ALGEBRAICITY OF THE ZETA FUNCTION ASSOCIATED TO A MATRIX OVER A FREE GROUP ALGEBRA

CHRISTIAN KASSEL AND CHRISTOPHE REUTENAUER

Abstract. Following and generalizing a construction by Kontsevich, we associate a zeta function to any matrix with entries in a ring of noncommutative Laurent polynomials with integer coefficients. We show that such a zeta function is an algebraic function.

1. Introduction

Fix a commutative ring K. Let F be a free group on a finite number of generators $X_1, \ldots, X_n$ and

$$KF = K\langle X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}\rangle$$

be the corresponding group algebra: equivalently, it is the algebra of noncommutative Laurent polynomials with coefficients in K. Any element $a \in KF$ can be uniquely written as a finite sum of the form

$$a = \sum_{g\in F} (a, g)\, g\,,$$

where $(a, g) \in K$.

Let M be a $d \times d$-matrix with coefficients in KF. For any $n \geq 1$ we may consider the n-th power M^n of M and its trace $\mathrm{Tr}(M^n)$, which is an element of KF. We define the integer $a_n(M)$ as the coefficient of 1 in the trace of M^n:

$$a_n(M) = (\mathrm{Tr}(M^n), 1)\ . \tag{1.1}$$

Let g_M and P_M be the formal power series

$$g_M = \sum_{n\geq 1} a_n(M)\, t^n \quad \text{and} \quad P_M = \exp\left(\sum_{n\geq 1} a_n(M)\, \frac{t^n}{n}\right). \tag{1.2}$$

They are related by

$$g_M = t\, \frac{d\log(P_M)}{dt}\,.$$

We call P_M the *zeta function* of the matrix M by analogy with the zeta function of a noncommutative formal power series (see Section 2.1); the two concepts will be related in Proposition 4.1.

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The motivation for the definition of P_M comes from the well-known identity expressing the inverse of the reciprocal polynomial of the characteristic polynomial of a matrix M with entries in a commutative ring

$$\frac{1}{\det(1-tM)} = \exp\left(\sum_{n\geq 1} \mathrm{Tr}(M^n)\,\frac{t^n}{n}\right).$$

Note that for any scalar $\lambda \in K$, the corresponding series for the matrix λM become

$$g_{\lambda M}(t) = g_M(\lambda t) \quad \text{and} \quad P_{\lambda M}(t) = P_M(\lambda t)\,. \tag{1.3}$$

Our main result is the following; it was inspired by Theorem 1 of [15].

Theorem 1.1. *For each matrix $M \in M_d(KF)$ where $K = \mathbb{Q}$ is the ring of rational numbers, the formal power series P_M is algebraic.*

The special case $d = 1$ is due to Kontsevich [15]. A combinatorial proof in the case $d = 1$ and F is a free group on one generator appears in [17].

Observe that by the rescaling equalities (1.3) it suffices to prove the theorem when $K = \mathbb{Z}$ is the ring of integers.

It is crucial for the veracity of Theorem 1.1 that the variables do not commute: for instance, if $a = x + y + x^{-1} + y^{-1} \in \mathbb{Z}[x, x^{-1}, y, y^{-1}]$, where x and y are commuting variables, then $\exp(\sum_{n\geq 1}(a^n, 1)\,t^n/n)$ is a formal power series with integer coefficients, but not an algebraic function (this follows from Example 3 in [7, Sect. 1]).

The paper is organized as follows. In Section 2 we define the zeta function ζ_S of a noncommutative formal power series S and show that it can be expanded as an infinite product under a cyclicity condition that is satisfied by the characterististic series of cyclic languages.

In Section 3 we recall the notion of an algebraic noncommutative formal power series and some of their properties.

In Section 4 we reformulate the zeta function of a matrix as the zeta function of a noncommutative formal power series before giving the proof of Theorem 1.1; the latter follows the steps sketched in [15] and relies on the results of the previous sections as well as on an algebraicity result by André [2] elaborating on an idea of D. and G. Chudnovsky.

We concentrate on two specific matrices in Section 5. We give a closed formula for the zeta function of the first matrix; its nonzero coefficients count the planar rooted bicubic maps as well as Chapoton's "new intervals" in a Tamari lattice (see [9, 22]).

