

# SINGULARITY ANALYSIS VIA THE ITERATED KERNEL METHOD

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ABSTRACT. In the quarter plane, five lattice path models with unit steps have resisted the otherwise general approach featured in recent works by Fayolle, Kurkova and Raschel. Here we consider these five models, called the singular models, and prove that the univariate generating functions marking the number of walks of a given length are not D-finite. Furthermore, we provide exact and asymptotic enumerative formulas for the number of such walks, and describe an efficient algorithm for exact enumeration.

*Dedicated to the remarkable Philippe Flajolet.*

## 1. INTRODUCTION

Lattice path models are classical objects, appearing very naturally in a variety of probabilistic and combinatorial contexts. Recent work has shown how they can help us better understand generating functions in a more general setting by addressing the question of predicting when the generating function of a combinatorial class will satisfy a ‘nice’ differential equation. Lattice path models restricted to the quarter plane have proved to be very useful in this regard – they offer a family of generating functions which are straightforward to manipulate, yet which possess some surprising structure. This, in turn, has led to some useful innovations in enumeration, including applications of boundary value methods [9, 11, 18], powerful and widely applicable variants of the kernel method [5, 6, 19], original computer algebra approaches [3, 16], and some freshened restatements of classic number theory results [4].

As has been remarked upon previously [8, 13], the property of being D-finite, that is, of satisfying a linear differential equation with polynomial coefficients, is exceptional and not expected of an arbitrary function. Indeed, in the case of combinatorial generating functions, the property of being D-finite appears to correlate with rich structure in the corresponding class – structure which we have yet to fully uncover. What can we learn from lattice path models?

A key observation of Bousquet-Mélou and Mishna [6] was that lattice path models with small steps restricted to the quarter plane appeared to be naturally partitioned according to the nature of their generating functions: specifically, they observed that D-finiteness of a model’s generating function appeared to be correlated to the finiteness of a group of plane transformations derived from the set of allowable steps. Furthermore, in two dimensions this property is further correlated with more combinatorial qualities of the step set: for example, symmetry across an axis or rotational symmetry, but *not*  $x \leftrightarrow y$  symmetry. In some cases the explanation is well understood, such as Theorem 1 in [7]. When the drift (that is, the vector sum of the allowable directions) is zero, Fayolle and Raschel [10] describe an arithmetic condition which begs a more combinatorial interpretation. Of the 79 non-isomorphic models, 23 are well studied with D-finite

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generating functions and 51 are highly suspected to be non D-finite: Kurkova and Raschel [18] proved that the trivariate generating functions marking endpoint are not D-finite by solving related boundary value problems, and Bostan, Raschel and Salvy [4] proved the excursion (walks returning to the origin) generating functions are not D-finite via an argument on the asymptotics of the coefficients. The remaining five models are called *singular* and resist both these strategies for different reasons. (For example, the excursion generating function is trivially 1 in these cases). Two of these models were previously considered [19], and both (univariate) generating functions were proven to be non D-finite. We apply this strategy to the final three models: it is an application of the *iterated kernel method* inspired by Bousquet-Mélou and Petkovšek [7], and Janse van Rensburg, Prellberg and Rechnitzer [15].

Specifically, the present work proves that the three remaining cases are not D-finite by analyzing an explicit generating function – asymptotic analysis and rapid exact enumeration are applications of this expression. In each case, we show that the number of singularities is infinite and far enough away from the dominating pole that they do not affect the first order asymptotics. The (essentially technical) challenge, as was the case in [19], is the justification that these singularities are true poles and are not somehow canceled by a quirk of the expression. This is significant because a D-finite function has a finite number of singularities, and so such a demonstration is a proof of non D-finiteness of the generating function. We diverge from [19] slightly and use a parameterization with its origins in the method of [9], as this yields a simpler process. In the course of our proofs we revisit some older theorems on polynomials that a reader faced with a similar problem may find useful. For a more general combinatorial discussion about why these models have D-finite generating functions, we direct the reader to [19].

In summary, for each of the five singular models we take a unified approach to prove formulas for asymptotic enumeration and determine an explicit expression for the generating function, information which cannot be determined using other known methods. In addition, we prove that the (univariate) counting generating function is not D-finite for the five models.

**1.1. The family of singular models.** A lattice path model is defined by a set of vectors – the allowable directions in which one can move along the sublattice  $\mathbb{N}^2$  of  $\mathbb{Z}^2$ . We are initially interested in models which permit only “small” steps; that is, the steps are contained in  $\{0, +1, -1\}^2$ . We use the notation  $NW \equiv (-1, 1)$ ,  $N \equiv (0, 1)$ ,  $NE \equiv (1, 1)$ , etc. The family of singular models consists of the following five models:

$$\begin{array}{l}
 \mathcal{A} = \begin{array}{c} \nearrow \\ \searrow \end{array} = \{NW, NE, SE\} \quad \mathcal{B} = \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} = \{NW, N, E, SE\} \quad \mathcal{C} = \begin{array}{c} \nearrow \\ \rightarrow \\ \rightarrow \\ \searrow \end{array} = \{NW, N, NE, E, SE\} \\
 \mathcal{D} = \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} = \{NW, N, SE\} \quad \mathcal{E} = \begin{array}{c} \nearrow \\ \rightarrow \\ \rightarrow \\ \searrow \end{array} = \{NW, N, NE, SE\}
 \end{array}$$

Models  $\mathcal{A}$  and  $\mathcal{D}$  are the two models considered by Mishna and Rechnitzer, and their strategy, known as the iterated kernel method, extends to all of these models. Note that the present work corrects an analytical error found in [19], which does not substantially change the stated results.

For each model  $S \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\}$  we address the following:

- ① What is the number  $S_n$  of walks of length  $n$  beginning at the origin and staying in  $\mathbb{N}^2$ ?
- ② How does  $S_n$  grow asymptotically when  $n$  is large?
- ③ Is the generating function  $S(t) = \sum_n S_n t^n$  D-finite?

| Model         | First 10 Terms in the counting sequence                       | (OEIS Tag) |
|---------------|---|------------|
| $\mathcal{A}$ | 1, 1, 3, 7, 21, 55, 165, 457, 1371, 3909, 11727               | (A151267)  |
| $\mathcal{B}$ | 1, 2, 6, 20, 70, 254, 942, 3550, 13532, 52030, 201386         | (A151284)  |
| $\mathcal{C}$ | 1, 3, 13, 59, 279, 1341, 6527, 31995, 157659, 779601, 3864985 | (A151321)  |
| $\mathcal{D}$ | 1, 1, 2, 4, 10, 23, 61, 153, 418, 1100, 3064                  | (A151256)  |
| $\mathcal{E}$ | 1, 2, 7, 24, 91, 339, 1316, 5064, 19876, 77655, 306653        | (A151294)  |

TABLE 1. The initial terms in the counting sequences for the number of walks of a given length with steps from the given model, restricted to the quarter plane. The OEIS Tag refers to the corresponding entry in the Online Encyclopedia of Integer Sequences [20]

The approach is to give an explicit expression for the generating function via the iterated kernel method, which entails describing a functional equation for a multivariate generating function, isolating its kernel. We generate a telescoping sum using a prescribed sequence of pairs which annihilate the kernel. The derived expression is useful to deduce asymptotic information and to demonstrate the source of an infinite collection of singularities.

The next section describes how to obtain generating function expressions. This is followed by the asymptotic analysis and non D-finiteness proofs for the symmetric models, and we conclude with a summary of the analysis of the asymmetric models.

## 2. AN EXPLICIT EXPRESSION FOR THE GENERATING FUNCTION

**2.1. The functional equation and its kernel.** Our central mathematical object is the multivariate generating function  $S_{x,y}(t) = \sum_{i,j,n} s_{ij}(n)x^i y^j t^n$ , where  $s_{ij}(n)$  counts the number of walks of length  $n$  with steps from  $\mathcal{S}$  which begin at the origin, end at the point  $(i, j)$ , and stay in  $\mathbb{N}^2$ . (Throughout,  $\mathcal{S}$  is our generic step set.) Our goal is to determine properties of  $S(t) \equiv S_{1,1}(t)$ , the generating function for the number of walks in the quarter plane.

To each of the five step sets, we associate a polynomial called the *kernel*; for the step set  $\mathcal{S}$ , define

$$K_{\mathcal{S}}(x, y) = xy - txy \sum_{(i,j) \in \mathcal{S}} x^i y^j.$$

As we restrict ourselves to small steps, the inventory of the steps has the form

$$(1) \quad \sum_{(i,j) \in \mathcal{S}} x^i y^j = xP_1(y) + P_0(y) + \frac{1}{x}P_{-1}(y) = yQ_1(x) + Q_0(x) + \frac{1}{y}Q_{-1}(x).$$

Thus,  $K_{\mathcal{S}}(x, y)$  can be regarded as a quadratic in  $y$  (respectively  $x$ ) whose coefficients contain  $t$ ,  $x$  and the  $Q_i(x)$  (resp.  $t$ ,  $y$ , and  $P_i(y)$ ):

$$(2) \quad K_{\mathcal{S}}(x, y) = -xtQ_1(x)y^2 + (x - txQ_0(x))y - xtQ_{-1}(x).$$

When the model is clear, we omit the subscript  $\mathcal{S}$ . One common property of the singular models is that they contain the steps NW and SE, and at least one other step, which prevents degeneracy in the quadratic.

Each model admits a functional equation for  $S_{x,y}(t)$ . We apply the common decomposition that a walk is either the empty walk, or a shorter walk followed by a single step. Taking into account the restrictions on walk location, as well as the fact that substituting  $x = 0$  (respectively  $y = 0$ ) into the function  $S_{x,y}(t)$  gives

the generating function of walks ending on the  $y$ -axis (respectively  $x$ -axis), we obtain, as many others have before us, the functional equation

$$(3) \quad K(x, y)S_{x,y}(t) = xy + K(x, 0)S_{x,0}(t) + K(0, y)S_{0,y}(t).$$

We are interested in the solutions to the kernel equation of the form:

$$(4) \quad K(x, Y_+(x; t)) = K(x, Y_-(x; t)) = K(X_+(y; t), y) = K(X_-(y; t), y) = 0,$$

and these algebraic functions are easily determined since the kernel is a quadratic:

$$(5) \quad Y_{\pm}(x; t) = \frac{(1 - tQ_0(x)) \mp \sqrt{(Q_0(x)^2 - 4Q_1(x)Q_{-1}(x))t^2 - 2Q_0(x)t + 1}}{2tQ_1(x)}$$

$$(6) \quad X_{\pm}(y; t) = \frac{(1 - tP_0(y)) \mp \sqrt{(P_0(y)^2 - 4P_1(y)P_{-1}(y))t^2 - 2P_0(y)t + 1}}{2tP_1(y)}.$$

There are other function pairs which annihilate the kernel, as we shall see. Remark that the boundary value method begins as we have, with the functional equation (3), but ultimately uses a different parametrization for the roots of the kernel, and from there a very different means to get access to the generating function.

The generating function has a natural expression in terms of iterated compositions of the  $Y$  and  $X$ , hence the name *iterated kernel method*.

