STRUCTURE AND ENUMERATION OF $(3+1)$-FREE POSETS

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Abstract. A poset is $(3+1)$-free if it does not contain the disjoint union of chains of length 3 and 1 as an induced subposet. These posets play a central role in the $(3+1)$-free conjecture of Stanley and Stembridge. Lewis and Zhang have enumerated *graded* $(3+1)$-free posets by decomposing them into bipartite graphs, but until now the general enumeration problem has remained open. We give a finer decomposition into bipartite graphs which applies to all $(3+1)$-free posets and obtain generating functions which count $(3+1)$-free posets with labelled or unlabelled vertices. Using this decomposition, we obtain a decomposition of the automorphism group and asymptotics for the number of $(3+1)$-free posets.

1. Introduction

A poset P is $(i+j)$*-free* if it contains no induced subposet that is isomorphic to the poset consisting of two disjoint chains of lengths i and j. In particular, P is $(3+1)$-free if there are no vertices $a, b, c, d \in P$ such that $a < b < c$ and d is incomparable to a, b, and c.

Posets that are (3+1)-free play a role in the study of Stanley's chromatic symmetric function [14, 15], a symmetric function associated with a poset that generalizes the chromatic polynomial of a graph. Namely, a well-known conjecture of Stanley and Stembridge [18] is that the chromatic symmetric function of a $(3+1)$-free poset has positive coefficients in the basis of elementary symmetric functions. As evidence toward this conjecture, Stanley [14] verified the conjecture for the class of 3-free posets, and Gasharov [8] has shown the weaker result that the chromatic symmetric function of a $(3+1)$-free poset is Schur-positive.

To make more progress toward the Stanley–Stembridge conjecture, a better understanding of $(3+1)$-free posets is needed. Reed and Skandera [11, 12] have given structural results and a characterization of $(3+1)$-free posets in terms of their antiadjacency matrix. In addition, certain families of $(3+1)$-free posets have been enumerated. For example, the number of $(3+1)$-and-$(2+2)$-free posets with n vertices is the nth Catalan number [17, Ex. 6.19(ddd)]; Atkinson, Sagan and Vatter [1] have enumerated the permutations that avoid the patterns 2341 and 4123, which give rise to the $(3+1)$-free posets of dimension two; and Lewis and Zhang [9] have made significant progress by enumerating *graded* $(3+1)$-free posets in terms of bicoloured graphs[1] using a new structural decomposition. However, until now the general enumeration problem for $(3+1)$-free posets remained open [16, Ex. 3.16(b)].

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[1]Throughout this paper, a *bicoloured* graph is a bipartite graphs with a specified ordered bipartition. For example, there are 2 bicoloured graphs with 1 vertex, 6 bicoloured graphs with 2 labelled vertices, and 4 bicoloured graphs with 2 unlabelled vertices.

In this paper, we give generating functions for $(3+1)$-free posets with unlabelled and labelled vertices in terms of the generating functions for bicoloured graphs with unlabelled and labelled vertices, respectively. As in the graded case, the two problems are equally hard, although the enumeration problem for bicoloured graphs has received more attention.

In the unlabelled case, let $p_{\mathrm{unl}}(n)$ be the number of $(3+1)$-free posets with n unlabelled vertices, and let $S(c,t)$ be the unique formal power series solution (in c and t) of the cubic equation

$$S(c,t) = 1 + \frac{c}{1+c} S(c,t)^2 + t S(c,t)^3. \tag{1.1}$$

We show that the ordinary generating function for unlabelled $(3+1)$-free posets is

$$\sum_{n \geq 0} p_{\mathrm{unl}}(n) x^n = S\big(x/(1-x), 1 - 2x - B_{\mathrm{unl}}(x)^{-1}\big), \tag{1.2}$$

where $B_{\mathrm{unl}}(x) = 1 + 2x + 4x^2 + 8x^3 + 17x^4 + \cdots$ is the ordinary generating function for unlabelled bicoloured graphs. Before our investigation, the On-Line Encyclopedia of Integer Sequences [13] had 22 terms in the entry [13, A049312] for the coefficients of $B_{\mathrm{unl}}(x)$, but only 7 terms in the entry [13, A079146] for the numbers $p_{\mathrm{unl}}(n)$. Using (1.2), we have closed this gap; the numbers $p_{\mathrm{unl}}(n)$ for $n = 0, 1, 2, \ldots, 22$ are

1, 1, 2, 5, 15, 49, 173, 639, 2469, 9997, 43109, 205092, 1153646, 8523086, 91156133, 1446766659, 32998508358, 1047766596136, 45632564217917, 2711308588849394, 219364550983697100, 24151476334929009951, 3618445112608409433287.

Similarly, in the labelled case, let $p_{\mathrm{lbl}}(n)$ be the number of $(3+1)$-free posets with n labelled vertices. We show that the exponential generating function for labelled $(3+1)$-free posets is

$$\sum_{n \geq 0} p_{\mathrm{lbl}}(n) \frac{x^n}{n!} = S\big(e^x - 1, 2e^{-x} - 1 - B_{\mathrm{lbl}}(x)^{-1}\big), \tag{1.3}$$

where $B_{\mathrm{lbl}}(x) = \sum_{n \geq 0} \sum_{i=0}^{n} \binom{n}{i} 2^{i(n-i)} \frac{x^n}{n!}$ is the exponential generating function for labelled bicoloured graphs. Such bicoloured graphs are easy to count, but before our investigation the OEIS had only 9 terms in the entry [13, A079145] for $p_{\mathrm{lbl}}(n)$. Using (1.3), arbitrarily many terms $p_{\mathrm{lbl}}(n)$ can be computed.

Our main tool is a new decomposition of (3+1)-free posets into *parts* (called *clone sets* and *tangles*). This *tangle decomposition* is compatible with the automorphism group, in the sense that for a $(3+1)$-free poset P, $\mathrm{Aut}(P)$ breaks up as the direct product of the automorphism groups of its parts. The tangle decomposition also generalizes a decomposition of Reed and Skandera [12] for $(3+1)$-and-$(2+2)$-free posets given by *altitudes* of vertices. In terms of generating functions, the restriction of our results to (3+1)-and-(2+2)-free posets corresponds to the specialization $t = 0$ in (1.1). Indeed, one can see that $S(x/(1-x), 0)$ satisfies the functional equation for the Catalan generating function, which is consistent with the enumeration result stated earlier for $(3+1)$-and-$(2+2)$-free posets [17, Ex. 6.19(ddd)].

Remark 1.1. Using the tangle decomposition it is possible to quickly generate all $(3+1)$-free posets of a given size up to isomorphism in a straightforward way (see Corollary 3.10). With this approach, we were able to list all $(3+1)$-free posets on up to 11 vertices in a few minutes on modest hardware. Note that this technique can accommodate the generation of interesting subclasses of $(3+1)$-free posets (*e.g.*, $(2+2)$-free, weakly graded, strongly graded, co-connected, fixed height) or constructing these posets from the bottom up, level by level (which can help compute invariants like the chromatic symmetric function).

Remark 1.2. Comparing the list of numbers above with data provided by Joel Brewster Lewis for the number of graded $(3+1)$-free posets [13, A222863, A222865], it appears that, asymptotically, almost all $(3+1)$-free posets are graded. We prove this in Section 5, building on the asymptotic analysis of Lewis and Zhang for the graded $(3+1)$-free posets. In fact, almost all $(3+1)$-free posets are 3-free, so their Hasse diagrams are bicoloured graphs.

Outline. In Section 2, we describe the tangle decomposition of a $(3+1)$-free poset into clone sets and tangles and use it to compute the poset's automorphism group. In Section 3, we describe the relationships between the different clone sets and tangles of a $(3+1)$-free poset as parts of a structure called the *skeleton* and enumerate the possible skeleta. In Section 4, we enumerate tangles in terms of bicoloured graphs, and as a result we obtain generating functions for $(3+1)$-free posets. In Section 5, we give asymptotics for the number of $(3+1)$-free posets.

2. The tangle decomposition

Throughout the paper, we assume that P is a $(3+1)$-free poset. We write $a \parallel b$ if vertices a and b in a poset are incomparable. In this section, we describe the tangle decomposition of a $(3+1)$-free poset.

Given a vertex $a \in P$, we write $D_a = \{x \in P : x < a\}$ and $U_a = \{x \in P : x > a\}$ for the (strict) downset and upset of a. The set $\mathcal{J}(P)$ of all downsets of P (that is, all downward closed subsets of P, not just those of the form D_a for some $a \in P$) forms a distributive lattice, and in particular a poset, under set inclusion. Similarly, the set of upsets of P forms a poset under set inclusion, but it will be convenient for us to consider instead the complements $P \setminus U_a \in \mathcal{J}(P)$ of the upsets of vertices $a \in P$.