2. Cyclic formal power series

2.1. **General definitions.** As usual, if A is a set, we denote by A^* the free monoid on A: it consists of all words on the alphabet A, including the empty word 1. Let $A^+ = A - \{1\}$.

Recall that $w \in A^+$ is *primitive* if it cannot be written as u^r for any integer $r \geq 2$ and any $u \in A^+$. Two elements $w, w' \in A^+$ are *conjugate* if $w = uv$ and $w' = vu$ for some $u, v \in A^*$.

Given a set A and a commutative ring K, let $K\langle\langle A\rangle\rangle$ be the algebra of noncommutative formal power series on the alphabet A. For any element

$S \in K\langle\langle A\rangle\rangle$ and any $w \in A^*$, we define the coefficient $(S,w) \in K$ by

$$S = \sum_{w\in A^*} (S,w)\, w\,.$$

As an example of such noncommutative formal power series, take the characteristic series $\sum_{w\in L} w$ of a language $L \subseteq A^*$. In the sequel we shall identify a language with its characteristic series.

The *generating series* g_S of an element $S \in K\langle\langle A\rangle\rangle$ is the image of S under the algebra map $\varepsilon : K\langle\langle A\rangle\rangle \to K[[t]]$ sending each $a \in A$ to the variable t. We have

$$g_S - (S,1) = \sum_{w\in A^+} (S,w)\, t^{|w|} = \sum_{n\geq 1} \left(\sum_{|w|=n} (S,w) \right) t^n\,, \tag{2.1}$$

where $|w|$ is the length of w.

The *zeta function* ζ_S of $S \in K\langle\langle A\rangle\rangle$ is defined by

$$\zeta_S = \exp\left(\sum_{w\in A^+} (S,w)\, \frac{t^{|w|}}{|w|} \right) = \exp\left(\sum_{n\geq 1} \left(\sum_{|w|=n} (S,w) \right) \frac{t^n}{n} \right). \tag{2.2}$$

The formal power series g_S and ζ_S are related by

$$t\, \frac{d\log(\zeta_S)}{dt} = t\, \frac{\zeta_S'}{\zeta_S} = g_S - (S,1)\,, \tag{2.3}$$

where ζ_S' is the derivative of ζ_S with respect to the variable t.

2.2. **Cyclicity.**

Definition 2.1. *An element $S \in K\langle\langle A\rangle\rangle$ is cyclic if*

(i) $\forall u,v \in A^*,\ \ (S,uv) = (S,vu)$ *and*
(ii) $\forall w \in A^+, \forall r \geq 2,\ \ (S,w^r) = (S,w)^r\,.$

Cyclic languages provide examples of cyclic formal power series. Recall from [4, Sect. 2] that a language $L \subseteq A^*$ is *cyclic* if

(1) $\forall u,v \in A^*,\ \ uv \in L \Longleftrightarrow vu \in L\,,$
(2) $\forall w \in A^+, \forall r \geq 2,\ \ w^r \in L \Longleftrightarrow w \in L\,.$

The characteristic series of a cyclic language is a cyclic formal power series in the above sense.

Let L be any set of representatives of conjugacy classes of primitive elements of A^+.

Proposition 2.2. *If $S \in K\langle\langle A\rangle\rangle$ is a cyclic formal power series, then*

$$\zeta_S = \prod_{\ell\in L} \frac{1}{1-(S,\ell)\, t^{|\ell|}}\,.$$

Proof. Since both sides of the equation have the same constant term 1, it suffices to prove that they have the same logarithmic derivative. The logarithmic derivative of the RHS multiplied by t is equal to

$$\sum_{\ell\in L} \frac{|\ell|\,(S,\ell)\, t^{|\ell|}}{1-(S,\ell)\, t^{|\ell|}}\,,$$

which in turn is equal to

$$\sum_{\ell\in L,\, k\geq 1} |\ell|\,(S,\ell)^k\, t^{k|\ell|}\,.$$

In view of (2.1) and (2.3) it is enough to check that for all $n \geq 1$,

$$\sum_{|w|=n} (S,w) = \sum_{\ell\in L,\, k\geq 1,\, k|\ell|=n} |\ell|\,(S,\ell)^k\,. \tag{2.4}$$

