gröbner bases, see Borges-Trenard et al. [23], Gerritzen [52], Green et al. [60], and Kang et al. [70]. For a connection between commutative and noncommutative Gröbner bases, see Eisenbud et al. [44]. For an extension to noncommutative power series, see Gerritzen and Holtkamp [55]. For textbooks on noncommutative Gröbner bases, see Bokut and Kukin [21], Bueso et al. [34], and Li [73].

11.3. **The nonassociative case.** The most important branch of the nonassociative theory deals with Gröbner-Shirshov bases for free Lie algebras; see Bokut and Chibrikov [19] and Bokut and Chen [13]. A theory of Gröbner-Shirshov bases in free nonassociative algebras has been developed by Gerritzen [53, 54] and Rajaee [95]. For related work on Sabinin algebras, see Shestakov and Umirbaev [97], Pérez-Izquierdo [92], and Chibrikov [39].

11.4. **Loday algebras.** An active area of current research is extending the Composition (Diamond) Lemma from associative algebras to the dialgebras and dendriform algebras introduced by Loday [76, 77, 78]. For associative dialgebras, see Bokut et al. [17]. For dendriform algebras, see Bokut et al. [15], Chen and Wang [38], as well as the papers on Rota-Baxter algebras by Bokut et al. [14, 18], Chen and Mo [37], Qiu [94], and Guo et al [63]. It is an open problem to extend these results further to the quadri-algebras of Aguiar and Loday [3], and to the Koszul dual of quadri-algebras introduced by Vallette [102, §5.6]. For Leibniz algebras, which are the analogues of Lie algebras in the setting of dialgebras, see Loday and Pirashvili [79], Aymon and Grivel [4], Casas et al. [36], Insua and Ladra [68]. For pre-Lie algebras, which are the analogue of Lie algebras in the setting of dendriform algebras, see Bokut et al. [16]. For L-dendriform algebras, which are the analogue of Lie algebras in the setting of quadri-algebras, see Bai et al. [5]. (For corresponding generalizations of Jordan algebras, see Hou et al. [66, 67].)

11.5. **Survey papers.** A survey of commutative and noncommutative Gröbner bases from the point of view of theoretical computer science has been written by T. Mora [87]. For an introduction to noncommutative Gröbner bases from the point of view of computer algebra, see Green [57, 58] and Ufnarovski [101]. A number of introductory surveys of Gröbner-Shirshov bases in associative and nonassociative algebras have been written by Bokut and his co-authors: see Bokut [12], Bokut and Kolesnikov [20], and Bokut and Shum [22].

Acknowledgements

I thank NSERC (Natural Sciences and Engineering Research Council of Canada) for financial support through a Discovery Grant, and the faculty and staff of CIMAT (Centro de Investigación en Matemáticas, Guanajuato, Mexico) for their hospitality during the Research School on *Associative and Nonassociative Algebras and Dialgebras: Theory and Algorithms - In Honour of Jean-Louis Loday (1946–2012)* from

February 17 to March 2, 2013 which was sponsored by CIMPA (Centre International de Mathématiques Pures et Appliquées). I thank my former Ph.D. student Hader Elgendy for pointing out some errors in an earlier version of these notes.

References

[1] M. P. Abramson: Historical background to Gröbner's paper. *ACM Commun. Comput. Algebra* 43 (2009) no. 1-2, 22–23.

[2] W. W. Adams, P. Loustaunau: *An Introduction to Gröbner Bases*. Graduate Studies in Mathematics, 3. American Mathematical Society, Providence, RI, 1994.

[3] M. Aguiar, J.-L. Loday: Quadri-algebras. *J. Pure Appl. Algebra* 191 (2004) no. 3, 205–221.

[4] M. Aymon, P.-P. Grivel: Un théorème de Poincaré-Birkhoff-Witt pour les algèbres de Leibniz. *Comm. Algebra* 31 (2003) no. 2, 527–544.

[5] C. Bai, L. Liu, X. Ni: Some results on L-dendriform algebras. *J. Geom. Phys.* 60 (2010) no. 6-8, 940–950.

[6] S. Bashir: *Automorphisms of Simple Anti-Jordan Pairs*. Ph.D. Thesis, University of Ottawa, Canada, 2008.

[7] T. Becker, V. Weispfenning: *Gröbner Bases: A Computational Approach to Commutative Algebra*. Graduate Texts in Mathematics, 141. Springer-Verlag, New York, 1993.