**2.2. Summary: The Symmetric Models.** We define the sequence of functions  $Y_i$  by the recurrence  $Y_{n+1}(x) = Y_n(Y_+(x; t))$  with base case  $Y_0(x) = x$ . Remark that  $Y_1 \equiv Y_+$ . In Section 3 we show that if  $S$  is a model symmetric about the line  $x = y$ , i.e.  $S \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ , then one has the explicit form

$$(7) \quad S(t) = \frac{1}{1 - |S|t} \left( 1 - 2 \sum_{n=0}^{\infty} (-1)^n Y_n(1) Y_{n+1}(1) \right).$$

Theorem 1 gives the first order asymptotics of the symmetric models, extracted from this expression. Table 4 provides polynomial equations that are satisfied by the remaining poles.

Section 3.3 outlines the proof that these models are not D-finite. For each model we provide an infinite family of poles, which is sufficient because D-finite series have at most a finite number of poles. We prove that an infinite collection of  $Y_n(1)$  contribute singularities; the main difficulty lies in ensuring that the singularities in the terms are genuinely singularities of the sum.

**2.3. Summary: The Asymmetric Models.** The remaining two models are treated similarly, but require a bit more work. Here we require two function sequences:

$$\chi_n(x) = X_+(Y_+(\chi_{n-1}(x))), \quad \chi_0(x) = x \quad \text{and} \quad \Upsilon_n(y) = Y_+(X_+(\Upsilon_{n-1}(y))), \quad \Upsilon_0(y) = y.$$

We show in Section 4 that the generating function for the asymmetric walks is

$$(8) \quad S(t) = \frac{1}{1 - |S|t} \left( 1 - t \sum_{n \geq 0} \chi_n(1) \cdot (Y_+(\chi_n(1)) - Y_+(\chi_{n-1}(1))) - t \sum_{n \geq 0} X_+(\Upsilon_n(1)) \cdot (\Upsilon_n(1) - \Upsilon_{n+1}(1)) \right).$$

Of course, this expression is also valid for the symmetric models. In this case,  $X_+ = Y_+$  and expression (8) reduces to equation (7)

The asymptotics are considered in Section 4.2, and D-finiteness results are considered in Section 4.3.

**2.4. What makes this family special?** Consider the lowest order terms of the roots of the kernel as a power series in  $t$ . They are

$$Y_+ = P_{-1}(x)t + O(t^2) \quad \text{and} \quad Y_- = \frac{1}{tP_1(x)} - \frac{P_0(x)}{P_1(x)} + O(t),$$

where  $P_r(x) = \sum_{(i,r) \in \mathcal{S}} x^i$ . Of the 56 (conjectured) non D-finite models only 5 models, precisely the singular family we are studying, have a lowest order term with a positive power in  $x$  and  $t$ , implying that the infinite sum obtained by the iterated kernel method converges. This prevents the method from being applied to a broader range of models in this context.

**2.5. Fast enumeration.** The focus of the present work is to prove that the generating functions of singular models are not D-finite. Nonetheless, we should not lose sight of the fact that our generating function expressions are also useful for enumeration.

In fact, we can use the series expression to generate the first  $N$  terms of  $S_{1,0}(t)$  and  $S_{1,1}(t)$  for each model with  $\tilde{O}(N^3)$  bit-complexity (where the notation  $\tilde{O}(\cdot)$  suppresses logarithmic factors), which is an order of magnitude faster than the  $\tilde{O}(N^4)$  bit-complexity of the naive generation algorithm. The key fact which gives this speedup is that we can form a linear recurrence for  $1/Y_n$  (see Table 3). To be more specific, we define  $Z_n := 1/(Y_n/t^n)$  and:

- (a) Generate  $Z_0$  and  $Z_1$  to precision  $2N$
- (b) Use the recurrences in Table 3 to form a linear recurrence for  $Z_n$
- (c) Exploit this recurrence to generate  $Z_2, \dots, Z_{\lfloor \frac{N}{2} \rfloor}$  to precision  $2N$  using only shifts and additions
- (d) Recover  $Y_n = t^n/Z_n \pmod{t^N}$  for  $n = 1, \dots, \lfloor \frac{N}{2} \rfloor$ .

The series expressions then allows us to generate the first  $N$  terms of the generating function. The cost of generating these terms is dominated by the inversion of the  $Z_n$ , which have summands whose bit-size grows linearly.

Although the generating functions are not D-finite, and hence the coefficients do not satisfy a fixed length linear recurrence with polynomial coefficients, we are able to generate the terms in a more efficient manner. In order to generate a large number of terms, one may wish to use modular methods in order to prevent a memory overflow.

### 3. SYMMETRIC MODELS: $\mathcal{A}, \mathcal{B}, \mathcal{C}$

**3.1. An explicit generating function expression.** We focus first on the three models  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$ , as these models are symmetric about the line  $x = y$ . As such, these models benefit from the relation  $S_{x,0} = S_{0,x}$ , and Equation (3) can be rewritten as

$$(9) \quad K(x, y)S_{x,y}(t) = xy + K(0, y)S_{0,y}(t) + K(0, x)S_{0,x}(t).$$

Our iterates satisfy  $Y_{n+1}(x) = Y_+(Y_n(x))$ ,  $Y_0(x) = x$ , and hence we see

$$K(Y_n, Y_{n+1}) = K(Y_n(x), Y_+(Y_n(x))) = 0$$

for all  $n$  by substituting  $x = Y_n(x)$  into the kernel relation  $K(x, Y_+(x)) = 0$ . Thus, when we make this substitution into Equation (9) we find for each  $n$ :

$$0 = Y_n(x)Y_{n+1}(x) + K(0, Y_{n+1}(x))S_{0, Y_{n+1}(x)}(t) + K(0, Y_n(x))S_{0, Y_n(x)}(t).$$

We can determine an expression for  $K(0, x)S_{0, x}(t)$  by taking an alternating sum of these equations since all of the  $K(0, Y_n(x))S_{0, Y_n(x)}(t)$  terms are canceled for  $n > 0$  in a telescoping sum:

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (-1)^n \left( Y_n(x)Y_{n+1}(x) + K(0, Y_{n+1}(x))S_{0, Y_{n+1}(x)}(t) + K(0, Y_n(x))S_{0, Y_n(x)}(t) \right) \\ &= K(0, x)S_{0, x}(t) + \sum_{n=0}^{\infty} (-1)^n Y_n(x)Y_{n+1}(x). \end{aligned}$$

We rearrange this and evaluate at  $x = 1$  to express the counting generating function for walks returning to the axis:

$$(10) \quad S_{0,1}(t) = \frac{1}{t} \sum_{n=0}^{\infty} (-1)^n Y_n(1)Y_{n+1}(1),$$

as  $K(0, 1) = -t$  for each case considered here. This converges as a power series because in each of these cases  $Y_n(x) = O(t^n)$ .

Furthermore, substituting  $x = 1$  and  $y = 1$  into Equation (9) gives the full counting generating function

$$(11) \quad S(t) = \frac{1 - 2tS_{0,1}(t)}{1 - t|S|} = \frac{1}{1 - t|S|} \left( 1 - 2 \sum_n (-1)^n Y_n(1)Y_{n+1}(1) \right).$$

We address the robustness of this expression as a complex function in Theorem 3, after we determine an explicit expression for  $Y_n(1)$  as a rational function of  $Y_1(1)$ .

**3.2. Asymptotic Enumeration.** Next we show that the sum  $\sum_n (-1)^n Y_n(1)Y_{n+1}(1)$  is convergent at  $t = \frac{1}{|S|}$ , and that its radius of convergence is bounded below by  $t = \frac{1}{p_0 + 2\sqrt{p_1 p_{-1}}}$  where  $p_i = P_i(1) = |\{(i, r) : -1 \leq r \leq 1, (i, r) \in S\}|$ . In our three symmetric models, the singularity of  $S(t)$  at  $\frac{1}{|S|}$  is thus dominant.

**Theorem 1.** For each model  $S$  in  $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ , the number of walks  $S_n = [t^n]S(t)$  grows asymptotically like

$$(12) \quad S_n \sim \kappa_S \left( \frac{1}{|S|} \right)^n + O\left( (p_0 + 2\sqrt{p_1 p_{-1}})^n \right),$$

where each  $\kappa_S$  is a constant which can be calculated to arbitrary precision using Equation (10).

*Proof.* We proceed by basic singularity analysis [14], aided by estimates from the related models restricted to the upper-half plane. As  $S(t) = \left( \frac{1}{1 - |S|t} \right) (1 - 2tS_{0,1}(t))$ , we show that  $S(t)$  admits a simple pole at  $|S|^{-1}$ , and that this is the dominant singularity. The asymptotic expression in Equation (12) is then a consequence of evaluating the residue at this value, and bounding the dominant singularity of  $S_{0,1}(t)$ .

To accomplish this, we first consider the class of walks with steps from  $S$  which remain in the upper half plane, and return to the  $x$ -axis. This is a well studied class, and the methods of Banderier and Flajolet [1] yield the following expression for the generating function  $H(t)$ :

$$H(t) = \frac{(1 - tp_0) - \sqrt{(1 - tp_0)^2 - 4t^2 p_{-1} p_1}}{2tp_1 p_{-1}}.$$

| Model | Asymptotic estimate for number of walks of length $n$     |                              |
|-------|---|------------------------------|
| $A$   | $A_n \sim \kappa_A 3^n + O\left((2\sqrt{2})^n\right)$     | $\kappa_A = 0.17317888\dots$ |
| $B$   | $B_n \sim \kappa_B 4^n + O\left((1 + 2\sqrt{2})^n\right)$ | $\kappa_B = 0.15194581\dots$ |
| $C$   | $C_n \sim \kappa_C 5^n + O\left((1 + 2\sqrt{3})^n\right)$ | $\kappa_C = 0.38220125\dots$ |

TABLE 2. Asymptotic estimates for number of walk of length  $n$ . On a modern computer the  $\kappa_S$  can be calculated to a thousand decimal places in seconds.

In particular, the dominant singularity of  $H(t)$  is  $t = \frac{1}{p_0 + 2\sqrt{p_1 p_{-1}}}$ . Now, the set of quarter plane walks with steps from  $\mathcal{S}$  which return to the  $x$ -axis is a subset of this set, and so

$$[t^n]S_{0,1}(t) \leq [t^n]H(t).$$

Consequently,  $S_{0,1}(t)$  is convergent for  $0 \leq t \leq p_0 + 2\sqrt{p_1 p_{-1}} < |\mathcal{S}|$ , where the latter inequality is a result of the fact that  $p_{-1} = 1$ ,  $p_0 \in \{0, 1\}$  and  $p_1 \in \{1, 2, 3\}$  in these cases. Thus, the singularity at  $|\mathcal{S}|^{-1}$  is indeed dominant.

