

Topological and symmetry broken phases of \mathbb{Z}_N parafermions in one dimension

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Abstract. We classify the gapped phases of \mathbb{Z}_N parafermions in one dimension and construct a representative of each phase. Even in the absence of additional symmetries besides parafermionic parity, parafermions may be realized in a variety of phases, one for each divisor n of N . The phases can be characterized by spontaneous symmetry breaking, topology, or a mixture of the two. Purely topological phases arise if n is a unitary divisor, i.e. if n and N/n are co-prime. Our analysis is based on the explicit realization of all symmetry broken gapped phases in the dual \mathbb{Z}_N -invariant quantum spin chains.

1. Introduction

Topological phases of matter received a lot of interest recently due to their peculiar properties such as the existence of robust edge modes, non-abelian excitations and their potential applications to quantum computation [1, 2]. While one-dimensional bosonic systems do not exhibit intrinsic topological order [3], the situation is very different for fermions. According to the general theory, there should exist two phases of fermions which are described by a \mathbb{Z}_2 topological invariant [4, 5, 6]. An explicit realization of both phases is provided by Kitaev's Majorana chain which models the dynamics of spinless fermions in a quantum wire, in proximity with a p-wave superconductor [7].

The most remarkable feature of Kitaev's model is the emergence of stable gapless Majorana edge modes in the topologically non-trivial phase [7]. In view of their non-abelian statistics, being able to create and to manipulate such isolated Majorana fermions would be a major step towards the realization of topological quantum computers [8]. While Kitaev's chain merely serves as a proof of principle, by now more realistic experimental setups for the creation and detection of Majorana fermions have been suggested [9, 10]. The basic ingredients are spin orbit quantum wires in proximity with an s-wave superconductor and in a magnetic field. Recent experiments based on this setup provide first evidence that Majorana fermions indeed exist [11, 12] even though at least part of the observed features could also be explained by disorder effects [13, 14, 15].

More recently, it has been investigated whether similar phenomena can be realized in heterostructures involving edges of *fractional* topological insulators. It was found that such systems can be used to isolate parafermions which may be regarded as fractionalized Majorana fermions [16, 17, 18, 19]. These efforts received their motivation from notes of P. Fendley on the existence of edge modes in parafermionic chains, unpublished for a long time but eventually made available in Ref. [20]. Bulk lattice defects with parafermionic statistics emerged independently in the study of fractional Chern insulators [21] and \mathbb{Z}_N rotor models [22], and a general framework characterizing the statistics of all these defects was developed in [23, 24]. Furthermore, two-dimensional arrays of parafermions have been studied in [25].

The study of parafermions has a long history. The exploration of para-particles began in Refs. [26, 27] in an attempt to generalize the canonical commutation relations of bosons and fermions. In contrast, these days the predominant notion of parafermions arises in the context of statistical physics, in connection with the so-called \mathbb{Z}_N clock model [28]. This model generalizes the \mathbb{Z}_2 symmetry of the Ising model in a natural way to the \mathbb{Z}_N symmetry of a clock with N marks.‡

Just as the Ising model may be mapped to a problem of free fermions, using a variation of the non-local Jordan-Wigner transformation, the \mathbb{Z}_N clock model exhibits a dual description in terms of fundamental parafermionic degrees of freedom. The parafermionic operators satisfy a \mathbb{Z}_N -graded variant of the usual Clifford algebra relations. This duality generalizes the well-known duality between the Ising model and free fermions for $N = 2$. It is known that the topologically trivial and the non-trivial phase of the Majorana chain are equivalent to the ordered and the disordered phase of the Ising model, respectively [31].

The previous paragraphs suggest two natural avenues to the classification of parafermionic chains. One first of all can generalize the group theoretical approach of Ref. [6] that has been applied successfully to enumerate and characterize the different phases of Majorana chains with or without time-reversal symmetry. This approach has the advantage of yielding the maximal set of *possibilities* in a clean and conceptually clear way. On the other hand, the proof that all topological phases can actually be realized requires a method of combining phases such that their topological charges add up. This procedure might not necessarily lead to the most natural representative of the new phase. Moreover, the approach does not reveal the possible patterns of symmetry breaking. This approach has been followed recently in Ref. [32] for purely topological phases.

The second approach which shall be followed in this paper is a complete characterization of gapped symmetric and symmetry broken phases of \mathbb{Z}_N quantum spin chains. These can be engineered systematically on the level of ground states and their associated frustration free parent Hamiltonians [33]. Via the duality mapping mentioned before, this classification provides an explicit characterization of all parafermionic phases which cannot be connected without crossing a phase transition, i.e. closing the gap.

We shall find that the distinct parafermionic phases are labeled by the possible

‡ The name “parafermions” is sometimes also used for the basic fields in a family of \mathbb{Z}_N symmetric conformal field theories that arise at critical points of so-called clock models [29]. These parafermions may be used to construct the Read-Rezayi states, i.e. they are of relevance for the study of fractional quantum Hall systems [30]. In the present paper, we shall be concerned exclusively with the gapped phases of \mathbb{Z}_N parafermions.

divisors of N . Generically, they may exhibit a coexistence of both ferromagnetic and topological order. In practice, this means that only part of the ground state degeneracy of an open chain is topologically protected and due to gapless edge modes, while the remaining degeneracy can be lifted by spontaneous symmetry breaking. When the chain is closed, only the topological degeneracy disappears, but a non-trivial ground state degeneracy persists in general. To be more specific, for a given divisor n of N , the phase will be purely topological if N and N/n are co-prime, and it will exhibit a unique ground state in a closed system and gapless edge modes in an open system. If instead $N/n = \text{gcd}(n, N/n)$ the phase is topologically trivial and entirely characterized by the existence or absence of symmetry breaking. The details of the classification will be explained in the main text through the concrete realization of each individual phase.

We would like to stress that our classification distinguishes only phases which cannot be connected by any local \mathbb{Z}_N -invariant interaction. Clearly, specific models with an explicit Hamiltonian depending on a given set of coupling constants may have arbitrarily complicated phase diagrams.

The paper is organized as follows. In Section 2 we introduce the kinematical setup that we are working with. This includes a detailed discussion of the symmetry algebra and of the Hilbert spaces in question. The possible symmetry breaking patterns and order parameters of the \mathbb{Z}_N quantum spin chain are analyzed in Section 3 and a representative Hamiltonian is constructed for each phase. These results form the heart of our paper. Finally, we perform a non-local duality transformation in Section 4 and interpret our previous results from a parafermionic perspective. As it turns out, the original symmetry broken phases of the spin chains generally exhibit a mixture of symmetry breaking and topological protection. We end with concluding remarks and an outlook to open problems.

2. Parafermions and \mathbb{Z}_N spin chains

We introduce the mathematical framework within which our considerations take place. This includes a detailed discussion of the symmetry algebra and of the Hilbert spaces involved.

2.1. \mathbb{Z}_N spin chains

\mathbb{Z}_N quantum spin chains are \mathbb{Z}_N symmetric models where the spins can show in any of N directions in the plane which can be thought of as corresponding to the N distinct N^{th} roots of unity on the unit circle. This setup generalizes the familiar \mathbb{Z}_2 case of the Ising model where one deals with spins pointing either up or down, and at the same time, provides a specialization of the XY model in which the spins are restricted to take discrete values.

