Proofs of some conjectures on monotonicity of number-theoretic and combinatorial sequences *

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Abstract

We develop techniques to deal with monotonicity of sequences $\{z_{n+1}/z_n\}$ and $\{\sqrt[n]{z_n}\}$. A series of conjectures of Zhi-Wei Sun and of Amdeberhan *et al.* are verified in certain unified approaches.

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1 Introduction

Let p_n denote the nth prime. In 1982, F. Firoozbakht conjectured that the sequence $\{\sqrt[n]{p_n}\}_{n\geq 1}$ is strictly decreasing, which has been confirmed for n up to 4×10^{18}. This conjecture implies the inequality $p_{n+1}-p_n<\log^2 p_n-\log p_n$ for large n, which is even stronger than Cramér's conjecture $p_{n+1}-p_n=O(\log^2 p_n)$. See Sun [17] for details. Motivated by this, Sun [18] posed a series of conjectures about monotonicity of sequences of the form $\{\sqrt[n]{z_n}\}_{n\geq 1}$, where $\{z_n\}_{n\geq 0}$ is a familiar number-theoretic or combinatorial sequence. Now partial progress has been made, including Chen *et al.* [4] for the Bernoulli numbers, Hou *et al.* [10] for the Fibonacci numbers and derangements numbers, Luca and Stănică [14] for the Bernoulli, Tangent and Euler numbers. The main object of this paper is to develop techniques to deal with monotonicity of $\{z_{n+1}/z_n\}$ and $\{\sqrt[n]{z_n}\}$ in certain unified approaches.

Two concepts closely related to monotonicity are log-convexity and log-concavity. Let $\{z_n\}_{n\geq 0}$ be a sequence of positive numbers. It is called *log-convex* if $z_{n-1}z_{n+1}\geq z_n^2$ for all $n\geq 1$ and *strictly log-convex* if the inequality is strict. The sequence is called *log-concave* if the inequality changes its direction. Clearly, a sequence $\{z_n\}_{n\geq 0}$ is log-convex

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(log-concave, resp.) if and only if the sequence $\{z_{n+1}/z_n\}_{n\geq 0}$ is increasing (decreasing, resp.). The log-convex and log-concave sequences arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated. We refer the reader to [16, 2, 19] for log-concavity and [13, 20] for log-convexity.

On the other hand, there are certain natural links between these two sequences $\{z_{n+1}/z_n\}$ and $\{\sqrt[n]{z_n}\}$. For example, it is well known that if the sequence $\{z_{n+1}/z_n\}$ is convergent, then so is the sequence $\{\sqrt[n]{z_n}\}$. There is a similar result for monotonicity: if the sequence $\{z_{n+1}/z_n\}$ is increasing (decreasing), then so is the sequence $\{\sqrt[n]{z_n}\}$ when $z_0 \leq 1$ ($z_0 \geq 1$). See Theorem 2.1 for the details. Thus we may concentrate our attention on log-convexity and log-concavity of sequences.

In the next section we present our main results about monotonicity of $\{z_{n+1}/z_n\}$ and $\{\sqrt[n]{z_n}\}$. As applications we verify some conjectures of Sun [18] and Amdeberhan *et al.* [1]. In Section 3 we propose a couple of problems for further work.

2 Theorems and applications

We first show that the monotonicity of $\{z_{n+1}/z_n\}$ implies that of $\{\sqrt[n]{z_n}\}$.

Theorem 2.1. *Let $\{z_n\}_{n\geq 0}$ be a sequence of positive numbers.*

(i) *Assume that $\{z_n\}_{n\geq 0}$ is log-convex. If $z_0 \leq 1$ (and $z_1^2 < z_0z_2$), then the sequence $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is (strictly) increasing.*

(ii) *Assume that $\{z_n\}_{n\geq 0}$ is log-concave and $z_0 \geq 1$. Then the sequence $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is decreasing. If $z_0 > 1$ or $z_1^2 > z_0z_2$, then $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is strictly decreasing.*

(iii) *Assume that $\{z_n\}_{n\geq N}$ is log-convex and $\sqrt[N]{z_N} < \sqrt[N+1]{z_{N+1}}$ for some $N \geq 1$. Then $\{\sqrt[n]{z_n}\}_{n\geq N}$ is strictly increasing. The similar result holds for log-concave sequences.*

