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FROM SEQUENCES TO POLYNOMIALS AND BACK, VIA OPERATOR ORDERINGS

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Abstract. C. M. Bender and G. V. Dunne showed that linear combinations of words $q^k p^n q^{n-k}$, where p and q are subject to the relation $qp - pq = \imath$, may be expressed as a polynomial in the symbol $z = \frac{1}{2}(qp + pq)$. Relations between such polynomials and linear combinations of the transformed coefficients are explored. In particular, examples yielding orthogonal polynomials are provided.

1. Introduction

Operator algebras provide a mathematical setting upon which many physical theories are built. Stellar among these theories is quantum mechanics with operator formalism at its core heart. The transition from Classical to Quantum Mechanics includes replacing the position and momentum by operators p and q acting on a function f by

$$pf = xf(x) \text{ and } qf = \imath \frac{df}{dx}, \tag{1.1}$$

called the annihilation and creation operators, respectively. The canonical commutation relation of these operators is

$$[q, p] := qp - pq = \imath, \text{ with } \imath = \sqrt{-1}. \tag{1.2}$$

Non-commutativity is a common feature in mathematical modeling of reality which, in quantum mechanics, introduces the so-called Heisenberg-Weyl algebra. This new quality does not come without a price − the order of components in operator successions is now relevant and has to be carefully traced in calculations. A traditional solution to this problem is to standardize the notation by fixing the order of operators; that is, to use the normally ordered expansion in powers of the form $q^k p^j$, in which all creation operators stand to the left of the annihilation operators.

A *word* in the letters p's and q's is called *balanced* if it contains the same number of p and q. Theorem 2.5 shows that every balanced word has a representation as a polynomial in $z = \frac{1}{2}(qp + pq)$. C. M. Bender and G. V. Dunne [4] studied operators in symmetrized form ($a_{n,k} = a^*_{n,n-k}$, where $*$ denotes complex conjugation),

$$\sum_{k=0}^{n} a_{n,k} q^k p^n q^{n-k}, \tag{1.3}$$

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and [4] refers the sequence $\{a_{n,k}\}$ as a *pyramid*. In this special setup, Theorem 2.5 associates the polynomial

$$P_n(z) = \sum_{k=0}^{n} b_{n,k} z^k \tag{1.4}$$

to the sequence $\{a_{n,k}\}$. The relation between $\{a_{n,k}\}$ and $\{b_{n,k}\}$ is explicitly given in Theorems 3.1 and 3.8 by

$$a_{n,k} = \frac{1}{\imath^n n!} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n+1}{n-k-j} \sum_{r=0}^{n} b_{n,r} \imath^r \left(j+\tfrac{1}{2}\right)^r \text{ for } 0 \leq k \leq n, \tag{1.5}$$

and

$$b_{n,k} = (-1)^k \imath^{n+k} \sum_{\ell=0}^{n} a_{n,\ell} \sum_{j=0}^{n-k} s(n,j+k) \binom{j+k}{k} \left(\ell - \tfrac{1}{2}\right)^j \text{ for } 0 \leq k \leq n. \tag{1.6}$$

Here $s(n,k)$ is the Stirling number of the first kind.

Some results pertinent to these two sequences include:

Proposition 1.1. *The polynomial $P_n(z)$ is monic if and only if $\{a_{n,k}\}$ is normalized by $a_{n,0} + a_{n,1} + \cdots + a_{n,n} = 1$.*

Theorem 1.2. *The coefficients $\{b_{n,r}\}$ of $P_n(z)$ are real if and only if the coefficients $\{a_{n,k}\}$ are hermitian-symmetric; that is if $a_{n,k} = a^*_{n,n-k}$.*

Parity of the polynomials $P_n(z)$ appears from symmetries in the pyramid $\{a_{n,k}\}$.

Proposition 1.3. *Assume the coefficients $\{a_{n,k}\}$ are real and symmetric. Then $P_n(z)$ has the same parity as n.*

The next question remains open.

Question 1.4. *Determine conditions on a real symmetric pyramid $\{a_{n,k}\}$ in order to obtain polynomials $\{P_n(z)\}$ which are orthogonal with respect to a positive weight function $w(x)$.*

It is simple to produce algebraic equations for the first few coefficients of a family of polynomials $C_n(z) = \sum_{k=0}^{n} c_{n,k} z^k$ in order to be orthogonal.

Lemma 1.5. *Assume $\{C_n(z)\}$ is a family of monic orthogonal polynomials, with C_n of the same parity as n. Then $c_{4,0} + c_{2,0}c_{3,1} - c_{2,0}c_{4,2} = 0$ and*

$$c_{2,0}c_{5,1} + c_{4,0}c_{5,3} - c_{2,0}c_{4,2}c_{5,3} + c_{6,0} - c_{2,0}c_{6,2} - c_{4,0}c_{6,4} + c_{2,0}c_{4,2}c_{6,4} = 0.$$

As shown in [4], the first condition may be used to prove that certain classical pyramids, such as the *symmetric ordering* $a_{n,k} = \delta_{n,k}$ and the *Born-Jordan ordering* $a_{n,k} = 1$ do not produce orthogonal polynomials. On the other hand, the *Weyl-ordering* $a_{n,k} = \binom{n}{k}$ and the case $a_{n,k} = \binom{n}{k}^2$ satisfy the conditions of Lemma 1.5. The polynomials coming from the Weyl-ordering may be expressed in terms of the *continuous Hahn polynomials* (see Example 5.3) and those obtained from $a_{n,k} = \binom{n}{k}^2$ can be expressed in terms of the *Bateman polynomials* (see Example 5.8). It is curious that these seem to be the only powers of binomial coefficients that give orthogonal polynomials. Experimental evidence on the basis of the first condition of Lemma 1.5 rules out the first 50000 power functions $\binom{n}{k}^r$.

Partial results on the pyramids $\{a_{n,k}\}$ associated to the Legendre and Hermite polynomials, as examples of orthogonal families, and also for the sequence $P_n(x) = x^n$, yield pyramids with combinatorial flavor. The complete characterization of these pyramids, as well as those corresponding to other classical orthogonal polynomials, remains an open question.