If we denote the size of the chain by L , the Hilbert space of the chain will be given by

$$\mathcal{H} = (\mathbb{C}^N)^{\otimes L}. \quad (2.1)$$

We choose a basis in \mathcal{H} given by $|q_1, \dots, q_L\rangle$, $q_i \in \mathbb{Z}_N$. (This means that $|q\rangle \equiv |q+N\rangle$.)

Define the operators

$$\sigma_i = \mathbf{1} \otimes \cdots \otimes \underbrace{\sigma}_i \otimes \mathbf{1} \cdots \otimes \mathbf{1} \quad (2.2)$$

$$\tau_i = \mathbf{1} \otimes \cdots \otimes \underbrace{\tau}_i \otimes \mathbf{1} \cdots \otimes \mathbf{1}, \quad (2.3)$$

where σ and τ are

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \omega^{N-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 1 \\ 1 & \dots & 0 & 0 & 0 \end{pmatrix}. \quad (2.4)$$

σ, τ satisfy the relations:[§]

$$X^N = 1, X^\dagger = X^{-1}, \quad \text{for } X = \sigma, \tau \quad (2.5)$$

$$\sigma^r \tau^s = \omega^{rs} \tau^s \sigma^r, \quad \omega = e^{2\pi i/N}, \quad r, s \in \mathbb{Z}_N, \quad (2.6)$$

and as obvious from the definition, σ_i, τ_j commute if $i \neq j$. Note that for $N = 2$: $\sigma = \sigma^z, \tau = \sigma^x$. The space of linear operators on \mathcal{H} is spanned by monomials $\prod_{i=1}^L \sigma_i^{p_i} \tau_i^{q_i}, p_i, q_i \in \mathbb{Z}_N$.

The global symmetry $G = \mathbb{Z}_N$ is generated by the operator shifting the labels by one unit on every site:

$$P = \prod_{i=1}^L \tau_i^\dagger, \quad (2.7)$$

and each tensorand \mathbb{C}^N in the Hilbert space \mathcal{H} is direct sum of N one-dimensional irreducible representations of \mathbb{Z}_N . The Hamiltonians H we are going to consider will be invariant under the action of \mathbb{Z}_N . In other words, $[H, P] = 0$.

2.2. Jordan Wigner transformation

Now we perform a generalized Jordan Wigner transformation to define the parafermions [28]. First we define operators living on the dual lattice as:

$$\mu_{i+1/2} = \sigma_i \sigma_{i+1}^\dagger, \quad \rho_{i+1/2} = \prod_{k=1}^i \tau_k^\dagger, \quad (2.8)$$

so that $\tau_i = \rho_{i-1/2} \rho_{i+1/2}^\dagger$, and μ, ρ satisfy the same commutation relations as respectively τ, σ . ρ is the disorder operator dual to the order operator σ , implementing the action of symmetries on spins between sites 1 and i , namely shifting those spins

[§] These are the relations of the Heisenberg group which is defined by exponentiation of the Weyl algebra $[Q, P] = QP - PQ = i\mathbf{1}$. In terms of the generators $\sigma = \exp(iP), \tau = \exp(iQ)$ and the central element $Z = \exp(i\mathbf{1})$ the group multiplication reduces to $\sigma^r \tau^s = Z^{rs} \tau^s \sigma^r$ for any parameters $r, s \in \mathbb{R}$.

by one unit. Parafermions are then defined on middle points between vertices of the original and dual lattice, as the product of order and disorder variables:

$$\gamma_{2i-1} = \left(\prod_{k=1}^{i-1} \tau_k \right) \sigma_i, \quad \gamma_{2i} = \omega^{(N-1)/2} \left(\prod_{k=1}^{i-1} \tau_k \right) \sigma_i \tau_i. \quad (2.9)$$

They satisfy the following commutation relations

$$\gamma_i^N = 1, \quad \gamma_i \gamma_j = \omega \gamma_j \gamma_i \quad (\text{for } 1 \leq i < j \leq 2L), \quad (2.10)$$

and the definition can be inverted to yield:

$$\sigma_i \sigma_{i+1}^\dagger = \omega^{(N+1)/2} \gamma_{2i} \gamma_{2i+1}^{-1} \quad (2.11)$$

$$\tau_i = \omega^{-(N-1)/2} \gamma_{2i-1}^{-1} \gamma_{2i}. \quad (2.12)$$

γ 's are the generators of an algebra which generalize the Clifford algebra of "Majorana fermions" occurring at $N = 2$ [34]. Elements of a generalized Clifford algebra (with even number of generators, as in the case at hand, where the algebra is isomorphic to the complex matrices acting on the Hilbert space of the chain) are the sum of homogeneous elements X which have a well defined \mathbb{Z}_N -grading $|X| \in \mathbb{Z}_N$ given by:

$$PXP^{-1} = \omega^{|X|} X, \quad (2.13)$$

with P the symmetry generator expressed in terms of the parafermions as

$$P = \prod_{i=1}^L \omega^{(N-1)/2} \gamma_{2i} \gamma_{2i-1}^{-1}. \quad (2.14)$$

In particular, the parafermionic generators themselves are elements of degree 1.

3. Symmetry broken phases in \mathbb{Z}_N spin chains

The general theory developed in [33, 35] classifies gapped phases of spin chains with symmetry group G in terms of classes of projective representations of subgroups H to which the symmetry is spontaneously broken. The subgroups of \mathbb{Z}_N are given by \mathbb{Z}_n , where n is a divisor of N . \mathbb{Z}_N only has trivial projective representations for any N , since the phase $e^{i\phi}$ appearing in the projective version of the defining relation, $P^N = e^{i\phi} \mathbf{1}$, can always be absorbed in the definition of P . Therefore gapped phases of \mathbb{Z}_N spin chains are indexed by divisors of N and fully characterized by spontaneous symmetry breaking (SSB).

Statistical systems with \mathbb{Z}_N symmetry have been studied extensively over the years, and their phase diagrams have been discussed in a variety of places, see e.g. [36, 37, 29, 38]. In this section, we explicitly construct representatives of all gapped phases we expect to have. On the way, we shall emphasize important aspects related to order parameters which will play a key role in the description of parafermionic phases in Section 4.

3.1. Construction of the phases

We first briefly recall the defining features of SSB in \mathbb{Z}_N spin chains. The breaking of \mathbb{Z}_2 symmetry in the ordered phase of the quantum Ising chain is an elementary textbook example [39]: the phase is characterized by a non-vanishing expectation value of the order parameter σ . The generalization to the breaking of \mathbb{Z}_N down to \mathbb{Z}_n , n divisor of N , is straightforward.

In the thermodynamic limit, the local order parameters describing the properties of the spin chain are given by the powers $\sigma^{\alpha n}$ with $\alpha \in \mathbb{Z}_m \setminus \{0\}$ and $m = N/n$. The group of residual symmetries which leave the order parameter invariant is $\{1, P^m, \dots, P^{(n-1)m}\} = \mathbb{Z}_n$. It is generated by P^m , the m^{th} power of the global parity. (If $n = N$ the phase is disordered.) Non-trivial elements of $\mathbb{Z}_N/\mathbb{Z}_n = \{1, P, \dots, P^{m-1}\} = \mathbb{Z}_m$ permute the ground states cyclically among themselves, so that the ground state manifold is m -fold degenerate. We will refer to \mathbb{Z}_m as the broken symmetries. The degeneracy of the ground state encodes a permutation representation of the symmetries on the ground state manifold. This representation corresponds to an invariant of the phase and cannot change unless a phase transition occurs [33]. The symmetry breaking perturbation $h \sum_i (\sigma_i^{\beta n} + \text{h.c.})$ (h small and positive) of the Hamiltonian will select a single ordered ground state, leading to a non-zero expectation value of the order parameter even when the external field h is set to zero.