Definition 2.1. The *view* $v(a)$ from a vertex $a \in P$ is the pair $(D_a, P \setminus U_a) \in \mathcal{J}(P) \times \mathcal{J}(P)$. If $v(a) = v(b)$, then we say a and b are *clones* and write $a \approx b$.

Note that the set $v(P)$ of views of all vertices of P inherits a poset structure from the set $\mathcal{J}(P) \times \mathcal{J}(P)$, where $v(a) \leq v(b)$ if and only if $D_a \subseteq D_b$ and $U_a \supseteq U_b$.

Also note that two vertices $a, b \in P$ are clones precisely when they are *interchangeable*, in the sense that the permutation of the vertices of P which only exchanges a and b is an automorphism of P.

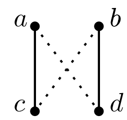
Example 2.2. Figure 1 shows a $(3+1)$-free poset P and the poset $v(P)$ of views. Since $v(d) = v(e)$, we have $d \approx e$.

Remark 2.3. The notion of clones is related to the notion of *trimming* of Lewis and Zhang [9]. Also, Zhang [20] has used techniques involving clones and $(2+2)$-avoidance to prove enumeration results about families of graded posets.

Definition 2.4. Let $a, b \in P$. We write $a \mathrel{\overset{\bullet\;\bullet}{\bowtie}} b$ if $D_a \parallel D_b$, and we write $a \mathrel{\underset{\bullet\;\bullet}{\bowtie}} b$ if $U_a \parallel U_b$.

The idea behind the notation is the following. If $a \mathrel{\overset{\bullet\;\bullet}{\bowtie}} b$, then there is some vertex $c \in D_a \setminus D_b$, so that $c < a$ and $c \not< b$, and there is some $d \in D_b \setminus D_a$, so that $d \not< a$

and $d < b$. Then, it can be checked that a, b, c, d are distinct vertices, and that they are incomparable except for the two relations $c < a$ and $d < b$. Hence we have the following induced $(2+2)$ subposet with a and b on the top:



Dually, if $a \underset{\cdot\cdot}{\bowtie} b$ then there is an induced $(2+2)$ subposet with a and b on the bottom.

Example 2.5. For the poset in Figure 1, $f \overset{\cdot\cdot}{\bowtie} g$ and $b \underset{\cdot\cdot}{\bowtie} c$. However, it is not the case that $a \underset{\cdot\cdot}{\bowtie} b$, since $U_b \subseteq U_a$.

The following lemma records basic properties of the relations $\approx$, $\overset{\cdot\cdot}{\bowtie}$, and $\underset{\cdot\cdot}{\bowtie}$ and their interactions.

Lemma 2.6. *Let P be a $(3+1)$-free-poset, and let a, b, c be any vertices of P.*

(i) If $a \approx b$ and $b \approx c$, then $a \approx c$.
(ii) If $a \overset{\cdot\cdot}{\bowtie} b$ and $b \approx c$, then $a \overset{\cdot\cdot}{\bowtie} c$.
(iii) If $a \underset{\cdot\cdot}{\bowtie} b$ and $b \approx c$, then $a \underset{\cdot\cdot}{\bowtie} c$.
(iv) If $a \overset{\cdot\cdot}{\bowtie} b$, then $U_a = U_b$.
(v) If $a \underset{\cdot\cdot}{\bowtie} b$, then $D_a = D_b$.
(vi) We have $v(a) \parallel v(b)$ if and only if $a \overset{\cdot\cdot}{\bowtie} b$ or $a \underset{\cdot\cdot}{\bowtie} b$.
(vii) It is not the case that $a \overset{\cdot\cdot}{\bowtie} b$ and $b \underset{\cdot\cdot}{\bowtie} c$.

Proof. Parts (i)–(iii) follow immediately from definitions. To show (iv), observe that if $a \overset{\cdot\cdot}{\bowtie} b$ then each vertex in $U_a \setminus U_b$ or $U_b \setminus U_a$ is the maximal element of a chain of length 3 in P and results in a $(3+1)$; therefore both of these sets are empty. The proof of (v) is analogous.

Next we prove (vi). From definitions it follows that $v(a) \parallel v(b)$ if and only if $a \overset{\cdot\cdot}{\bowtie} b$ or $a \underset{\cdot\cdot}{\bowtie} b$ or $(D_a \nsubseteq D_b$ and $U_a \nsubseteq U_b)$ or $(U_b \nsubseteq U_a$ and $D_b \nsubseteq D_a)$. In a $(3+1)$-free poset, however, $(D_a \nsubseteq D_b$ and $U_a \nsubseteq U_b)$ and $(U_b \nsubseteq U_a$ and $D_b \nsubseteq D_a)$ are both false.

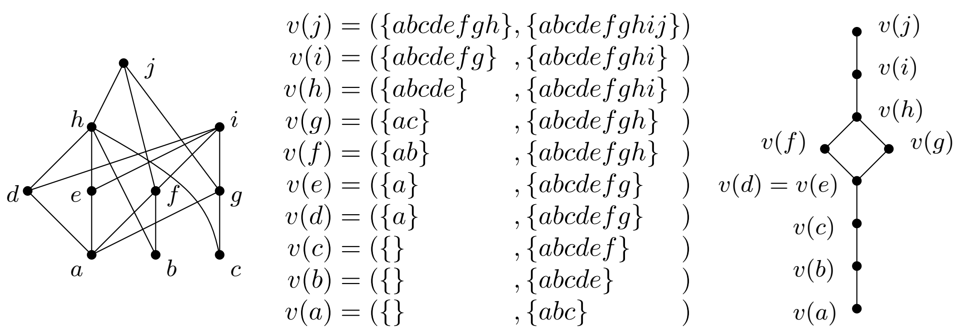


FIGURE 1. Left: the Hasse diagram of a $(3+1)$-free poset P with 10 vertices. Centre: the list of views of the vertices of P. Right: the view poset $v(P)$.

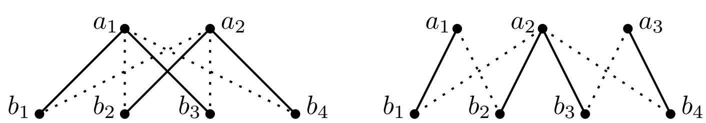
For (vii), suppose $a \mathrel{\overset{\bullet\,\bullet}{\bowtie}} b$ and $b \mathrel{\underset{\bullet\,\bullet}{\bowtie}} c$. Note that by (iv) we have $a \neq c$. Since $b \mathrel{\underset{\bullet\,\bullet}{\bowtie}} c$, the set $U_b \setminus U_c$ is not empty; let $f \in U_b \setminus U_c$. Since $a \mathrel{\overset{\bullet\,\bullet}{\bowtie}} b$, then $U_a = U_b$ by (iv) and hence $f \in U_a \setminus U_c$. Since $a \mathrel{\overset{\bullet\,\bullet}{\bowtie}} b$, the set $D_a \setminus D_b$ is not empty; let $e \in D_a \setminus D_b$. Therefore P has a chain $e < a < f$ of length 3. We will show that this chain, together with c, forms $(3+1)$. We already know $c \parallel f$. Since $c \not< f$, we have $c \not< e$ and $c \not< a$. Also, since $e \not< b$ and $D_b = D_c$, we have $e \not< c$ and hence also $a \not< c$. Therefore $c \parallel e$ and $c \parallel a$. We have shown that c is incomparable to f, a, and e, contradicting the assumption that P is $(3+1)$-free. $\square$

Now, consider a graph Γ on the vertices of P with edge set $\{(a,b) : a \mathrel{\overset{\bullet\,\bullet}{\bowtie}} b\}$. We say that a subset $A \subseteq P$ is the *top of a tangle* if $|A| \geq 2$ and A, when viewed as a subset of $V(\Gamma)$, is a connected component of Γ. Analogously, a subset $B \subseteq P$ is the *bottom of a tangle* if $|B| \geq 2$ and B is a connected component under the relation $\mathrel{\underset{\bullet\,\bullet}{\bowtie}}$.

By conclusion (vii) of Lemma 2.6, if A is the top of a tangle and B is the bottom of a tangle, then $A \cap B = \emptyset$. Let us say that a top of a tangle A and a bottom of a tangle B are *matched* if there is an induced $(2+2)$ subposet whose top two vertices are in A, and whose bottom two vertices are in B.

Proposition 2.7. *In a $(3+1)$-free poset P, every top of a tangle is matched to a unique bottom of a tangle, and every bottom of a tangle is matched to a unique top of a tangle. That is, there is a perfect matching between tops of tangles and bottoms of tangles of P.*

Proof. To prove that a top of a tangle A is matched to a unique bottom of a tangle, it suffices to show that any two induced $(2+2)$ subposets whose tops intersect in A have bottoms which are connected by the $\mathrel{\underset{\bullet\,\bullet}{\bowtie}}$ relation. This follows by case analysis; the two most general cases to check are shown below, and the other cases occur when some of the b_i coincide.