[8] G. Benkart, T. Roby: Down-up algebras. *J. Algebra* 209 (1998) no. 1, 305–344. Addendum: "Down-up algebras". *J. Algebra* 213 (1999), no. 1, 378.

[9] G. M. Bergman: The diamond lemma for ring theory. *Adv. in Math.* 29 (1978) no. 2, 178–218.

[10] G. Birkhoff, P. M. Whitman: Representation of Jordan and Lie algebras. *Trans. Amer. Math. Soc.* 65 (1949) 116–136.

[11] L. A. Bokut: Imbeddings into simple associative algebras. *Algebra i Logika* 15 (1976) no. 2, 117–142.

[12] L. A. Bokut: The method of Gröbner-Shirshov bases. *Siberian Adv. Math.* 9 (1999) no. 3, 1–16.

[13] L. A. Bokut, Y. Chen: Gröbner-Shirshov bases for Lie algebras: after A. I. Shirshov. *Southeast Asian Bull. Math.* 31 (2007) no. 6, 1057–1076.

[14] L. A. Bokut, Y. Chen, X. Deng: Gröbner-Shirshov bases for Rota-Baxter algebras. *Sibirsk. Mat. Zh.* 51 (2010) no. 6, 1237–1250.

[15] L. A. Bokut, Y. Chen, J. Huang: Gröbner-Shirshov bases for L-algebras. *Internat. J. Algebra Comput.* (to appear). `arXiv:1005.0118 [math.RA]`

[16] L. A. Bokut, Y. Chen, Y. Li: Gröbner-Shirshov bases for Vinberg-Koszul-Gerstenhaber right-symmetric algebras. *Fundam. Prikl. Mat.* 14 (2008) no. 8, 55–67.

[17] L. A. Bokut, Y. Chen, C. Liu: Gröbner-Shirshov bases for dialgebras. *Internat. J. Algebra Comput.* 20 (2010) no. 3, 391–415.

[18] L. A. Bokut, Y. Chen, X. Deng: Gröbner-Shirshov bases for associative algebras with multiple operators and free Rota-Baxter algebras. *J. Pure Appl. Algebra* 214 (2010) no. 1, 89–100.

[19] L. A. Bokut, E. S. Chibrikov: Lyndon-Shirshov words, Gröbner-Shirshov bases, and free Lie algebras. *Non-associative Algebra and Its Applications*, pages 17–39. Lect. Notes Pure Appl. Math., 246, Chapman & Hall/CRC, Boca Raton, 2006.

[20] L. A. Bokut, P. S. Kolesnikov: Gröbner-Shirshov bases: from their incipiency to the present. *J. Math. Sci.* Vol. 116, No. 1, 2003.

[21] L. A. Bokut, G. P. Kukin: *Algorithmic and Combinatorial Algebra*. Mathematics and its Applications, 255. Kluwer Academic Publishers Group, Dordrecht, 1994.

[22] L. A. Bokut, K. P. Shum: Gröbner and Gröbner-Shirshov bases in algebra: an elementary approach. *Southeast Asian Bull. Math.* 29 (2005) no. 2, 227–252.

[23] M. A. Borges-Trenard, M. Borges-Quintana, T. Mora: Computing Gröbner bases by FGLM techniques in a non-commutative setting. *J. Symbolic Comput.* 30 (2000) no. 4, 429–449.

[24] M. R. Bremner: How to compute the Wedderburn decomposition of a finite-dimensional associative algebra. *Groups Complex. Cryptol.* 3 (2011) no. 1, 47–66.

[25] M. R. Bremner: Algebras, dialgebras, and polynomial identities. *Serdica Math. J.* 38 (2012) 91–136.