For later convenience we will now explicitly construct representatives of each phase. This includes both the desired states and an associated parent Hamiltonian. According to the above discussion, a phase labeled by n should have m ordered ground states $|\psi_\alpha\rangle$ ($\alpha \in \mathbb{Z}_m$) which satisfy

$$P|\psi_\alpha\rangle = |\psi_{\alpha+1}\rangle, \quad P^{km}|\psi_\alpha\rangle = |\psi_\alpha\rangle, \quad (3.1)$$

for all $k \in \mathbb{Z}_n$. A simple solution of these constraints is given by the product states

$$|\psi_\alpha\rangle = \bigotimes_{i=1}^L |\alpha\rangle_i, \quad \text{where} \quad |\alpha\rangle := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} |\alpha + mk\rangle. \quad (3.2)$$

The states $|\alpha\rangle$ are orbits under the subgroup \mathbb{Z}_n of \mathbb{Z}_N . As such they span the subspace $\mathbb{C}^m \subseteq \mathbb{C}^N$ and they satisfy the periodicity condition $|\alpha\rangle \equiv |\alpha + m\rangle$.

The associated Hamiltonian will be defined in such a way that the m -fold degeneracy of the ground state is already exact in finite systems and not only in the thermodynamic limit. This is achieved by choosing a frustration free combination of specific projectors which are localized on two neighboring sites. Taking $H = -\sum_i h_{i,i+1}$, the following choice of the two-body term $h_{i,i+1}$ guarantees the \mathbb{Z}_N -invariance and the projector property ($h_{i,i+1}^2 = h_{i,i+1}$, $h_{i,i+1}^\dagger = h_{i,i+1}$)

$$h_{i,i+1} = \sum_{\alpha=0}^{m-1} |\alpha\rangle_i \langle\alpha|_{i+1} \langle\alpha|_i \langle\alpha|_{i+1}. \quad (3.3)$$

The Hamiltonian H is a sum of commuting projectors, so that it is possible to diagonalize all terms simultaneously. The resulting spectrum is given by the integers between $-L$ and 0, and hence the chain is gapped as desired. Assuming periodic or open boundary conditions will make no difference for the conclusions we want to draw

in this section. In both cases the space of ground states is spanned by the vectors $|\psi_\alpha\rangle$ with $\alpha \in \mathbb{Z}_m$, for which $h_{i,i+1} = 1$.

It remains to find a more explicit form of the Hamiltonian (3.3) in terms of the spin operators σ_i and τ_i . Using the single-site matrix elements

$$\langle q|\alpha\rangle\langle\langle\alpha|p\rangle = \frac{1}{N}\langle q|\sum_{\beta=0}^{m-1}(\sigma\omega^{-\alpha})^{n\beta}\sum_{j=0}^{n-1}\tau^{mj}|p\rangle, \quad (3.4)$$

it can easily be verified that Eq. (3.3) reduces to

$$h_{i,i+1} = \frac{1}{mn^2}\sum_{\beta=0}^{m-1}(\sigma_i\sigma_{i+1}^\dagger)^{\beta n}\sum_{j,k=0}^{n-1}\tau_i^{jm}\tau_{i+1}^{km}. \quad (3.5)$$

We have thus succeeded in constructing gapped Hamiltonians realizing each possible SSB phase of \mathbb{Z}_N spin chains.

Before moving on, we would like to comment on the specific form of the Hamiltonians obtained. The cases $n = 1$ and $n = N$ correspond, respectively, to the situations with no residual symmetry and full symmetry of the ground state. In these two extreme cases, the interaction simplifies to the expressions

$$(n, m) = (1, N) : \quad h_{i,i+1} = \frac{1}{N}\sum_{\alpha=0}^{N-1}(\sigma_i\sigma_{i+1}^\dagger)^\alpha \quad (3.6)$$

$$(n, m) = (N, 1) : \quad h_{i,i+1} = \frac{1}{N^2}\sum_{j,k=0}^{N-1}\tau_i^j\tau_{i+1}^k. \quad (3.7)$$

It is now evident that the general formula (3.5) is obtained by multiplying together the Hamiltonians of a \mathbb{Z}_m chain whose symmetry is completely broken, $1/m\sum_{\beta=0}^{m-1}(\sigma_i^n(\sigma_{i+1}^\dagger)^n)^\beta$, and that of a \mathbb{Z}_n chain without symmetry breaking, $1/n^2\sum_{j,k=0}^{n-1}(\tau_i^m)^j(\tau_{i+1}^m)^k$.

While our definition reproduces the conventional form of the ferromagnetic clock model deep in the symmetry broken phase [20] for $n = 1$, the case $n = N$ leads to a more complicated expression than the one used in [20], namely $\sum_{j=0}^{N-1}\tau_i^j$ (even though both Hamiltonians have the same symmetric ground state). Moreover, the duality transformations (2.8) between the ordered and the disordered phase do not preserve the structure of the Hamiltonians (3.5). It is therefore reasonable to search for a slight modification of the Hamiltonian (3.5) such that it reduces to the form of the spin chains considered in [20], while retaining the ground states above. This goal is achieved by considering the expression

$$\hat{H} = -\sum_i\left(\frac{1}{m}\sum_{\beta=0}^{m-1}(\sigma_i\sigma_{i+1}^\dagger)^{\beta n} + \frac{1}{n}\sum_{j=0}^{n-1}\tau_i^{jm}\right). \quad (3.8)$$

This operator is Hermitian, \mathbb{Z}_N invariant, and each single term is a projector commuting with all the other terms. Therefore the chain is still gapped after the modification, albeit with a different spectrum in comparison to H .

The ground states are such that each term is 1 and \hat{H} has exactly m ground states which are given by (3.2). Indeed, the terms $1/n\sum_{j=0}^{n-1}\tau_i^{jm}$ act as identity on states

$\bigotimes_{i=1}^L |\alpha_i\rangle_i$ ($\alpha_i \in \mathbb{Z}_m$). Demanding, in addition, that also $1/m \sum_{\beta=0}^{m-1} (\sigma_i \sigma_{i+1}^\dagger)^{\beta n}$ acts as identity selects states with $\alpha_i = \alpha_{i+1}$. Hence \hat{H} is another representative of the SSB phase under consideration. Note that the Hamiltonians \hat{H} in (3.8) admit a simple action (in the thermodynamics limit) under the duality transformations (2.8). The order-disorder duality simply swaps the phases labeled by n with those labeled by m , and at the same time interchanges σ with ρ and τ with μ . Nonetheless, we will stick to our original definition of the Hamiltonian, see Eq. (3.5), in what follows.