In the first case, for example, $b_1 \mathrel{\underset{\bullet\,\bullet}{\bowtie}} b_2$ and $b_3 \mathrel{\underset{\bullet\,\bullet}{\bowtie}} b_4$ imply $b_1 \mathrel{\underset{\bullet\,\bullet}{\bowtie}} b_4$ (and also $b_2 \mathrel{\underset{\bullet\,\bullet}{\bowtie}} b_3$). By a symmetric argument, connected components under $\mathrel{\underset{\bullet\,\bullet}{\bowtie}}$ induce connected components under $\mathrel{\overset{\bullet\,\bullet}{\bowtie}}$. $\square$

Proposition 2.7 justifies the terms 'top of a tangle' and 'bottom of a tangle' and the following definition.

Definition 2.8. A *tangle* is a matched pair $T = (A, B)$ of a top of a tangle A and a bottom of a tangle B.

In other words, a tangle is a subposet of P that is connected by induced $(2+2)$ subposets. In particular, P is $(2+2)$-free exactly when it has no tangles.

Example 2.9. Very often, a two-level poset which is not connected consists of a single tangle. For example, let P be the poset with vertices $\{a_1, a_2, a_3, c_1, c_2\} \cup \{b, d\}$ and relations $a_i > c_j$, $b > d$. Then, the connected components of P are $\{a_1, a_2, a_3, c_1, c_2\}$ and $\{b, d\}$. Every subset of the form $\{a_i, b, c_j, d\}$ forms an induced $(2+2)$ subposet, so $\{a_1, a_2, a_3, b\}$ is the top of a tangle, $\{c_1, c_2, d\}$ is the bottom of a tangle, and the whole poset P is a single tangle.

Example 2.10. In the poset P of Figure 1, the connected component of f under $\overset{\cdot\cdot}{\bowtie}$ is $\{f, g\}$, and the connected component of b under $\underset{\cdot\cdot}{\bowtie}$ is $\{b, c\}$. Therefore P contains the tangle $T = (\{f, g\}, \{b, c\})$. One can check that in fact this is the only tangle of P.

Definition 2.11. Let $T_1 = (A_1, B_1), \ldots, T_s = (A_s, B_s)$ be the tangles of P. A *clone set* is an equivalence class, under $\approx$, of vertices in $P \setminus \bigcup_{j=1}^{s}(A_j \cup B_j)$. We refer to tangles and clone sets as *parts* of P. The set of parts is the *tangle decomposition* of P.

Example 2.12. The tangle decomposition of the poset in Figure 1 appears in Figure 2. It consists of six parts—five clone sets and one tangle.

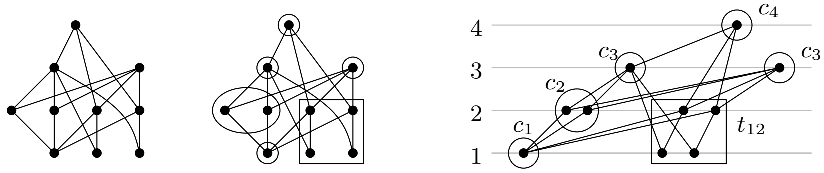


Figure 2. Left: the Hasse diagram of the poset P from Figure 1. Centre: the tangle decomposition of P into its parts. Right: a compatible listing of the parts. Clone sets are enclosed in circles, and tangles are enclosed in boxes.

The tangle decomposition provides a decomposition of a $(3+1)$-free poset from which the automorphism group, among other properties, can be computed. To show this, it will be useful to have a different characterization of the tops of tangles, bottoms of tangles, and clone sets of P which gives a natural ordering of these subsets of P, as follows. A *co-connected component* of a poset Q is a connected component of the incomparability graph of Q.

Proposition 2.13. *Let $v(P) \subseteq \mathcal{J}(P) \times \mathcal{J}(P)$ be the poset of views of all vertices of the $(3+1)$-free poset P. Then, there is a listing $(S_1, S_2, \ldots, S_k)$ of the co-connected components of $v(P)$ such that for every $x \in S_i$ and every $y \in S_{i+1}$, we have $x < y$. Moreover, the preimages $v^{-1}(S_i)$ for $i = 1, 2, \ldots, k$ are exactly the tops of tangles, bottoms of tangles, and clone sets of P.*

Proof. Any poset Q can be decomposed into its co-connected components, and then it is easy to show that for any two co-connected components S, S' of Q, we either have $x < y$ for all $x \in S$ and all $y \in S'$, or we have $x > y$ for all $x \in S$ and all $y \in S'$. In other words, the set of co-connected components of Q is totally ordered.

We can apply this to the poset $v(P)$ to obtain the list $(S_1, S_2, \ldots, S_k)$ described in the statement of the proposition. Then, for vertices $a, b \in P$, we have that $v(a) = v(b)$ if and only if $a \approx b$, and we have $v(a) \parallel v(b)$ if and only if $a \overset{\cdot\cdot}{\bowtie} b$ or $a \underset{\cdot\cdot}{\bowtie} b$ by Lemma 2.6. Thus, for each S_i, it follows that either $|S_i| = 1$ and $v^{-1}(S_i)$ is a clone set, or that $|S_i| \geq 2$ and $v^{-1}(S_i)$ is the top or the bottom of a tangle. $\square$

Let $\mathrm{Aut}(P)$ be the automorphism group of the poset P. Any part X_i of P gives an induced subposet of P, and we write $\mathrm{Aut}(X_i)$ for its automorphism group as a poset. In particular, if X_i is a clone set with k vertices, then $\mathrm{Aut}(X_i)$ is the symmetric group on these k vertices; if X_i is a tangle, then it can be seen as a bicoloured graph (with colour classes 'top' and 'bottom'), and $\mathrm{Aut}(X_i)$ is the group of colour-preserving automorphisms of this graph.

Theorem 2.14. *Let P be a $(3+1)$-free poset, decomposed into its clone sets $C_1, C_2, \ldots, C_r$ and its tangles $T_1, T_2, \ldots, T_s$. Then, the automorphism group of P is*

$$\mathrm{Aut}(P) = \prod_{i=1}^{r} \mathrm{Aut}(C_i) \times \prod_{j=1}^{s} \mathrm{Aut}(T_j).$$

Proof. Let C_i be a clone set of P. Then, all the vertices of C_i have the same downset and the same upset, so any permutation of P which acts trivially on $P \setminus C_i$ is an automorphism of P. That is, we have $\mathrm{Aut}(C_i) \subseteq \mathrm{Aut}(P)$.

Let T_j be a tangle of P, and let A_j be its top and B_j be its bottom. Then, all the vertices of B_j have the same downset, and their upsets differ only by vertices of A_j. Similarly, all the vertices of A_j have the same upset, and their downsets differ only by vertices of B_j. Thus, any permutation of P which acts trivially on $P \setminus T_j$ and preserves T_j is an automorphism of P. That is, we have $\mathrm{Aut}(T_j) \subseteq \mathrm{Aut}(P)$.

Together, these facts show that $\prod_{i=1}^{r} \mathrm{Aut}(C_i) \times \prod_{j=1}^{s} \mathrm{Aut}(T_j) \subseteq \mathrm{Aut}(P)$.

For the reverse inclusion, consider an automorphism $\tau \colon P \to P$. Then, τ acts on the poset of views $v(P)$, and this action preserves the list of co-connected components $(S_1, \ldots, S_k)$ of $v(P)$, so τ preserves each of the parts $v^{-1}(S_1), \ldots, v^{-1}(S_k)$ of the tangle decomposition of P. Thus, τ restricts to an automorphism of each clone set and each tangle of P. $\square$

Note that the tangle decomposition of a $(3+1)$-free poset P into its parts generalizes the decomposition considered by Reed and Skandera [12] of a $(3+1)$-and-$(2+2)$-free poset given by the *altitude* $\alpha(a) = |D_a| - |U_a|$ of the vertices $a \in P$, since the altitude $\alpha(a)$ is a function of the view $v(a)$. Of course, even in a $(3+1)$-free poset P with an induced $(2+2)$ subposet, the altitude is well-defined, and it gives a finer decomposition of P than the tangle decomposition. However, the altitude decomposition is too fine, as the example in Figure 3 shows. Namely, there is an automorphism τ which swaps the two vertices with altitude -1, the two vertices with altitude -2, and two of the three vertices with altitude 2, as illustrated. But there is no automorphism of the poset which acts nontrivially on a single block of the altitude decomposition.