2. Balanced words in p and q are polynomials in z

Let $\mathcal{X} = \{p, q\}$ be an alphabet. A *word* over $\mathcal{X}$ is an expression of the form

$$w = w_1 w_2 \cdots w_k, \tag{2.1}$$

with $w_j \in \mathcal{X}$. The set of all words is denoted by $W(\mathcal{X})$. The multiplication of words is defined by concatenation. Every word w over $\mathcal{X}$, with $w_1 = p$, has a unique representation in the form

$$w = p^{n_1} q^{m_1} p^{n_2} q^{m_2} \cdots p^{n_j} q^{m_j} \tag{2.2}$$

with $n_i, m_i \in \mathbb{N}$ with the possibility that $m_j = 0$ if the last letter in w is p. A similar unique representation exists if $w_1 = q$.

Definition 2.1. A word $w \in \mathcal{X}$ is called *balanced* if it has the same number of p's and q's, that is, $n_1 + n_2 + \cdots + n_j = m_1 + m_2 + \cdots + m_j$.

Definition 2.2. The *free algebra* $F_{\mathbb{C}}(\mathcal{X})$ is the set

$$F_{\mathbb{C}}(\mathcal{X}) = \left\{ \sum_{j=1}^{m} \alpha_j w^{(j)} : m \in \mathbb{N},\, \alpha_j \in \mathbb{C} \text{ and } w^{(j)} \text{ is a word over } \mathcal{X} \right\} \tag{2.3}$$

The results presented here are related to the algebra $\mathcal{A}$ obtained from $F_{\mathbb{C}}(\mathcal{X})$ after the identification $qp - pq = \imath$. The fact that this is a non-homogenous element leads to difficulties in defining a degree. For instance, the elements qp^2q, $2\imath pq + p^2q^2$, and $\frac{1}{4}(qp + pq)^2 + \frac{1}{4}$ all represent the same element in $\mathcal{A}$.

Definition 2.3. The *Heisenberg-Weyl algebra* $\mathcal{A}$ is the quotient algebra

$$\mathcal{A} = F_{\mathbb{C}}(\mathcal{X}) / \{qp - pq = \imath\}. \tag{2.4}$$

Every element $w \in \mathcal{A}$ has a representation in the form $w = \sum_{j=1}^{m} \alpha_j w^{(j)}$, where $m \in \mathbb{N}$, $\alpha_j \in \mathbb{C}$, $w^{(j)}$ is a word in $\{p, q\}$ and $qp - pq = \imath$. In general, there are many such representations.

Definition 2.4. Define the special subsets of $\mathcal{A}$ by

$$\mathcal{B}[p, q] = \left\{ \sum_{k=0}^{n} \alpha_{n,k} w^{(j)} : \text{ for some } n \in \mathbb{N},\, \alpha_{n,k} \in \mathbb{C} \text{ and } w^{(j)} \text{ balanced word in } p, q \right\}$$

and

$$\mathcal{H}[p, q] = \left\{ \sum_{k=0}^{n} a_{n,k} q^k p^n q^{n-k} : \text{ for some } n \in \mathbb{N}, a_{n,k} \in \mathbb{R} \text{ and } a_{n,k} = a_{n,n-k} \right\}.$$

Observe that $\mathcal{H}[p, q] \subset \mathcal{B}[p, q]$, elements in the latter are hermitian.

The main result of this section is stated next.

Theorem 2.5. *Every element of $\mathcal{B}[p,q]$ is a polynomial in $z = \frac{1}{2}(qp+pq)$.*

Proof. It suffices to prove the result for a balanced word

$$w = p^{a_1} q^{b_1} p^{a_2} q^{a_2} \cdots p^{a_t} q^{b_t}, \tag{2.5}$$

where $a_j,\, b_j \geq 1$ for $2 \leq j \leq t-1$ and $a_1,\, b_t \geq 0$. An easy induction argument proves the recurrences

$$p^c q = qp^c - [(c-1)\imath + 1]\, p^{c-1} \text{ and } q^c p = pq^c + [(c-1)\imath + 1]\, q^{c-1}. \tag{2.6}$$

Without loss of generality, assume that $a_1 \geq 1$. Then

$$\begin{aligned} w &= p\left(p^{a_1-1} q\right) q^{b_1-1} p^{a_2} q^{b_2} \cdots p^{a_t} q^{b_t} \\ &= p\left[qp^{a_1-1} - ((a_1-2)\imath + 1)\, p^{a_1-2}\right] q^{b_1-1} p^{a_2} \cdots q^{b_t} \\ &= pq\left[p^{a_1-1} q^{b_1-1} p^{a_2} \cdots q^{b_t}\right] - ((a_1-2)\imath + 1)\, p^{a_1-1} q^{b_1-1} p^{a_2} \cdots q^{b_t}. \end{aligned}$$

The result now follows by induction on the number of p's and q's in the word starting with $qp = z + \imath/2$ and $pq = z - \imath/2$. □

Theorem 4.3 and Proposition 4.4 show that the polynomials associated to elements of $\mathcal{H}[p,q]$ have real coefficients and have the same parity as n.

The next corollary is called a *useful identity* in [4].

Corollary 2.6. *The identity $q^k p^n q^{n-k} = p^{n-k} q^n p^k$ holds.*

Proof. Simply write

$$q^k p^n q^{n-k} = \left(q^k p^k\right)\left(p^{n-k} q^{n-k}\right). \tag{2.7}$$

The words $q^k p^k$ and $p^{n-k} q^{n-k}$ are balanced, so they commute. This gives the result. □

3. An expression for a polynomial in two different bases

Let $n \in \mathbb{N}$ and $a_{n,k} \in \mathbb{C}$. Theorem 2.5 gives a polynomial map $\mathcal{O} : \mathcal{H}[p,q] \to \mathbb{C}[x]$:

$$\sum_{k=0}^{n} a_{n,k} q^k p^n q^{n-k} \mapsto P_n(z) := \sum_{r=0}^{n} b_{n,r} z^r. \tag{3.1}$$

Explicit formulas connecting $\{a_{n,k}\}$ and $\{b_{n,k}\}$ are given in this section.

Theorem 3.1. *The sequence $\{a_{n,k}\}$ is given by*

$$a_{n,k} = \frac{1}{\imath^n n!} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n+1}{n-k-j} P_n\left(\imath\left(j + \tfrac{1}{2}\right)\right) \text{ for } 0 \leq k \leq n. \tag{3.2}$$

Expanding P_n gives (1.5).