Now any word $w = u^k$ is the k-th power of a unique primitive word u, which is the conjugate of a unique element $\ell \in L$. Moreover, w has exactly $|\ell|$ conjugates and since S is cyclic, we have

$$(S,w) = (S,u^k) = (S,u)^k = (S,\ell)^k.$$

From this Equation (2.4) follows immediately. □

Corollary 2.3. *If a cyclic formal power series S has integer coefficients, i.e., $(S,w) \in \mathbb{Z}$ for all $w \in A^*$, then so has ζ_S.*

3. Algebraic noncommutative series

This section is essentially a compilation of well-known results on algebraic noncommutative series.

Recall that a *system of proper algebraic noncommutative equations* is a finite set of equations

$$\xi_i = p_i \quad i = 1,\dots,n\,,$$

where $\xi_1,\dots,\xi_n$ are noncommutative variables and $p_1,\dots,p_n$ are elements of $K\langle \xi_1,\dots,\xi_n, A\rangle$, where A is some alphabet. We assume that each p_i has no constant term and contains no monomial ξ_j. One can show that such a system has a unique solution $(S_1,\dots,S_n)$, i.e., there exists a unique n-tuple $(S_1,\dots,S_n) \in K\langle\langle A\rangle\rangle^n$ such that $S_i = p_i(S_1,\dots,S_n,A)$ for all $i = 1,\dots,n$ and each S_i has no constant term (see [20], [18, Th. IV.1.1], or [21, Prop. 6.6.3]).

If a formal power series $S \in K\langle\langle A\rangle\rangle$ differs by a constant from such a formal power series S_i, we say that S is *algebraic*.

Example 3.1. Consider the proper algebraic noncommutative equation

$$\xi = a\xi^2 + b\,.$$

(Here $A = \{a,b\}$.) Its solution is of the form

$$S = b + abb + aabbb + ababb + \cdots\,.$$

One can show (see [3]) that S is the characteristic series of Łukasiewicz's language, namely of the set of words $w \in \{a,b\}^*$ such that $|w|_b = |w|_a + 1$ and $|u|_a \geq |u|_b$ for all proper prefixes u of w.

Recall also that $S \in K\langle\langle A\rangle\rangle$ is *rational* if it belongs to the smallest subalgebra of $K\langle\langle A\rangle\rangle$ containing $K\langle A\rangle$ and closed under inversion. By a theorem of Schützenberger (see [5, Th. I.7.1]), a formal power series $S \in K\langle\langle A\rangle\rangle$ is rational if and only if it is *recognizable*, i.e., there exist an integer $n \geq 1$, a representation μ of the free monoid A^* by matrices with entries in K, a row-matrix α and a column-matrix β such that for all $w \in A^*$,

$$(S,w) = \alpha\mu(w)\beta\,.$$

We now record two well-known theorems.

Theorem 3.2. *(1) If $S \in K\langle\langle A\rangle\rangle$ is algebraic, then its generating series $g_S \in K[[t]]$ is algebraic in the usual sense.*

(2) The set of algebraic power series is a subring of $K\langle\langle A\rangle\rangle$.

(3) A rational power series is algebraic.

(4) The Hadamard product of a rational power series and an algebraic power series is algebraic.

(5) Let $A = \{a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}\}$ and L be the language consisting of all words on the alphabet A whose image in the free group on $a_1, \dots, a_n$ is the identity element. Then the characteristic series of L is algebraic.

Items (1)–(4) of the previous theorem are due to Schützenberger [20], Item (5) to Chomsky and Schützenberger [10] (see [21, Example 6.6.8]).

The second theorem is a criterion due to Jacob [13].

Theorem 3.3. *A formal power series $S \in K\langle\langle A\rangle\rangle$ is algebraic if and only if there exist a free group F, a representation μ of the free monoid A^* by matrices with entries in KF, indices i, j and $\gamma \in F$ such that for all $w \in A^*$,*

$$(S, w) = \big((\mu w)_{i,j}, \gamma\big).$$

The following is an immediate consequence of Theorem 3.3.