Proof. (i) Let $x_n = z_n/z_{n-1}$ for $n \geq 1$. Then by the log-convexity of $\{z_n\}$, the sequence $\{x_n\}$ is increasing:

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq x_{n+1} \leq \ldots. \tag{2.1}$$

Write

$$z_n = \frac{z_n}{z_{n-1}}\frac{z_{n-1}}{z_{n-2}}\cdots\frac{z_1}{z_0}z_0 = x_nx_{n-1}\ldots x_1z_0.$$

Then

$$\frac{\sqrt[n+1]{z_{n+1}}}{\sqrt[n]{z_n}} = \frac{\sqrt[n+1]{x_{n+1}x_nx_{n-1}\ldots x_1z_0}}{\sqrt[n]{x_nx_{n-1}\ldots x_1z_0}} = \frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n(n+1)]{x_nx_{n-1}\ldots x_1z_0}} \geq \sqrt[n+1]{\frac{x_{n+1}}{x_n}} \geq 1 \tag{2.2}$$

since $z_0 \leq 1$ and (2.1). Thus the sequence $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is increasing. Clearly, if $z_1^2 < z_0z_2$, i.e., $x_1 < x_2$, then the first inequality in (2.2) is strict, and so the sequence $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is strictly increasing.

(ii) Note that a sequence $\{z_n\}$ is log-concave if and only if the sequence $\{1/z_n\}$ is log-convex. Hence (ii) can be proved as did in (i).

(iii) Let $y_n = z_N^{N+1-n}/z_{N+1}^{N-n}$ for $0 \le n \le N-1$ and $y_n = z_n$ for $n \ge N$. Then (iii) follows by applying (i) and (ii) to the sequence $\{y_n\}_{n\ge 0}$ respectively. □

Remark 2.2. (A) Although $\{n\}_{n\ge 0}$ is log-concave, $\{\sqrt[n]{n}\}_{n\ge 1}$ is not decreasing since $\sqrt{2} = \sqrt[4]{4}$. However, $\{\sqrt[n]{n}\}_{n\ge 3}$ is decreasing by Theorem 2.1 (iii).

(B) We can replace the condition $\sqrt[N]{z_N} < \sqrt[N+1]{z_{N+1}}$ ($\sqrt[N]{z_N} > \sqrt[N+1]{z_{N+1}}$, resp.) by $z_N^2 < z_{N+1}$ ($z_N^2 > z_{N+1}$, resp.) in Theorem 2.1 (iii). In this case we define $y_n = 1$ for $0 \le n \le N-1$ and $y_n = z_n$ for $n \ge N$.

(C) It is possible that $\{\sqrt[n]{z_n}\}_{n\ge 1}$ is monotonic but $\{z_{n+1}/z_n\}_{n\ge 0}$ is not. For example, let F_n be the nth Fibonacci number: $F_0 = 0, F_1 = 1$ and $F_{n+1} = F_{n-1} + F_n$. It is showed that $\{\sqrt[n]{F_n}\}_{n\ge 2}$ is strictly increasing [10, Theorem 1.1]. However, $\{F_n\}$ is neither log-concave nor log-convex since $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ for $n \ge 2$.

We next apply Theorem 2.1 to verify some conjectures posed by Sun in [18].

The Bell number $B(n)$ counts the number of partitions of the set $\{1,\ldots,n\}$ into disjoint nonempty subsets. It is known that

$$\{B(n)\}_{n\ge 0} = \{1, 1, 2, 5, 15, 52, 203, 877, \ldots\}. \qquad [15, A000110]$$

Engel [8] showed that the sequence $\{B(n)\}$ is log-convex. So by Theorem 2.1 we have the following result, which was conjectured by Sun [18, Conjecture 3.2].