Proof. The realization $p = x$ and $q = \imath \frac{d}{dx}$ gives $\frac{1}{2}(qp+pq)\,(x^m) = \imath\left(m + \frac{1}{2}\right) x^m$ and $q^k p^n q^{n-k}\,(x^m) = \imath^n n! \binom{m+k}{m+k-n} x^m$, with the usual convention that $\binom{a}{b} = 0$ if $b < 0$. It follows that

$$\sum_{k=0}^{n} a_{n,k} q^k p^n q^{n-k}\,(x^m) = \imath^n n! \sum_{\ell=0}^{m} \binom{n+\ell}{\ell} a_{n,n-m+\ell} x^m \tag{3.3}$$

and

$$P_n\left(\tfrac{1}{2}(qp+pq)\right)x^m = P_n\left(\imath\left(m+\tfrac{1}{2}\right)\right)x^m. \tag{3.4}$$

Therefore

$$P_n\left(\imath\left(m+\tfrac{1}{2}\right)\right) = \imath^n n! \sum_{\ell=0}^{m} \binom{n+\ell}{\ell} a_{n,n-m+\ell}, \quad \text{for } 0 \le m \le n. \tag{3.5}$$

Then (3.2) is obtained by solving the linear system (3.5) for $a_{n,m}$, and using the formula for matrix M_n^{-1} given in the next statement. □

Lemma 3.2. *The inverse of the Hankel matrix* $M_n = \left[\binom{i+j}{n}\right]_{0\le i,j\le n}$ *is*
$M_n^{-1} = \left[(-1)^{n-i-j}\binom{n+1}{i+j+1}\right]_{0\le i,j\le n}$.

Proof. The claim is equivalent to the identity

$$\sum_{k=0}^{n}(-1)^{n-i-j}\binom{i+k}{n}\binom{n+1}{k+j+1} = \begin{cases} 0 & \text{if } i \ne j, \\ 1 & \text{if } i = j. \end{cases} \tag{3.6}$$

The proof of (3.6) is a routine application of WZ.

An alternative proof. Consider the Jordan block matrix $J_{n+1} = J_{n+1}(i,j)$ with zero entries except for 1 when $j = i+1$. Then $J_{n+1}^k(i,j)$ is zero except a shifted diagonal with $1's$ at $j = i+k$. In particular, $J_{n+1}^k = 0$ for $k > n$. Thus $\tilde{M}_n$ can be expressed as $\tilde{M}_n = \sum_{k=0}^{n}\binom{n+k}{k}J_{n+1}^k = \sum_{k=0}^{+\infty}\binom{n+k}{k}J_{n+1}^k$. Now use $\sum_{k=0}^{+\infty}\binom{n+k}{k}z^k = (1-z)^{-n-1}$ to conclude that $\tilde{M}_n = (I_{n+1}-J_{n+1})^{-n-1}$. Thus

$$\tilde{M}_n^{-1} = (I_{n+1}-J_{n+1})^{n+1} = \sum_{k=0}^{n}\binom{n+1}{k}(-1)^k J_{n+1}^k$$

which proves the result. □

The expression for $a_{n,k}$ is particularly simple in the outer diagonal $\{a_{n,n}\}$.

Proposition 3.3. *Let* $\{a_{n,k}\}$ *be a pyramid with corresponding polynomials* $\{P_n\}$. *The outer diagonal of the pyramid is given by*

$$a_{n,n} = \frac{1}{\imath^n n!}P_n\left(\frac{\imath}{2}\right). \tag{3.7}$$

Therefore, if the polynomials P_n *have an exponential generating function*

$$G(z,t) = \sum_{n=0}^{\infty}\frac{P_n(z)}{n!}t^n \tag{3.8}$$

then the horizontal generating function for the outer diagonal is

$$\sum_{n=0}^{\infty}a_{n,n}t^n = G\left(\frac{\imath}{2},\frac{t}{\imath}\right). \tag{3.9}$$

The authors of [4] state that *apparently, the classical orthogonal polynomials give pyramids with ugly entries.* The result of Proposition 3.3 gives expressions for the outer diagonal $\{a_{n,n}\}$.

Example 3.4. The Legendre polynomials

$$P_n(x) = \frac{1}{\binom{2n}{n}} \sum_{m\geq 0} (-1)^m \binom{n}{m}\binom{2n-2m}{n} x^{n-2m}, \tag{3.10}$$

normalized to be monic, form a sequence of orthogonal polynomials. The corresponding (non-normalized) pyramid is

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 7 & & 10 & & 7 & & & \\ & & 17 & & 103 & & 103 & & 17 & & \\ & 203 & & 2948 & & 7138 & & 2948 & & 203 & \\ 583 & & 20091 & & 100286 & & 100286 & & 20091 & & 583 \end{array}$$

Proposition 3.3 gives

$$a_{n,n} = \frac{n!}{2^n(2n)!} \sum_{j=0}^{n} 2^{2j}\binom{n}{j}\binom{2n-2j}{n}. \tag{3.11}$$

Example 3.5. The (monic) Hermite polynomials are defined by

$$H_n(x) = \frac{n!}{2^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}. \tag{3.12}$$

The corresponding pyramid is

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 3 & & 2 & & 3 & & & \\ & & 7 & & 17 & & 17 & & 7 & & \\ & 25 & & 76 & & 182 & & 76 & & 25 & \\ 27 & & 159 & & 454 & & 454 & & 159 & & 27 \end{array}$$

The information to state the next Lemma came from OEIS, entry *A*047974.

Lemma 3.6. *The outer diagonal sequence* $\{a_{n,n}\}$ *of the pyramid corresponding to the Hermite polynomials is given by* $a_{n,n} = h_n/2^n n!$*, where* $\{h_n\}$ *satisfies the recurrence* $h_n = h_{n-1} + 2(n-1)h_{n-2}, \quad h_1 = 1,\ h_2 = 3$*. An explicit representation of* $\{h_n\}$ *is given by*

$$h_n = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} x^n e^{-(x-1)^2/4}\, dx, \tag{3.13}$$

so that h_n *can be interpreted as the moment of order* n *of a Gaussian random variable* X*, with mean* 1 *and variance* 2.