In contrast, for the tangle decomposition, every automorphism of the poset can be factored as a product of automorphisms which only act nontrivially on a single part.

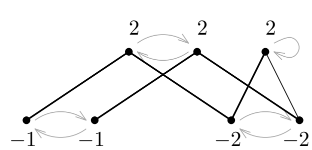


FIGURE 3. A poset P consisting of a single tangle. The vertices are labelled by their altitude α, and the arrows describe an automorphism τ of P.

3. Skeleta

Any finite poset P can be decomposed into *levels* as follows: take L_1 to be the set of minimal vertices of P, L_2 to be the set of subminimal vertices (that is, the set of minimal vertices of $P \setminus L_1$), and so on up to the set L_h of sub$^{(h-1)}$minimal vertices of P, where h is the *height* of P. We say that the *level* of a vertex $a \in P$ is $\ell(a)$, where $a \in L_{\ell(a)}$.

If P is $(3+1)$-free, then the only interesting part of the poset structure occurs between adjacent levels, as the following proposition shows.

Proposition 3.1 (Lewis and Zhang [9])**.** *Let P be a $(3+1)$-free poset and $a, b \in P$ be two vertices with $\ell(a) \leq \ell(b) - 2$. Then, we have $a < b$.*

Note that the covering relations of P may include relations $a < b$ for which $\ell(a) = \ell(b) - 2$. This occurs for the poset in Figure 1, for example, where $b < h$, $c < h$, $f < j$, and $g < j$ are covering relations.

The following proposition gives a partial converse of Proposition 3.1.

Proposition 3.2 (Reed and Skandera [12])**.** *Let P be a poset such that for any two vertices $a, b \in P$ with $\ell(a) \leq \ell(b) - 2$, we have $a < b$. Then, P is $(3+1)$-free if and only if for any two vertices $c, d \in P$ with $\ell(c) = \ell(d)$, we have $U_c \subseteq U_d$ or $D_c \subseteq D_d$ (and symmetrically, $U_c \supseteq U_d$ or $D_c \supseteq D_d$).*

Note that the vertices of a clone set C_i all have the same downset, so they are on the same level. Also, any copy of the $(2+2)$ poset must be contained in two adjacent levels, so any tangle T_j must be contained in two adjacent levels. Thus, we can speak of the level of a clone set or the (adjacent) levels of a tangle.

By construction, the poset structure between two parts of P is fairly restricted. If C_i and C_j are distinct clone sets, then C_i is either completely above, completely below, or completely incomparable with C_j (meaning that every vertex of C_i has the same relationship with every vertex of C_j). If C_i is a clone set and T_j is a tangle, then C_i can be

- completely above T_j;
- completely above the bottom of T_j and incomparable with the top;
- completely below the top of T_j and incomparable with the bottom;
- completely below T_j; or
- completely incomparable with T_j.

Similarly, there are only six possible ways for two tangles T_i and T_j to relate to each other. The following theorem shows how all of these relationships between different parts of P can be put together.

Theorem 3.3. *Let P be a $(3+1)$-free poset, decomposed into clone sets $C_1, \ldots, C_r$ and tangles $T_1, \ldots, T_s$. Then, there exists a listing $(X_1, \ldots, X_{r+s})$ of the clone sets and the tangles of P such that, for any two vertices $a \in X_i$ and $b \in X_j$ with $i \neq j$, we have $a < b$ exactly when*

(i) $\ell(a) \leq \ell(b) - 2$; or
(ii) $\ell(a) = \ell(b) - 1$ and $i < j$.

Definition 3.4. A listing which satisfies the conditions of Theorem 3.3 is called a *compatible* listing.

Example 3.5. A compatible listing for the poset in Figure 1 is

$$\big(\{a\},\{d,e\},\{h\},(\{f,g\},\{b,c\}),\{j\},\{i\}\big).$$

This compatible listing is shown in Figure 2.

Proof of Theorem 3.3. By Proposition 3.1, for any vertices $a, b \in P$, we have the implications

$$\ell(a) \leq \ell(b) - 2 \implies a < b \implies \ell(a) \leq \ell(b) - 1,$$

so the only interesting case is when $\ell(a) = \ell(b) - 1$.

Let $(S_1, S_2, \ldots, S_k)$ be the list of co-connected components of the view poset $v(P)$ as in Proposition 2.13, and consider the preimages $v^{-1}(S_i)$ for $i = 1, 2, \ldots, k$; these preimages are the clone sets, the tops of tangles, and the bottoms of tangles of P. For each level L_i of P, we can obtain a partial listing of the parts X_j of P which intersect L_i by ordering them according to the position of the co-connected component $v(L_i \cap X_j)$ in the list $(S_1, S_2, \ldots, S_k)$.

Thus we have a partial listing of the parts which intersect L_i according to the order of their images on the view poset $v(P)$, and it remains to show that these listings can be reconciled.

It can be shown that for any two adjacent levels L_i and L_{i+1}, the partial listings for L_i and L_{i+1} can be interleaved in a unique way to satisfy condition (ii) of the theorem between the two levels. Since the order of parts at each level L_i is preserved by these interleavings, a standard argument (see, *e.g.*, [6, Chapter 3]) shows that they can be combined into a listing of all the parts of P which satisfies condition (ii) for all pairs of adjacent levels. □

Note that the listing $(X_1, X_2, \ldots, X_{r+s})$ from Theorem 3.3 is not unique in general. In particular, if $(\ldots, X_i, X_{i+1}, \ldots)$ is a compatible listing, then the listing $(\ldots, X_{i+1}, X_i, \ldots)$ obtained by swapping the parts X_i and X_{i+1} is compatible exactly when X_i and X_{i+1} contain no vertices on the same or on adjacent levels of P. We call such a swap *valid*.

Example 3.6. In Figure 2 we can swap the clone set $\{j\}$ on level 4 with the tangle $(\{f,g\},\{b,c\})$ on levels 1 and 2 to obtain another compatible listing for the poset.

Therefore the natural setting for compatible listings is that of free partially commuting monoids [5], also known as trace monoids [6].

Definition 3.7. Let Σ be the countable alphabet

$$\Sigma = \{c_1, c_2, \ldots, c_i, \ldots\} \cup \{t_{12}, t_{23}, \ldots, t_{i\,i+1}, \ldots\},$$

let Σ^* be the free monoid generated by Σ, and let M be the free partially commuting monoid with commutation relations

$$\begin{aligned} c_i c_j &= c_j c_i, && \text{if } |i-j| \geq 2, \\ c_i t_{j\,j+1} &= t_{j\,j+1} c_i, && \text{if } i \leq j-2 \text{ or } i \geq j+3, \\ t_{i\,i+1} t_{j\,j+1} &= t_{j\,j+1} t_{i\,i+1}, && \text{if } |i-j| \geq 3. \end{aligned}$$

Definition 3.8. If P is a $(3+1)$-free poset, then for each compatible listing $(X_1, X_2, \ldots, X_{r+s})$ of its clone sets and tangles, we can obtain a word in Σ^* by replacing each clone set at level i by the letter c_i and each tangle straddling levels $\{i, i+1\}$ by the letter $t_{i\,i+1}$. It can be seen that any two compatible listings for P are related by a sequence of valid swaps, so the set of these words is an equivalence class under the commutation relations for M (see, *e.g.*, [6, Chapter 1]), and the corresponding element of M is called the *skeleton* of P.

Example 3.9. The two representatives in Σ^* for the skeleton of the poset in Figure 2 are $c_1c_2c_3t_{12}c_4c_3$ and $c_1c_2c_3c_4t_{12}c_3$.

The point of a skeleton is that it exactly captures the relationships between different parts of P. More precisely, two posets with the same skeleton and isomorphic parts are themselves isomorphic; conversely, given a skeleton, any set of parts (with the right number of clone sets and tangles) can be plugged into the skeleton. Together, Corollary 3.10, Theorem 3.11, and Theorem 3.12 below show this and give a characterization of the elements of M which are skeleta.

Corollary 3.10. *Let P be a $(3+1)$-free poset. Then, P is uniquely determined (up to isomorphism) by its skeleton together with, for each letter c_i or $t_{i\,i+1}$ of the skeleton, the cardinality of the corresponding clone set or the isomorphism class of the corresponding tangle.*

Proof. Note that the notion of a "clone set or tangle of P corresponding to a letter of the skeleton of P" is only well-defined if we take the convention that two copies of the same letter don't commute with each other. With this caveat in mind, the result follows from the fact that the conditions in Theorem 3.3 completely specify the order relations between vertices in different parts of P in terms of the skeleton of P. □

Theorem 3.11. *Let m be an element of the monoid M. Then, m is the skeleton of some $(3+1)$-free poset if and only if*

(i) every representative $w \in \Sigma^*$ for m starts with the letter c_1 or t_{12}; and
(ii) no representative $w \in \Sigma^*$ for m contains a factor of the form c_ic_i, $i \geq 1$.