We also need to verify that  $S_{0,1}(|\mathcal{S}|^{-1}) \neq 0$  to justify that  $|\mathcal{S}|^{-1}$  is not a removable singularity. In Section 3.3.1, we determine an explicit expression for  $1/Y_n$ . Substituting  $t = 1/|\mathcal{S}|$  into this expression proves that  $Y_n(1)Y_{n+1}(1)$  is monotonically decreasing, so that the error on the  $N^{\text{th}}$  partial sum of the alternating series is bounded by  $Y_{N+1}(1)Y_{N+2}(1)$ . Numerically evaluating the  $10^{\text{th}}$  partial sum is sufficient to bound  $1 - 2 \sum_n (-1)^n Y_n(1)Y_{n+1}(1)$  away from 0 in each case. The results are summarized in Table 2.

We have shown that the dominant singularity of  $S(t)$  is indeed the simple pole at  $|\mathcal{S}|^{-1}$ . The residue is  $S_{0,1}(|\mathcal{S}|^{-1})$  which, when evaluated with suitable precision, gives the stated constant in Table 2. The sub-dominant factor comes from the inverse of the dominant singularity of  $S_{0,1}(t)$ , which is bounded by  $\frac{1}{p_0 + 2\sqrt{p_1 p_{-1}}}$ .

□

**3.3. The generating functions  $A(t)$ ,  $B(t)$  and  $C(t)$  are not D-finite.** The set of D-finite functions are closed under algebraic substitution. Thus, since our goal is to prove that the generating functions  $A(t)$ ,  $B(t)$  and  $C(t)$  are not D-finite, it is sufficient to consider these functions evaluated at  $t = q/(1 + q^2)$ . These turn out to be easier to analyze as the transformation concentrates the singularities around the unit circle. *As such, we shall re-interpret the notation we have introduced thus far to be functions of  $q$  directly.*

For each model, the  $Y_n(1)$  terms contribute singularities. A quick glance at an example is very suggestive; see Figure 1 for the singularities of  $Y_{20}(1)$  in the  $q$ -plane for the three different models. The main difficulty is proving that the singularities are genuinely present in the generating function. To prove this we follow these steps:

**Step 1:** Determine an explicit expression for  $Y_n(1)$ ;

**Step 2:** Determine a polynomial  $\sigma_n(q)$  whose set of roots contains the poles of  $Y_n(1)$ ;

**Step 3:** Determine a region where there are roots of  $\sigma_n(q)$  that are truly poles of  $Y_n(1)$ ;

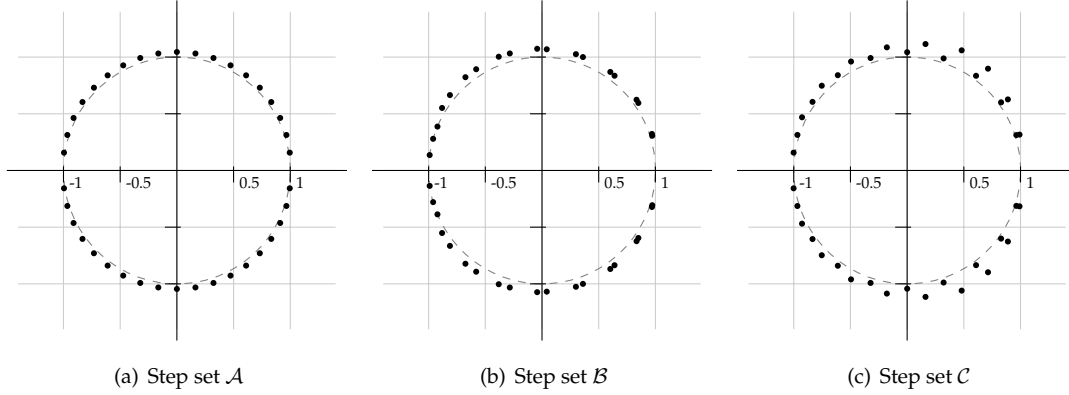


FIGURE 1. Plots of the singularities of  $Y_{20}(1)|_{t=\frac{q}{1+q^2}}$  for the three symmetric models.

**Step 4:** Show that there is no point  $\rho$  in that region that is a root of both  $\sigma_n(q)$  and  $\sigma_k(q)$  for different  $n$  and  $k$ ;

**Step 5:** Demonstrate that  $S_{1,0}(q/(1+q^2))$  has an infinite number of singularities and, consequently, is not  $D$ -finite. It follows that  $S(t)$  is not  $D$ -finite, by closure under algebraic substitution and the expression in Equation (11).

3.3.1. *Step 1: An explicit expression for  $Y_n$ .* In this section we find an explicit, non-iterated expression for the functions  $Y_n$ . We follow the method of [19] very closely, with the exception that we make the variable substitution earlier in the process. As such, we repeat, we view all functions as functions of  $q$  in this section.

We begin by performing the variable substitution  $t = q/(1+q^2)$  directly into Equation (3), and re-solve the kernel to ensure control over the choice of the branch in the solution. The kernels are:

$$\begin{aligned} \text{Model } \mathcal{A} \quad K(x, y) &= -q(x^2 + 1)y^2 + x(1 + q^2)y - qx^2 \\ \text{Model } \mathcal{B} \quad K(x, y) &= -q(x + 1)y^2 + x(-qx + 1 + q^2)y - qx^2 \\ \text{Model } \mathcal{C} \quad K(x, y) &= -q(1 + x + x^2)y^2 + x(-qx + 1 + q^2)y - qx^2. \end{aligned}$$

Recall we denote this generically as  $K(x, y) = a_2y^2 + a_1y + a_0$ , adapting the  $a_i$  to each particular model. Each is solved as before to get our initial solutions to  $K(x, Y(x)) = 0$ . Great care is taken here to ensure that the branch as written remains analytic at 0:

$$\begin{aligned} \text{Model } \mathcal{A} \quad Y_{\pm 1}(x; q) &= \frac{x}{2q(1+x^2)} \cdot \left( 1 + q^2 \mp \sqrt{1 - 2(2x^2 + 1)q^2 + q^4} \right) \\ \text{Model } \mathcal{B} \quad Y_{\pm 1}(x; q) &= \frac{x}{2q(1+x)} \cdot \left( 1 - qx + q^2 \mp \sqrt{q^4 - 2q^3x + (x^2 - 4x - 2)q^2 - 2qx + 1} \right) \\ \text{Model } \mathcal{C} \quad Y_{\pm 1}(x; q) &= \frac{x}{2q(1+x+x^2)} \cdot \left( 1 - qx + q^2 \mp \sqrt{q^4 - 2q^3x - (3x^2 + 4x + 2)q^2 - 2qx + 1} \right). \end{aligned}$$

We define the sequence of iterates  $\{Y_n(x)\}_{(n)}$  as before:  $Y_{n+1}(x) = Y_+(Y_n(x); q)$ ,  $Y_1(x) = Y_+(x; q)$ .

For each of these models, examining the coefficients of  $y$  in the kernel implies

$$(13) \quad \frac{1}{Y_-(x; q)} + \frac{1}{Y_+(x; q)} = \frac{Y_-(x; q) + Y_+(x; q)}{Y_-(x; q) \cdot Y_+(x; q)} = \frac{-a_1/a_2}{a_0/a_2} = -\frac{a_1}{a_2}.$$

Furthermore, the iterates compose nicely due to the following lemma.



| Model                      | Recurrence   | $\bar{Y}_n(q)$   |
|----------------------------|--|--|
| $\mathcal{A}$              | $\bar{Y}_n = (q + \frac{1}{q})\bar{Y}_{n-1} - \bar{Y}_{n-2}$     | $\frac{(q^2 - q^{2n}) + q(q^{2n} - 1)\bar{Y}_1}{q^n(q^2 - 1)}$   |
| $\mathcal{B}, \mathcal{C}$ | $\bar{Y}_n = (q + \frac{1}{q})\bar{Y}_{n-1} - \bar{Y}_{n-2} - 1$ | $\frac{q(q-1)(q^{2n}-1)\bar{Y}_1^{\mathcal{B},\mathcal{C}} + (q-q^n)(2q^{n+1}-q^n+q^2-2q)}{q^n(q+1)(q-1)^2}$ |

TABLE 3. The recurrences and solutions for models  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$ .

**Lemma 2.** For each of the symmetric models  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  we have

$$Y_-(Y_+(x)) = Y_+(Y_-(x)) = x.$$

*Proof.* For any given model, expanding the polynomial

$$p(z) = \prod_{(j,k) \in \{\pm 1\}^2} (z - Y_j(Y_k(z))),$$

implies  $p(z) = (x - z)^2 r(z)$ , where  $r(z) \in \mathbb{R}(x, t)[z]$  is given by

$$\begin{aligned}
& r(z) \\
\mathcal{A} & ((t^2 + x^2)z^2 - x(1 - 2t^2)z + t^2x^2) (t^2 + x^2)^{-1} \\
\mathcal{B} & ((t^2x^2 + 2t^2x + tx + tx^2 + t^2)z^2 + (-x + 2t^2x^2 + tx^2 + 2t^2x)z + t^2x^2) (t(x+1)(tx+x+t))^{-1} \\
\mathcal{C} & ((2t^2x + t^2 + tx^2 + tx + x^2 + t^2x^2)z^2 + (2t^2x^2 + tx^2 + 2t^2x - x)z + t^2x^2) (2t^2x + t^2 + tx^2 + tx + x^2 + t^2x^2)^{-1}
\end{aligned}$$

Thus, two of  $p(z)$ 's roots are equal to  $x$  and examination of the initial terms of a Taylor series in  $t$  shows that  $Y_+(Y_+(x))$  and  $Y_-(Y_-(x))$  are not.  $\square$

It turns out to be easier to work with the reciprocal of  $Y_n$ , so we define  $\bar{Y}_n = \frac{1}{Y_n(1)}$  and view this as a function of  $q$ . Equation (13) then converts into a recurrence after the substitution  $x = Y_{n-1}(x)$ . Specifically, this gives a linear recurrence for the reciprocal function,  $\frac{1}{Y_n(x)}$ ; we are interested in this evaluated at  $x = 1$ , and the resulting recurrences and their solutions in terms of  $\bar{Y}_1$  are summarized in Table 3.

Following the same procedure as above, we obtain a generic expression for  $S_{1,0}(t)$ , the generating function for the number of walks which return to the axis for model  $\mathcal{S}$ , which can be applied to all three symmetric walks:

$$(14) \quad S_{1,0} \left( \frac{q}{1+q^2} \right) = (q+1/q) \sum_{n=0}^{\infty} (-1)^n Y_n(1) Y_{n+1}(1).$$

Our careful choice of branches now implies that this is a formal power series. (Remark, this was not the case in [19].) Our expression is robust – the sum converges everywhere, except possibly on the unit circle and at the poles of the  $Y_n$ .

**Proposition 3.** For each model in  $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$  the sum  $(q+1/q) \sum_{n=0}^{\infty} (-1)^n Y_n(1) Y_{n+1}(1)$  is convergent for all  $q \in \mathbb{C}$  with  $|q| \neq 1$ , except possibly at the set of points defined by the singularities of the  $Y_n(1)$  for all  $n$ .

