The number of {1243, 2134}-avoiding permutations

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Abstract

We show that the counting sequence for permutations avoiding both of the (classical) patterns 1243 and 2134 has the algebraic generating function supplied by Vaclav Kotesovec for sequence A164651 in The On-Line Encyclopedia of Integer Sequences.

1 Introduction

Several authors have developed methods to count permutations avoiding a given set of patterns; see, for example, the references in the Wikipedia entry [1]. In particular, enumeration schemes have been developed for automated counting [2, 3]. When successful, an automated method produces an enumeration scheme that yields the initial terms of the counting sequence (perhaps 20 or more) and sometimes these terms appear to have an algebraic generating function that does not follow readily from the enumeration scheme. Here, we treat one such case. The counting sequence for permutations that avoid both of the (classical) patterns 1243 and 2134 begins 1, 2, 6, 22, 87, 354, ..., sequence A164651 in The On-Line Encyclopedia of Integer Sequences [4]. In a comment on this sequence dated Oct 24 2012, Vaclav Kotesovec observed that the generating function

$$\frac{3x^2 - 9x + 2 + x(1-x)\sqrt{1-4x}}{2(x-1)(x^2+4x-1)}$$

fits the known terms of the sequence.

We will show that {1243, 2134}-avoiders do indeed have this generating function. Defining a *start-small* permutation to be one that does not start with its largest entry, the proof rests on a bijection ϕ from start-small {1243, 2134}-avoiders of length n to lists of start-small 123-avoiders whose total length is $n-1+$ the length of the list. Under this bijection ϕ, for example, $12345 \rightarrow (12, 12, 12, 12)$, a list of length 4 with total length of its entries $= 8$, and $3412 \rightarrow (3412)$, a singleton list. Now, 123-avoiders are famously counted by the Catalan numbers and the well known combinatorial interpretation of the transform $(a_n)_{n\geq 1} \rightarrow (b_n)_{n\geq 1}$, defined via generating functions by

$$1 + B(x) = \frac{1}{1 - A(x)},$$

permits counting these lists of start-small 123-avoiders. The generating function for start-small {1243, 2134}-avoiders thus obtained readily yields the generating function for unrestricted {1243, 2134}-avoiders.

In Section 2, we show that ϕ is given by iteration of a more basic bijection, which is presented in Section 3. Finally, Section 4 gives the bookkeeping details to obtain the desired generating function from ϕ.

2 Reduction of problem

A *mid-123* entry in a permutation is an entry that serves as the "2" in a 123 pattern. A *key* mid-123 entry is a mid-123 entry b whose immediate predecessor is either $< b$ or a right-to-left maximum (max for short). For example, the mid-123 entries in 1 3 4 5 2 6 are $3, 4, 5, 2$ but only $3, 4$, and 5 are key.

Lemma 1. *A permutation with no key mid-*123 *entries has no mid-*123 *entries at all and so is a* 123-*avoider.*

Proof. Suppose the ith entry π_i of a permutation π is a mid-123 entry but not key. Then, by definition of key, we have $\pi_{i-1} > \pi_i$ and π_{i-1} is not a right-to-left max. So π_{i-1} is also a mid-123 entry. Again, if π_{i-1} is not key, then $\pi_{i-2} > \pi_{i-1}$ and π_{i-2} is not a right-to-left max. Iterating this process, we must eventually arrive at a key mid-123 entry. Hence, every permutation with a mid-123 entry has a key mid-123 entry . □

Lemma 2. *Suppose b is the last mid-*123 *entry in a* {1243, 2134}-*avoider. Then there is a unique c such that bc forms the "23" of a* 123 *pattern.*

Proof. Suppose abc and abc' are 123 patterns with $c \neq c'$, say $c < c'$. If c precedes c' in the permutation, then c is a mid-123, violating the hypothesis on b. If c follows c', then $abc'c$ is a proscribed 1243. □

Let $\mathcal{A}_n$ denote the set of start-small {1243, 2134}-avoiders on $[n]$, and $\mathcal{A}_{n,k}$ the subset with k key mid-123 entries.