Note that condition (i) of Theorem 3.11 corresponds to the requirement that every vertex of P on level L_{i+1} be greater than some vertex on the previous level L_i, while condition (ii) forbids pairs of clone sets that could be merged into a single clone set.

Proof. Let m be the skeleton of a $(3+1)$-free poset P. Then, every representative $w \in \Sigma^*$ for m corresponds to a compatible listing $(X_1, X_2, \ldots, X_{r+s})$ for P. If $a \in X_i$ is a vertex on level $\ell(a) > 1$, then there exists a vertex b in its downset D_a which is on the previous level, $\ell(b) = \ell(a) - 1$. By Theorem 3.3, it follows that either $b \in X_i$ as well, in which case X_i is a tangle with a on top and b on the bottom, or $b \in X_j$ with $j < i$. By induction, X_1 must contain a vertex on level 1, which implies condition (i).

Now, suppose w contains c_ic_i as a factor for some $i \geq 1$, and let X_j, X_{j+1} be the corresponding entries in the compatible listing for P, and let $a \in X_j$ and $b \in X_{j+1}$. Then, by Theorem 3.3, the vertices a and b have the same downset and the same upset, so $a \approx b$. However, this contradicts the fact that X_j and X_{j+1} are distinct clone sets of P. Thus, condition (ii) holds.

Conversely, it is straightforward to construct a poset P from a skeleton m and a list of parts. This always yields a $(3+1)$-free poset, but note that condition (ii) is necessary to ensure that the clone sets of P are as specified, and condition (i) is necessary to ensure that the levels of the clone sets and tangles of P match with the representative w. □

Theorem 3.12. *Let m be an element of the monoid M. Then, there exists a representative $w_0 \in \Sigma^*$ for m for which every pair of consecutive letters is either*

$$\begin{array}{ll} c_i c_j & \text{for } i \geq j-1; \text{ or} \\ c_i t_{j\,j+1} & \text{for } i \geq j-1; \text{ or} \\ t_{i\,i+1} c_j & \text{for } i \geq j-2; \text{ or} \\ t_{i\,i+1} t_{j\,j+1} & \text{for } i \geq j-2. \end{array}$$

Furthermore,

(i) this representative w_0 is unique and is the lexicographically maximal representative for m with respect to the total order $\{c_1 < t_{12} < c_2 < t_{23} < \cdots\}$ on Σ;

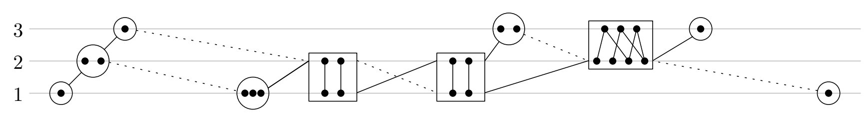
(ii) if w_0 starts with c_1 or t_{12}, then every representative $w \in \Sigma^*$ for m starts with c_1 or t_{12}; and

(iii) if w_0 does not contain a factor of the form $c_i c_i$, $i \geq 1$, then no representative $w \in \Sigma^*$ for m contains a factor of this form.

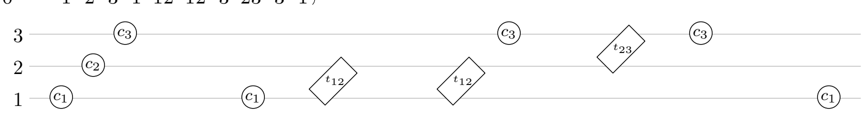
Proof. Note that the forbidden pairs of consecutive letters for w_0 are exactly the pairs of letters from Σ which commute and which are increasing with respect to the order $\{c_1 < t_{12} < c_2 < t_{23} < \cdots\}$. Thus, a representative w_0 can be constructed by taking any representative of $w \in \Sigma^*$ for m and repeatedly applying commutation relations to get rid of the forbidden pairs. This procedure eventually terminates, since the representative is made lexicographically larger at each step. The remaining properties of w_0 can be checked using standard arguments about partially commuting monoids (see, *e.g.*, [6, Chapter 4]). $\square$

Example 3.13. Of the two representatives given in Example 3.9, $c_1c_2c_3c_4t_{12}c_3$ is lexicographically maximal.

Consider the 26-vertex (3+1)-free poset P with 10 parts shown in the compatible listing below. Only some of the comparability and incomparability relations between parts are drawn, but the others can be determined from Theorem 3.3.



The lexicographically maximal representative for the skeleton of P is $w_0 = c_1c_2c_3c_1t_{12}t_{12}c_3t_{23}c_3c_1$, shown below.



The decorated Dyck path associated with w_0 is the following.

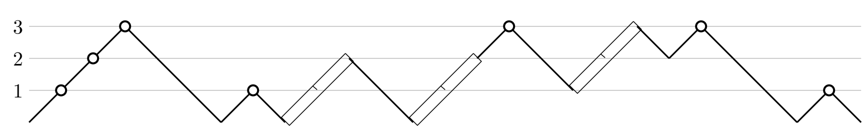


FIGURE 4. An example of the bijection given in Theorem 3.14.

Using this characterization of skeleta, we can enumerate them, and this will allow us to obtain generating functions for $(3+1)$-free posets.

Theorem 3.14. *There is a bijection between skeleta of $(3+1)$-free posets and certain decorated Dyck paths. (See Figure 4 for an example.)*

Proof. Given the lexicographically maximal representative w_0 for a skeleton, we can obtain a decorated Dyck path that starts at $(0,0)$, ends at $(2n,0)$ for some $n \geq 0$, and never goes below the x-axis as follows: replace each letter c_i by a $(1,1)$ step ending at height i, each letter $t_{i\,i+1}$ by a $(2,2)$ step ending at height $i+1$, and add $(1,-1)$ down steps as necessary. We call the result *decorated* since a $(2,2)$ step can be seen as a pair of consecutive decorated $(1,1)$ steps. Since w_0 not contain c_ic_i as a factor, the decorated Dyck path obtained from w_0 contains no sequence $(1,1),(1,-1),(1,1)$ of consecutive undecorated steps (up-down-up). Conversely, every decorated Dyck path avoiding this sequence can be obtained from a skeleton. □

Theorem 3.15. *Let $S(c,t) \in \mathbb{Q}[[c,t]]$ be the ordinary generating function for skeleta with respect to the number of clone sets and the number of tangles, that is, the formal power series*

$$S(c,t) = \sum_{r,s\geq 0} (\#\text{ of distinct skeleta with } r \text{ clone sets and } s \text{ tangles})\, c^r t^s.$$

Then, $S(c,t)$ is uniquely determined by the equation

$$S(c,t) = 1 + \frac{c}{1+c} S(c,t)^2 + tS(c,t)^3. \tag{3.1}$$

Proof. Theorems 3.11 and 3.12 give a correspondence between skeleta and some representative words w_0 in Σ^*. In turn, Theorem 3.14 gives a correspondence between these words and decorated Dyck paths with no up-down-up subsequence of undecorated steps. Thus, we can view the generating function $S(c,t)$ for skeleta with respect to the numbers of clone sets and tangles as the generating function for decorated Dyck paths with no undecorated up-down-up subsequence with respect to the numbers of undecorated up-steps and pairs of decorated up-steps.

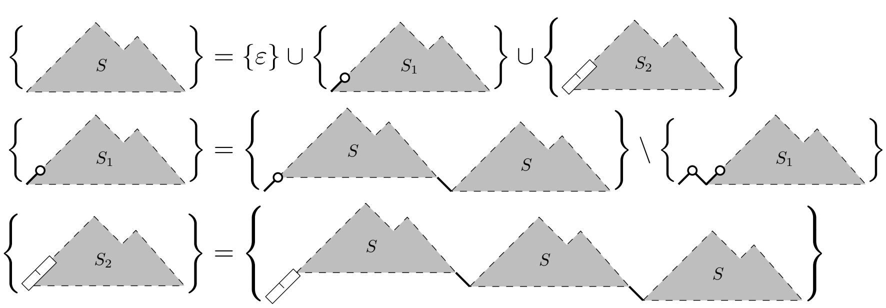


FIGURE 5. Equations relating the sets counted by $S(c,t)$, $S_1(c,t)$, and $S_2(c,t)$, where $S_1(c,t)$ and $S_2(c,t)$ are the generating functions for decorated Dyck paths beginning with $(1,1)$ and $(2,2)$, respectively.