Model  $\sigma_n(q)$

$$\mathcal{A} \quad \alpha_n(q) = q^{4n} + q^{2n+2} - 4q^{2n} + q^{2n-2} + 1$$

$$\mathcal{B} \quad \beta_n(q) = (q^{2n-1} + (q^3 - 2q^2 - 2q + 1)q^{n-2} + 1)(q^{2n+1} + (q^3 - 2q^2 - 2q + 1)q^{n-1} + 1)$$

$$\mathcal{C} \quad \gamma_n(q) = q^2(1 + q^2 - q)(1 + q^{4n}) + q(q^2 - 3q + 1)(q + 1)^2(q^n + q^{3n}) \\ + q^{2n}(1 - q^2 - 4q + 14q^3 - 4q^5 - q^4 + q^6)$$

TABLE 4. The singularities of  $Y_n$  in the  $q$ -plane satisfy the polynomial equation  $\sigma_n(q) = 0$ .

*Proof.* In all cases the ratio test is applied to the explicit formulas for  $\bar{Y}_n(1; q)$ . It is a mechanical exercise to verify that for each case when  $|q| < 1$ :

$$\lim_{n \rightarrow \infty} \left| \frac{Y_{n+1}Y_{n+2}}{Y_nY_{n+1}} \right| = \left| \frac{\bar{Y}_n}{\bar{Y}_{n+2}} \right| = |q|^2 < 1,$$

and when  $|q| > 1$ :

$$\lim_{n \rightarrow \infty} \left| \frac{Y_{n+1}Y_{n+2}}{Y_nY_{n+1}} \right| = \left| \frac{\bar{Y}_n}{\bar{Y}_{n+2}} \right| = \frac{1}{|q|^2} < 1.$$

□

3.3.2. *Step 2: The singularities of  $Y_n(1)$ .* In order to argue about the singularities, we find a family of polynomials  $\sigma_n(q)$  that the roots of  $\bar{Y}_n$  satisfy: the polynomials in Table 4 are obtained by manipulating the explicit expressions given above. Unfortunately, extraneous roots are introduced during the algebraic manipulation when an equation is squared to remove the square root present. In fact, the extraneous roots are exactly those which correspond to a negative sign in front of the square root. If one defines  $\bar{Y}_{-n} = \bar{Y}_{-1} \circ \bar{Y}_{-(n-1)}$  for  $n > 1$ , then using the same argument as above one can check that  $\bar{Y}_{-n}$  satisfies the same recurrence relation as  $\bar{Y}_n$ , up to a reversal of the sign in front of the square root. Thus, we see that the set of roots of  $\sigma_n(q)$  is simply the union of the sets of roots of  $\bar{Y}_n$  and  $\bar{Y}_{-n}$ .

Furthermore, we can show that these roots are dense around the unit circle using the results of Beraha, Kahane, and Weiss – specifically, a weakened statement of the main theorem of [2].

**Proposition 4** ([2]). *Given non-zero polynomials  $\mu_1, \dots, \mu_k, \lambda_1, \dots, \lambda_k$ , define*

$$P_n(q) = \sum_{j=1}^k \mu_j(q)\lambda_j(q)^n.$$

*If there does not exist a constant  $\omega$  such that  $|\omega| = 1$  and  $\lambda_j = \omega\lambda_k$  for  $j \neq k$ , and for some  $l \geq 2$*

$$|\lambda_1(x)| = |\lambda_2(x)| = \dots = |\lambda_l(x)| > |\lambda_j(x)|,$$

*for all  $l + 1 \leq j \leq k$ , then  $x$  is a limit point of the zeroes of  $\{P_n(q)\}$  – i.e., there exists a sequence  $q_n$  converging to  $x$  such that  $P_n(q_n) = 0$  for all  $n$ .*

As each of  $\alpha_n(q)$ ,  $\beta_n(q)$ , and  $\gamma_n(q)$  can be decomposed into the required form where the  $\lambda_j(q)$  are simply powers of  $q$ , and thus have the same modulus when  $q$  is on the unit circle, this immediately gives the following result.

**Corollary 5.** *The roots of the families of polynomials  $\{\alpha_n(q)\}$ ,  $\{\beta_n(q)\}$ , and  $\{\gamma_n(q)\}$  are dense around the unit circle.*

Furthermore, as our results on the convergence of the series in Equation (14) is only valid at points off of the unit circle, we use a Lemma of Konvalina and Matache to determine when the roots of  $\alpha_n, \beta_n$ , and  $\gamma_n$  may lie on the unit circle.

**Lemma 6** (Konvalina and Matache [17], Lemma 1). *Suppose  $F(x)$  is a palindromic polynomial (its coefficient sequence is the same when read from the left or right) of degree  $2N$ . Then the argument of any root of  $F(x)$  which lies on the unit circle satisfies*

$$\phi(\theta) = \epsilon_N + 2 \sum_{k=0}^{N-1} \epsilon_k \cos((N-k)\theta)$$

where  $\epsilon_j$  denotes the coefficient of  $x^j$  in  $F(x)$ .

Applied to our polynomials, it gives the following.

**Proposition 7.** *For all natural numbers  $n$ ,  $\alpha_n(q)$  and  $\gamma_n(q)$  have no roots on the unit circle, except possibly  $q = \pm 1$ . Furthermore, if  $q$  is a root of  $\beta_n(q)$  on the unit circle not equal to 1 then*

$$\arg q \in \left[ \pi - \arccos\left(\sqrt{2} - \frac{1}{2}\right), \pi \right) \cup \left[ -\pi, -\pi + \arccos\left(\sqrt{2} - \frac{1}{2}\right) \right).$$

*Proof.* As  $\alpha_n(q), \beta_n(q)$ , and  $\gamma_n(q)$  are palindromic, Lemma 6 implies, after some trigonometric simplification, that the argument of any root  $q$  on the unit circle satisfies

$$\phi_A(\theta) = \mathbf{X} + 2 \cos^2(\theta) - 3$$

$$\phi_B(\theta) = \mathbf{X}^2 + (2 \cos^2(\theta) - \cos(\theta) - 3) \mathbf{X} + 2 \cos^3(\theta) - 4 \cos^2(\theta) - \cos(\theta) + 4$$

$$\phi_C(\theta) = 2(2 \cos(\theta) - 1) \mathbf{X}^2 + (4 \cos^2(\theta) - 2 \cos(\theta) - 6) \mathbf{X} + 4 \cos^3(\theta) - 8 \cos^2(\theta) - 6 \cos(\theta) + 12$$

respectively, where  $\mathbf{X} = \cos(n\theta)$ .

It is easy to see that  $\phi_A(\theta) = 0$  only if  $\theta = 0$  or  $\theta = \pi$ . For the other models, in order to give a bound on where the roots of each  $\phi$  expression lie we treat  $\mathbf{X}$  as an independent real variable lying in the range  $[-1, 1]$  for some fixed value of  $\theta$ , and determine where the minimum value of the above expression is at most zero.

First consider  $\phi_B(\theta)$ . As this is a quadratic in  $\mathbf{X}$  it attains its minimum value either at  $\mathbf{X} = \pm 1$  or when  $\mathbf{X} = -(2 \cos^2(\theta) - \cos(\theta) - 3)/2$ .

Substituting  $\mathbf{X} = \pm 1$  into  $\phi_B(\theta)$  yields expressions which are always greater than zero when  $\theta \notin \{0, \pi\}$ . Furthermore, when  $\mathbf{X} = -(2 \cos(\theta)^2 - \cos(\theta) - 3)/2$  our expression for  $\phi_B(\theta)$  simplifies to

$$\frac{1}{4} (-4 \cos^4(\theta) + 12 \cos^3(\theta) - 5 \cos^2(\theta) - 10 \cos(\theta) + 7).$$

One can verify that this is at most zero only when  $\theta = 0$  or

$$\theta \in \left[ \pi - \arccos\left(\sqrt{2} - \frac{1}{2}\right), \pi \right) \cup \left[ -\pi, -\pi + \arccos\left(\sqrt{2} - \frac{1}{2}\right) \right).$$

As  $\phi_C(\theta)$  is also a quadratic in  $\mathbf{X}$  an analogous argument shows that  $\phi_C(\theta) = 0$  only if  $\theta \in \{0, \pi\}$ , as desired.  $\square$

Thus, every point on the unit circle is a limit point of each of the sets

$$\{q : \alpha_n(q) = 0 \text{ for some } n\}, \{q : \beta_n(q) = 0 \text{ for some } n\}, \text{ and } \{q : \gamma_n(q) = 0 \text{ for some } n\},$$

but no element of these sets lies on the unit circle (except in a special region when dealing with model  $\mathcal{B}$ ). In fact, as the polynomials are palindromic, a straightforward application of Rouché's theorem proves that all roots of  $\alpha_n, \beta_n,$  and  $\gamma_n$  converge to the unit circle as  $n$  approaches infinity.

3.3.3. *Step 3: Verify that  $Y_n(1)$  has some singularities.* At this point we have not yet completely established that the  $Y_n(1)$  actually have singularities. Theoretically, it is possible that all the roots were added in our manipulations to determine  $\sigma_n(q)$  for the different models (as mentioned above, the roots of  $\sigma_n(q)$  are either singularities of  $Y_n(1)$  or singularities of  $Y_{-n}(1)$ ). Thus, we prove Lemma 8 which describes at least some region where we are certain to find roots of  $\bar{Y}_n$ . Experimentally, it seems that the roots are evenly partitioned so that those outside the unit circle belong to  $\bar{Y}_n$  and those inside the unit circle belong to  $\bar{Y}_{-n}$ , but we do not prove this.

**Lemma 8.** *For each model, if  $\arg(q) \in (-\pi/2, -3\pi/8) \cup (3\pi/8, \pi/2)$  then  $\bar{Y}_n = \bar{Y}_{-n}|_{q \rightarrow 1/q}$  for all  $n$ . Consequently  $Y_n$  admits at least one singularity in the complex  $q$ -plane in that region, for an infinite number of  $n$ .*

The proof requires only basic manipulations of the formulas. We offer the proof for the  $\mathcal{A}$  case, the other two are similar.

*A case.* First, note that it is sufficient to prove the result for  $\arg(q) \in (-\pi/2, -3\pi/8)$ . We claim that in this region the identity  $\bar{Y}_1 = \bar{Y}_{-1}|_{q \rightarrow 1/q}$  holds, which is equivalent to proving  $q^2 \sqrt{(q^4 - 6q^2 + 1)/q^4} = -\sqrt{q^4 - 6q^2 + 1}$  (using the standard branch-cut of the square root).

If  $q = re^{i\theta}$  in polar form it is straight forward to verify  $\Re(q^4 - 6q + 1), \Im(q^4 - 6q + 1) \geq 0$  for the values of  $\theta$  under consideration, so  $\arg(q^4 - 6q^2 + 1) \in [0, \pi/2]$ . Furthermore, for these values of  $\theta$  we have  $\arg(1/q^4) \in (-\pi/2, 0)$  so  $\arg(q^4 - 6q^2 + 1) + \arg(1/q^4) \in (-\pi/2, \pi/2]$ , and

$$q^2 \sqrt{(q^4 - 6q^2 + 1)/q^4} = q^2 \sqrt{1/q^4} \sqrt{(q^4 - 6q^2 + 1)}.$$

By our choice of region,  $\arg(1/q^4) = -4\theta - 2\pi$  and thus  $q^2 \sqrt{1/q^4} = r^2 e^{2i\theta} \cdot \frac{1}{r^2} e^{i(-4\theta - 2\pi)/2} = e^{-\pi i} = -1$ , proving the result on  $\bar{Y}_1$ .