To produce the promised bijection ϕ from $\mathcal{A}_n$ to lists of start-small 123-avoiders whose total length is $n-1+$ the length of the list, it suffices to exhibit, for $0 \le k \le n-2$, a bijection

$$\mathcal{A}_{n,k} \to \big\{(\sigma_1,\ldots,\sigma_{k+1}) : \begin{array}{l}\text{each } \sigma_i \text{ is a start-small 123-avoider,} \\ \quad \text{lengths of the } \sigma_i\text{'s sum to } n+k\end{array}\big\}.$$

For $k=0$, naturally the bijection is $\pi \to (\pi)$, a singleton list, because, by Lemma 1, π is already 123-avoiding. For $1 \le k \le n-2$ and $k+1 \le j \le n-1$ let

$$\mathcal{A}_{n,k,j} = \{\pi \in \mathcal{A}_{n,k} : \text{ the last mid-123 entry of } \pi \text{ is in position } j\}.$$

The result of the next Section extracts from a start-small {1243, 2134}-avoider that *does* contain 123's two {1243, 2134}-avoiders, both start-small, the first with *one fewer* key mid-123 entries than the original and the second with *no* key mid-123 entries. Iteration then gives the desired bijection ϕ.

3 The crucial bijection

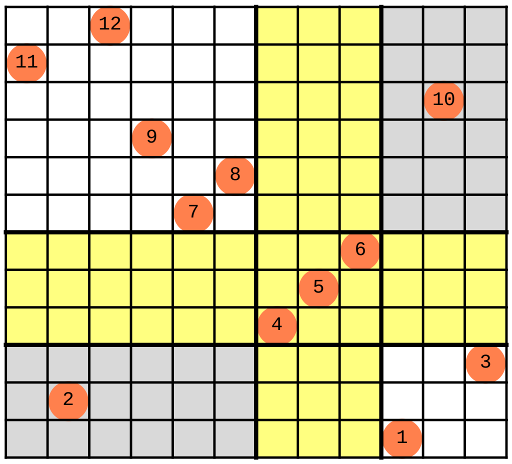
Proposition 3. *For $1 \le k < j \le n-1$, there is a bijection*

$$\mathcal{A}_{n,k,j} \to \mathcal{A}_{j,\,k-1} \times \mathcal{A}_{n+1-j,\,0}\,.$$

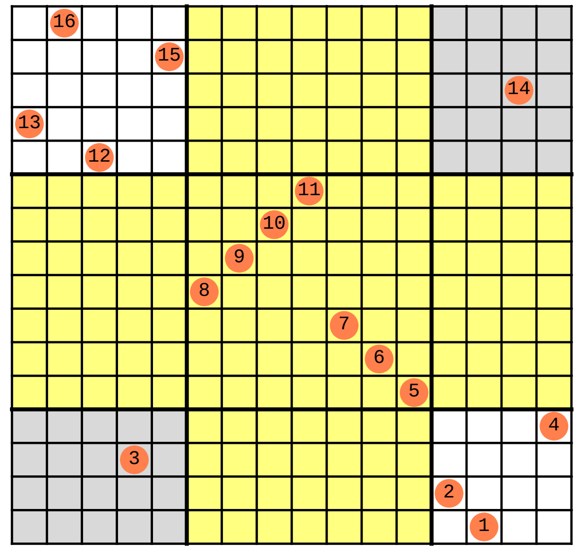
Proof. Given $\pi \in \mathcal{A}_{n,k,j}$, we need to reversibly produce $\sigma_1 \in \mathcal{A}_{j,\,k-1}$ and $\sigma_2 \in \mathcal{A}_{n+1-j,\,0}$. Write π as $\tau_1 b \tau_2$ where b is the last mid-123 entry in π, and let abc be the 123 pattern in π with smallest a (c is uniquely determined by Lemma 2). Concatenate a and τ_2 and standardize (replace smallest entry by 1, next smallest by 2, and so on) to get the desired σ_2 with no 123's.

Concatenate τ_1 and c to get a {1243, 2134}-avoider ρ—a candidate (after standardization) for σ_1. This ρ may need further processing because of two glitches: ρ may still have k key mid-123's instead of the required $k-1$ and ρ cannot end with its smallest entry (which must be possible in σ_1). But these glitches cancel out.

If b is a key mid-123 in π, then ρ has $k-1$ key mid-123's because b has been lost. In this case, just standardize ρ to get σ_1. Otherwise, the last entry of τ_1 exceeds b but is not a right-to-left max. Delete from ρ the longest terminal string of τ_1 that is decreasing but does not contain a right-to-left max of π (equivalently, does not contain an entry $> c$);



A {1243, 2134}-avoider,
last mid-123, b = 6, is key



A {1243, 2134}-avoider,
last mid-123, b = 5, is not key

say $r \geq 1$ entries are deleted. Add r to each remaining entry of ρ, append the entries $r, r-1, \ldots, 1$ and standardize to obtain σ_1.