Let $S_1(c,t)$ and $S_2(c,t)$ be the generating functions for decorated Dyck paths with no undecorated up-down-up subsequence beginning with $(1,1)$ and $(2,2)$, respectively. As shown in Figure 5, we can decompose the sets counted by $S(c,t)$, $S_1(c,t)$, and $S_2(c,t)$ to obtain the relations

$$\begin{aligned} S(c,t) &= 1 + S_1(c,t) + S_2(c,t) \\ S_1(c,t) &= cS(c,t)^2 - cS_1(c,t) \\ S_2(c,t) &= tS(c,t)^3. \end{aligned}$$

Solving this system of equations yields Equation (3.1). □

4. Enumeration

In this section, we carry out the enumeration of unlabelled and labelled $(3+1)$-free posets by reducing it to the enumeration of unlabelled and labelled bicoloured graphs. Our approach is to consider such a bicoloured graph as a $(3+1)$-free poset in the natural way (with colour classes 'top' and 'bottom') and to apply the machinery of Section 3, as shown in the following lemma.

Lemma 4.1. *The ordinary generating function for skeleta of bicoloured graphs is given by*

$$\sum_{r_1,r_2,s\geq 0} \begin{pmatrix} \textit{\# of skeleta of bicoloured graphs} \\ \textit{with } r_1 \textit{ clone sets on level } 1,\ r_2 \\ \textit{clone sets on level } 2, \textit{ and } s \textit{ tangles} \end{pmatrix} c_1^{r_1} c_2^{r_2} t_{12}^s = \left(1 - \frac{c_1}{1+c_1} - \frac{c_2}{1+c_2} - t_{12}\right)^{-1}. \tag{4.1}$$

Proof. By a slight abuse, we can consider a bicoloured graph as a poset with two levels L_1 and L_2 given by the colour classes; the abuse being that any isolated vertices of the graph should be in L_1, since they are minimal vertices of the poset, but here we allow them to be in L_1 or L_2 according to the bicolouring of the graph. Such a poset is necessarily $(3+1)$-free, and allowing this abuse corresponds exactly to dropping condition (i) from the characterization of skeleta from Theorem 3.11. Note that only the letters c_1, c_2, t_{12} can appear in the skeleton, and none of them commute, so the possible skeleta in this case are exactly the strings of letters from the alphabet $\{c_1, c_2, t_{12}\}$ with no factor equal to c_1c_1 or c_2c_2. The statement then follows by standard generating function techniques. □

Now that we have an explicit expression for the generating function of skeleta of bicoloured graphs, we can perform appropriate substitutions to get equations relating the generating functions for tangles and for bicoloured graphs.

Theorem 4.2. *Let $B_{\mathrm{unl}}(x,y) \in \mathbb{Q}[[x,y]]$ be the ordinary generating function for unlabelled bicoloured graphs, up to isomorphism. Then, the ordinary generating function for unlabelled tangles is*

$$T_{\mathrm{unl}}(x,y) = 1 - x - y - B_{\mathrm{unl}}(x,y)^{-1}.$$

Proof. This follows from Lemma 4.1 by plugging in the values $c_1 = x/(1-x)$ and $c_2 = y/(1-y)$ for the clone sets of unlabelled vertices and $t = T_{\mathrm{unl}}(x,y)$ for the tangles in (4.1). □

Theorem 4.3. *Let $B_{\mathrm{lbl}}(x,y) \in \mathbb{Q}[[x,y]]$ be the exponential generating function for labelled bicoloured graphs, that is, the formal power series*

$$B_{\mathrm{lbl}}(x,y) = \sum_{i,j\geq 0} 2^{ij} \frac{x^i y^j}{i!j!}.$$

Then, the exponential generating function for labelled tangles is

$$T_{\mathrm{lbl}}(x,y) = e^{-x} + e^{-y} - 1 - B_{\mathrm{lbl}}(x,y)^{-1}.$$

Proof. This follows from Lemma 4.1 by plugging in the values $c_1 = e^x - 1$ and $c_2 = e^y - 1$ for the clone sets of labelled vertices and $t = T_{\mathrm{lbl}}(x,y)$ for the tangles in (4.1). □

With these expressions for the generating functions $T_{\mathrm{unl}}(x,y)$ and $T_{\mathrm{lbl}}(x,y)$ in hand, the following corollaries of Theorem 3.15 yield the equations (1.2) and (1.3) from the introduction.

Corollary 4.4. *Let $S(c,t)$ be the generating function of Theorem 3.15 for skeleta. Then, the ordinary generating function for unlabelled $(3+1)$-free posets is*

$$\sum_{n\geq 0} p_{\mathrm{unl}}(n) x^n = S\big(x/(1-x), T_{\mathrm{unl}}(x,x)\big).$$

Corollary 4.5. *Let $S(c,t)$ be the generating function of Theorem 3.15 for skeleta. Then, the exponential generating function for labelled $(3+1)$-free posets is*

$$\sum_{n\geq 0} p_{\mathrm{lbl}}(n) \frac{x^n}{n!} = S\big(e^x - 1, T_{\mathrm{lbl}}(x,x)\big).$$

Remark 4.6. François Bergeron has pointed out to us that the results of this section can be generalized to obtain the cycle index series (see [2]) for the species of $(3+1)$-free posets.

5. Asymptotics

In this section we determine the asymptotics for the number of labelled and unlabelled $(3+1)$-free posets. Recall that the (univariate) exponential generating function for labelled bicoloured graphs is $B_{\mathrm{lbl}}(x) = \sum_{n\geq 0}\sum_{i=0}^{n} \binom{n}{i} 2^{i(n-i)} \frac{x^n}{n!}$. Let

$$b_{\mathrm{lbl}}(n) = [x^n/n!]B_{\mathrm{lbl}}(x) = \sum_{i=0}^{n} \binom{n}{i} 2^{i(n-i)}$$

be the number of bicoloured graphs on n labelled vertices. Lewis and Zhang [9, Proposition 9.1] gave asymptotics for these coefficients.

Proposition 5.1 (Lewis and Zhang)**.** *There exist constants C_1 and C_2 such that*

$$b_{\mathrm{lbl}}(2k) \sim C_1 \binom{2k}{k} 2^{k^2} \qquad \textit{and} \qquad b_{\mathrm{lbl}}(2k+1) \sim C_2 \binom{2k+1}{k} 2^{k(k+1)}.$$

Recall that the ordinary generating function for unlabelled bicoloured graphs up to isomorphism is $B_{\mathrm{unl}}(x) = 1 + 2x + 4x^2 + 8x^3 + 17x^4 + \cdots$. Let

$$b_{\mathrm{unl}}(n) = [x^n]B_{\mathrm{unl}}(x)$$

be the number of such graphs with n vertices. From the theory of automorphisms of random graphs [4, Chapter 9] or by standard methods (see the Appendix for a proof in the spirit of [7]), almost all unlabelled bicoloured graphs have a trivial automorphism group, so we can relate the asymptotics of $b_{\mathrm{unl}}(n)$ and $b_{\mathrm{lbl}}(n)$ as follows.

Proposition 5.2. *If $b_{\mathrm{unl}}(n)$ is the number of bicoloured graphs with n unlabelled vertices and $b_{\mathrm{lbl}}(n)$ is the number of bicoloured graphs with n labelled vertices then*

$$n! \cdot b_{\mathrm{unl}}(n) \sim b_{\mathrm{lbl}}(n).$$

Lewis and Zhang [9, Theorem 9.2] also gave the asymptotics for the number of (weakly) graded $(3+1)$-free posets with n labelled vertices[2]. Using Proposition 5.2 and their method of proof one gets the asymptotics for the number of (weakly) graded $(3+1)$-free posets with n unlabelled vertices.

Theorem 5.3 (Lewis and Zhang)**.** *Let $p^{\mathrm{g}}_{\mathrm{lbl}}(n)$ and $p^{\mathrm{g}}_{\mathrm{unl}}(n)$ be the number of graded $(3+1)$-free posets with n labelled vertices and n unlabelled vertices respectively, and let $p^{\mathrm{w}}_{\mathrm{lbl}}(n)$ and $p^{\mathrm{w}}_{\mathrm{unl}}(n)$ be the corresponding numbers for weakly graded posets. Then*

(i) $p^{\mathrm{g}}_{\mathrm{lbl}}(n) \sim p^{\mathrm{w}}_{\mathrm{lbl}}(n) \sim b_{\mathrm{lbl}}(n)$, *and*
(ii) $p^{\mathrm{g}}_{\mathrm{unl}}(n) \sim p^{\mathrm{w}}_{\mathrm{unl}}(n) \sim b_{\mathrm{unl}}(n)$.

We are ready to state the main result of this section, which gives the asymptotics for the number of $(3+1)$-free posets with labelled and unlabelled vertices respectively.