Proof. The sequence $\{a_{n,n}\}$ is related to the (monic) Hermite polynomials by the statement of Proposition 3.3. The details follow from the recurrence for the Hermite polynomials: $2H_{n+1}(x) = 2xH_n(x) - nH_{n-1}(x)$. □

Example 3.7. The Chebyshev polynomials T_n are given by

$$T_n(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right]. \tag{3.14}$$

In this case (3.7) gives

$$a_{n,n} = \frac{1}{\imath^n n!} T_n\left(\frac{\imath}{2}\right) = \frac{1}{2n!}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

This yields $a_{n,n} = L_n/2n!$ where L_n is the *Lucas number*. The corresponding result for the Chebyshev polynomials of the second kind $U_n(x)$ gives $a_{n,n} = F_{n+1}/n!$, with F_n is the *Fibonacci number*. The proof is based on Binet's formula

$$F_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}. \tag{3.15}$$

Theorem 3.1 provided an expression for $\{a_{n,k}\}$ in terms of the associated polynomial sequence $\{P_n\}$. The next result gives the polynomial $P_n(x)$ in terms of the sequence $\{a_{n,k}\}$.

Theorem 3.8. *The polynomials $P_n(z)$ associated to the sequence $\{a_{n,k}\}$ are given by*

$$P_n(z) = \imath^n n! \sum_{k=0}^{n} a_{n,k} \binom{-\imath z - \frac{1}{2} + k}{n} = \imath^n \sum_{k=0}^{n} a_{n,n-k}\left(-\imath z + \tfrac{1}{2} - k\right)_n, \tag{3.16}$$

where $(x)_n$ is the shifted factorial defined by $(x)_n = x(x+1)\cdots(x+n-1)$. Expanding in powers of z gives (1.6).

Proof. The result follows directly from formula (3.5), that gives $P_n(\imath(m+\frac{1}{2}))$ for $0 \le m \le n$ (a total of $n+1$ points), remarking that (3.5) holds in fact for all $m \in \mathbb{R}$ when written in the equivalent form

$$P_n\left(\imath\left(m+\tfrac{1}{2}\right)\right) = \imath^n n! \sum_{k=0}^{n} a_{n,k}\binom{m+k}{n}. \tag{3.17}$$

To obtain (1.6), use $(y)_k = \sum_{j=0}^{k} s(k,j)(y+k-1)^j$, the generating function of the Stirling numbers. □

The 1-dimensional *Weyl algebra* $\mathcal{A}_1$ is the free algebra with two generators R and D together with the commutation relation $RD - DR = 1$. This is a parallel version of the Heisenberg-Weyl algebra $\mathcal{A}$ discussed in Section 2. Each $u \in \mathcal{A}_1$ can be expressed uniquely in the normal form $u = \sum_{j,k} \alpha_{j,k} R^j D^k$. The connection to lattice paths or Ferrer diagrams is natural and well-known. To see this, assume u has n and m letters of R and D, respectively. Construct a walk from $(0,m)$ to $(n,0)$ as follows: by reading u from left to right, move a unit right (resp. down) step if the letter is R (resp. D). See references [5, 6, 9, 14] for details.

The algebra $\mathcal{A}_1$ allows alternative proofs for some of the results given in this section. Among the properties of $\mathcal{A}_1$ used, the following ones are easy to establish by induction:

$$D^k R^k = \prod_{j=1}^{k}(DR - j + 1), \qquad R^k D^n = \sum_{j=0}^{k}\binom{k}{j}(n-j+1)_j D^{n-j} R^{k-j}. \tag{3.18}$$

If $D = \imath q$ and $R = p$, then

$$RD - DR = 1, \quad \mathcal{O}(q^n p^n) = \imath^n \mathcal{O}(D^n R^n). \tag{3.19}$$

In this setting, the proof of Theorem 3.8 begins with the introduction of $x := RD + DR = 2DR + 1$, so that $DR = \frac{x-1}{2}$. Note that $D = iq$ and $R = p$ yields $x = \imath(pq + qp) = 2\imath z$. To complete the proof, use (3.18) to find

$$\begin{aligned}
\mathcal{O}(q^n p^n) &= \imath^n \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} \binom{k}{j} (n-j+1)_j D^{n-j} R^{k-j} R^{n-k} \\
&= \imath^n \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} \binom{k}{j} (n-j+1)_j D^{n-j} R^{n-j} \\
&= \imath^n \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} \binom{k}{j} (n-j+1)_j \prod_{m=1}^{k} \left(\tfrac{1}{2}x - m + \tfrac{1}{2}\right).
\end{aligned}$$

The transformation of the last expression to (3.16) is automatic with the WZ method.

Definition 3.9. The polynomials appearing in Theorem 3.8 are denoted by

$$Q_{n,k}(z) := \binom{-\imath z - \frac{1}{2} + k}{n} = \frac{1}{n!} \prod_{\ell=0}^{n-1} (-\imath z - \tfrac{1}{2} + k - \ell). \tag{3.20}$$

These are the polynomials associated to the homogeneous elementary words considered by Bender-Dunne:

$$Q_{n,k}(z) = \frac{1}{\imath^n n!} \mathcal{O}\left(q^k p^n q^{n-k}\right). \tag{3.21}$$

Theorem 3.10. *The polynomials $\{Q_{n,k}(z) : 0 \leq k \leq n\}$ form a basis for the vector space of polynomials of degree at most n.*

Proof. Each $Q_{n,k}(z)$ is a polynomial of degree n, so it suffices to establish their linear independence. Fix $n \in \mathbb{N}$. From (1.6) and (3.21) it follows that the coefficient of x^r in $Q_{n,k}(x)$ is

$$\frac{(-\imath)^r}{n!} \sum_{j=0}^{n-r} s(n, j+r) \binom{j+r}{r} \left(k - \tfrac{1}{2}\right)^j. \tag{3.22}$$