3.2. Order parameters

In a macroscopic system with broken symmetry one will not observe a quantum superposition of ground states. Fluctuations will rather single out a unique ground state with a well-defined magnetization. In order to select a single ground state $|\psi_0\rangle$, we add a symmetry breaking perturbation $h \sum_i (\sigma_i^{\beta n} + \text{h.c.})$ to the Hamiltonian (3.5). Afterwards we take the thermodynamic limit and set $h = 0$. It is easy to check that this procedure implies a spontaneous magnetization

$$\langle \sigma_i^{\beta n} \rangle := \langle \psi_0 | \sigma_i^{\beta n} | \psi_0 \rangle \neq 0, \quad (3.9)$$

and also $\langle \tau_i^{km} \rangle \neq 0$ for all $k \in \mathbb{Z}_n$. More importantly, we also have

$$\langle \rho_{i+1/2}^{km} \rangle \neq 0. \quad (3.10)$$

In contrast, all other powers of σ and ρ have vanishing expectation values. Now we expect that slight modifications of Hamiltonians of gapped phases cannot change abruptly the expectation values unless a phase transition is reached. Then Eqs. (3.9, 3.10) hold not only for the fine tuned Hamiltonians we have constructed but in a whole region of non-zero measure in the phase diagram, which constitutes the SSB phase.

The mechanism of symmetry breaking is well understood for the Ising chain and it applies similarly to \mathbb{Z}_N spins. The operator $\rho_{i+1/2}$ disorders an ordered state by creating kink excitations, topological objects rotating all spins from site 1 to i by one unit. The disordered phase can be characterized by the condensation of these kinks [40].

We are now prepared to appreciate that a particular situation is encountered when there exist two integers α, k such that $\alpha n = km$. Indeed, under these circumstances we are able to construct another *non-local* order parameter

$$(\rho_{i-1/2}^\dagger \sigma_i)^{\alpha n}. \quad (3.11)$$

(The non-zero expectation value in the state $|\psi_0\rangle$ is implied by Eqs. (3.9, 3.10).) With the definition $g := \text{gcd}(n, m)$, the solutions of the equation $\alpha n = km$ can be parametrized as follows,

$$\alpha = k \frac{m}{n} \quad \text{with} \quad k \in \left\{ 0, \frac{n}{g}, 2\frac{n}{g}, \dots, (g-1)\frac{n}{g} \right\} = \frac{n}{g} \mathbb{Z}_g. \quad (3.12)$$

We conclude that we have additional (non-local) order parameters whenever $g > 1$. Their existence is a direct consequence of being in a SSB phase of the \mathbb{Z}_N spin chains. (Of course the presence of non-local order parameters does not mean that the phase

at hand is topological, since the degeneracy of the ground state of the spin chain can be completely lifted through SSB.) In Section 4.2, we will shed more light onto the physical implications of the presence of this new non-local order parameter. It is no coincidence that the product (3.11) of order and disorder variables that we have just encountered agrees with powers of the parafermion, see Eq. (2.9).

4. Classification of parafermionic phases

In this section we will translate the classification of symmetry breaking patterns in \mathbb{Z}_N spin chains into the dual language of parafermions. It will be found that the resulting parafermionic Hamiltonians generally exhibit a mixture of symmetry breaking and non-trivial topology which can be made manifest in the ground state degeneracies, in the occurrence of protected edge modes and in the possibility to define an integer valued topological invariant. We shall point out essential differences in the ground state degeneracy that arise when considering closed boundary conditions as opposed to open ones.

4.1. Duality transformation of the Hamiltonian

Our first goal is to rewrite the Hamiltonian Eq. (3.5) in terms of the parafermions. Using the transformation rules (2.11) and the relation

$$(\gamma_i^a \gamma_j^b)^c = \omega^{-abc(c-1)/2} \gamma_i^{ac} \gamma_j^{bc} \quad (\text{for } i < j), \quad (4.1)$$

we immediately obtain the two-site Hamiltonian

$$\begin{aligned} h_{i,i+1} &= \frac{1}{m} \sum_{\beta=0}^{m-1} \omega^{\beta n(N+\beta n)/2} \gamma_{2i}^{\beta n} \gamma_{2i+1}^{-\beta n} \\ &\times \frac{1}{n^2} \sum_{j,k=0}^{n-1} \omega^{(m^2(j^2+k^2)-Nm(j+k))/2} \gamma_{2i-1}^{-jm} \gamma_{2i}^{jm} \gamma_{2i+1}^{-km} \gamma_{2i+2}^{km}. \end{aligned} \quad (4.2)$$

Despite the non-locality of the transformation, the Hamiltonian still remains local when expressed in terms of the parafermionic variables. The only complication arises for periodic boundary conditions since the naive application of the Jordan Wigner transformation generates a non-local term in this case. In other words, locality implies that the Hamiltonian for the periodic parafermionic chain needs to be different from that of the periodic spin model, see Section 4.3.2 for details.

In the following we will use a basis of parity eigenvectors in the space of ground states which is more natural to work with in the parafermionic picture:

$$|\phi_\alpha\rangle := \sum_{\beta=0}^{m-1} \omega^{\beta\alpha n} |\psi_\beta\rangle, \quad (4.3)$$

so that

$$\sigma_i^{\beta n} |\phi_\alpha\rangle = |\phi_{\alpha+\beta}\rangle, \quad \tau_i^{km} |\phi_\alpha\rangle = |\phi_\alpha\rangle, \quad P |\phi_\alpha\rangle = \omega^{-\alpha n} |\phi_\alpha\rangle. \quad (4.4)$$

4.2. Quantum phases

In Section 3 we characterized SSB phases of \mathbb{Z}_N spin chains. However, the local order parameter $\sigma^{\alpha n}$ of the spin chain becomes non-local when expressed in terms of parafermionic variables. Since physical perturbations are assumed to be local, this observation suggests that there is no way to spontaneously break the symmetry in the parafermionic chain, and that the phases are characterized by topological order instead of ferromagnetic order. In this case, the states emerging at the macroscopic level would be those with definite \mathbb{Z}_N parity. In the fermionic case $N = 2$, for example, it is well-established that there is no local order parameter, and correspondingly no local perturbation which can break the symmetry [6, 41].

While valid for a considerable subset of symmetry breaking patterns in the \mathbb{Z}_N spin chains, the previous expectation fails to be true in general. This exactly happens in the phases labeled by a divisor n such that $g = \gcd(n, N/n) > 1$. Also in the dual picture, parafermionic parity can be broken (partly) in these cases and, depending on the precise values of g and n , this leads to phases which, in addition to ferromagnetic order, might as well exhibit topological order.

In order to substantiate our claim, let us briefly recall our results from Section 3.2. For $g > 1$ we have pointed out the existence of a second set of order parameters (3.11), which become *local* when rewritten in terms of parafermions. More precisely, they correspond exactly to mutually commuting powers of the parafermions,

$$\left[\gamma_i^{pN/g}, \gamma_j^{p'N/g} \right] = 0 \quad (\text{with } p, p' \in \mathbb{Z}_g). \quad (4.5)$$

Note that, more generally, $\gamma_j^{pN/g}$ commutes with $\gamma_i^{\alpha n}$ and γ_i^{km} for every i, j , so that

$$\left[\gamma_j^{pN/g}, h_{i,i+1} \right] = 0, \quad (4.6)$$

implying a non-trivial ground state degeneracy. The parafermionic local order parameters transform non-trivially under the action of elements in the set $\{P, \dots, P^{g-1}\} \subset \mathbb{Z}_N$. Hence they can be used to detect a symmetry breaking from \mathbb{Z}_N to $\mathbb{Z}_{N/g}$.