[26] M. R. Bremner, I. R. Hentzel: Identities for generalized Lie and Jordan products on totally associative triple systems. *J. Algebra* 231 (2000) no. 1, 387–405.
[27] M. R. Bremner, S. Madariaga: Polynomial identities for tangent algebras of monoassociative loops. *Comm. Algebra* (to appear)
[28] M. R. Bremner, L. A. Peresi: Classification of trilinear operations. *Comm. Algebra* 35 (2007) no. 9, 2932–2959.
[29] M. R. Bremner, L. A. Peresi: An application of lattice basis reduction to polynomial identities for algebraic structures. *Linear Algebra Appl.* 430 (2009) no. 2-3, 642–659.
[30] M. R. Bremner, L. A. Peresi: Polynomial identities for the ternary cyclic sum. *Linear Multilinear Algebra* 57 (2009) no. 6, 595–608.
[31] B. Buchberger: *An Algorithm for Finding the Basis Elements of the Residue Class Ring of a Zero Dimensional Polynomial Ideal*. Translated from the 1965 German original by Michael P. Abramson. *J. Symbolic Comput.* 41 (2006) no. 3-4, 475–511.
[32] B. Buchberger: Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems. *Aequationes Math.* 4 (1970) 374–383.
[33] B. Buchberger: Comments on the translation of my PhD thesis: "An Algorithm for Finding the Basis Elements of the Residue Class Ring of a Zero Dimensional Polynomial Ideal" *J. Symbolic Comput.* 41 (2006) no. 3-4, 471–474.
[34] J. Bueso, J. Gómez-Torrecillas, A. Verschoren: *Algorithmic Methods in Noncommutative Algebra: Applications to Quantum Groups*. Mathematical Modelling: Theory and Applications, 17. Kluwer Academic Publishers, Dordrecht, 2003.
[35] R. Carlsson: n-ary algebras. *Nagoya Math. J.* 78 (1980) 45–56.
[36] J. M. Casas, M. A. Insua, M. Ladra: Poincaré-Birkhoff-Witt theorem for Leibniz n-algebras. *J. Symbolic Comput.* 42 (2007) no. 11-12, 1052–1065.
[37] Y. Chen, Q. Mo: Embedding dendriform algebra into its universal enveloping Rota-Baxter algebra. *Proc. Amer. Math. Soc.* 139 (2011) no. 12, 4207–4216.
[38] Y. Chen, B. Wang: Gröbner-Shirshov bases and Hilbert series of free dendriform algebras. *Southeast Asian Bull. Math.* 34 (2010) no. 4, 639–650.
[39] E. S. Chibrikov: On free Sabinin algebras. *Comm. Algebra* 39 (2011) no. 11, 4014–4035.
[40] J. M. Clifton: A simplification of the computation of the natural representation of the symmetric group S_n. *Proc. Amer. Math. Soc.* 83 (1981) no. 2, 248–250.
[41] A. M. Cohen, D. A. H. Gijsbers: *Documentation on the GBNP Package*. Available at: `http://www.win.tue.nl/~amc/pub/grobner/doc.html` (accessed 19 January 2013)
[42] D. Cox, J. Little, D. O'Shea: *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1992.
[43] W. A. de Graaf: *Lie Algebras: Theory and Algorithms*. North-Holland Mathematical Library, 56. North-Holland Publishing Co., Amsterdam, 2000.
[44] D. Eisenbud, I. Peeva, B. Sturmfels: Non-commutative Gröbner bases for commutative algebras. *Proc. Amer. Math. Soc.* 126 (1998) no. 3, 687–691.
[45] H. A. Elgendy: *Polynomial Identities and Enveloping Algebras for n-ary Structures*. Ph.D. thesis, University of Saskatchewan, Canada, 2012.
[46] H. A. Elgendy: Universal associative envelopes of nonassociative triple systems. *Comm. Algebra* (to appear). `arXiv:1211.4243 [math.RA]`
[47] H. A. Elgendy, M. R. Bremner: Universal associative envelopes of $(n{+}1)$-dimensional n-Lie algebras. *Comm. Algebra* 40 (2012) no. 5, 1827–1842.
[48] V. Ene, J. Herzog: *Gröbner Bases in Commutative Algebra*. Graduate Studies in Mathematics, 130. American Mathematical Society, Providence, RI, 2012.
[49] J. R. Faulkner: Identity classification in triple systems. *J. Algebra* 94 (1985) no. 2, 352–363.
[50] J. R. Faulkner, J. C. Ferrar: Simple anti-Jordan pairs. *Comm. Algebra* 8 (1980) no. 11, 993–1013.
[51] R. Fröberg: *An Introduction to Gröbner Bases*. Pure and Applied Mathematics (New York). John Wiley & Sons, Ltd., Chichester, 1997.
[52] L. Gerritzen: On infinite Gröbner bases in free algebras. *Indag. Math. (N.S.)* 9 (1998) no. 4, 491–501.
[53] L. Gerritzen: Hilbert series and non-associative Gröbner bases. *Manuscripta Math.* 103 (2000) no. 2, 161–167.

[54] L. Gerritzen: Tree polynomials and non-associative Gröbner bases. *J. Symbolic Comput.* 41 (2006) no. 3-4, 297–316.

[55] L. Gerritzen, R. Holtkamp: On Gröbner bases of noncommutative power series. *Indag. Math. (N.S.)* 9 (1998) no. 4, 503–519.