Now suppose $\sum_{k=0}^{n} u_k(n) Q_{n,k}(x) = 0$. The vanishing of the coefficient of x^r gives the system of equations

$$\sum_{k=0}^{n} u_k(n) \sum_{j=0}^{n-r} s(n, j+r) \binom{j+r}{r} \left(k - \tfrac{1}{2}\right)^j = 0 \tag{3.23}$$

for $0 \leq r, k \leq n$. Let $S = (S_{r,k})$ be the $(n+1) \times (n+1)$ matrix with entries

$$S_{r,k} = \sum_{j=0}^{n-r} s(n, j+r) \binom{j+r}{r} \left(k - \tfrac{1}{2}\right)^j, \tag{3.24}$$

so that (3.23) is $Su = 0$, where u is the vector $(u_k(n))$. The independence of $\{Q_{n,k}(x)\}$ is equivalent to the invertibility of S. Observe the factorzation $S = XY$

where $X = (X_{a,b})$ and $Y = (Y_{a,b})$ with $X_{a,b} = \binom{a+b}{b} s(n, a+b)$ and $Y_{a,b} = \left(b - \frac{1}{2}\right)^a$. The matrix X is upper triangular, $x_{a,b} = 0$ if $a + b > n$ and Y is a Vandermonde matrix. Therefore

$$\det X = \prod_{a=0}^{n} \binom{n}{a} \text{ and } \det Y = \prod_{j<k} \left[\left(k - \tfrac{1}{2}\right) - \left(j - \tfrac{1}{2}\right) \right] = \prod_{i=0}^{n} i!, \tag{3.25}$$

proving that $\det S = \prod_{k=1}^{n} k^k$ and $S = XY$ is invertible. □

Corollary 3.11. *The balanced words* $\left\{ q^k p^n q^{n-k} : 0 \leq k \leq n \right\}$ *form a basis for the class of balanced words of weight* n.

The next result phrases Theorem 3.10 in the language of the Weyl algebra $\mathcal{A}_1$.

Theorem 3.12. *The set* $S = \left\{ R^j D^k : j, k \geq 0 \right\}$ *is a basis for* $\mathcal{A}_1$.

Proof. As before, it suffices to verify linear independence. Let $K = \mathbb{C}[x, y]$ be a commutative polynomial ring, with basis $\{x^j y^k : j, k \geq 0\}$ over $\mathbb{C}$. Define the linear operators R and D on K by

$$R\left(x^j y^k\right) = x^{j+1} y^k \text{ and } D\left(x^j y^k\right) = j x^{j-1} y^k + x^j y^{k+1}. \tag{3.26}$$

A direct calculation shows that $RD - DR = 1$ on K, and hence $\mathbb{C}[x, y]$ is a representation of the Weyl algebra $\mathcal{A}_1$. To verify linear independence, suppose

$$L := \sum_{j,k} \alpha_{j,k} R^j D^k = 0. \tag{3.27}$$

A direct computation gives $D^k(1) = y^k$ and $R^j(y^k) = x^j y^k$. Thus, the value $L(1) = 0$ gives $\alpha_{j,k} = 0$ which proves independence. □

Problem. A word $w = w_0 w_1 \cdots w_n$ in $\mathcal{A}_1$ (where each $w_k = p$ or q) is a *palindrome* if $w_{n-k} = w_k$ for $0 \leq k \leq n$. The 'adjoint' of a monomial word $w = w_0 w_1 \cdots w_n$ is the word $w^* = w_n \cdots w_1 w_0$. This is extended to z in $\mathcal{A}_1$ by linearity. The element z is called Hermitian if $z = z^*$. For instance p and q are Hermitians, but pq is not, since $(pq)^* = q^* p^* = qp$. *Question*: Is it true that a monomial word w is Hermitian if and only if w is a palindrome? It is clear that if w is a palindrome, then w is Hermitian. The questions is to decide on the converse.

4. Polynomials versus pyramids

This section discusses how certain properties of the pyramids $\{a_{n,k}\}$ are reflected on the corresponding polynomials $P_n(z)$.

Proposition 4.1. *The polynomial* $P_n(z)$ *is monic if and only if* $\{a_{n,k}\}$ *is normalized by* $a_{n,0} + a_{n,1} + \cdots + a_{n,n} = 1$.

Proof. The polynomial

$$\binom{-\imath z - \frac{1}{2} + k}{n} = \frac{1}{n!} \prod_{j=0}^{n-1} \left(-\imath z - \tfrac{1}{2} + k - j\right) \tag{4.1}$$

has leading coefficient $(-\imath)^n / n!$. Theorem 3.8 now shows that the leading coefficient of $P_n(z)$ is the sum of $\{a_{n,k}\}$. □

The next statement clarifies the condition of hermitian-symmetry imposed on the pyramids [4]. The analysis begins the following observation.

Lemma 4.2. *The polynomial* $Q_{n,k}(x)$ *defined in* (3.20)*, with* $x \in \mathbb{R}$*, satisfies the symmetry identity* $Q^*_{n,k}(x) = (-1)^n Q_{n,n-k}(x)$*.*

Proof. This follows directly from

$$Q^*_{n,k}(x) = \frac{1}{n!}\prod_{\ell=0}^{n-1}\left(-\imath x - \tfrac{1}{2} + k - \ell\right)^* = \frac{(-1)^n}{n!}\prod_{\ell=0}^{n-1}\left(-\imath x + \tfrac{1}{2} - k + \ell\right).$$

∎

The next result characterizes real polynomials P_n.

Theorem 4.3. *The coefficients* $\{b_{n,r}\}$ *of* $P_n(z)$ *are real if and only if the coefficients* $\{a_{n,k}\}$ *are hermitian-symmetric; that is if* $a_{n,k} = a^*_{n,n-k}$*.*

Proof. The identity $P_n(x)^* = (-\imath)^n n! \sum_{k=0}^{n} a^*_{n,k} Q_{n,k}(x)$ and Lemma 4.2 show that $b_{n,r}$ real is equivalent to

$$\sum_{k=0}^{n} a_{n,k} Q_{n,k}(x) = \sum_{k=0}^{n} a^*_{n,n-k} Q_{n,k}(x). \tag{4.2}$$

The result now follows from Theorem 3.10. ∎

Proposition 4.4. *Assume the coefficients* $\{a_{n,k}\}$ *are real and symmetric. Then* P_n *has the same parity as* n*.*

Proof. Use the identity $Q_{n,k}(-x) = (-1)^n Q_{n,n-k}(x)$ established in the same way as in the proof of Lemma 4.2. ∎

5. Necessary conditions for orthogonality

Properties of the pyramid $\{a_{n,k}\}$ reflect on those of the associated sequence of polynomials $\{P_n\}$. For instance, if $\{a_{n,k}\}$ is normalized (total sum equal to 1), real and symmetric ($a_{n,k} = a_{n,n-k}$), then $\{P_n\}$ are monic, with real coefficients and P_n has the same parity as n. The question considered in this section is to determine conditions on $\{a_{n,k}\}$ that yield orthogonal polynomials.