To show explicitly how the symmetry breaking occurs we rely on the Hamiltonian (4.2). We suppose open boundary conditions, for which the Hamiltonians for the spins and the parafermions are identical. Let us then consider a $\mathbb{Z}_N \rightarrow \mathbb{Z}_{N/g}$ symmetry breaking perturbation of the original parafermionic Hamiltonian,

$$H \mapsto H + h \sum_i \left\{ \gamma_i^{pN/g} + \text{h.c.} \right\} \quad (\text{with } p \in \mathbb{Z}_g \setminus \{0\}). \quad (4.7)$$

The extra term in the Hamiltonian is local in the parafermionic variables. It acts on the ground states as

$$\gamma_{2i-1}^{pN/g} |\psi_\alpha\rangle = \left(\prod_{k=1}^{i-1} \tau_k \right)^{pN/g} \sigma_i^{pN/g} |\psi_\alpha\rangle = \omega^{\alpha pN/g} |\psi_\alpha\rangle, \quad (4.8)$$

and analogously for γ_{2i} , with an extra α -independent phase. As a consequence, the SSB induced by the perturbation (4.7) singles out a subset of the m ground states,

namely exactly those labeled by an element α such that

$$\alpha \frac{pN}{g} = 0 \pmod{N}. \quad (4.9)$$

It is evident that this equation has m/g solutions

$$\alpha \in \{0, g, \dots, m - g\} = g\mathbb{Z}_{m/g}. \quad (4.10)$$

Taking $h \rightarrow 0$ after passing to the thermodynamic limit gives $\langle \psi | \gamma_i^{pN/g} | \psi \rangle = \langle \psi | \psi \rangle \neq 0$, where $|\psi\rangle$ belongs to the m/g -dimensional space of residual ground states spanned by $\{|\psi_0\rangle, |\psi_g\rangle, \dots, |\psi_{m-g}\rangle\}$. In Section 4.3 we will show that the residual degeneracy is associated with the presence of edge modes and should therefore be regarded as topological. If however $m = g$ no such degeneracy is present, as e.g. when $N = 4$ and $n = 2$, and the system is in a purely symmetry breaking phase.

Before moving on, we discuss the implications for a system with periodic boundary conditions. Namely, in that case the ground state degeneracy due to topologically protected edge modes is expected to disappear while the degeneracy due to symmetry breaking should persist. The degeneracy due to symmetry breaking is g , and therefore one expects a closed parafermionic chain to have a g -fold degenerate ground state. Whenever $g < m$ our arguments imply the existence of topologically protected edge modes. We illustrate this phenomenology in the case of \mathbb{Z}_8 parafermions in Figure 1, see also Table 2 for a more abstract summary of our findings. Our expectations will be verified in Section 4.3 using the explicit realization of the phases.

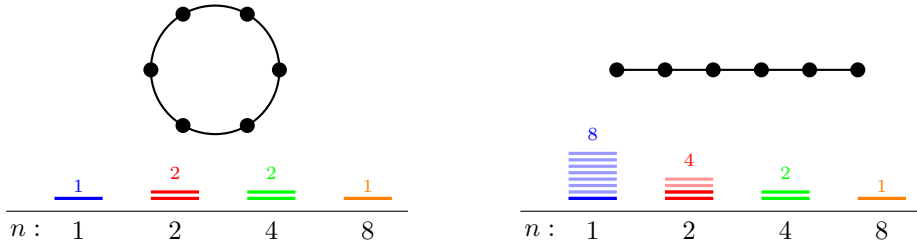


Figure 1. (Color online) Degeneracies of the ground states as a function of the divisor n labeling the phases of \mathbb{Z}_8 parafermions for both closed (left) and open (right) boundary conditions. In the closed chain the degeneracy is entirely due to symmetry breaking. The additional ground states (pale) in the open chain are accounted for by the appearance of symmetry protected edge modes.

4.3. Realization of the phases

In this section we will provide an explicit realization of all parafermionic phases discussed in the previous section, thereby putting our considerations on a concrete basis. It turns out to be convenient to discuss open and closed boundary conditions separately. We interpret a reduction in the ground state degeneracy when closing the chain as one of the signals for the presence of topological order.

4.3.1. Open boundaries. The Hamiltonian for the open chain is

$$H_{\text{op}} = - \sum_{i=1}^{L-1} h_{i,i+1}, \quad (4.11)$$

where $h_{i,i+1}$ is given in (4.2) for any fixed choice of phase n . H_{op} has m degenerate ground states $|\phi_\alpha\rangle$, $\alpha \in \mathbb{Z}_m$, see Eq. (4.3). Part of this degeneracy is due to a symmetry breaking $\mathbb{Z}_N \rightarrow \mathbb{Z}_{N/g}$ and can be understood in terms of zero modes (operators commuting with the Hamiltonian H_{op}) $\gamma_j^{pN/g}$ which are *delocalized* along the whole chain. Indeed, from Eq. (4.6) we see that $[\gamma_j^{pN/g}, H_{\text{op}}] = 0$ for every j , and acting with powers of $\gamma_j^{N/g}$ on a reference ground state with definite parity, say $|\phi_0\rangle$, will generate a g -dimensional subspace in the space of ground states. Local perturbations can be used to select a specific direction and lift this degeneracy, see Eq. (4.7).

When $m/g > 1$ the open system exhibits additional ground states. All of them can be obtained using the action of zero modes γ_1^m and γ_{2L}^m (with $\eta = 1, \dots, m/g - 1$), which are *localized* at the boundary. Note that the resulting edge states are distinguished by their degree. The degeneracy associated with the edge zero modes is topological and robust against local perturbations. This is a consequence of the following observation to be established below: The presence of edge modes implies that the residual symmetry transformations in $\mathbb{Z}_{N/g}$ can be effectively factorized as the product of two operators localized respectively at the left and right boundary. The fractionalization of these symmetries protects the topological part of the ground state degeneracy and allows to define a topological invariant. In this sense the parafermionic phases fit into the general framework developed for classifying symmetry protected topological phases of spin chains with broken symmetries [33, 35].

Before we comment on the general situation let us first of all stick to the setting of the fine tuned Hamiltonians (4.2). Consider the action of P^g , the generator of the unbroken symmetries, on the ground state manifold: $P^g|\psi_\alpha\rangle = |\psi_{\alpha+g}\rangle$. (Recall that also $P^m|\psi_\alpha\rangle = |\psi_\alpha\rangle$, since α is defined modulo m .) We postulate that this action can be replaced by that of the operator \hat{P}_g written in terms of edge zero modes as

$$\hat{P}_g = \omega^{\epsilon n(N+\epsilon n)/2} \gamma_1^{\epsilon n} \gamma_{2L}^{-\epsilon n}, \quad (4.12)$$

with a well-defined, yet hitherto undetermined, number $\epsilon \in \mathbb{Z}_{m/g}$ which only depends on the choice of divisor n . The phase factor is chosen such that $\hat{P}_g = \sigma_1^{\epsilon n} (\sigma_L^\dagger)^{\epsilon n} P^{\epsilon n}$, from which we compute $\hat{P}_g|\psi_\alpha\rangle = |\psi_{\alpha+\epsilon n}\rangle$. A comparison with the original expression $P^g|\psi_\alpha\rangle = |\psi_{\alpha+g}\rangle$ shows that we can then determine the value of ϵ by imposing

$$n\epsilon = g \pmod{m}. \quad (4.13)$$

Taking into account that $g = \text{gcd}(m, n)$, this equation has a unique solution for $\epsilon \in \mathbb{Z}_{m/g}$. Indeed if $\epsilon_1 = (g + p_1 m)/n$ and $\epsilon_2 = (g + p_2 m)/n$ are two solutions, then $\epsilon_1 - \epsilon_2 = (p_1 - p_2)m/n$ must be an integer, implying $(p_1 - p_2)$ is a multiple of n/g and so $\epsilon_1 - \epsilon_2 = 0 \pmod{m/g}$. It is also simple to show that the broken symmetries $\{P, \dots, P^{g-1}\}$ do not admit a representation on ground states in the factorized form of Eq. (4.12). For $g = 1$ the \mathbb{Z}_N symmetry is not broken and the phase is purely topological.