Corollary 2.3. *The sequence $\{\sqrt[n]{B(n)}\}_{n\ge 1}$ is strictly increasing.*

Let $p(n)$ denote the number of partitions of a positive integer n. Then

$$\{p(n)\}_{n\ge 1} = \{1, 2, 3, 5, 7, 11, 15, 22, 30, \ldots\}. \qquad [15, A000041]$$

Janoski [11] showed the sequence $\{p(n)\}_{n\ge 25}$ is log-concave, which was conjectured by Chen [3]. Note that $p(25) = 1958$ and $p(26) = 2436$. It follows from Theorem 2.1 (iii) that $\{\sqrt[n]{p(n)}\}_{n\ge 25}$ is strictly decreasing. Thus we have the following result (the case for $6 \le n \le 24$ may be confirmed directly), which was conjectured by Sun [18, Conjecture 2.14].

Corollary 2.4. *The sequence $\{\sqrt[n]{p(n)}\}_{n\ge 6}$ is strictly decreasing.*

Many combinatorial sequences satisfy a three-term recurrence. Došlić [7], Liu and Wang [13] gave some sufficient conditions for log-convexity of such sequences. The following result is a variation of Liu and Wang [13, Theorem 3.1].

Proposition 2.5. *Let* $\{z_n\}_{n\geq 0}$ *be a sequence of positive numbers and satisfy*

$$a_n z_{n+1} = b_n z_n + c_n z_{n-1}, \tag{2.3}$$

where a_n, b_n, c_n *are positive for all* $n \geq 1$*. Let*

$$\lambda_n := \frac{b_n + \sqrt{b_n^2 + 4a_n c_n}}{2a_n}$$

be the positive root of $a_n\lambda^2 - b_n\lambda - c_n = 0$*. Suppose that* z_0, z_1, z_2, z_3 *is log-convex. If there exists a sequence* $\{\nu_n\}_{n\geq 1}$ *of positive numbers such that* $\nu_n \leq \lambda_n$ *and*

$$\Delta_n(\nu) := a_n\nu_{n-1}\nu_{n+1} - b_n\nu_{n-1} - c_n \geq 0 \tag{2.4}$$

for $n \geq 2$*, then the sequence* $\{z_n\}_{n\geq 0}$ *is log-convex.*

Proof. In Liu and Wang [13, Theorem 3.1], it is shown that if $\Delta(\lambda) \geq 0$, then $\{z_n\}_{n\geq 0}$ is log-convex. So it suffices to show that $\Delta(\nu) \geq 0$ implies $\Delta(\lambda) \geq 0$.

Indeed, if $\Delta(\nu) \geq 0$, then $(a_n\nu_{n+1} - b_n)\nu_{n-1} \geq c_n$, which implies that $a_n\nu_{n+1} - b_n \geq 0$. Thus $a_n\lambda_{n+1} - b_n \geq 0$ and $(a_n\lambda_{n+1} - b_n)\lambda_{n-1} \geq (a_n\nu_{n+1} - b_n)\nu_{n-1} \geq c_n$, and so $\Delta(\lambda) \geq 0$, as required. □

The nth trinomial coefficient T_n is the coefficient of x^n in the expansion $(1 + x + x^2)^n$. It is known that

$$(n + 1)T_{n+1} = (2n + 1)T_n + 3nT_{n-1} \tag{2.5}$$

and

$$\{T_n\}_{n\geq 0} = \{1, 1, 3, 7, 19, 51, 141, 393, \ldots\}. \qquad [15, A002426]$$

We have the following result, which was conjectured by Sun [18, Conjecture 3.6]

Corollary 2.6. *The sequence* $\{\sqrt[n]{T_n}\}_{n\geq 1}$ *is strictly increasing.*

Proof. We first apply Proposition 2.5 to prove the log-convexity of the sequence $\{T_n\}_{n\geq 4}$. It is easy to verify that T_4, T_5, T_6, T_7 is log-convex. Note that

$$\lambda_n = \frac{2n + 1 + \sqrt{16n^2 + 16n + 1}}{2(n + 1)} = 1 + \frac{\sqrt{16n^2 + 16n + 1} - 1}{2(n + 1)} = 1 + \frac{8n}{\sqrt{16n^2 + 16n + 1} + 1}$$

and $\sqrt{16n^2 + 16n + 1} \leq 4n + 2$. Hence

$$\lambda_n \geq 1 + \frac{8n}{4n + 3} = \frac{12n + 3}{4n + 3}.$$

Let $\nu_n = (12n + 3)/(4n + 3)$. Then for $n \geq 2$,

$$\Delta_n = (n + 1)\frac{(12n - 9)(12n + 15)}{(4n - 1)(4n + 7)} - (2n + 1)\frac{12n - 9}{4n - 1} - 3n = \frac{36(n - 2)}{(4n - 1)(4n + 7)} \geq 0.$$