Theorem 5.4. *If $p_{\mathrm{lbl}}(n)$ is the number of $(3+1)$-free posets with n labelled vertices and $p_{\mathrm{unl}}(n)$ is the number of $(3+1)$-free posets with n unlabelled vertices then*

(i) $p_{\mathrm{lbl}}(n) \sim b_{\mathrm{lbl}}(n)$, *and*
(ii) $p_{\mathrm{unl}}(n) \sim b_{\mathrm{unl}}(n)$.

Combining Theorems 5.3 and 5.4, it follows that almost all $(3+1)$-free posets are (weakly) graded. This fact may be surprising at first, but it is actually a consequence of the stronger fact that almost all $(3+1)$-free posets have Hasse diagrams which are bicoloured graphs, meaning that they have exactly two levels.

Like the proof of Theorem 5.3, the proof of Theorem 5.4 relies on the following result of Bender [3, Theorem 1].

Theorem 5.5 (Bender)**.** *Suppose that $F(x) = \sum_{n\geq 1} f_n x^n$, that $H(x,y)$ is a formal power series in x and y, and that $G(x) = \sum_{n\geq 0} g_n x^n = H(x, F(x))$. Let $C = \frac{\partial H}{\partial y}\Big|_{(0,0)}$. Suppose that*

1. *$H(x,y)$ is analytic in a neighbourhood of $(0,0)$.*
2. $\lim_{n\to\infty} \frac{f_{n-1}}{f_n} = 0$,
3. $\sum_{k=1}^{n-1} |f_k f_{n-k}| = O(f_{n-1})$.

Then

$$g_n = C \cdot f_n + O(f_{n-1}),$$

and in particular $g_n \sim C \cdot f_n$.

[2]Recall that a poset P is *weakly graded* if there exists a rank function $\rho : P \to \{0,1,2,\ldots\}$ such that if $a < b$ is a covering relation then $\rho(b) - \rho(a) = 1$ and the minimal vertices of connected components of P have rank 0. A poset is *strongly graded* if it is weakly graded, minimal vertices have the same rank, and maximal vertices have the same rank (i.e., all maximal chains in the poset have the same number of vertices).

Proof of Theorem 5.4(i). Let $H_{\mathrm{lbl}}(x,y)$ be the formal power series in x and y defined by

$$H_{\mathrm{lbl}}(x,y) = S(e^x - 1, 2e^{-x} - 1 - (1+y)^{-1}), \tag{5.1}$$

where $S(c,t)$ is the unique formal power series solution of the cubic equation

$$S(c,t) = 1 + \frac{c}{1+c} S(c,t)^2 + tS(c,t)^3, \tag{5.2}$$

as defined in Theorem 3.15. From Equation (1.2), we have that

$$H_{\mathrm{lbl}}(x, B_{\mathrm{lbl}}(x) - 1) = \sum_{n \geq 0} p_{\mathrm{lbl}}(n) \frac{x^n}{n!}.$$

In order to apply Theorem 5.5 we first check its three conditions. Condition 1, that $H_{\mathrm{lbl}}(x,y)$ be analytic in a neighbourhood of $(0,0)$, follows by construction. Lewis and Zhang [9] verified that the coefficients $b_{\mathrm{lbl}}(n)/n!$ of the generating function $B_{\mathrm{lbl}}(x) - 1$ satisfy conditions 2 and 3. That is,

$$\lim_{n\to\infty} \frac{n \cdot b_{\mathrm{lbl}}(n-1)}{b_{\mathrm{lbl}}(n)} = 0$$

and

$$\sum_{k=1}^{n-1} \left| \frac{b_{\mathrm{lbl}}(k)}{k!} \right| \cdot \frac{b_{\mathrm{lbl}}(n-k)}{(n-k!)} = O\left(\frac{b_{\mathrm{lbl}}(n-1)}{(n-1)!} \right).$$

Using the chain rule on (5.1) and implicit differentiation on (5.2), we have

$$\frac{\partial}{\partial y} H_{\mathrm{lbl}}(x,y) = \left. \frac{S(c,t)^3}{(1+y)^2 \left(1 - \frac{2cS(c,t)}{1+c} - 3tS(c,t)^2\right)} \right|_{\substack{c=e^1-1 \\ t=2e^{-x}-1-(1+y)^{-1}}}$$

and at $(x,y) = (0,0)$, it follows that

$$C = \left. \frac{\partial}{\partial y} H_{\mathrm{lbl}}(x,y) \right|_{(x,y)=(0,0)} = 1.$$

So, by Theorem 5.5, we have

$$\frac{p_{\mathrm{lbl}}(n)}{n!} = \frac{b_{\mathrm{lbl}}(n)}{n!} + O\left(\frac{b_{\mathrm{lbl}}(n-1)}{(n-1)!} \right) \sim \frac{b_{\mathrm{lbl}}(n)}{n!}.$$ □

Proof of Theorem 5.4(ii). Let $H_{\mathrm{unl}}(x,y)$ be the formal power series in x and y defined by

$$H_{\mathrm{unl}}(x,y) = S(x(1-x)^{-1}, 1 - 2x - (1+y)^{-1}), \tag{5.3}$$

where $S(c,t)$ is again the formal power series solution of (5.2) as defined in Theorem 3.15. From Equation (1.3), we have that

$$H_{\mathrm{unl}}(x, B_{\mathrm{unl}}(x) - 1) = \sum_{n \geq 0} p_{\mathrm{unl}}(n) x^n.$$

Again we check the conditions of Theorem 5.5. As in the labelled case above, $H_{\mathrm{unl}}(x,y)$ is analytic by construction so Condition 1 holds. From Proposition 5.2, we have $b_{\mathrm{unl}}(n) \sim b_{\mathrm{lbl}}(n)/n!$, and since the coefficients $b_{\mathrm{lbl}}(n)/n!$ satisfy conditions 2 and 3, so do the coefficients $b_{\mathrm{unl}}(n)$.

Using the chain rule on (5.3) and implicit differentiation on (5.2), we have We can now apply Theorem 5.5. Using the chain rule

$$\frac{\partial}{\partial y} H_{\mathrm{unl}}(x,y) = \left. \frac{S(c,t)^3}{(1+y)^2 \left(1 - \frac{2cS(c,t)}{1+c} - 3tS(c,t)^2\right)} \right|_{\substack{c=x(1-x)^{-1} \\ t=1-2x-(1+y)^{-1}}}$$

and at $(x,y)=(0,0)$, it follows that

$$C = \left. \frac{\partial}{\partial y} H_{\mathrm{unl}}(x,y) \right|_{(x,y)=(0,0)} = 1.$$

So, by Theorem 5.5, we have

$$p_{\mathrm{unl}}(n) = b_{\mathrm{unl}}(n) + O(b_{\mathrm{unl}}(n-1)) \sim b_{\mathrm{unl}}(n). \qquad \square$$

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Appendix

In this appendix we show that almost all bicoloured graphs are asymmetric, since we are not aware of this result appearing (or being a direct consequence of a result appearing) in the literature.

A labelled bicoloured graph on a set of n vertices, say $V = \{1,2,\dots n\}$, can be defined as a triple $G = (A,B,E)$, where the sets $A \subseteq V$ and $B = V \setminus A$ are the colour classes of vertices, and $E \subseteq A \times B$ is the set of edges. The symmetric group S_n acts on these triples by relabelling the vertices, and for $\pi \in S_n$, we can write $\pi \cdot G = (\pi \cdot A, \pi \cdot B, \pi \cdot E)$. An equivalence class of these labelled graphs under relabelling (that is, an orbit under the action of S_n) is called an *unlabelled* bicoloured graph on n vertices.

Let $b_{\mathrm{lbl}}(n)$ and $b_{\mathrm{unl}}(n)$ be the numbers, respectively, of labelled and unlabelled bicoloured graphs on n vertices. We want to show that, as $n \to \infty$, we have $b_{\mathrm{lbl}}(n) \sim n! \cdot b_{\mathrm{unl}}(n)$. This is equivalent to the statement that, asymptotically, almost every unlabelled bicoloured graph has a trivial automorphism group (which is stronger than the statement that almost every *labelled* bicoloured graph has a trivial automorphism group). Although it is not a direct consequence, this result is not surprising in view of the corresponding result for the class of all graphs; this appendix gives a modification of the argument of Erdős and Rényi [7] to the case of bicoloured graphs.