Corollary 3.4. *If $S \in K\langle\langle A\rangle\rangle$ is an algebraic power series and $\varphi : B^* \to A^*$ is a homomorphism of finitely generated free monoids, then the power series*

$$\sum_{w\in B^*} (S, \varphi(w))\, w \in K\langle\langle B\rangle\rangle$$

is algebraic.

As a consequence of Theorem 3.2 (5) and of Corollary 3.4, we obtain the following.

Corollary 3.5. *Let $f : A^* \to F$ be a homomorphism from A^* to a free group F. Then the characteristic series of $f^{-1}(1) \in K\langle\langle A\rangle\rangle$ is algebraic.*

4. Proof of Theorem 1.1

Let M be a $d \times d$-matrix. As observed in the introduction, it is enough to establish Theorem 1.1 when all the entries of M belong to $\mathbb{Z}F$.

We first reformulate the formal power series g_M and P_M of (1.2) as the generating series and the zeta function of a noncommutative formal power series, respectively.

Let A be the alphabet whose elements are triples $[g, i, j]$, where i, j are integers such that $1 \le i, j \le d$ and $g \in F$ appears in the (i, j)-entry $M_{i,j}$ of M, i.e., $(M_{i,j}, g) \neq 0$. We define the noncommutative formal power series $S_M \in K\langle\langle A\rangle\rangle$ as follows: for $w = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n] \in A^+$, the scalar (S_M, w) vanishes unless we have

(a) $j_n = i_1$ and $j_k = i_{k+1}$ for all $k = 1, \dots, n-1$,

(b) $g_1 \cdots g_n = 1$ in the group F,

in which case (S_M, w) is given by

$$(S_M, w) = (M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n) \in K\,.$$

By convention, $(S_M, 1) = d$.

Proposition 4.1. *The generating series and the zeta function of S_M are related to the formal power series g_M and P_M of (1.2) by*

$$g_{S_M} - d = g_M \quad \text{and} \quad \zeta_{S_M} = P_M\,.$$

Proof. For $n \geq 1$ we have

$$\begin{aligned}\mathrm{Tr}(M^n) &= \sum M_{i_1,j_1} \cdots M_{i_n,j_n} \\ &= \sum (M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n)\, g_1 \cdots g_n\,,\end{aligned}$$

where the sum runs over all indices $i_1, j_1, \ldots, i_n, j_n$ satisfying Condition (a) above and over all $g_1, \ldots, g_n \in F$. Then

$$a_n(M) = (\mathrm{Tr}(M^n), 1) = \sum (M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n)\,,$$

where Conditions (a) and (b) are satisfied. Hence,

$$a_n(M) = \sum_{w \in A^*,\, |w|=n} (S, w)\,,$$

which proves the proposition in view of (1.2), (2.1) and (2.2). □

We next establish that S_M is both cyclic in the sense of Section 2 and algebraic in the sense of Section 3.

Proposition 4.2. *The noncommutative formal power series S_M is cyclic.*

Proof. (i) Conditions (a) and (b) above are clearly preserved under cyclic permutations. Hence we also have

$$(S_M, w) = (M_{i_2,j_2}, g_2) \cdots (M_{i_n,j_n}, g_n)\, (M_{i_1,j_1}, g_1)$$

when $w = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n]$ such that Conditions (a) and (b) are satisfied. It follows that $(S, uv) = S(vu)$ for all $u, v \in A^*$.

(ii) If w satisfies Conditions (a) and (b), so does w^r for $r \geq 2$. Conversely, if w^r $(r \geq 2)$ satisfies Condition (a), then since

$$w^r = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n]\, [g_1, i_1, j_1] \cdots$$

we must have $j_n = i_1$ and $j_k = i_{k+1}$ for all $k = 1, \ldots, n-1$, and so w satisfies Condition (a).

If w^r $(r \geq 2)$ satisfies Condition (b), i.e., $(g_1 \cdots g_n)^r = 1$, then $g_1 \cdots g_n = 1$ since F is torsion-free. Hence w satisfies Condition (b). It follows that $(S, w^r) = ((M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n))^r = (S, w)^r$. □

Proposition 4.3. *The noncommutative formal power series S_M is algebraic.*

Proof. We write S_M as the Hadamard product of three noncommutative formal power series S_1, S_2, S_3.