For example, for $\pi = 11\ 2\ 12\ 9\ 7\ 8\ 4\ 5\ 6\ 1\ 10\ 3$, we find that the last mid-123, $b = 6$, is a key mid-123, and $a = 2,\ c = 10$. So $\sigma_2 = \text{standardize}(2\ 1\ 10\ 3) = 2\ 1\ 4\ 3$ and $\sigma_1 = \text{standardize}(11\ 2\ 12\ 9\ 7\ 8\ 4\ 5\ 10) = 8\ 1\ 9\ 6\ 4\ 5\ 2\ 3\ 7$.

As another example, for $\pi = 13\ 16\ 12\ 3\ 15\ 8\ 9\ 10\ 11\ 7\ 6\ 5\ 2\ 1\ 14\ 4$, we find that the last mid-123, $b = 5$, is *not* key, and $a = 3,\ c = 14$. So $\sigma_2 = \text{standardize}(3\ 2\ 1\ 14\ 4) = 3\ 2\ 1\ 5\ 4$. Here, $\tau_1 = 13\ 16\ 12\ 3\ 15\ 8\ 9\ 10\ 11\ 7\ 6$ and the longest decreasing terminal string of τ_1 that does not contain a right-to-left max of π has length $r = 3$. So $\sigma_1 = \text{standardize}(3 + (13\ 16\ 12\ 3\ 15\ 8\ 9\ 10$ ~~11 7 6~~ $14)\ 3\ 2\ 1) = \text{standardize}(16\ 19\ 15\ 6\ 18\ 11\ 12\ 13\ 17\ 3\ 2\ 1) = 9\ 12\ 8\ 4\ 11\ 5\ 6\ 7\ 10\ 3\ 2\ 1$.

The invertibility of this map rests on structural properties of {1243, 2134}-avoiders evident in the matrix diagrams above: consider scanning the entries leftward from the last mid-123 entry b. In case b is key, as long as these entries decrease, they decrease by 1 (in square box at center of yellow region). In case b is not key, they increase by 1 until either an entry $> c$, a decrease, or a jump > 1 to an entry $< c$ occurs. In the latter case, the "missing" entries in the jump occur in increasing order immediately to the left of the jump (again in square box at center of yellow region). In both cases, the gray regions are empty except for the "a" and "c", and the yellow square box is bisected by the main diagonal.

The preceding observations, all immediate consequences of the patterns 1243 and 2134 being proscribed and b being the last mid-123, validate the following description of the inverse map. For a permutation σ, we use $\sigma[j]$ to denote the entry in position j and $\sigma[j,k]$ to denote the list of entries occupying positions j through k.

Suppose given a $\{1243,\ 2134\}$-avoider σ_1 and a 123-avoider σ_2 and we wish to recapture π. We have $n = \text{length}(\sigma_1) + \text{length}(\sigma_2) - 1$ and $j = \text{length}(\sigma_1)$. Define the integer r by writing σ_1 in the form $\mu, r, r-1, \ldots, 2, 1$ with r maximal, where it is understood that μ is σ_1 and r is 0 if σ_1 does not end with 1. Set $p = j - r$, the length of μ, and $q = \text{length}(\sigma_2) - 1$. The positions i and k of the "a" and "c" respectively in π are given by $i =$ position in μ of its minimum entry, and $k = j - 1 +$ position in σ_2 of its maximum entry. The values of "a" and "c" are recaptured as $a = \sigma_2[1]$ and $c = \sigma_1[p] + q$. Define s to be the length of the longest increasing terminal string in $\sigma_1[i+1, p]$.

Now concatenate the strings

$$\sigma_1[1, p-s] + q,\ \ \sigma_1[p-s+1, p-1] + q-1,\ \ n-j+r+s,\ \ \sigma_1[p+1, j] + q,\ \ \sigma_2[2, q+1],$$

where the next to last string is empty when $p = j$, equivalently, when $r = 0$.

Finally, to obtain π from the resulting string, overwrite, with a and c respectively, the entries in the positions that a and c should occupy, namely, positions i and k.

For instance, the second example above yielded

$$\sigma_1 = 9\ 12\ 8\ 4\ 11\ \overset{\leftarrow\ s\ \longrightarrow}{5\ 6\ 7\ 10}\ \overset{\leftarrow\ r\ \rightarrow}{3\ \ 2\ \ 1}\,, \qquad \sigma_2 = 3\ 2\ 1\ 5\ 4\,,$$

and, to reverse the map, we find $n = 16,\ j = 12,\ r = 3,\ p = 9,\ q = 4,\ s = 4,\ i = 4,\ k = (j-1) + 4 = 15,\ a = \sigma_2[1] = 3,\ c = \sigma_1[9] + 4 = 14$. The strings to be concatenated are

$$(9\ 12\ 8\ 4\ 11) + 4,\ \ (5\ 6\ 7) + 3,\ \ 11,\ \ (3\ 2\ 1) + 4,\ \ (2\ 1\ 5\ 4)\,,$$

yielding

$$13\ 16\ 12\ \cancel{8}\ 15\ 8\ 9\ 10\ 11\ 7\ 6\ 5\ 2\ 1\ \cancel{5}\ 4\,,$$

with entries in positions $i = 4$ and $k = 15$ crossed out. Replace them with $a = 3$ and $c = 14$, respectively, to recover π.