A.1. **Stating the problem.** To obtain an expression for $b_{\mathrm{lbl}}(n)$, note that a labelled bicoloured graph can be obtained by first choosing the colour class A freely among subsets of V, which uniquely determines the colour class $B = V \setminus A$, and then choosing the edge set E freely among subsets of $A \times B$. If we group these graphs by the size of A, we get

$$b_{\mathrm{lbl}}(n) = \sum_{A \subseteq V} \sum_{E \subseteq A \times B} 1 = \sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)}.$$

To obtain a similar expression for $b_{\mathrm{unl}}(n)$, we can use Burnside's lemma. Given colour classes A and B, the permutation in S_n which fix them are the permutations in the subgroup $S_A \times S_B$, where S_A is the symmetric group on the set A, and similarly for B. A permutation $\pi \in S_A \times S_B$ fixes an edge set $E \subseteq A \times B$ exactly when E is a union of orbits of the action of π on $A \times B$, so we get

$$\begin{aligned} b_{\mathrm{unl}}(n) &= \frac{1}{n!} \sum_{\pi \in S_n} |\{G = (A, B, E) \mid \pi \cdot G = G\}| \\ &= \frac{1}{n!} \sum_{A \subseteq V} \sum_{\pi \in S_A \times S_B} 2^{(\#\text{ of orbits of } \pi \text{ in } A \times B)}. \end{aligned}$$

If the colour class A has k elements, then the number of orbits of π in $A \times B$ is at most $k(n-k)$, which is the case for $\pi = \mathrm{id}$, the identity permutation. For an arbitrary permutation $\pi \in A \times B$, let

$$r(\pi) = k(n-k) - (\#\text{ of orbits of } \pi \text{ in } A \times B)$$

be the *redundancy* of π with respect to the bipartition (A, B). This number only depends on the (coloured) cycle type of π, and not on the labels of the vertices in A and B, so we can identify the group $S_A \times S_B$ with $S_k \times S_{n-k}$ and write

$$b_{\mathrm{unl}}(n) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} \sum_{\pi \in S_k \times S_{n-k}} 2^{-r(\pi)}.$$

To show that $b_{\mathrm{lbl}}(n) \sim n! \cdot b_{\mathrm{unl}}(n)$, our approach will be to show that most bicoloured graphs are *balanced*, in the sense that each half of the bipartition (A, B) contains roughly half of the n vertices; and then to show that for balanced bicoloured graphs, most permutations have a high redundancy.

Before proceeding, however, let us rephrase the problem slightly. Showing that $b_{\mathrm{lbl}}(n) \sim n! \cdot b_{\mathrm{unl}}(n)$ as $n \to \infty$ is equivalent to showing that

$$\frac{n! b_{\mathrm{unl}}(n)}{b_{\mathrm{lbl}}(n)} = \frac{\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} \sum_{\pi \in S_k \times S_{n-k}} 2^{-r(\pi)}}{\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)}} \longrightarrow 1.$$

Since the identity permutation has redundancy $r(\mathrm{id}) = 0$, this is in turn equivalent to showing that

$$\text{(A.1)} \qquad \frac{n! b_{\mathrm{unl}}(n) - b_{\mathrm{lbl}}(n)}{b_{\mathrm{lbl}}(n)} = \frac{\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} \sum_{\pi \in S_k \times S_{n-k} \setminus \{\mathrm{id}\}} 2^{-r(\pi)}}{\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)}} \longrightarrow 0.$$

Since this is clearly a non-negative quantity, it suffices to provide a vanishing upper bound.

A.2. **Reduction to the balanced case.** We will consider a bicoloured graph $G = (A, B, E)$ on n vertices to be balanced if the size k of the colour class A (and hence the size $n-k$ of the colour class B) is in the range $\left[\frac{1}{2}(n-n^{2/3}), \frac{1}{2}(n+n^{2/3})\right]$. Then, for unbalanced graphs, we have $k(n-k) \leq \frac{1}{4}(n^2 - n^{4/3})$. Also, since $2^{-r(\pi)} \leq 1$, we have the bound

$$\binom{n}{k} \sum_{\pi \in S_k \times S_{n-k}} 2^{-r(\pi)} \leq n!.$$

Then, the contribution of the unbalanced graphs to the numerator $n! b_{\mathrm{unl}}(n) - b_{\mathrm{lbl}}(n)$ in (A.1) can be bounded above by

$$\sum_{k \notin \left[\frac{n-n^{2/3}}{2}, \frac{n+n^{2/3}}{2}\right]} \binom{n}{k} 2^{k(n-k)} \sum_{\pi \in S_k \times S_{n-k} \setminus \{\mathrm{id}\}} 2^{-r(\pi)} \leq 2^{(n^2 - n^{4/3})/4} n!.$$

For the denominator $b_{\mathrm{lbl}}(n)$, we have a lower bound given by looking at the term for $k = \lfloor \frac{1}{2} n \rfloor$, which is bounded below by

$$\binom{n}{k} 2^{k(n-k)} \geq 2^{(n^2-1)/4}.$$

The ratio of these two bounds is $2^{(1-n^{4/3})/4} n!$, which vanishes as $n \to \infty$, as can be seen by using Stirling's approximation for $n!$.

A.3. **Redundancy in the balanced case.** Given a size k for the colour class A in the range $\left[\frac{1}{2}(n-n^{2/3}), \frac{1}{2}(n+n^{2/3})\right]$, we will now bound the quantity

$$\sum_{\pi \in S_A \times S_B \setminus \{\mathrm{id}\}} 2^{-r(\pi)}$$

by giving an estimate for the redundancy $r(\pi)$. Given a permutation $\pi \in S_A \times S_B$, let $\mathrm{Orb}(\pi, A \times B)$ be the set of orbits of the action of π on the set $A \times B$. Then, we have

$$r(\pi) = k(n-k) - |\mathrm{Orb}(\pi, A \times B)| = \sum_{\mathcal{O} \in \mathrm{Orb}(\pi, A \times B)} |\mathcal{O}| - 1.$$

Let $(a_1\, a_2\, \ldots\, a_p)$ be one of the cycles of π on A, so that $\pi(a_1) = a_2$, $\pi(a_2) = a_3$, $\ldots$, $\pi(a_p) = a_1$. The action of π on $A \times B$ restricts to an action on the subset $\{a_1, \ldots, a_p\} \times B$, and every orbit of this action contains at least one element in $\{a_1\} \times B$. As long the cycle is non-trivial, so that $p \geq 2$, this gives a contribution of at least

$$\sum_{\mathcal{O} \in \mathrm{Orb}(\pi, \{a_1, \ldots, a_p\} \times B)} |\mathcal{O}| - 1 \geq (p-1)\,|B| \geq \frac{p}{2} \cdot \frac{n - n^{2/3}}{2}$$

to the redundancy $r(\pi)$. In particular, if $d_A(\pi)$ is the number of elements $a \in A$ with $\pi(a) \neq a$, we have $r(\pi) \geq \frac{1}{4} d_A(\pi)(n - n^{2/3})$. Symmetrically, we have $r(\pi) \geq \frac{1}{4} d_B(\pi)(n - n^{2/3})$, where $d_B(\pi)$ is the number of elements $b \in B$ with $\pi(b) \neq b$. If $d(\pi)$ is the total number of non-fixed points of π, then we have $\max\{d_A(\pi), d_B(\pi)\} \geq \frac{1}{2} d(\pi)$, so we have the bound

$$r(\pi) \geq \tfrac{1}{8} d(\pi)(n - n^{2/3}).$$

Now, consider the permutations $\pi \in S_A \times S_B$ with $d(\pi)$ in the interval $[\ell, u]$ for some parameters ℓ and u. It is straightforward to show that the number of these permutations is less than n^u. Together with the estimate above for $r(\pi)$, this gives the bound

$$\log_2 \sum_{\substack{\pi \in S_A \times S_B \setminus \{\mathrm{id}\} \\ d(\pi) \in [\ell, u]}} 2^{-r(\pi)} \leq u \log_2 n - \frac{1}{8} \ell (n - n^{2/3}).$$

For each of the intervals $[\ell, u] = [1, n^{1/3}]$ and $[\ell, u] = [n^{1/3}, n]$, this bound goes to $-\infty$ as $n \to \infty$. It follows that there is a bound $F(n)$ such that, as $n \to \infty$, we have

$$\sum_{\pi \in S_k \times S_{n-k} \setminus \{\mathrm{id}\}} 2^{-r(\pi)} \leq F(n) \longrightarrow 0$$

for k in the range $\left[\frac{n - n^{2/3}}{2}, \frac{n + n^{2/3}}{2}\right]$. Thus, we have

$$\frac{\sum_{k \in \left[\frac{n-n^{2/3}}{2}, \frac{n+n^{2/3}}{2}\right]} \binom{n}{k} 2^{k(n-k)} \sum_{\pi \in S_k \times S_{n-k} \setminus \{\mathrm{id}\}} 2^{-r(\pi)}}{\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)}} \leq F(n) \longrightarrow 0$$

as $n \to \infty$, which completes the proof that $b_{\mathrm{lbl}}(n) \sim n! \cdot b_{\mathrm{unl}}(n)$ asymptotically.

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