The series $S_1 \in K\langle\langle A \rangle\rangle$ is defined for $w = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n] \in A^+$ by

$$(S_1, w) = (M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n)$$

and by $(S_1, 1) = 1$. This is a recognizable, hence rational, series with one-dimensional representation $A^* \to K$ given by $[g, i, j] \mapsto (M_{i,j}, g)$.

Next consider the representation μ of the free monoid A^* defined by

$$\mu([g, i, j]) = E_{i,j}\,,$$

where $E_{i,j}$ denotes as usual the $d \times d$-matrix with all entries vanishing, except the (i,j)-entry which is equal to 1. Set

$$S_2 = \sum_{w \in A^*} \mathrm{Tr}\,((\mu w))\; w \in K\langle\langle A \rangle\rangle\,.$$

The power series S_2 is recognizable, hence rational. Let us describe S_2 more explicitly. For $w = 1$, $\mu(w)$ is the identity $d \times d$-matrix; hence $(S_2, 1) = d$. For $w = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n] \in A^+$, we have

$$\mathrm{Tr}\,((\mu w)) = \mathrm{Tr}(E_{i_1,j_1} \cdots E_{i_n,j_n})\,.$$

It follows that $\mathrm{Tr}\,((\mu w)) \neq 0$ if and only if $\mathrm{Tr}(E_{i_1,j_1} \cdots E_{i_n,j_n}) \neq 0$, which is equivalent to $j_n = i_1$ and $j_k = i_{k+1}$ for all $k = 1, \dots, n-1$, in which case $\mathrm{Tr}\,((\mu w)) = 1$. Thus,

$$S_2 = d + \sum_{n \geq 1} \sum [g_1, i_1, i_2]\,[g_2, i_2, i_3] \cdots [g_n, i_n, i_1]\,,$$

where the second sum runs over all elements $g_1, \dots, g_n \in F$ and all indices $i_1, \dots, i_n$.

Finally consider the homomorphism $f : A^* \to F$ sending $[g, i, j]$ to g. Then by Corollary 3.5 the characteristic series $S_3 \in K\langle\langle A \rangle\rangle$ of $f^{-1}(1)$ is algebraic.

It is now clear that S_M is the Hadamard product of S_1, S_2, and S_3:

$$S_M = S_1 \odot S_2 \odot S_3\,.$$

Since by [5, Th. I.5.5] the Hadamard product of two rational series is rational, $S_1 \odot S_2$ is rational as well. It then follows from Theorem 3.2 (4) and the algebraicity of S_3 that $S_M = S_1 \odot S_2 \odot S_3$ is algebraic. □

Since M has entries in $\mathbb{Z}F$, the power series $g_{S_M} = g_M + d$ belongs to $\mathbb{Z}[[t]]$. It follows by Corollary 2.3 and Proposition 4.2 that the power series $P_M = \zeta_{S_M}$ has integer coefficients as well. Moreover, by Theorem 3.2 (1) and Proposition 4.3,

$$t\,\frac{d\log(P_M)}{dt} = g_M$$

is algebraic.

To complete the proof of Theorem 1.1, it suffices to apply the following algebraicity theorem.

Theorem 4.4. *If $f \in \mathbb{Z}[[t]]$ is a formal power series with integer coefficients such that $t\,d\log f/dt$ is algebraic, then f is algebraic.*

Note that the integrality condition for f is essential: for the transcendental formal power series $f = \exp(t)$, we have $t\,d\log f/dt = t$, which is even rational.

Proof. This result follows from cases of the Grothendieck–Katz conjecture proved in [2] and in [6]. The conjecture states that if $Y' = AY$ is a linear system of differential equations with $A \in M_d(\mathbb{Q}(t))$, then far from the poles of A it has a basis of solutions that are algebraic over $\mathbb{Q}(t)$ if and only if for almost all prime numbers p the reduction $\bmod\, p$ of the system has a basis of solutions that are algebraic over $\mathbb{F}_p(t)$.