Thus $\{T_n\}_{n\geq 4}$ is log-convex by Proposition 2.5. Now $\sqrt[4]{19} < \sqrt[5]{51}$ since $19^5 = 2476099 < 51^4 = 6765201$. It follows that $\{\sqrt[n]{T_n}\}_{n\geq 4}$ is strictly increasing by Theorem 2.1 (iii). Clearly, $\{\sqrt[n]{T_n}\}_{1\leq n\leq 4}$ is strictly increasing, so is the total sequence $\{\sqrt[n]{T_n}\}_{n\geq 1}$. □

We refer the reader to [7] for another proof of the log-convexity of $\{T_n\}_{n\geq 4}$.

The derangements number d_n counts the number of permutations of n elements with no fixed points. It is known that $d_{n+1} = nd_n + nd_{n-1}$ and

$$\{d_n\}_{n\geq 0} = \{1, 0, 1, 2, 9, 44, 265, 1854, \ldots\}. \qquad [15, A000166]$$

The Motzkin number M_n counts the number of lattice paths starting from $(0, 0)$ to $(n, 0)$, with steps $(1, 0), (1, 1)$ and $(1, -1)$, and never falling below the x-axis. It is known that $(n + 3)M_{n+1} = (2n + 3)M_n + 3nM_{n-1}$ and

$$\{M_n\}_{n\geq 0} = \{1, 1, 2, 4, 9, 21, 51, 127, \ldots\}. \qquad [15, A001006]$$

The (large) Schröder number S_n counts the number of king walks, from $(0, 0)$ to (n, n), and never rising above the line $y = x$. It is known that $(n+2)S_{n+1} = 3(2n+1)S_n-(n-1)S_{n-1}$ and

$$\{S_n\}_{n\geq 0} = \{1, 2, 6, 22, 90, 394, 1806, \ldots\}. \qquad [15, A006318]$$

It is shown [13, §3] by means of recurrence relations that three sequences $\{d_n\}_{n\geq 2}, \{M_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ are log-convex respectively. So we have the following result, which was conjectured by Sun [18, Conjectures 3.3, 3.7 and 3.11].

Corollary 2.7. *Three sequences* $\{\sqrt[n]{d_n}\}_{n\geq 2}, \{\sqrt[n]{M_n}\}_{n\geq 1}$ *and* $\{\sqrt[n]{S_n}\}_{n\geq 1}$ *are strictly increasing respectively.*

Davenport and Pólya [6] showed that the binomial convolution preserves log-convexity: if both $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are log-convex, then so is the sequence $\{z_n\}_{n\geq 0}$ defined by

$$z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}, \qquad n = 0, 1, 2, \ldots.$$

Let $\{a(n, k)\}_{0\leq k\leq n}$ be a triangle of nonnegative numbers. A general problem is in which case the operator $z_n = \sum_{k=0}^{n} a(n, k)x_k y_{n-k}$ preserves log-convexity. Wang and Yeh [19] developed techniques to deal with such a problem for log-concavity. For example, if the triangle $\{a(n, k)\}$ has the LC-positivity property and $a(n, k) = a(n, n - k)$, then $z_n = \sum_{k=0}^{n} a(n, k)x_k y_{n-k}$ preserves log-concavity. There is a similar result for log-convexity. The following result follows from Liu and Wang [13, Conjecture 5.3], which has been shown by Chen *et al.* [5]. For the sake of brevity we here omit the details of the proof.