Given this, we note

$$\bar{Y}_{-n} = \frac{(q^2 - q^{2n}) + q(q^{2n} - 1)\bar{Y}_{-1}}{q^n(q^2 - 1)} = \frac{q^{1-n} - q^{n-1}}{q - q^{-1}} + \frac{q^n - q^{-n}}{q - q^{-1}} \bar{Y}_{-1},$$

so that

$$\bar{Y}_{-n}|_{q \rightarrow 1/q} = \frac{q^{1-n} - q^{n-1}}{q - q^{-1}} + \frac{q^n - q^{-n}}{q - q^{-1}} \bar{Y}_1,$$

by the base case, as the rest is invariant, and thus  $\bar{Y}_{-n}(1/q) = \bar{Y}_n(q)$ .  $\square$

As the region considered above is disjoint from the region where the roots of  $\beta_n(q)$  lie on the unit circle, all the singularities we have found lie off of the unit circle.

3.3.4. *Step 4: The singularities are distinct.* We prove that the poles are distinct when they lie off of the unit circle by determining expressions for the powers of  $q$  at the poles of the  $Y_n$ .

**Proposition 9.** *For models  $\mathcal{A}$  and  $\mathcal{C}$ , if  $q_n$  is a pole of  $Y_n$  which lies off of the unit circle then it is not a pole of  $Y_k$  for  $k \neq n$ . For model  $\mathcal{B}$ , if  $q_n$  is a pole of  $Y_n$  off of the unit circle then it is not a pole of  $Y_k$  for  $|k - n| > 1$ .*

*Proof.* For each of the three models we find the roots of the numerators of our explicit expressions in Table 3 as quadratics in  $q^n$ . This determines functions  $r_1(q)$  and  $r_2(q)$ , independent of  $n$ , such that  $q_n^n = r_1(q_n)$  or  $q_n^n = r_2(q_n)$  at any pole  $q_n$  of  $Y_n$ .

Now, suppose  $q_n$  is also a pole of  $Y_k$  for  $k \neq n$ , so that  $q_n^k = r_1(q_n)$  or  $q_n^k = r_2(q_n)$ . If  $q_n^k = r_1(q_n) = q_n^n$  or  $q_n^k = r_2(q_n) = q_n^n$  then it is immediate that  $q_n$  must be on the unit circle. Thus we may assume, without loss of generality, that  $q_n^n = r_1(q_n)$  and  $q_n^k = r_2(q_n)$  – we consider each model separately.

- (Model A) Here,

$$r_1(q), r_2(q) = \pm \frac{2\sqrt{-q^2}}{q^2 + \sqrt{1 - 6q^2 + q^4 - 1}},$$

so that  $q_n^{n-k} = r_1(q_n)/r_2(q_n) = -1$ , implying that  $q_n$  must lie on the unit circle.

- (Model B) Here,

$$r_1(q) = \frac{2q^2}{2q^2 + \sqrt{q^4 - 2q^3 - 5q^2 - 2q + 1} - 2q + q\sqrt{q^4 - 2q^3 - 5q^2 - 2q + 1} - 1 - q^3},$$

and  $r_2(q) = r_1(q)/q$ , so that  $q_n^{n-k} = r_1(q_n)/r_2(q_n) = q_n$ . Thus, either  $n = k + 1$  or  $q_n$  lies on the unit circle.

- (Model C) In this slightly trickier case we have

$$r_1(q) = \frac{q(-1 - q - i\sqrt{3} + \sqrt{3}q)}{-2q^2 - \sqrt{1 - 2q - 9q^2 - 2q^3 + q^4} - 2q + q\sqrt{1 - 2q - 9q^2 - 2q^3 + q^4} + 1 + q^3}$$

$$r_2(q) = \frac{q(-1 - q + i\sqrt{3} - \sqrt{3}q)}{-2q^2 - \sqrt{1 - 2q - 9q^2 - 2q^3 + q^4} - 2q + q\sqrt{1 - 2q - 9q^2 - 2q^3 + q^4} + 1 + q^3},$$

which implies

$$q_n^{n-k} = r_1(q_n)/r_2(q_n) = e^{-2\pi i/3} + \sqrt{3} \frac{e^{-\pi i/6}}{q - e^{\pi i/3}}.$$

Substituting  $q_n = re^{i\theta}$  into this expression allows one to see that the right hand side has modulus greater than or equal to one when  $\theta \in [0, \pi)$  and modulus less than or equal to one when  $\theta \in (-\pi, 0]$ .

Suppose now that  $n > k$ . If  $|q_n| < 1$  then  $|q_n|^{n-k} < 1$  and  $q_n$  cannot lie above the real axis (as the modulus of the right hand side would be greater than or equal to 1). Similarly, if  $|q_n| > 1$  then  $|q_n|^{n-k} > 1$  and  $q_n$  cannot lie beneath the real axis. As we take the principal branch of the square root in the definition of  $\bar{Y}_n(q)$ , we have  $\bar{Y}_n(q)^* = \bar{Y}_n(q^*)$  (where  $q^*$  denotes the complex conjugate of  $q$ ) so there are, in fact, no solutions off of the unit circle or real axis. One can easily verify that there are no non-unit real solutions, and when  $n < k$  the argument is analogous. □

3.3.5. *Step 5: The generating function is not D-finite.* Now we tie up all the arguments.

**Theorem 10.** *The generating functions  $A(t)$ ,  $B(t)$ ,  $C(t)$  of walks in the quarter plane with steps from  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  respectively, are all not D-finite.*

*Proof.* We now show that the infinite set of poles described in Proposition 9 are indeed poles of the generating function. The D-finiteness result follows from this and the fact that D-finite functions have only a finite number of singularities.

Fix a model. For each  $n$ , there is at least one choice of  $q_n$  amongst the poles of  $Y_n(1)$  which is not a pole of any  $Y_k(1)$  for  $|n - k| > 1$ . This is a direct consequence of Proposition 9.

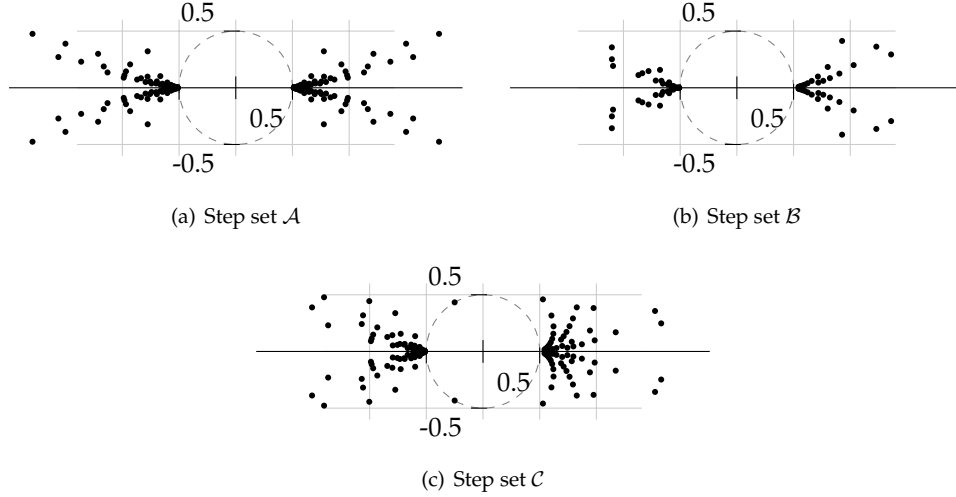


FIGURE 2. All singularities from  $Y_1(t), \dots, Y_{15}(t)$  for the symmetric models; these form the sub-dominant singularities of  $S_{1,1}(t)$ . The curve  $|t| = 1/2$  is sketched.

Next, we break the main sum of Equation (14) into three parts, and examine the behaviour at  $q_n$  – let us first consider the cases of models  $\mathcal{A}$  and  $\mathcal{C}$ . The sum is decomposed as follows:

$$\frac{q}{1+q^2} \cdot S_{0,1} = \underbrace{\sum_{k=0}^{n-2} (-1)^k Y_k Y_{k+1}}_{\text{a finite sum}} + \underbrace{(-1)^{n-1} Y_n (Y_{n-1} - Y_{n+1})}_{\text{pole contribution}} + \underbrace{\sum_{k \geq n+1} (-1)^k Y_k Y_{k+1}}_{\text{convergent at } q=q_n}.$$

The initial and terminal sums do not admit poles at  $q_n$  since Proposition 9 implies in these two cases  $Y_k$  does not have a pole at  $q_n$  for  $k \neq n$ , and an argument identical to the proof of Proposition 3 implies the second summation is convergent at this point. Furthermore, if we substitute  $q_n$  into the corresponding recurrence from Table 3, and recall it is a zero of  $\bar{Y}_n$ , we derive  $\bar{Y}_{n+1}(q_n) = -\bar{Y}_{n-1}(q_n) + \epsilon$  (where  $\epsilon = 0$  for model  $\mathcal{A}$  and  $\epsilon = 1$  for models  $\mathcal{B}$  and  $\mathcal{C}$ ), and so  $Y_{n-1} - Y_{n+1} \neq 0$ . We can then conclude that  $q_n$  is a pole of the series, for all  $n \geq 1$ .

Thus, we have shown that both  $A\left(\frac{q}{1+q^2}\right)$  and  $C\left(\frac{q}{1+q^2}\right)$  have an infinite number of poles, and are not D-finite. The stated result follows immediately from the fact that the class of D-finite functions is closed under algebraic substitution.

The remaining case of model  $\mathcal{B}$  is almost identical, save for the fact that  $Y_n$  and  $Y_{n-1}$  share some, but not all, of their poles. If  $q_n$  is not a pole of  $Y_{n+1}$ , then the argument above shows that it is a pole of  $B\left(\frac{q}{1+q^2}\right)$ . If  $q_n$  is a pole of  $Y_{n+1}$ , then the summand  $(-1)^n Y_n Y_{n+1}$  has a pole of a larger order than the other two summands in which that pole appears, and it cannot be cancelled by the rest of the summation. This means that  $q_n$  is again a pole of  $B\left(\frac{q}{1+q^2}\right)$ , and the remainder of the argument is as for the other two cases.  $\square$

**3.4. Return to the  $t$ -plane.** It is useful to visualize the singularities in the  $t$ -plane as well, as they control the sub-dominant asymptotics. Figure 2 contains precisely such a plot.

The sub-dominant singularities in the  $t$ -plane appear to converge to  $t = 1/2$ ; in fact for model  $\mathcal{C}$  there are two singularities of  $Y_2(t)$  which have modulus exactly  $1/2$ :  $t = -1/4 \pm \sqrt{3}i/4$ .