We have already seen that some of the parafermionic phases differ in the symmetry group that is preserved by their ground states. We next wish to establish that $x = \epsilon n \in \mathbb{Z}_{N/g}$ is a topological invariant which can distinguish phases n_1 and n_2 corresponding to the *same* symmetry breaking pattern $\mathbb{Z}_N \rightarrow \mathbb{Z}_{N/g}$. We note that the phases n_1 and n_2 come with a different number of zero modes, N/n_1 and N/n_2 , respectively. One may hence argue that both phases are already distinguishable.

However, as we shall show in the following section, the previous observation is an artifact of the open chain. In the closed chain, both phases will exhibit g ground states and hence an independent invariant is required to discriminate between the two. We have verified on the computer up to $N = 5 \cdot 10^5$ that two different phases $n_1 \neq n_2$ (with the same g) never give rise to the same invariant x . At the moment we still lack an analytical proof for this assertion.

For a fixed phase n , our definition yields a topological invariant with values in $\mathbb{Z}_{N/g}$. However, comparing these invariants for phases n_1 and n_2 with different values of g makes no sense, at least not a priori, since the preserved symmetries protecting the non-trivial topology and therefore also the groups of topological invariants are different. One could nevertheless attempt to make them comparable by working with hierarchies of topological phases, see [42]. In order to develop a better intuition for the topological invariant, we collected the values of $x = \epsilon n$ for several types of parafermions and phases in Table 1.

Let us finally discuss whether our arguments rely on the particular choice of fine tuned Hamiltonians (4.2) or whether they remain true if one permits small but finite deformations. For the special case of Majorana chains ($N = 2$) it has been established by Kitaev that the zero modes persist [7], at least in the limit of an infinite chain where a potential level splitting is exponentially suppressed. The same statement has been derived by Fendley for the purely topological phase of \mathbb{Z}_N parafermions with $n = 1$ [20]. While the extension of Fendley's argument to our more general setting is beyond the scope of our paper, it seems plausible that localized edge modes remain stable in a gapped system. If this is the case, the factorization of the generator P^g of the $\mathbb{Z}_{N/g}$ symmetry on the low energy spectrum then again leads to a representation $\hat{P}_g = P_l P_r$ with P_l and P_r localized at the left and right boundary, respectively, but now over a distance of the correlation length (see [6, 41] for the case of Majorana fermions). Just as before, this fractionalization is characterized by the $\mathbb{Z}_{N/g}$ -degree x of P_l . By its mere definition as an integer the value of x cannot be changed smoothly and thus constitutes a $\mathbb{Z}_{N/g}$ -valued topological invariant that can be used to characterize the (symmetry protected) topological phase. In the special case $N = 2$ it reduces to the familiar expression for interacting Majorana fermions [6, 41].

We now show that the existence of the phases realized by our Hamiltonians (4.2) can be predicted on general grounds by discussing in how many possible ways any potential residual symmetry $\mathbb{Z}_{N/g}$ can be "fractionalized". The appealing feature of this approach is that it does not make any reference to a concrete Hamiltonian, thus giving additional support to the fact that the classification developed so far is indeed complete. The following discussion generalizes previous insights for Majorana chains [6, 41], and also a recent work on parafermions [32] which focuses on purely topological phases ($g = 1$ in our notation). Suppose again that the action on low energy states of the symmetry transformation P^g can be represented effectively as $\hat{P}_g = P_l P_r$, with P_l, P_r operators of definite $\mathbb{Z}_{N/g}$ degree acting respectively on the left and right boundaries. Here g is fixed and taken from the set of possible values of $\gcd(n', N/n')$, where n' is running through the divisors of N . In the thermodynamic limit, the existence of non-trivial factorizations implies topological degeneracies due to edge modes, and hence allows to distinguish different topological phases. We will next derive and solve consistency conditions for the existence of such a factorization.

Denoting by x the $\mathbb{Z}_{N/g}$ degree of P_l (and by $-x$ that of P_r), the consistency of

Type: N	Phase: n	Symmetry	Ground state degeneracy		
			SSB: g	Edge: N/ng	Invariant: x
Prime p	1	\mathbb{Z}_p	1	p	1
	p	\mathbb{Z}_p	1	1	0
8	1	\mathbb{Z}_8	1	8	1
	2	\mathbb{Z}_4	2	2	2
	4	\mathbb{Z}_4	2	1	0
	8	\mathbb{Z}_8	1	1	0
12	1	\mathbb{Z}_{12}	1	12	1
	2	\mathbb{Z}_6	2	3	2
	3	\mathbb{Z}_{12}	1	4	9
	4	\mathbb{Z}_{12}	1	3	4
	6	\mathbb{Z}_6	2	1	0
	12	\mathbb{Z}_{12}	1	1	0

Table 1. List of some \mathbb{Z}_N parafermionic phases together with their properties. The number n runs through the divisors of the parafermionic type N and labels the possible phases, g and N/ng are respectively symmetry breaking and topological degeneracies of the ground state of a system with open boundaries, and x is a topological invariant allowing to distinguish different phases with the same broken symmetries. Rows in boldface correspond to purely topological phases ($g = 1$).

the assigned grades with the factorization requires that

$$\hat{P}P_l = \omega^{gx} P_l \hat{P} = P_l P_r P_l = \omega^{x^2} P_l \hat{P}. \quad (4.14)$$

In other words, the degree of P_l must satisfy the equation

$$x(x - g) = 0 \pmod{N}, \quad \text{with } x \in \mathbb{Z}_{N/g}. \quad (4.15)$$

The solutions to this equation label all non-equivalent factorizations of P^g and correspondingly all the possible topological phases with a given pattern of symmetry breaking $\mathbb{Z}_N \rightarrow \mathbb{Z}_{N/g}$.