Recall that a family of polynomials $\{C_n\}$ is called orthogonal if there is a positive weight function $w(z)$ such that

$$\langle C_n, C_m\rangle := \int_{\mathbb{R}} C_n(z) C_m(z) w(z)\, dz = \begin{cases} w_n > 0 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \tag{5.1}$$

Now assume that $C_n(z) = \sum_{k=0}^{n} c_{n,k} z^k$ is a family of monic, orthogonal polynomials with C_n of the same parity as n. The orthogonality of $\{C_n\}$ yields a sequence of algebraic equations that the coefficients $\{c_{n,k}\}$ must satisfy. For instance, $\langle C_0, C_2\rangle = 0$ gives

$$\left[z^2\right] + c_{2,0}\,[1] = 0, \tag{5.2}$$

where

$$[z^a] = \int_{\mathbb{R}} z^a w(z)\, dz. \tag{5.3}$$

Similarly, the orthogonality of the pairs $\{C_0, C_4\}$ and $\{C_1, C_3\}$ yield

$$\begin{aligned} \left[z^4\right] + c_{4,2}\left[z^2\right] + c_{4,0}\left[1\right] &= 0 \\ \left[z^4\right] + c_{3,1}\left[z^2\right] &= 0. \end{aligned} \tag{5.4}$$

The vanishing of the corresponding determinant gives

$$c_{4,0} + c_{2,0}c_{3,1} - c_{2,0}c_{4,2} = 0. \tag{5.5}$$

Looking at the first six polynomials gives, among many, the relation

$$c_{2,0}c_{5,1} + c_{4,0}c_{5,3} - c_{2,0}c_{4,2}c_{5,3} + c_{6,0} - c_{2,0}c_{6,2} - c_{4,0}c_{6,4} + c_{2,0}c_{4,2}c_{6,4} = 0. \tag{5.6}$$

Example 5.1. The *symmetric ordering* has $a_{n,0} = a_{n,n} = \frac{1}{2}$ and $a_{n,k} = 0$ if $k \neq 0, n$. Theorem 3.8 gives

$$P_n(z) = \frac{\imath^n}{2}\left[(-\imath z + \tfrac{1}{2} - n)_n + (-\imath z + \tfrac{1}{2})_n\right]. \tag{5.7}$$

The first few values appear in [4]:

$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = z^2 - \frac{3}{4},\ P_3(z) = z^3 - \frac{23}{4}z,\ P_4(z) = z^4 - \frac{43}{4}z^2 + \frac{105}{16}.$

This sequence of polynomials is not orthogonal since condition (5.5) is not satisfied.

Example 5.2. The polynomials corresponding to the *Born-Jordan ordering* have $a_{n,k} = \frac{1}{n+1}$ for $0 \leq k \leq n$. Theorem 3.8 gives

$$P_n(z) = \frac{\imath^n}{n+1}\sum_{k=0}^{n}\left(-\imath z + \tfrac{1}{2} - k\right)_n. \tag{5.8}$$

The first few values also appear in [4]:

$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = z^2 - \frac{5}{12},\ P_3(z) = z^3 - \frac{11}{4}z,\ P_4(z) = z^4 - \frac{19}{2}z^2 + \frac{189}{80}.$

Condition (5.5) does not hold, so this sequence is not orthogonal.

Example 5.3. The *Weyl ordering* has $a_{n,k} = 2^{-n}\binom{n}{k}$, and (3.16) give the polynomials

$$P_n(z) = \frac{\imath^n}{2^n}\sum_{k=0}^{n}\binom{n}{k}(-\imath z + \tfrac{1}{2} - k)_n. \tag{5.9}$$

The first few values may be found in [4]:

$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = z^2 - \frac{1}{4},\ P_3(z) = z^3 - \frac{5}{4}z,\ P_4(z) = z^4 - \frac{7}{2}z^2 + \frac{9}{16}.$

The condition (5.5) is now satisfied, so one might expect that these polynomials form an orthogonal family. It is stated in [4] that

$$P_n(z) = \frac{n!}{(2\imath)^n}\,{}_3F_2\left(\left.\begin{matrix} -n, n+1, \frac{1}{4} - \frac{\imath z}{2} \\ \frac{1}{2}, 1 \end{matrix}\right| 1\right). \tag{5.10}$$

To verify (5.10) from (5.9) is equivalent to the identity

$$\sum_{k=0}^{n}\binom{n}{k}(-\imath z + \tfrac{1}{2} - k)_n = (-1)^n n!\,{}_3F_2\left(\left.\begin{matrix} -n, n+1, \frac{1}{4} - \frac{\imath z}{2} \\ \frac{1}{2}, 1 \end{matrix}\right| 1\right). \tag{5.11}$$

The left-hand side of (5.11) is written now in hypergeometric form.