#### 4. ASYMMETRIC MODELS

The asymmetric models are not substantially different, but when we iterate we have more functions to track. Aside from some irritating bookkeeping, there is no main obstacle to following the strategy of the symmetric models.

**4.1. An explicit generating function expression.** To obtain the generating function expressions we follow the same path as in the symmetric case: we generate a sequence of equations, each which annihilates the kernel. This opens up the possibility of a telescoping sum expression from which we can find an expression for the generating function of walks returning to the axis in terms of iterates. An explicit expression for these iterates is obtained by solving some very simple recurrences. We complete these steps for the asymmetric models in this section.

As before, we begin with the main functional equation (3), make the substitution  $t = q/(1 + q^2)$ , and re-arrange to get the kernel equations:

$$\begin{aligned} (xy(1 + q^2) - qy^2 - qxy^2 - qx^2) D_{x,y}(t) &= xy(1 + q^2) - qx^2 D_{x,0}(t) - qy^2 D_{0,y}(t) \\ (xy(1 + q^2) - qy^2 - qxy^2 - qx^2 y^2 - qx^2) E_{x,y}(t) &= xy(1 + q^2) - qx^2 E_{x,0}(t) - qy^2 E_{0,y}(t), \end{aligned}$$

with kernels

$$\begin{aligned} \text{Model } \mathcal{D} : \quad K(x, y) &= -q(1 + x)y^2 + (1 + q^2)xy - qx^2 \\ \text{Model } \mathcal{E} : \quad K(x, y) &= -q(1 + x + x^2)y^2 + (1 + q^2)xy - qx^2. \end{aligned}$$

As there is no longer an  $x = y$  symmetry, we solve the kernels as functions of both  $x$  and  $y$ ; that is, we find  $Y(x)$  satisfying  $K(x, Y(x)) = 0$  and also  $X(y)$  satisfying  $K(X(y), y) = 0$ . We have some choice over how we split the solutions over different branches. One such choice of branches is:

Model  $\mathcal{D}$  :

$$\begin{aligned} X_{\pm}(y; q) &= \frac{y}{2q} \cdot \left( 1 - qy + q^2 \mp \sqrt{q^4 - 2q^3y + (y^2 - 2)q^2 - 2qy + 1} \right) \\ Y_{\pm}(x; q) &= \frac{x}{2q(1 + x)} \cdot \left( 1 + q^2 \mp \sqrt{q^4 - 4q^2x - 2q^2 + 1} \right), \end{aligned}$$

Model  $\mathcal{E}$  :

$$\begin{aligned} X_{\pm}(y; q) &= \frac{y}{2q(1 + y^2)} \cdot \left( 1 - qy + q^2 \mp \sqrt{q^4 - 2q^3y - (3y^2 + 2)q^2 - 2qy + 1} \right) \\ Y_{\pm}(x; q) &= \frac{x}{2q(1 + x + x^2)} \cdot \left( 1 + q^2 \mp \sqrt{q^4 - 2(2x^2 + 2x + 1)q^2 + 1} \right). \end{aligned}$$

Next, as we described in the introductory summary, we repeatedly alternate the substitution of the  $X$  and  $Y$  and create two related sequences of functions:

$$\chi_n(x) = X_+(Y_+(\chi_{n-1}(x); q); q), \quad \chi_0(x) = x \quad \text{and} \quad \Upsilon_n(y) = Y_+(X_+(\Upsilon_{n-1}(y); q); q), \quad \Upsilon_0(y) = y.$$

Simple substitutions yield the kernel relations

$$K(\chi_n(x), Y_+(\chi_n(x))) = K(X_+(\Upsilon(y)), \Upsilon(y)) = 0,$$

amongst others. As before, we generate an infinite list of relations by substituting  $x = \chi_n(x)$ ,  $y = Y_+(\chi_n(x))$ , and then a second infinite list using the substitutions  $x = X_+(\Upsilon(y))$ ,  $y = \Upsilon(y)$ . Again, we form a telescoping

sum, and after some manipulation this results in an expression for the generating functions of the walks returning to the axis. For  $S \in \{D, E\}$  we have:

$$(15) \quad S_{x,0} \left( \frac{q}{1+q^2} \right) = \frac{q}{1+q^2} \sum_{n \geq 0} \chi_n(x) \cdot \underbrace{(Y_+ \circ \chi_n(x) - Y_+ \circ \chi_{n-1}(x))}_{\Delta_{L,n}(x)}$$

$$(16) \quad S_{0,y} \left( \frac{q}{1+q^2} \right) = \frac{q}{1+q^2} \sum_{n \geq 0} X_+ \circ \Upsilon_n(y) \cdot \underbrace{(\Upsilon_n(y) - \Upsilon_{n+1}(y))}_{\Delta_{R,n}(y)}$$

The two models have identical structure in their generating function, and differ only in their respective functions  $X_+$  and  $Y_+$ . Our greatest challenge at this point is keeping track of the various parts:

$$\begin{aligned} \Delta_{L,n}(x) &= Y_+ \circ \chi_n(x) - Y_+ \circ \chi_{n-1}(x) & \Delta_{R,n}(y) &= \Upsilon_n(y) - \Upsilon_{n+1}(y) \\ \Delta_{L,0}(x) &= Y_+(x) & \Delta_{R,0}(y) &= \Upsilon_0(y). \end{aligned}$$

For each model we isolate the left and right hand sides, defining  $L(x, q) = qx^2 S_{x,0}(q/(1+q^2))$  and  $R(y, q) = qy^2 S_{0,y}(q/(1+q^2))$ , so that

$$S_{x,y}(q/(1+q^2)) = \frac{xy(1+q^2) - L(x, q) - R(y, q)}{K(x, y)},$$

and the counting generating function has the form

$$(17) \quad S(q/(1+q^2)) = \frac{(1+q^2) - L(1, q) - R(1, q)}{1 - K(1, 1)}.$$

For both asymmetric models we find an infinite set of points at which  $L(1, q)$  is singular, but  $R(1, q)$  is convergent.

Similar to previous cases, we can use the coefficients of  $K(x, y)$  and the facts that

$$Y_{\pm}(X_{\mp}(y)) = y \quad X_{\pm}(Y_{\mp}(x)) = x,$$

to form paired up recurrences for the multiplicative inverses of these functions. Here we again use the notation that  $\overline{F} = \frac{1}{F}$ :

$$(18) \quad \begin{aligned} \overline{\chi}_n &= (q+1/q)\overline{Y_+ \circ \chi_{n-1}} - \overline{\chi}_{n-1} - 1, & \overline{Y_+ \circ \chi}_n &= (q+1/q)\overline{\chi}_n - \overline{Y_+ \circ \chi}_{n-1} \\ \overline{\Upsilon}_n &= (q+1/q)\overline{X_+ \circ \Upsilon_{n-1}} - \overline{\Upsilon}_{n-1}, & \overline{X_+ \circ \Upsilon}_n &= (q+1/q)\overline{\Upsilon}_n - \overline{X_+ \circ \Upsilon}_{n-1} - 1. \end{aligned}$$

Solving these recurrences, we obtain the closed form expressions:

$$(19) \quad \begin{aligned} \overline{\chi}_n &= \frac{(q^{4n+3} - q^{4n+1} - q^3 + q)\overline{Y_+} - 2q^{4n+2} + q^{4n} + 2q^{2n+2} + q^4 - 2q^2}{q^{2n}(q^2 - 1)^2} \\ \overline{Y_+ \circ \chi}_n &= \frac{(q^{4n+4} - q^{4n+2} - q^2 + 1)\overline{Y_+} - 2q^{4n+3} + q^{4n+1} + q^{2n+3} + q^{2n+1} + q^3 - 2q}{q^{2n}(q^2 - 1)^2} \\ \overline{\Upsilon}_n &= \frac{(q^{4n+3} - q^{4n+1} - q^3 + q)\overline{X_+} - q^{4n+2} - q^{4n+1} + q^{4n} + q^{2n+3} + q^{2n+1} + q^4 - q^3 - q^2}{q^{2n}(q^2 - 1)^2} \\ \overline{X_+ \circ \Upsilon}_n &= \frac{(q^{4n+4} - q^{4n+2} - q^2 + 1)\overline{X_+} - q^{4n+3} - q^{4n+2} + q^{4n+1} + 2q^{2n+2} + q^3 - q^2 - q}{q^{2n}(q^2 - 1)^2}. \end{aligned}$$

We next show that our expressions for  $L(1, q)$  and  $R(1, q)$  in terms of  $\chi_n, Y_+ \circ \chi_n, \Upsilon_n$ , and  $X_+ \circ \Upsilon_n$  are valid for almost all of the complex plane.



**Proposition 11.** For either  $\mathcal{D}$  or  $\mathcal{E}$ , let  $q \in \mathbb{C}$  such that  $|q| \neq 1$  and  $\chi_n, \Delta_{L,n}, \Delta_{R,n}$ , and  $X_+ \circ \Upsilon_n$  are all analytic. Then the related series  $L(1, q)$  and  $R(1, q)$  both converge for  $q \in \mathbb{C}$ .

*Proof.* Using our explicit expressions above, it can easily be shown that for both models,

$$\lim_{n \rightarrow \infty} \left| \frac{\chi_n(x) \Delta_{L,n}(x)}{\chi_{n-1}(x) \Delta_{L,n-1}(x)} \right| = \lim_{n \rightarrow \infty} \left| \frac{X_+ \circ \Upsilon_n(x) \Delta_{R,n}^S(x)}{X_+ \circ \Upsilon_{n-1}(x) \Delta_{R,n-1}(x)} \right| = \begin{cases} 1/|q|^4 & : |q| > 1 \\ |q|^4 & : |q| < 1 \end{cases},$$

which proves the convergence where the functions  $\chi_n, \Delta_{L,n}, \Delta_{R,n}$ , and  $X_+ \circ \Upsilon_n$  are analytic.  $\square$

**4.2. Asymptotic enumeration of models  $\mathcal{D}$  and  $\mathcal{E}$ .** Both of the generating functions for the asymmetric models have a dominant singularity at  $t = 1/|S|$ , although proving this is more complicated than in the symmetric case. For model  $\mathcal{D}$ , the numerator of the generating function has a residue of zero (indeed a square-root singularity appears) and one must do a more careful analysis. For model  $\mathcal{E}$ , the numerator of the generating function has a non-zero residue as before, but we must consider two series in the proof. Because of this, we simply get a bound on the growth constant at the dominant singularity – we do not provide a mechanism for its calculation to arbitrary precision as in the symmetric cases. We make use of combinatorial arguments, so we return to the  $t$ -plane for the remainder of this section.

**4.2.1. Model  $\mathcal{D}$ .** This case was completely considered by Mishna and Rechnitzer, and we restate their results.

**Theorem 12** (Mishna and Rechnitzer [19]; Proposition 16). If  $D_n$  denotes the number of walks with steps from  $\mathcal{D}$  and staying in the positive quarter plane, then  $D_n \sim \kappa_D \frac{3^n}{\sqrt{n}} (1 + o(1))$ , where  $\kappa_D \in \left[0, \sqrt{\frac{3}{\pi}}\right]$ .