Proposition 2.8. *If both* $\{x_n\}_{n\geq 0}$ *and* $\{y_n\}_{n\geq 0}$ *are log-convex, then so is the sequence* $\{z_n\}_{n\geq 0}$ *defined by*

$$z_n = \sum_{k=0}^{n} \binom{n}{k}^2 x_k y_{n-k}, \qquad n = 0, 1, 2, \ldots.$$

Now we apply Proposition 2.8 to verify some conjectures of Sun. Let $g_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$. Then

$$\{g_n\}_{n\geq 0} = \{1, 3, 15, 93, 639, 4653, 35169, \ldots\}. \qquad [15, A002893]$$

Let $D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$ be the Domb numbers. Then

$$\{D(n)\}_{n\geq 0} = \{1, 4, 28, 256, 2716, 31504, \ldots\}. \qquad [15, A002895]$$

Clearly, the center binomial coefficients $\binom{2k}{k}$ is log-convex in k (see [13] for instance). So the sequences $\{g_n\}_{n\geq 0}$ and $\{D(n)\}_{n\geq 0}$ are log-convex respectively by Proposition 2.8. Thus we have the following result, which was conjectured by Sun [18, Conjectures 3.9 and 3.12].

Corollary 2.9. *The sequences $\{\sqrt[n]{g_n}\}_{n\geq 1}, \{D(n+1)/D(n)\}_{n\geq 0}$ and $\{\sqrt[n]{D(n)}\}_{n\geq 1}$ are strictly increasing respectively.*

Another main result of this paper is the following criterion for log-convexity.

Theorem 2.10. *Suppose that*

$$z_n = \sum_{k\geq 1} \frac{\alpha_k}{\lambda_k^n} \qquad n = 0, 1, 2, \ldots,$$

where $\{\alpha_k\}_{k\geq 1}, \{\lambda_k\}_{k\geq 1}$ are two nonnegative sequences and λ_k is not constant. Then the sequence $\{z_n\}_{n\geq 0}$ is log-convex.

Proof. We have

$$\begin{aligned} z_{n+1}z_{n-1} - z_n^2 &= \sum_{k\geq 1} \frac{\alpha_k}{\lambda_k^{n+1}} \sum_{k\geq 1} \frac{\alpha_k}{\lambda_k^{n-1}} - \sum_{k\geq 1} \frac{\alpha_k}{\lambda_k^n} \sum_{k\geq 1} \frac{\alpha_k}{\lambda_k^n} \\ &= \sum_{j>i\geq 1} \frac{\alpha_i\alpha_j(\lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j)}{\lambda_i^{n+1}\lambda_j^{n+1}} \\ &= \sum_{j>i\geq 1} \frac{\alpha_i\alpha_j(\lambda_i - \lambda_j)^2}{\lambda_i^{n+1}\lambda_j^{n+1}}. \end{aligned}$$

Thus $z_{n+1}z_{n-1} - z_n^2 \geq 0$, and the sequence $\{z_n\}_{n\geq 0}$ is therefore log-convex. ☐

Taking $\lambda_k = k$, then z_n is precisely the Dirichlet generating function of the sequence $\{\alpha_k\}_{k\geq 1}$. In particular, z_n coincides with Riemann zeta function $\zeta(n)$ when $\alpha_k = 1$ for all k. Thus the sequence $\{\zeta(n)\}_{n\geq 1}$ is strictly log-convex. On the other hand, taking $\lambda_k = k^2$ and $\alpha_k = 1$ for all k, then the sequence $\{\zeta(2n)\}_{n\geq 1}$ is also strictly log-convex. These two results have been obtained by Chen *et al.* [4] in an analytical approach.

The classical Bernoulli numbers are defined by

$$B_0 = 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0, \qquad n = 1, 2, \ldots.$$

It is well known that $B_{2n+1} = 0, (-1)^{n-1}B_{2n} > 0$ for $n \geq 1$ and

$$(-1)^{n-1}B_{2n} = \frac{2(2n)!\zeta(2n)}{(2\pi)^{2n}}$$

(see [9, (6.89)] for instance). It immediately follows that the sequence $\{(-1)^{n-1}B_{2n}\}_{n\geq 1}$ is log-convex, and the sequence $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}_{n\geq 1}$ is therefore strictly increasing, which was conjectured by Sun [18, Conjecture 2.15] and has been verified by Chen *et al.* [4] and by Luca and Stănică [14] respectively.