Note that Eq. (4.13) can be rewritten as $n(n\epsilon - g) = 0 \pmod{N}$, so that any of its solutions gives rise to a solution of Eq. (4.15) with $x = n\epsilon$. In other words, the factorizations (4.12) discussed for our fine tuned Hamiltonians (4.2) are indeed predicted by Eq. (4.15). In fact, we claim that the solutions of Eq. (4.13) realize all the possible solutions to that equation. To support our assertion, we first associate to each solution of Eq. (4.15) a divisor n_x of N with $\gcd(n_x, N/n_x) = g$. This can be achieved by setting $n_x := \gcd(x, N/g)$. The proof of this fact can be easily carried out using simple properties of the greatest common divisor:

$$\gcd(n_x, N/n_x) = \gcd\left(\gcd(x, N/g), \frac{N}{\gcd(x, N/g)}\right) = \frac{\gcd(\gcd(x, N/g)^2, N)}{\gcd(x, N/g)} \quad (4.16)$$

$$= \frac{\gcd(\gcd(x^2, (N/g)^2), N)}{\gcd(x, N/g)} = \frac{\gcd(x^2, \gcd((N/g)^2, N))}{\gcd(x, N/g)} \quad (4.17)$$

$$\stackrel{(4.15)}{=} \frac{\gcd(xg \pmod{N}, N)}{\gcd(x, N/g)} = g \cdot \frac{\gcd(x, N/g)}{\gcd(x, N/g)} = g. \quad (4.18)$$

By the definition of n_x , the number $\epsilon_x := x/n_x$ is an integer between 0 and $N/(n_x g) - 1$. The final step would consist of establishing the relation

$$\epsilon_x n_x = g \pmod{N/n_x}, \quad (4.19)$$

which is exactly Eq. (4.13). Unfortunately at present we do not know how to prove this last equation, but its validity has been checked numerically case by case up to $N = 5 \cdot 10^5$. Therefore the solutions of (4.15) can be expressed by those of (4.13) in the form $x = \epsilon n$, and our Hamiltonians (4.2) provide an explicit realization of all the phases predicted by demanding a consistent factorization of unbroken symmetries.

We conclude with some remarks on the number of distinct phases which preserve the full symmetry \mathbb{Z}_N . In this case solutions of the factorization constraint are given by so-called unitary divisors, namely divisors n of N such that $g = \gcd(n, N/n) = 1$. In the prime factorization $N = \prod p_i^{a_i}$, $n = \prod p_i^{c_i}$ is a unitary divisor if each c_i is 0 or a_i . For example the divisors of $N = 4$ are 1, 2, 4 but only 1, 4 are unitary. The number of parafermionic phases characterized by purely topological order (i.e. without symmetry breaking) is given by the number of unitary divisors. For $N = 1, 2, 3, \dots$ their numbers are (Sloane's A034444)

$$1, 2, 2, 2, 2, 4, 2, 2, 2, 4, 2, 4, 2, 4, 4, 2, 2, 4, 2, 4, 4, 4, 2, 4, 2, 4, 2, 4, 2, 4, 2, 8, \dots \quad (4.20)$$

These numbers can also be identified with 2^q , where q is the number of different primes dividing N .

Actually, it can easily be shown that a similar counting is at work if one considers a symmetry breaking $\mathbb{Z}_N \rightarrow \mathbb{Z}_{N/g}$ with $g \neq 1$. Indeed, our goal is to enumerate all phases n with a specific value of $g = \gcd(n, N/n)$. Writing $N = \prod p_i^{a_i}$ and $n = \prod p_i^{c_i}$ as in the previous paragraph, one easily finds $g = \prod p_i^{\min(c_i, a_i - c_i)}$. All the values of n which give rise to the same value of g thus arise from the original one by either keeping the same value of c_i or replacing it by $a_i - c_i$, factor by factor. As long as $c_i \neq a_i - c_i$ for all indices i (as is the case for $g = 1$), this reasoning again yields 2^q phases, the number of unitary divisors of N . If however the equation $c_i = a_i - c_i$ is satisfied for one or more of the indices i , the exponent will be reduced accordingly.

4.3.2. Closed boundaries. The parafermionic closed chain is defined by the local Hamiltonian

$$H_{\text{per}}^P = H_{\text{op}} - h_{L,1}^P. \quad (4.21)$$

In specifying $h_{L,1}^P$ we need to pay attention to the order of indices since parafermions on different sites do not commute. The substitution $(2i, 2i+1) \rightarrow (2L, 1)$ in the term $\gamma_{2i}^{\beta n} \gamma_{2i+1}^{-\beta n}$ of Eq. (4.2) inverts the order of indices, and two different Hamiltonians are obtained depending on whether we commute the parafermions before or after the replacement of indices. Indeed performing the above substitution and then commuting the parafermions gives $\gamma_{2L}^{\beta n} \gamma_1^{-\beta n} = \omega^{(\beta n)^2} \gamma_1^{-\beta n} \gamma_{2L}^{\beta n}$. If instead we commute the parafermions in Eq. (4.2) before replacing the indices we will get $\omega^{-(\beta n)^2} \gamma_1^{-\beta n} \gamma_{2L}^{\beta n}$. The ambiguous extra phase produced by inverting the operations signals an ambiguity in the definition of $h_{L,1}^P$. The correct definition of the coupling instead keeps the order of labels and implements the replacement $(2i, 2i+1) \rightarrow (1, 2L)$ in $\gamma_{2i}^{\beta n} \gamma_{2i+1}^{-\beta n}$. Following

this prescription we get

$$\begin{aligned}
 h_{L,1}^P &= \frac{1}{m} \sum_{\beta=0}^{m-1} \omega^{\beta n(N+\beta n)/2} \gamma_1^{\beta n} \gamma_{2L}^{-\beta n} \\
 &\times \frac{1}{n^2} \sum_{j,k=0}^{n-1} \omega^{(m^2(j^2+k^2)-Nm(j+k))/2} \gamma_{2L-1}^{-jm} \gamma_{2L}^{jm} \gamma_1^{-km} \gamma_2^{km}.
 \end{aligned} \tag{4.22}$$

When rewritten in terms of spins, this operator turns out to be non-local. More precisely, it contains a twisting by the inverse global parity P^\dagger :

$$h_{L,1}^P = \frac{1}{mn^2} \sum_{\beta=0}^{m-1} (P^\dagger \sigma_L \sigma_1^\dagger)^{\beta n} \sum_{j,k=0}^{n-1} \tau_L^{jm} \tau_1^{km}. \tag{4.23}$$

While completely equivalent when regarded with open boundary conditions, the spin model and the parafermionic chain thus exhibit essential differences when considered on a closed ring.

In order to understand the physical implications of our previous comment, we now study the spectrum of the periodic chain, with particular attention to the ground state and its degeneracy. Recall that H_{op} has m ground states which are given by $|\phi_\alpha\rangle$ as defined in Eq. (4.3). Since $[H_{\text{op}}, h_{L,1}^P] = 0$, the operators H_{op} and $h_{L,1}^P$ admit a common basis of eigenvectors, and our problem simply amounts to diagonalizing the projector $h_{L,1}^P$ on the space spanned by the vectors $|\phi_\alpha\rangle$.