**4.2.2. Model  $\mathcal{E}$ .** In this case, we separately consider the two generating functions of walks returning to the axis, and bound their convergence at the point  $t = 1/4$ .

**Lemma 13.** The function  $E_{1,0}(t)$  is analytic for  $|t| \leq \frac{1}{2\sqrt{3}}$ , while the function  $E_{0,1}(t)$  is analytic for  $|t| \leq \frac{1}{1+2\sqrt{2}}$ .

*Proof.* We use the same approach as in Theorem 1, bounding the exponential growth factor by considering walks in the half plane that end at the  $x$  and  $y$  axis respectively. This proves the coefficient of  $t^n$  in  $E_{1,0}(t)$  has growth bounded above by  $O((2\sqrt{3})^n)$  and the coefficient of  $t^n$  in  $E_{0,1}(t)$  has growth bounded above by  $O((1+2\sqrt{2})^n)$ . The exponential growth in the asymptotic expression corresponds to the inverse of the dominant singularity, and the result follows.  $\square$

**Lemma 14.** The function  $E(t)$  has a simple singularity at  $t = 1/4$  where it has a residue of value  $\kappa_E \in \left[\frac{122}{525}, \frac{7}{10}\right]$ .

*Proof.* To compute the residue, it suffices to substitute the required value into the explicit expressions for  $\chi_n, Y_+ \circ \chi_n, \Upsilon_n$ , and  $X_+ \circ \Upsilon_n$  for all  $n$ . We treat the generating functions returning to the axes separately, with convergence established by the ratio test. Furthermore, we can tightly bound the series in the numerator using the values of some initial terms and two telescoping series, and compute that, as desired:

$$1 - \frac{1}{4}E_{1,0}(1/4) - \frac{1}{4}E_{0,1}(1/4) \in \left[\frac{122}{525}, \frac{7}{10}\right] \subset [0.232, 0.7].$$

$\square$

$$\begin{array}{l|l}
\chi_n & \omega_n^1 = (q^{4n+2} + q^{2n+4} - 4q^{2n+2} + q^{2n} + q^2)^2 \\
Y_+ \circ \chi_n & \omega_n^2 = (q^{4n+2} + q^{2n+4} - 4q^{2n+2} + q^{2n} + q^2) (q^{4n+4} + q^{2n+4} - 4q^{2n+2} + q^{2n} + 1) \\
\Upsilon_n & \omega_n^3 = (q^{4n+3} + q^{2n+4} - q^{2n+3} - 2q^{2n+2} - q^{2n+1} + q^{2n} + q) \\
& \quad \cdot (q^{4n+1} + q^{2n+4} - q^{2n+3} - 2q^{2n+2} - q^{2n+1} + q^{2n} + q^3) \\
X_+ \circ \Upsilon_n & \omega_n^3 = (q^{4n+3} + q^{2n+4} - q^{2n+3} - 2q^{2n+2} - q^{2n+1} + q^{2n} + q)^2
\end{array}$$

TABLE 5. The minimal polynomials of the singularities for functions defined by Eqn. (19) (Model  $\mathcal{D}$ ).

One may note that the location of the singularity is predicted, but not proven, by the results of [12]. Lemmas 13 and 14 combine to give us the leading term asymptotics of Model  $\mathcal{E}$ .

**Corollary 15.** *The number,  $E_n$ , of walks taking steps in  $\mathcal{E}$  and staying in the positive quarter plane grows asymptotically as*

$$E_n = \kappa_E \cdot 4^n + O\left((1 + 2\sqrt{2})^n\right),$$

where  $\kappa_E \in \left[\frac{122}{525}, \frac{7}{10}\right] \subset [0.232, 0.7]$ .

Computational evidence given by calculating the series for  $E_{1,0}(1/4)$  and  $E_{0,1}(1/4)$  to a large number of terms implies that the value of the growth constant is approximately 0.2636, which is consistent with the growth of computationally generated values of  $E_n$  for large  $n$ .

**4.3. The generating functions  $D(t)$  and  $E(t)$  are not D-finite.** The additional sums that arise in our expressions for  $D(t)$  and  $E(t)$  do not change our fundamental argument. In the symmetric examples we found a set of singularities associated to each  $Y_n$  and proved that they do not cancel. Here, although the same structure is undoubtedly present, we prove the existence and non cancellation of singularities along a single line. The set of singularities is infinite, and thus the generating functions are not D-finite. Indeed, this argument is simpler and we would have emulated it in the symmetric cases had we found a ray or line which contained an infinite number of singularities.

More specifically, for both asymmetric models, we demonstrate an infinite source of singularities in  $L(1, q)$  and prove that  $R_1(q)$  converges at those points. As in the previous cases, we find polynomials  $\omega_n^1, \omega_n^2, \omega_n^3$ , and  $\omega_n^4$  that the roots of  $\overline{\chi_n}, \overline{Y_+ \circ \chi_n}, \overline{\Upsilon_n}$ , and  $\overline{X_+ \circ \Upsilon_n}$  must satisfy – note that the orders of the roots of the polynomials match the orders of the roots of our functions. These polynomials are summarized in Tables 5 and 6. To be more precise,  $\omega_n^1(q)$  contains the poles of  $\chi_n(q)$ ,  $\omega_n^2(q)$  contains the poles of  $Y_+ \circ \chi_n(q)$ , etc.

We next prove that for even  $n$ , each  $\chi_n$  has a distinct singularity on the imaginary axis, and we prove that it is indeed a singularity of the generating function. We prove this separately for each model, but the arguments (indeed the computations!) are almost identical in both cases. In order to manipulate the unwieldy formulas which arise we used the Groebner package in Maple version 16 to calculate the relevant Gröbner bases.

$$\begin{array}{l|l}
\chi_n & \omega_n^1 = q^2 (q^4 - q^2 + 1) (q^{8n} + 1) + 2q^2 (q^4 - 4q^2 + 1) (q^{6n} + q^{2n}) + (q^8 - 10q^6 + 24q^4 - 10q^2 + 1)q^{4n} \\
Y_+ \circ \chi_n & \omega_n^2 = (q^4 - q^2 + 1) (q^{8n+4} + 1) + (q^6 - 3q^4 - 3q^2 + 1) (q^{6n+2} + q^{2n}) + (q^8 - 9q^6 + 22q^4 - 9q^2 + 1)q^{4n} \\
\Upsilon_n & \omega_n^3 = q^2 (q^4 - q^2 + 1) (q^{8n} + 1) + q(q^6 - q^5 - q^4 - 2q^3 - q^2 - q + 1) (q^{6n} + q^{2n}) \\
& \quad + (q^8 - 2q^7 - 4q^6 + 2q^5 + 12q^4 + 2q^3 - 4q^2 - 2q + 1)q^{4n} \\
X_+ \circ \Upsilon_n & \omega_n^4 = (q^4 - q^2 + 1) (q^{8n+4} + 1) + 2q(q^4 - q^3 - 2q^2 - q + 1) (q^{6n+2} + q^{2n}) \\
& \quad + (q^8 - 2q^7 - 5q^6 + 2q^5 + 14q^4 + 2q^3 - 5q^2 - 2q + 1)q^{4n}
\end{array}$$

TABLE 6. The minimal polynomials of the singularities for functions defined by Eqn. (19) (Model  $\mathcal{E}$ ).

Both cases also invoke  $\chi_{-n}(q)$  to prove that certain solutions of the polynomial are actually solutions of the model. These are defined by rolling the recurrence in reverse, as before.

**4.4. Model  $\mathcal{D}$ .** First, note that in the case of model  $\mathcal{D}$  the poles of  $\chi_n(q)$  are contained in the roots of  $\omega_n^1 = (q^{4n+2} + q^{2n+4} - 4q^{2n+2} + q^{2n} + q^2)^2$ , by Table 5.

**Lemma 16.** *The function  $\bar{\chi}_n(q)$  has a root on the imaginary axis between  $i$  and  $2i$ , when  $n$  is even.*

*Proof.* Suppose  $r \in \mathbb{R}$  and substitute  $q = ri$  into  $\omega_n^1(q)$ :

$$\omega_n^1(ri) = R^2 - r^2 + 4r^2R^2 - r^2R^4 + r^4R^2,$$

where  $R = r^n$ . We remark that this is a real valued function of  $r$ , and if  $r = 1$  then  $R = 1$  and  $\omega_n^1(i) = 4$ . However,  $\omega_n^1(2i) = 33R^2 - 4 - 4R^4$  which is negative for  $R \geq 2$ . Thus, the Intermediate Value Theorem implies  $\omega_n^1(ri)$  has a zero on the imaginary axis between  $i$  and  $2i$ . Denote this value by  $r_c$ . The expression for  $\omega_n^1(ri)$  is palindromic, so  $r = 1/r_c$  is also a root of  $\omega_n^1(ri)$ .

For  $r > 0$  it can easily be shown that  $\chi_n(i/r) = \chi_{-n}(ri)$ , and thus one of  $ir_c$  and  $i/r_c$  must be a root of  $\bar{\chi}_n$ . As the numerator of  $\bar{\chi}_n(ri)$  is

$$4R^4r^2 + (r^4 + 1)(1 + R^4) + (r^2 + 1)(1 - R^4)\sqrt{1 + 6r^2 + r^4} + 4r^2(1 - R^2),$$

which is strictly positive for  $0 < r < 1$ , the root of  $\omega_n^1$  between  $i$  and  $2i$  is in fact a root of  $\bar{\chi}_n$ .  $\square$

Furthermore, the poles of  $\chi_n$  and the poles of  $\chi_k$  are distinct when  $n \neq k$ . To see this, we note that we can re-write the expression above for  $\omega_n^1(ri)$  as  $-R^4r^2 + (1 + 4r^2 + r^4)R^2 - r^2$ . Treating  $r$  and  $R$  as independent variables, for a fixed positive value of  $r$  this expression is a polynomial in  $R$  whose coefficients have signs negative, positive, negative, respectively. Descartes's Rule of Signs then implies that there are at most 2 roots to this equation, one of which we know to be inside the unit circle and one of which lies outside of it. The result then follows from the observation that different values of  $n$  each yield a different value of  $(ri)^n = R^n$  for  $r > 1$ .

Next we show that these singularities are present in the term  $\chi_n \Delta_{L,n}(q)$ . To do this, we let  $o \geq 1$  denote the multiplicity of the root  $q = ri$  of  $q^{4n+2} + q^{2n+4} - 4q^{2n+2} + q^{2n} + q^2$  we have found above, so that  $\chi_n(q)$  has a pole of order  $2o$  at  $q = ri$ .

**Lemma 17.** For an infinite number of  $n$  there exists a distinct purely imaginary number  $ri$  with  $1 < r < 2$ , such that  $\chi_n (Y_+ \circ \chi_n - Y_+ \circ \chi_{n-1})$  has a pole (of order  $3o \geq 3$ ) at  $q = ri$ . Furthermore, for  $k \neq n$  the summand  $\chi_k (Y_+ \circ \chi_k - Y_+ \circ \chi_{k-1})$  of  $L(1, q)$  is analytic at  $ri$ .