Let us now sketch a proof of the theorem (see also Exercise 5 of [1, p. 160]). Set $g = tf'/f$ and consider the system $y' = (g/t)\, y$; it defines a differential form ω on an open set S of the smooth projective complete curve $\overline{S}$ associated to g. We now follow [2, § 6.3], which is inspired from [11]. First, extend ω to a section (still denoted ω) of $\Omega^1_{\overline{S}}(-D)$, where D is the divisor of poles of ω. For any $n \geq 2$, we have a differential form $\sum_{i=1}^n p_i^*(\omega)$ on S^n, where $p_i : S^n \to S$ is the i-th canonical projection; this form goes down to the symmetric power $S^{(n)}$. Now let J be the generalized Jacobian of S which parametrizes the invertible fibre bundles over $\overline{S}$ that are rigidified over D. There is a morphism $\varphi : S \to J$ and a unique invariant differential form ω_J on J such that $\omega = \varphi^*(\omega_J)$. For any $n \geq 2$, φ induces a morphism $\varphi^{(n)} : S^{(n)} \to J$ such that $(\varphi^{(n)})^*(\omega_J) = \sum_{i=1}^n p_i^*(\omega)$. For n large enough, $\varphi^{(n)}$ is dominant and if ω_J is exact, then so is ω. To prove that ω_J is exact, we note that J, being a scheme of commutative groups, is uniformized by $\mathbb{C}^n$. We can now apply Theorem 5.4.3 of *loc. cit.*, whose hypotheses are satisfied because the solution f of the system has integer coefficients.

Alternatively, one can use a special case of a generalized conjecture of Grothendieck-Katz proved by Bost, namely Corollary 2.8 in [6, Sect. 2.4]: the vanishing of the p-curvatures in Condition (i) follows by a theorem of Cartier from the fact that the system has a solution in $\mathbb{F}_p(t)$, namely the reduction mod p of f for all prime numbers p for which such a reduction of the system exists (see Exercise 3 of [1, p. 84] or Theorem 5.1 of [14]); Condition (ii) is satisfied since $\mathbb{C}^n$ satisfies the Liouville property. □

A nice overview of such algebraicity results is given in Chambert-Loir's Bourbaki report [8]; see especially Theorem 2.6 and the following lines.

5. Examples

Kontsevich [15] computed P_ω when $\omega = X_1 + X_1^{-1} + \cdots + X_n + X_n^{-1}$ considered as a 1×1-matrix, obtaining

$$P_\omega = \frac{2^n}{(2n-1)^{n-1}} \cdot \frac{\big(n - 1 + n\big(1 - 4(2n-1)t^2\big)^{1/2}\big)^{n-1}}{\big(1 + \big(1 - 4(2n-1)t^2\big)^{1/2}\big)^n}, \tag{5.1}$$

which shows that P_ω belongs to a quadratic extension of $\mathbb{Q}(t)$.

We now present similar results for the zeta functions of two matrices, the first one of order 2, the second one of order $d \geq 3$.

5.1. **Computing P_M for a 2×2-matrix.** Consider the following matrix with entries in the ring $\mathbb{Z}\langle a, a^{-1}, b, b^{-1}, d, d^{-1}\rangle$, where a, b, d are noncommuting variables:

$$M = \begin{pmatrix} a + a^{-1} & b \\ b^{-1} & d + d^{-1} \end{pmatrix}. \tag{5.2}$$

Proposition 5.1. *We have*

$$g_M = 3\,\frac{(1 - 8t^2)^{1/2} - 1 + 6t^2}{1 - 9t^2} \tag{5.3}$$

and

$$P_M = \frac{(1-8t^2)^{3/2} - 1 + 12t^2 - 24t^4}{32t^6}\,. \tag{5.4}$$

Expanding P_M as a formal power series, we obtain

$$P_M = 1 + \sum_{n\geq 1} \frac{3\cdot 2^n}{(n+2)(n+3)} \binom{2n+2}{n+1} t^{2n}\,.$$

Proof. View the matrix M under the form of the graph of Figure 1, with two vertices $1, 2$ and six labeled oriented edges. We identify paths in this graph and words on the alphabet $A = \{a, a^{-1}, b, b^{-1}, d, d^{-1}\}$. Let B denote the set of nonempty words on A which become trivial in the corresponding free group on a, b, d and whose corresponding path is a closed path. Then the integer $a_n(M)$ is the number of words in B of length n. We have $\varepsilon(B) = g_M$, where $\varepsilon : K\langle\langle A\rangle\rangle \to K[[t]]$ is the algebra map defined in Section 2.1.