[56] C. M. Glennie: Some identities valid in special Jordan algebras but not valid in all Jordan algebras. *Pacific J. Math.* 16 (1966) 47–59.

[57] E. L. Green: An introduction to noncommutative Gröbner bases. *Computational Algebra*, pp. 167–190, Lecture Notes in Pure and Appl. Math., 151. Dekker, New York, 1994.

[58] E. L. Green: Noncommutative Gröbner bases, and projective resolutions. *Computational Methods for Representations of Groups and Algebras*, pp. 29–60. Progr. Math., 173. Birkhäuser, Basel, 1999.

[59] E. L. Green, L. S. Heath, B. J. Keller: Opal: a system for computing noncommutative Gröbner bases. *Rewriting Techniques and Applications*, pp. 331–334. Lecture Notes in Computer Science, 1232. Springer, 1997.

[60] E. L. Green, T. Mora, V. Ufnarovski: The non-commutative Gröbner freaks. *Symbolic Rewriting Techniques (Ascona, 1995)*, 93104, Progr. Comput. Sci. Appl. Logic, 15, Birkhäuser, Basel, 1998.

[61] W. Gröbner: Über die algebraischen Eigenschaften der Integrale von linearen Differentialgleichungen mit konstanten Koeffizienten. *Monatsh. Math. Phys.* 47 (1939) no. 1, 247–284.

[62] W. Gröbner: On the algebraic properties of integrals of linear differential equations with constant coefficients. Translated from the German by Michael Abramson. *ACM Commun. Comput. Algebra* 43 (2009) no. 1-2, 24–46.

[63] L. Guo, W. Sit, R. Zhang: Differential type operators and Gröbner-Shirshov bases. *J. Symbolic Comput.* 52 (2013) 97–123.

[64] M. R. Hestenes: A ternary algebra with applications to matrices and linear transformations. *Arch. Rational Mech. Anal.* 11 (1962) 138–194.

[65] T. L. Hodge, B. J, Parshall: On the representation theory of Lie triple systems. *Trans. Amer. Math. Soc.* 354 (2002) no. 11, 4359–4391.

[66] D. Hou, C. Bai: J-dendriform algebras. *Front. Math. China* 7 (2012) no. 1, 29–49.

[67] D. Hou, X. Ni, C. Bai: Pre-Jordan algebras. *Math. Scand.* (to appear)

[68] M. A. Insua, M. Ladra: Gröbner bases in universal enveloping algebras of Leibniz algebras. *J. Symbolic Comput.* 44 (2009) no. 5, 517–526.

[69] N. Jacobson: *Structure and Representations of Jordan Algebras*. American Mathematical Society Colloquium Publications, Vol. XXXIX. American Mathematical Society, Providence, R.I., 1968.

[70] S.-J. Kang, D.-I. Lee, K.-H. Lee, H. Park: Linear algebraic approach to Gröbner-Shirshov basis theory. *J. Algebra* 313 (2007) no. 2, 988–1004.

[71] B. J. Keller: *Algorithms and Orders for Finding Noncommutative Gröbner Bases*. Ph.D. Thesis, Virginia Polytechnic Institute and State University, 1997.

[72] B. J. Keller: Alternatives in implementing noncommutative Gröbner basis systems. *Symbolic Rewriting Techniques*, pp. 105–126. Progr. Comput. Sci. Appl. Logic, 15. Birkhüser, Basel, 1998.

[73] H. Li: *Gröbner Bases in Ring Theory*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[74] W. G. Lister: A structure theory of Lie triple systems. *Trans. Amer. Math. Soc.* 72 (1952) 217–242.

[75] W. G. Lister: Ternary rings. *Trans. Amer. Math. Soc.* 154 (1971) 37–55.

[76] J.-L. Loday: Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *Enseign. Math.* (2) 39 (1993) no. 3-4, 269–293.

[77] J.-L. Loday: Algèbres ayant deux opérations associatives (digèbres). *C. R. Acad. Sci. Paris Sér. I Math.* 321 (1995) no. 2, 141–146.

[78] J.-L. Loday: Dialgebras. *Dialgebras and Related Operads*, pp. 7–66. Lecture Notes in Math., 1763, Springer, Berlin, 2001.

[79] J.-L. Loday, T. Pirashvili: Universal enveloping algebras of Leibniz algebras and (co)homology. *Math. Ann.* 296 (1993) no. 1, 139–158.