*Proof.* Let  $ri$  be the root of  $\bar{\chi}_n$  described above. Then using the identity

$$Y_+ \circ \chi_n - Y_+ \circ \chi_{n-1} = \frac{\overline{Y_+ \circ \chi_{n-1}} - \overline{Y_+ \circ \chi_n}}{Y_+ \circ \chi_{n-1} \overline{Y_+ \circ \chi_n}},$$

we find that the zeroes of  $Y_+ \circ \chi_n - Y_+ \circ \chi_{n-1}$  are also roots (with the same multiplicity) of the polynomial equation

$$(q^{2n+1} + q^{n+2} - q^n + q) (q^{2n+1} + q^n + q - q^{n+2}) (q^{4n+2} + q^{2n+4} - 4q^{2n+2} + q^{2n} + q^2) = 0,$$

while the poles of  $Y_+ \circ \chi_n - Y_+ \circ \chi_{n-1}$  are also roots (with the same multiplicity) of the polynomial equation

$$(q^{4n+4} + q^{2n+4} - 4q^{2n+2} + q^{2n} + 1) (q^{4n} + q^{2n+4} - 4q^{2n+2} + q^{2n} + q^4) (q^{4n+2} + q^{2n+4} - 4q^{2n+2} + q^{2n} + q^2)^2 = 0.$$

Note that the last factor of each of these polynomials is the factor that appears in  $\omega_n^1$ ! In fact, by treating  $q^n$  as an independent variable  $Q$  and taking a Gröbner basis with respect to a lexicographical ordering one can show that for all values of  $q$  not satisfying  $(q^4 - 4q^2 + 1)(q^2 + 3q + 1)(q^2 - 3q + 1) = 0$  these are the only factors of the above polynomials that can share roots with  $\omega_n^1$ . As  $(q^4 - 4q^2 + 1)(q^2 + 3q + 1)(q^2 - 3q + 1)$  has no zeroes on the imaginary axis, we can see that  $Y_+ \circ \chi_n - Y_+ \circ \chi_{n-1}$  has a pole of order  $o$  at  $ri$ , which combines with the pole of  $\chi_n$ , which has order  $2o$ , to yield the result.

We have already noted above that  $\chi_k$  shares no poles with  $\chi_n$  when  $k \neq n$ , so it is sufficient to show  $Y_+ \circ \chi_k - Y_+ \circ \chi_{k-1}$  does not have a pole at  $ri$ . Indeed, since the poles of  $Y_+ \circ \chi_k - Y_+ \circ \chi_{k-1}$  are roots of  $(q^{4k+4} + q^{2k+4} - 4q^{2k+2} + q^{2k} + 1) (q^{4k} + q^{2k+4} - 4q^{2k+2} + q^{2k} + q^4) (q^{4k+2} + q^{2k+4} - 4q^{2k+2} + q^{2k} + q^2)^2$ , we can substitute  $q = ri$ , set  $P = r^k$ , and factor the result to see that the only possible way this polynomial can be zero for  $|r| > 1$  is if  $rP^2 - P - r - Pr^2 = 0$ . Taking a Gröbner basis (again with respect to a lexicographical ordering) of this polynomial with the one obtained by substituting  $R = r^n$  in  $\omega_n^1(ri)$ , we get  $(PR - 1)(PR + 1)(R - P)(R + P)$  as a generator, which cannot equal 0 when  $|r| > 1$ .  $\square$

**Theorem 18.** The generating function  $D(t)$  for walks in the quarter plane with steps from  $\mathcal{D}$  is not  $D$ -finite.

*Proof.* First, recall that  $D(q/(1 + q^2)) = \frac{(1+q^2)L(1,q) - R(1,q)}{q^2 - 3q + 1}$ . The proof is similar to the symmetric case, except we restrict our attention to poles located on the imaginary axis. For each  $n$ , Lemma 17 describes  $q_n$ , a purely imaginary pole of the  $n$ -th summand in  $L(1, q)$ . We show that it is also a pole of  $L(1, q)$ , and then verify that it is not a pole of  $R(1, q)$ . Consequently it is not cancelled in the full expression, and is also a pole of  $D(q/(1 + q^2))$ .

Lemma 17 proves that the purely imaginary poles of the term  $\chi_n (Y_+ \circ \chi_n - Y_+ \circ \chi_{n-1})$  are not poles of any other term of the main summation in our expression for  $D_{1,0} \left( \frac{q}{1+q^2} \right)$ . The arguments of the symmetric case can be used almost verbatim to show that the poles are also poles of  $L(1, q)$ .

Next, we show that none of the summands of  $R(1, q)$  have purely imaginary poles. We consider the family of polynomials that the poles satisfy, and show that none of them have purely imaginary roots via a Gröbner basis computation, where we replace the  $q^n$  terms by a single variable in order to make the computation generic.

Specifically, the computation is done as follows: substitute  $q = ri$  into  $\omega_n^3$ , set  $R = r^n$ , and take a Gröbner basis of the real and imaginary parts of the resulting polynomial with respect to a lexicographical ordering. The result has  $r^5(r-1)(r+1)(r^2+1)^6$  as an element, so  $\omega_n^3$  has no root of the form  $q = ri$  with  $r > 1$ . Similarly, the analogous Gröbner basis computation on the real and imaginary parts of  $\omega_n^4(ri)$  has  $r^5(r^2+1)^6$  as an element. Thus,  $\Upsilon_n^S$  and  $X_+ \circ \Upsilon_n^S$  have no poles of the form  $q = ri$  with  $r > 1$ , so  $R(1, q)$  contains no pole located at any of the singularities of  $L(1, q)$  described above.

By now we are almost on autopilot: any singularity of  $L(1, q)$  not cancelled by  $R(1, q)$  is a singularity of the complete expression. Since we have an infinite family of poles,  $D(1/(1+q^2))$  is not D-finite since it has an infinite family of singularities on the imaginary axis. As a consequence,  $D(t)$  is also not D-finite since it can be obtained from this function by algebraic substitution.  $\square$

4.4.1. *Model  $\mathcal{E}$ .* The argument to show that this model is not D-finite is identical, save for the actual location of the singularities on the imaginary axis. We highlight the differences.

**Lemma 19.** *The function  $\bar{\chi}_n^E(q)$  has a root on the imaginary axis between  $i$  and  $2i$ .*

*Proof.* As before, we substitute  $q = ri$  into the equation for  $\bar{\chi}_n(q)$  to get

$$\bar{\chi}_n(ri) = 4R^4r^2 + 4r^2 - 4R^2r^2 + R^4r^4 + R^4 + r^4 + 1 + (r^2 + 1 - R^4 - R^4r^2)\sqrt{1 + 10r^2 + r^4},$$

where  $R = r^n$ . If we substitute  $r = R = 1$  into this expression, we get the value 8. If we substitute  $r = 2$ , we get the expression  $33R^4 + 33 - 16R^2 + (5 - 5R^4)\sqrt{57}$  which is negative for  $R \geq 2$ . Thus, the Intermediate Value Theorem implies that  $\bar{\chi}_n(ri)$  has a root for  $r \in (1, 2)$ .  $\square$

As was the case with Model  $\mathcal{D}$ , Descartes's Rule of Signs allows us to conclude this root is unique as  $\bar{\chi}_n(ri) = -r^2(1+r^2+r^4)R^8 - 2r^2(1+4r^2+r^4)R^6 + (1+10r^2+24r^4+10r^6+r^8)R^4 - 2r^2(1+4r^2+r^4)R^2 - r^2(1+r^2+r^4)$  has two sign changes in its coefficients when viewed as a polynomial in  $R$ . Now, we prove that the other functions under consideration have no imaginary poles.

**Lemma 20.** *The functions  $Y_+ \circ \chi_n(q)$ ,  $\Upsilon_n(q)$ , and  $X_+ \circ \Upsilon_n(q)$  have no poles  $q$  on the imaginary axis.*

*Proof.* Substituting  $q = ri$  into our expression for  $\omega_n^2(q)$  gives

$$\begin{aligned} \omega_n^2(ri) &= r^8R^8 + r^4 + r^6R^8 + r^2 + r^4R^8 + 1 + R^2 + 3R^2r^2 + R^4 + 9R^4r^2 + 22R^4r^4 \\ &\quad + 3R^6r^4(r^2 - 1) + 3R^2r^4(3R^2r^2 - 1) + R^6r^2(r^6 - 1) + R^2r^6(R^2r^2 - 1), \end{aligned}$$

which is strictly positive for  $r > 1$ . To prove that  $\bar{\Upsilon}_n(q)$  and  $\bar{X}_+ \circ \bar{\Upsilon}_n(q)$  have no roots on the imaginary axis, we substitute  $q = ri$  into  $\omega_n^3(q)$  and  $\omega_n^4(q)$  obtaining expressions which have real and imaginary components which are non-zero polynomials in  $r$  and  $R$ . Taking a Gröbner basis of these real and imaginary components allows us to eliminate  $R$  in each case and prove the result.  $\square$

This has the immediate corollary that  $R(1, q)$  admits no polar singularities at these poles of  $\chi_n(q)$ , of which we have found an infinite number. Next, we show the poles do not cancel.

**Lemma 21.** *At each of the poles found above,  $Y_+ \circ \chi_n(q) - Y_+ \circ \chi_{n-1}(q) \neq 0$ .*

*Proof.* Substituting  $q = ri$  into our explicit expression for  $\overline{Y_+ \circ \chi_n}(q)$  allows us to determine that any root of  $Y_+ \circ \chi_n(q) - Y_+ \circ \chi_{n-1}(q)$  must satisfy the polynomial equation

$$(4r^6 + 9r^4 + 4r^2)R^8 - (r^8 + 1)R^4 + 4r^6 + 9r^4 + 4r^2 = 0.$$

The Gröbner Basis of this polynomial and the one we found for  $\overline{\chi_n}(ri)$  with respect to a lexicographical ordering has

$$r^2(9r^{16} + 236r^{14} + 2148r^{12} + 7684r^{10} + 11974r^8 + 7684r^6 + 2148r^4 + 236r^2 + 9)(1 + r^2)^4$$

as one of its generators. This has no positive real roots in  $r$ , so the result holds. □

This allows us to conclude our study of Model  $\mathcal{E}$  with the following theorem.

**Theorem 22.** *The generating function  $E(t)$  for walks in the quarter plane with steps from  $\mathcal{E}$  is not D-finite.*

## 5. CONCLUSION

This work addresses a family of lattice path models that have resisted other powerful approaches. There could also be other models, with larger step sizes or in higher dimensions, to which this method is suitable. In three dimensions the challenge is to set up the equations in such a way that unknowns are canceled at the same rate in which they are generated. Ideally, we would like to automate as much as is possible.

In this model, the connection between the infiniteness of the group and the infinite number of singularities is quite transparent. Is there any hope to transport this concept to the remaining small step models in the quarter plane, and show that their counting functions are also not D-finite (in addition to the provably non-D-finite multivariate generatingfunction)?

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