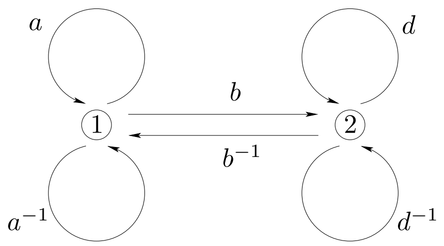

Figure 1. *A graph representing M*

We define B_i $(i = 1, 2)$ as the set of paths in B starting from and ending at the vertex i; we have $B = B_1 + B_2$. Each set B_i is a free subsemigroup of A^*, freely generated by the set C_i of closed paths not passing through i (except at their ends). The sets C_i do not contain the empty word. We have

$$B_i = C_i^+ = \sum_{n\geq 1} C_i^n \qquad (i = 1, 2)$$

Given a letter x, we denote by $C_i(x)$ the set of closed paths in C_i starting with x. Any word of $C_i(x)$ is of the form xwx^{-1}, where $w \in B_j$ when $i \xrightarrow{x} j$; such w does not start with x^{-1}. Identifying a language with its characteristic series and using the standard notation $L^* = 1 + \sum_{n\geq 1} L^n$ for any language L, we obtain the following two equations:

$$C_1(a) = a(C_1(a) + C_1(b))^* a^{-1} \tag{5.5}$$

and

$$C_1(b) = b(C_2(d) + C_2(d^{-1}))^* b^{-1}\,. \tag{5.6}$$

Applying the algebra map ε and taking into account the symmetries of the graph, we see that the four noncommutative formal power series $C_1(a)$, $C_1(a^{-1})$, $C_2(d)$, $C_2(d^{-1})$ are sent to the same formal power series $u \in \mathbb{Z}[[t]]$, while $C_1(b)$, $C_2(b^{-1})$ are sent to the same formal power series v. It follows from (5.5) and (5.6) that u and v satisfy the equations

$$u = t^2(u+v)^* = \frac{t^2}{1-u-v} \quad \text{and} \quad v = t^2(2u)^* = \frac{t^2}{1-2u}\,, \tag{5.7}$$

from which we deduce

$$t^2 = u(1-u-v) = v(1-2u)\,.$$

The second equality is equivalent to $(u-v)(u-1) = 0$. Since $C_1(a)$ does not contain the empty word, the constant term of u vanishes, hence $u - 1 \neq 0$. Therefore, $u = v$.

Since $C_1 = C_1(a) + C_1(a^{-1}) + C_1(b)$ and $C_2 = C_2(d) + C_2(d^{-1}) + C_2(b^{-1})$, we have $\varepsilon(C_1) = \varepsilon(C_2) = 2u + v = 3u$. Therefore, $\varepsilon(B_1) = \varepsilon(B_2) = 3u/(1-3u)$ and

$$g_M = \varepsilon(B) = \frac{6u}{1-3u}\,. \tag{5.8}$$

Let us now compute u using (5.7) and the equality $u = v$. The formal power series u satisfies the quadratic equation $2u^2 - u + t^2 = 0$. Since u has zero constant term, we obtain

$$u = \frac{1-(1-8t^2)^{1/2}}{4}\,.$$

From this and (5.8), we obtain the desired form for g_M.

Let $P(t)$ be the right-hand side in Equation (5.4). To prove $P_M = P(t)$, we checked that $tP'(t)/P(t) = g_M$ and the constant term of $P(t)$ is 1. □

Remark 5.2. We found Formula (5.4) for $P(t)$ as follows. We first computed the lowest coefficients of g_M up to degree 10:

$$g_M = 6\,(t^2 + 5t^4 + 29t^6 + 181t^8 + 1181t^{10}) + O(t^{12})\,.$$

From this it was not difficult to find that

$$P_M = 1 + 3\,t^2 + 12\,t^4 + 56\,t^6 + 288\,t^8 + 1584\,t^{10} + O(t^{12})\,. \tag{5.9}$$