Lemma 5.4. *For* $n \in \mathbb{N}$

$$\sum_{k=0}^{n} \binom{n}{k} (-\imath z + \tfrac{1}{2} - k)_n = (-\imath z - n + \tfrac{1}{2})_n \, {}_2F_1 \left(\begin{matrix} -n, \frac{1}{2} - \imath z \\ \frac{1}{2} - n - \imath z \end{matrix} \middle| -1 \right). \tag{5.12}$$

Proof. It suffices to verify

$$\sum_{k=0}^{n} \binom{n}{k} (m + 1 - k)_n = (m - n + 1)_n \, {}_2F_1 \left(\begin{matrix} -n, m+1 \\ m - n + 1 \end{matrix} \middle| -1 \right) \tag{5.13}$$

obtained from (5.12) with $z = \imath(m + \frac{1}{2})$, as both sides are polynomials in z. This is accomplished by writing the left-hand side as

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!\, k!} \frac{(m + 1 - n + k)_n}{(m - n + 1)_n} \tag{5.14}$$

and using

$$\frac{n!}{(n-k)!} = (-1)^n (-n)_k \text{ and } \frac{(m + 1 - n + k)_n}{(m - n + 1)_n} = \frac{(m+1)_k}{(m + 1 - n)_k}. \tag{5.15}$$

This proves the result. ∎

Therefore (5.10) is equivalent to

$$(-\imath z - n + \tfrac{1}{2})_n \, {}_2F_1 \left(\begin{matrix} -n, \frac{1}{2} - \imath z \\ \frac{1}{2} - n - \imath z \end{matrix} \middle| -1 \right) = (-1)^n n! \, {}_3F_2 \left(\begin{matrix} -n, n+1, \frac{1}{4} - \frac{\imath z}{2} \\ \frac{1}{2}, 1 \end{matrix} \middle| 1 \right). \tag{5.16}$$

This is proved by observing again that both sides are polynomials in z, so it suffices to verify (5.16) when $z = \imath \left(m + \frac{1}{2} \right)$ and $m \in \mathbb{N}$. The identity becomes

$$\frac{(m - n + 1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, m+1 \\ m - n + 1 \end{matrix} \middle| -1 \right) = (-1)^n {}_3F_2 \left(\begin{matrix} -n, n+1, \frac{m+1}{2} \\ \frac{1}{2}, 1 \end{matrix} \middle| 1 \right). \tag{5.17}$$

In detail,

$$\frac{(m - n + 1)_n}{n!} \sum_{k=0}^{n} \frac{(-1)^k (-n)_k (m+1)_k}{k! (m - n + 1)_k} = (-1)^n \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k \left(\frac{m+1}{2} \right)_k}{(1)_k k! \left(\frac{1}{2} \right)_k}. \tag{5.18}$$

The method of Wilf-Zeilberger (WZ) described in [11] shows that both sides of (5.18), with m fixed, satisfy the recurrence

$$(n+2) u(n+2, m) - (2m+1) u(n+1, m) - (n+1) u(n, m) = 0. \tag{5.19}$$

Upon verifying the value for $n = 0$ and $n = 1$, the assertion (5.18) follows.

Note 5.5. The result (5.17) also follows from entries 102 and 103 in page 540 of [12]. The stated identity is

$${}_3F_2 \left(\begin{matrix} -n, a, b \\ \frac{a-n}{2}, \frac{1+a-n}{2} \end{matrix} \middle| 1 \right) = \frac{(2b - a + 1)_n}{(1-a)_n} {}_2F_1 \left(\begin{matrix} -n, 2b \\ 2b - a + 1 \end{matrix} \middle| -1 \right), \tag{5.20}$$

and (5.17) is obtained by choosing $a = n + 1$, $b = \frac{m+1}{2}$.

Note 5.6. The left-hand side of (5.18) can be reduced to

$$c(n,m) := \sum_{k=0}^{n} \binom{m+k}{m}\binom{m}{n-k} = \sum_{k=0}^{m} \binom{n+k}{m}\binom{m}{k}. \tag{5.21}$$

The equality of the two versions of $c(n,m)$ follows from (5.19). These coefficients have remarkable properties which is a topic deferred to [2].

Note 5.7. The hypergeometric representation of the polynomials $P_n(z)$ given in (5.10) shows that $P_n(z)$ may be expressed in terms of the *continuous Hahn polynomials*

$$p_n(z;a,b,c,d) = \imath^n \frac{(a+c)_n(a+d)_n}{n!}\, {}_3F_2\left(\left. \begin{matrix} -n, n+a+b+c+d-1, a+\imath z \\ a+c, a+d \end{matrix} \right| 1 \right).$$

The identity (5.10) shows that

$$P_n(z) = \frac{(-1)^n 2^n}{\binom{2n}{n}} p_n\left(-\frac{z}{2}; \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, n\right). \tag{5.22}$$

This is discussed in [7] and [8].

The continuous Hahn polynomials satisfy the orthogonality condition

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(a+\imath z)\Gamma(b+\imath z)\Gamma(c-\imath z)\Gamma(d-\imath z) p_n(z;a,b,c,d) p_m(z;a,b,c,d)\, dz$$
$$= \delta_{n,m} \frac{\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+a+c)\Gamma(n+b+d)}{n!(2n+a+b+c+d-1)\Gamma(n+a+b+c+d-1)}.$$

In the special case appearing here, using the identity $\left|\Gamma\left(\frac{1}{2}+\imath z\right)\right|^2 = \pi\ \mathrm{sech}(\pi z)$, the polynomials $P_n(z)$ are orthogonal on $\mathbb{R}$ with weight function $w(z) = \mathrm{sech}(\pi z)$ and generating function

$$\sum_{n=0}^{\infty} \frac{(2t)^n}{n!} P_n(z) = \frac{\exp(2z)\ \arctan t}{\sqrt{1+t^2}}. \tag{5.23}$$

Example 5.8. The second example corresponds to the coefficients $a_{n,k} = \binom{2n}{n}^{-1}\binom{n}{k}^2$. Theorem 3.8 now gives

$$P_n(z) = \frac{\imath^n (n!)^3}{(2n)!} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{-\imath z - \frac{1}{2} + k}{n}. \tag{5.24}$$

To identify this class of polynomials it is convenient to convert them to hypergeometric form to produce

$$P_n(z) = \frac{\imath^n (n!)^3}{(2n)!}\, {}_3F_2\left(\left. \begin{matrix} -n, -n, \frac{1}{2}-\imath z \\ 1, \frac{1}{2}-\imath z - n \end{matrix} \right| 1 \right), \tag{5.25}$$

as remarked in [4].

An alternative form of these polynomials is given next.