Now consider the tangent numbers

$$\{T(n)\}_{n\geq 0} = \{1, 2, 16, 272, 7936, 353792, \ldots\}, \qquad [15, A000182]$$

which are defined by

$$\tan x = \sum_{n\geq 1} T(n)\frac{x^{2n-1}}{(2n-1)!}$$

and are closely related to the Bernoulli numbers:

$$T(n) = (-1)^{n-1}B_{2n}\frac{(4^n-1)}{2n}4^n$$

(see [9, (6.93)] for instance). It is not difficult to verify that $(4^n - 1)/n$ is log-convex in n (we leave the details to the reader). On the other hand, the product of log-convex sequences is still log-convex. So the sequence $\{T(n)\}_{n\geq 0}$ is log-convex. Thus we have the following result, which was conjectured by Sun [18, Conjecture 3.5].

Corollary 2.11. *Both $\{T(n+1)/T(n)\}_{n\geq 0}$ and $\{\sqrt[n]{T(n)}\}_{n\geq 1}$ are strictly increasing.*

Let A_n be defined by the recurrence relation

$$(-1)^{n-1}A_n = C_n + \sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-1}A_jC_{n-j},$$

with $A_1 = 1$ and $C_n = \frac{1}{n+1}\binom{2n}{n}$ the Catalan number. Let $a_n = 2A_n/C_n$. Lasalle [12] and Amdeberhan *et al.* [1] showed that both $\{A_n\}_{n\geq 1}$ and $\{a_n\}_{n\geq 2}$ are increasing sequences of positive integers. The latter also obtained the recurrence

$$2na_n = \sum_{k=1}^{n-1}\binom{n}{k-1}\binom{n}{k+1}a_ka_{n-k}, \qquad n = 2, 3, \ldots$$

with $a_1 = 1$, and defined another sequence $\{b_n\}_{n\geq 1}$ by the recurrence

$$b_n = \sum_{k=1}^{n-1}\binom{n-1}{k-1}\binom{n-1}{k+1}b_kb_{n-k}, \qquad n = 2, 3, \ldots$$

with $b_1 = 1$. They [1, Conjecture 9.1] conjectured that both $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are log-convex.

Let $\{j_{\mu,k}\}_{k\geq 1}$ be the (nonzero) zeros of the Bessel function of the first kind

$$J_\mu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\mu+1)} \left(\frac{x}{2}\right)^{2m+\mu}$$

and let

$$\zeta_\mu(s) = \sum_{k=1}^{\infty} \frac{1}{j_{\mu,k}^s}$$

be the Bessel zeta function. Then

$$\begin{aligned} A_n &= 2^{2n+1}(2n-1)!\zeta_1(2n), \\ a_n &= 2^{2n+1}(n+1)!(n-1)!\zeta_1(2n), \\ b_n &= 2^{2n-1}(n-1)!n!\zeta_0(2n). \end{aligned}$$

See [1, Corollary 5.3, 5.4 and (7.14)] for details. Now note that both $J_0(x)$ and $J_1(x)$ have only real zeros. Hence both sequences $\{\zeta_1(2n)\}_{n\geq 0}$ and $\{\zeta_0(2n)\}_{n\geq 0}$ are log-convex by Theorem 2.10. This leads to an affirmation answer to [1, Conjecture 9.1].

Corollary 2.12. *Three sequences $\{A_n\}$, $\{a_n\}$ and $\{b_n\}$ are log-convex respectively.*

3 Further work

Sun [18] also proposed a series of conjectures about monotonicity of sequences of the form $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$. Roughly speaking, he conjectured that $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$ has the reverse monotonicity to $\{\sqrt[n]{z_n}\}$ for certain number-theoretic and combinatorial sequences $\{z_n\}$. Clearly, if $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$ is decreasing (increasing, resp.) with the limit 1, then $\{\sqrt[n]{z_n}\}$ is increasing (decreasing, resp.). It is a challenging problem to study monotonicity of $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$, which is equivalent to log-concavity and log-convexity of $\{\sqrt[n]{z_n}\}$. A natural problem is to ask in which case the log-convexity (log-concavity, resp.) of z_n implies the log-concavity (log-convexity, resp.) of $\sqrt[n]{z_n}$.

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