Up to a shift, the sequence (5.9) of nonzero coefficients of P_M is the same as the sequence of numbers of “new” intervals in a Tamari lattice computed by Chapoton in [9, Sect. 9]. (We learnt this from [16] where this sequence is listed as A000257.) Chapoton gave an explicit formula for the generating function ν of these “new” intervals (see Eq. (73) in *loc. cit.*). Rescaling ν, we found that $P(t) = (\nu(t^2) - t^4)/t^6$ has up to degree 10 the same expansion as (5.9). It then sufficed to check that $tP'(t)/P(t) = g_M$.

By [16] the integers in the sequence A000257 also count the number of planar rooted bicubic maps with $2n$ vertices (see [22, p. 269]). Planar maps also come up in the combinatorial interpretation of (5.1) given in [17, Sect. 5] for $n = 2$.

Note that the sequence of nonzero coefficients of $g_M/6$ is listed as A194723 in [16].

5.2. **A similar $d \times d$-matrix.** Fix an integer $d \geq 3$ and let M be the $d \times d$-matrix with entries $M_{i,j}$ defined by

$$M_{i,i} = a_i + a_i^{-1} \qquad \text{and} \qquad M_{i,j} = \begin{cases} b_{ij} & \text{if } i < j\,, \\ b_{ji}^{-1} & \text{if } j < i\,, \end{cases}$$

where $a_1, \ldots, a_d$, b_{ij} $(1 \leq i < j \leq d)$ are noncommuting variables. This matrix is a straightforward generalization of (5.2).

Proceeding as above, we obtain two formal power series u and v satisfying the following equations similar to (5.7):

$$u = t^2(u + (d-1)v)^* = \frac{t^2}{1 - u - (d-1)v}$$

and

$$v = t^2(2u + (d-2)v)^* = \frac{t^2}{1 - 2u - (d-2)v},$$

We deduce the equality $u = v$ and the quadratic equation $u(1 - du) = t^2$. We finally have

$$g_M = \frac{d(d+1)u}{1 - (d+1)u},$$

which leads to

$$g_M = \frac{d(d+1)}{2} \frac{(1 - 4dt^2)^{1/2} - 1 + 2(d+1)t^2}{1 - (d+1)^2t^2}.$$

Its expansion as a formal power series is the following:

$$\begin{aligned} g_M &= d(d+1)\left\{t^2 + (2d+1)\,t^4 + (5d^2 + 4d + 1)\,t^6 \right.\\ &\quad + (14d^3 + 14d^2 + 6d + 1)\,t^8 \\ &\quad \left. + (42d^4 + 48d^3 + 27d^2 + 8d + 1)\,t^{10}\right\} + O(t^{12}). \end{aligned}$$

When $d = 2, 3, 4$, the sequence of nonzero coefficients of $g_M/d(d+1)$ is listed respectively as A194723, A194724, A194725 in [16] (it is also the d-th column in Sequence A183134). These sequences count the d-ary words either empty or beginning with the first letter of the alphabet, that can be built by inserting n doublets into the initially empty word.

We were not able to find a closed formula for P_M analogous to (5.4). Using Maple, we found that, for instance up to degree 10, the expansion of P_M is

$$\begin{aligned} &1 + \frac{d(d+1)}{2}\,t^2 + \frac{d(d+1)(d^2+5d+2)}{8}\,t^4 \\ &+ \frac{d(d+1)(d^4 + 14d^3 + 59d^2 + 38d + 8)}{48}\,t^6 \\ &+ \frac{d(d+1)(d^6 + 27d^5 + 271d^4 + 1105d^3 + 904d^2 + 332d + 48)}{384}\,t^8. \end{aligned}$$

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Christian Kassel: Institut de Recherche Mathématique Avancée, CNRS & Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg, France
E-mail address: kassel@math.unistra.fr
URL: www-irma.u-strasbg.fr/~kassel/

Christophe Reutenauer: Mathématiques, Université du Québec à Montréal, Montréal, CP 8888, succ. Centre Ville, Canada H3C 3P8
E-mail address: reutenauer.christophe@uqam.ca
URL: www.lacim.uqam.ca/~christo/