A brief calculation shows that the operator $h_{L,1}^P$ acts on the ground states of the open chain as

$$h_{L,1}^P |\phi_\alpha\rangle = \frac{1}{m} \sum_{\beta=0}^{m-1} \omega^{\beta n^2 \alpha} |\phi_\alpha\rangle = \delta_{\alpha n=0}^{(\text{mod } m)} |\phi_\alpha\rangle. \tag{4.24}$$

Put differently, the state $|\phi_\alpha\rangle$ remains in the ground state manifold of the closed parafermionic chain if and only if

$$\alpha n = 0 \pmod{m}. \tag{4.25}$$

In order to enumerate the solutions to this equation, let us first introduce the symbols $\tilde{n} = n/g$ and $\tilde{m} = m/g$ where we used the previous abbreviation $g = \text{gcd}(n, m)$. With this notation, we can write the solutions of Eq. (4.25) as

$$\alpha = p \frac{\tilde{m}}{\tilde{n}} \quad \text{with} \quad p \in \tilde{n} \mathbb{Z}_g. \tag{4.26}$$

Hence, the possible values of p are $p = 0, \tilde{n}, \dots, \tilde{n}(g-1)$, so that there are g distinct solutions.

It may easily be shown that the previous g states exhaust the ground state manifold of the periodic Hamiltonian. Note that we can associate the degeneracy of the closed chain to parafermionic zero modes as in the case of the open chain. Indeed Eq. (4.6) implies that $[\gamma_i^{pN/g}, H_{\text{per}}^P] = 0$, and acting with powers of $\gamma_i^{N/g}$ on a reference ground state with definite parity will produce g mutually orthogonal ground states which can be distinguished by their parity eigenvalues.

Phase: n (a divisor of N)	\mathbb{Z}_N spin model	\mathbb{Z}_N parafermions
Symmetry breaking	$\mathbb{Z}_N \rightarrow \mathbb{Z}_n$	$\mathbb{Z}_N \rightarrow \mathbb{Z}_{N/g}$
Topological invariant	\emptyset	$x = \epsilon n \in \mathbb{Z}_{N/g}$
Ground state degeneracy	open chain	$g \cdot (N/ng)$
	closed chain	g

Table 2. Classification of gapped phases in \mathbb{Z}_N spin models and in the dual parafermionic theories. In both cases, the phases are labeled by a divisor n of N but the physical interpretation is rather different since the two theories are non-locally related. In particular, the ground state degeneracy in the open parafermionic chain factorizes into contributions from spontaneous symmetry breaking (g) and from topologically protected edge modes (N/ng). We also note that the closed chains are not equivalent since periodic boundary conditions lead to different Hamiltonians for the spin chain and the parafermions. The number g is determined by $g = \gcd(n, N/n)$ while the constant ϵ is the unique solution to the equation $n\epsilon = g \pmod{N/n}$, see Eq. (4.13).

Our analysis implies that the closed parafermionic chains have a unique ground state for n being a unitary divisor, while there exists a g -fold degenerate ground state when n fails to be a unitary divisor. In the former case, there is no spontaneous symmetry breaking and the ground state degeneracy in the open chain may be completely attributed to the non-trivial topology of the system. On the other hand, in the latter case, the ground state degeneracy is only partially lifted when making the transition from open to closed boundary conditions. In that case, we observe a combination of non-trivial topology and spontaneous symmetry breaking if $m/g > 1$. For $g = m$ there is no non-trivial topology and only symmetry breaking. These results are in complete agreement with our expectations from Section 4.2.

Phases of the closed parafermionic chain with the same symmetry breaking pattern and the same number of ground states can be characterized and distinguished by the topological invariant defined in Section 4.3.1. Even though the definition of the topological invariant requires an open chain, it should be clear that the latter can be realized virtually using the tool of entanglement spectroscopy [6], i.e. by dividing the closed chain into two segments and tracing out the degrees of freedom associated with one of them.

5. Conclusions and outlook

In the current paper we have classified the massive phases of \mathbb{Z}_N parafermionic chains. This was achieved by constructing all gapped symmetry broken phases in the dual quantum spin chains. The number of distinct phases and their specific characteristics depend crucially on the number-theoretic properties of N . First of all, it is easy to construct one SSB phase of the spin model for each divisor n of N . These phases are characterized by the (co-)existence of local and non-local order parameters. Since \mathbb{Z}_N does not lead to symmetry protected topological phases [3, 35], the enumeration of divisors exhausts all possibilities for gapped phases.

In a second step we have then interpreted the resulting ground states and Hamiltonians from a parafermionic perspective. Of course, this leads to the same number of phases. However, in contrast to the spin model the parafermionic phases can now exhibit features of both, symmetry breaking and non-trivial topology. Whether

the resulting parafermionic phase should be interpreted as topological first of all depends on whether n and N/n still have common divisors. A parafermionic phase is purely topological if n is a unitary divisor of N . By definition, this is the case if n and the quotient N/n are co-prime or, in our notation, if $g = \gcd(n, N/n) = 1$. The number of unitary divisors is known to be 2^q where q is the number of distinct prime factors in N . In contrast, a parafermionic phase described by $n < N$ not being a unitary divisor (i.e. $g > 1$) definitively exhibits spontaneous symmetry breaking. If, in addition, $gn < N$ then the phase is also protected by topology. Topology protected phases in open systems exhibit gapless parafermionic edge modes. These edge modes are gapped out when the chain is closed, while a potential degeneracy due to symmetry breaking persists. We also proposed a topological invariant characterizing the topological properties of open chains. Table 2 provides a compact, yet exhaustive, summary of our findings.

While our current results are only concerned with the intrinsic \mathbb{Z}_N symmetry of parafermionic chains, it would be interesting to extend our analysis to systems which are required to respect additional symmetries such as inversion or time-reversal symmetry. Alternatively, one could also add internal degrees of freedom which transform under a discrete or continuous symmetry group. In all these cases one is lead to so-called symmetry protected topological phases which, for bosonic systems, have been fully classified in Refs. [3, 33, 35]. It should be noted that models with continuous symmetry lead to a few peculiarities that have been addressed in Ref. [42].

For fermionic chains it is known that the inclusion of additional symmetries has profound consequences. Most importantly, the restriction to time-reversal invariant systems enhances the group of topological invariants to \mathbb{Z} or \mathbb{Z}_8 , respectively, depending on whether interactions are allowed [41, 31, 6] or not [4, 5]. A general mathematical formalism for treating fermionic symmetry protected topological systems has been developed in Ref. [43] and it appears feasible to extend it to the \mathbb{Z}_N -graded generalization of the Clifford algebra [34] needed to describe parafermions.

Finally, the classification scheme used in this paper is, of course, not limited to \mathbb{Z}_N spin models. It should rather be applicable to any model of statistical physics which exhibits a Kramers-Wannier type of duality. While abelian dualities are well-explored (see, e.g., [44]) and close to our exposition, recent progress on non-abelian dualities [45] might offer a vast playground for the exploration of new topological phases.

Needless to say, knowledge about the classification of topological phases, symmetry protected or not, and a representative Hamiltonian for each phase should merely be the starting point for a more thorough investigation. Indeed, the ultimate goal should be to gain control over the complete phases diagram of physically realistic Hamiltonians, including the location and the nature of phase transitions. For the Majorana chain with interactions such an analysis was provided in [46], based on the known phase diagram of the dual anisotropic next-nearest-neighbor Ising (ANNNI) model. It would be interesting to extend this analysis to the case of \mathbb{Z}_N spin models. Also, with regard to the potential physical realization and detection of parafermions one will need to understand the effects of adding disorder to the couplings.

*** Note ***

While in the process of completing this work, the paper [32] appeared which has considerable overlap with our own results. We gratefully acknowledge useful discussions with J. Motruk and A. Turner about aspects of entanglement and

symmetry fractionalization in parafermionic chains prior to the publication of their paper.

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