[80] O. Loos: *Lectures on Jordan Triples*. The University of British Columbia, Vancouver, Canada, 1971.

[81] O. Loos: Assoziative Tripelsysteme. *Manuscripta Math.* 7 (1972) 103–112.

[82] F. S. Macaulay: *The Algebraic Theory of Modular Systems.* Revised reprint of the 1916 original. With an introduction by Paul Roberts. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1994. http://archive.org/details/algebraictheoryo00macauoft
[83] K. McCrimmon: *A Taste of Jordan Algebras.* Universitext. Springer-Verlag, New York, 2004.
[84] K. Meyberg: *Lectures on Algebras and Triple Systems.* Notes on a course of lectures given during the academic year 1971-1972. The University of Virginia, Charlottesville, 1972.
[85] A. A. Mikhalev, A. A. Zolotykh: Standard Gröbner-Shirshov bases of free algebras over rings. I. Free associative algebras. *Internat. J. Algebra Comput.* 8 (1998) no. 6, 689–726.
[86] F. Mora: Groebner bases for noncommutative polynomial rings. *Algebraic Algorithms and Error-Correcting Codes*, Lecture Notes in Computer Science, 229, pp. 353–362. Springer, Berlin, 1986.
[87] T. Mora: An introduction to commutative and noncommutative Gröbner bases. *Theoret. Comput. Sci.* 134 (1994) 131–173.
[88] I. M. Musson: *Lie Superalgebras and Enveloping Algebras.* Graduate Studies in Mathematics, 131. American Mathematical Society, Providence, 2012.
[89] M. H. A. Newman: On theories with a combinatorial definition of "equivalence". *Annals of Math.* 43, 2 (1942) 223–243.
[90] *On-line Encyclopedia of Integer Sequences*: http://oeis.org/
[91] J.-M. Pérez-Izquierdo: An envelope for Bol algebras. *J. Algebra* 284 (2005) no. 2, 480–493.
[92] J.-M. Pérez-Izquierdo: Algebras, hyperalgebras, nonassociative bialgebras and loops. *Adv. Math.* 208 (2007) no. 2, 834–876.
[93] J.-M. Pérez-Izquierdo, I. P. Shestakov: An envelope for Malcev algebras. *J. Algebra* 272 (2004) no. 1, 379–393.
[94] J. Qiu: Gröbner-Shirshov bases for commutative algebras with multiple operators and free commutative Rota-Baxter algebras. arXiv:1301.5018
[95] S. Rajaee: Non-associative Gröbner bases. *J. Symbolic Comput.* 41 (2006) no. 8, 887–904.
[96] D. E. Rutherford: *Substitutional Analysis.* Edinburgh, at the University Press, 1948.
[97] I. P. Shestakov, U. U, Umirbaev: Free Akivis algebras, primitive elements, and hyperalgebras. *J. Algebra* 250 (2002) no. 2, 533–548.
[98] A. I. Shirshov: Some algorithmic problems for ϵ-algebras. *Sibirsk. Mat. Zh.* 3 (1962) 132–137.
[99] A. I. Shirshov: On a hypothesis in the theory of Lie algebras. *Sibirsk. Mat. Zh.* 3 (1962) 297–301 (1962).
[100] A. I. Shirshov: *Selected Works of A. I. Shirshov.* Translated from the Russian by M. R. Bremner and M. V. Kotchetov. Edited by L. A. Bokut, V. N. Latyshev, I. P. Shestakov and E. Zelmanov. Contemporary Mathematicians. Birkhäuser Verlag, Basel, 2009.
[101] V. S. Ufnarovski: Introduction to noncommutative Gröbner bases theory. *Gröbner Bases and Applications*, pp. 259–280. London Math. Soc. Lecture Note Ser., 251. Cambridge Univ. Press, Cambridge, 1998.
[102] B. Vallette: Manin products, Koszul duality, Loday algebras and Deligne conjecture. *J. Reine Angew. Math.* 620 (2008) 105–164.
[103] A. Young: *The Collected Papers of Alfred Young (1873–1940).* With a foreword by G. de B. Robinson and a biography by H. W. Turnbull. Mathematical Expositions, No. 21. University of Toronto Press, 1977.
[104] A. I. Zhukov: Reduced systems of defining relations in non-associative algebras. *Mat. Sbornik N.S.* 27(69) (1950) 267–280.

Department of Mathematics and Statistics, University of Saskatchewan, Canada
E-mail address: bremner@math.usask.ca
URL: math.usask.ca/~bremner