4 Putting it all together

A *k-list* is a list of length k. Recall that if $\mathcal{A}$ is a class (species) of combinatorial structures with a_n structures of size n ($n \geq 1$), and a *compositional* $\mathcal{A}$-structure of size n is one obtained by taking a composition $(n_1, n_2, \ldots, n_k)$ of n and forming a k-list of $\mathcal{A}$-structures of respective sizes $n_1, n_2, \ldots, n_k$, then the counting sequence $(b_n)_{n\geq 1}$ for compositional $\mathcal{A}$-structures has generating function $B(x) := \sum_{n\geq 1} b_n x^n$ given by

$$1 + B(x) = \frac{1}{1 - A(x)},$$

where $A(x)$ is the generating function for $\mathcal{A}$-structures. This is known as the INVERT transform [5] but bearing the combinatorial interpretation in mind, it could as well be called the Compositional transform. We will apply it to the class of (nonempty) start-small 123-avoiders with size measured as "length minus 1" (note that there is no start-small 123-avoider of length 1).

With $C(x) := \frac{1-\sqrt{1-4x}}{2x}$ denoting the generating function for the Catalan numbers C_n, which are well known to count Dyck paths of n upsteps, we compute

$$\begin{aligned}
[x^n]\, C(x)^3 \;&=\; \#\text{ ordered triples } (P, Q, R) \text{ of Dyck paths with a total of } n \text{ upsteps} \\
&=\; \#\text{ Dyck paths of } n+2 \text{ upsteps that start } UU \;\big((P,Q,R) \to UUPDQDR\big) \\
&=\; \#\text{ start-small 123-avoiders on } [n+2] \text{ (via Krattenthaler's bijection [6])}\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
[x^n]\, x\, C(x)^3 \;=\; [x^{n-1}]C(x)^3 \;&=\; \#\text{ start-small 123-avoiders on } [n+1] \\
&=\; \#\text{ start-small 123-avoiders of size } n\,,
\end{aligned}$$

and, using the Compositional transform, we have for $n \geq 1$,

$$\begin{aligned}
[x^n]\, \frac{1}{1 - x\, C(x)^3} \;&=\; \#\text{ lists of start-small 123-avoiders of total size } n \\
&=\; \sum_{k=1}^{n} \#\; k\text{-lists of start-small 123-avoiders of total length } n+k\,.
\end{aligned}$$

Hence, for $n \geq 2$,

$$\begin{aligned}
[x^n]\, \frac{x}{1 - x\, C(x)^3} \;&=\; \sum_{k=1}^{n-1} \#\; k\text{-lists of start-small 123-avoiders of total length } n-1+k \\
&=\; \#\text{ start-small } \{1243,\, 2134\}\text{-avoiders of length } n \text{ (via the bijection } \phi)\,,
\end{aligned}$$

from which it is immediate that the generating function for start-small {1243, 2134}-avoiders (with x marking length) is

$$G(x) = 1 + \frac{x}{1 - x\, C(x)^3} - x\,.$$

Now let $(u_n)_{n\geq 0} = (1, 1, 2, 6, \dots)$ and $(v_n)_{n\geq 0} = (1, 0, 1, 4, \dots)$ denote the counting sequences for {1243, 2134}-avoiders and start-small {1243, 2134}-avoiders respectively. Clearly, $v_n = u_n - u_{n-1}$ for $n \geq 1$, (consider deletion of the first entry from a {1243, 2134}-avoider on $[n]$ that starts n). So the generating functions $F(x) = \sum_{n\geq 0} u_n x^n$ and $G(x) = \sum_{n\geq 0} v_n x^n$ are related by $F(x) = G(x)/(1-x)$. Thus

$$F(x) = \frac{G(x)}{1-x} = \frac{1 + \frac{x}{1-x\,C(x)^3} - x}{1-x}$$

which, after expansion, agrees with Kotesovec's formula.

Losonczy [7] has counted permutations that avoid 3421, 4312 and 4321 or equivalently (by reversal) both of the patterns treated here and 1234.

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