Lemma 5.9. *The polynomials* $P_n(z)$ *in* (5.25) *are given by*

$$P_n(z) = \frac{(-\imath)^n (n!)^3}{(2n)!}\, {}_3F_2\left(\left. \begin{matrix} -n, n+1, \frac{1}{2}-\imath z \\ 1, 1 \end{matrix} \right| 1 \right). \tag{5.26}$$

Proof. Both sides are polynomials in z, so it suffices to verify the identity

$$(5.27)\qquad {}_3F_2\left(\begin{matrix}-n, n+1, m+1\\ 1,1\end{matrix}\middle| 1\right) = (-1)^n\binom{m}{n}{}_3F_2\left(\begin{matrix}-n, -n, m+1\\ 1, m-n+1\end{matrix}\middle| 1\right),$$

obtained from the value $z = \left(m+\frac{1}{2}\right)\imath$. This may be written in the equivalent form

$$(5.28)\qquad \sum_{k=0}^{n}(-1)^k\binom{n+k}{k}\binom{m+k}{k}\binom{n}{k} = (-1)^n\sum_{k=0}^{n}\binom{n}{k}^2\binom{m+k}{n}.$$

A direct calculation shows that the identity holds for $n = 0$, 1 and the WZ-method shows that both sides satisfy the recurrence

$$(5.29)\quad (n+2)^2u(n+2,m) + (2m+1)(2n+3)u(n+1,m) - (n+1)^2u(n,m) = 0.$$

□

Note 5.10. The polynomials in Lemma 5.9 can be expressed in terms of the *Bateman polynomials* (see Section 18.19 in [10]):

$$(5.30)\qquad F_n(z) = {}_3F_2\left(\begin{matrix}-n, n+1, \frac{1+z}{2}\\ 1,1\end{matrix}\middle| 1\right) \text{ as } P_n(z) = \frac{(-\imath)^n(n!)^3}{(2n)!}F_n(-2\imath z).$$

It is curious that these seem to be the only powers of binomial coefficients that give orthogonal polynomials. The powers up to 50000 have been excluded using the first condition in Lemma 1.5.

6. Pyramid for the monomial z^n

For enumerative purposes, denote $[n] := \{1,\dots,n\}$ and $[-n,n] := \{\pm1,\dots,\pm n\}$. The *symmetric group* S_n is the set of permutations of $[n]$. The *signed permutation* group B_n (or, *hyperoctahedral group*) is the permutations π of $[-n,n]$, provided $\pi(-k) = -\pi(k)$. In the literature, S_n (resp. B_n) is a Coxeter group type A (resp. type B). A *descent* is a position k where the permutation value has decreased: $\pi(k-1) > \pi(k)$. Convention: $\pi(0) := 0$. The classical Eulerian sequence $A_{n,k}$ of type A (resp. Eulerian sequence $B_{n,k}$ of type B) enumerates S_n (resp. B_n) with k descents. The corresponding Eulerian polynomials of type A and type B are defined, respectively, by $A_n(x) = \sum_{k=0}^{n} A_{n,k}x^k$ and $B_n(x) = \sum_{k=0}^{n} B_{n,k}x^k$.

The polynomials $A_n(x)$ and $B_n(x)$ have the following rational generating functions:

$$\frac{A_n(x)}{(1-x)^{n+1}} = \sum_{k=0}^{\infty}(k+1)^n x^k, \qquad \text{and} \qquad \frac{B_n(x)}{(1-x)^{n+1}} = \sum_{k=0}^{\infty}(2k+1)^n x^k.$$

As a direct consequence, we find the connections

$$B_{2n}(x) = (1-x)^n A_n(x), \qquad \text{and} \qquad B_{2n+1}(x) = (1-x)^n A_{n+1}(x).$$

Several authors (see [1, 3, 13]) considered some *quantum* extensions of Eulerian polynomials.

The pyramid corresponding to the polynomials $P_n(z) = z^n$ is given by Theorem 3.1 as

$$(6.1)\qquad a_{n,k} = \frac{1}{n!2^n}\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n+1}{n-k-j}(2j+1)^n.$$

Define $B_{n,k} = 2^n n! a_{n,k}$. The rest of the section shows that $B_{n,k}$ are the coefficients of type B polynomials.

Lemma 6.1. *The numbers* $B_{n,k}$ *are integers with bivariate generating function*

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} B_{n,k}x^k\right)\frac{z^n}{n!} = \frac{(1-x)e^{(1-x)z}}{1-xe^{2z(1-x)}}. \tag{6.2}$$

Letting $x \to 1$ *shows that* $\{a_{n,k}\}$ *are normalized; that is* $a_{n,0}+a_{n,1}+\cdots+a_{n,n}=1$.

Letting $x=-1$ in (6.2) gives a relation between the numbers $B_{n,k}$ and the *Euler numbers* E_n defined by the generating function $\displaystyle\sum_{n=0}^{\infty} E_n\frac{z^n}{n!} = \frac{1}{\cosh z}$.

Corollary 6.2. *The numbers* $B_{n,k}$ *satisfy* $\displaystyle\sum_{k=0}^{n}(-1)^k B_{n,k} = 2^n E_n$.

The numbers $B_{n,k}$ resemble the *Eulerian numbers* $\left\langle {n \atop k} \right\rangle$, the coefficients of the type A polynomials, with bivariate generating function

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k\right)\frac{z^n}{n!} = \frac{(1-x)e^{(1-x)z}}{1-xe^{z(1-x)}}. \tag{6.3}$$

The similarity extends to the explicit expressions

$$B_{n,n-k} = (-1)^n\sum_{j=0}^{k}(-1)^j\binom{n+1}{2k+1-j}(2k+1-2j)^n \tag{6.4}$$

and

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+1-j)^n. \tag{6.5}$$

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References

[1] T. Amdeberhan. Theorems, problems and conjectures. Available at `http://www.math.tulane.edu/~tamdeberhan/conjectures.html`.

[2] T. Amdeberhan, V. De Angelis, A. Dixit, V. Moll, and C. Vignat. Arithmetic properties of a sequence arising from operator orderings. *In preparation.*

[3] M. Beck and B. Braun. Euler-Mahonian statistics via polyhedral geometry. *arXiv:1109.3353v2*, 2012.

[4] C. M. Bender and G. V. Dunne. Polynomials and operator orderings. *J. Math. Phys.*, 29:1727–1731, 1988.

[5] P. Blasiak, G. H. E. Duchamp, A. Horzela, K. A. Penson, and A. I Solomon. Heisenberg-Weyl algebra revisited: combinatorics of words and paths. *J. Phys. A.*, 41:415204, 2008.

[6] R. Graham, D. Knuth, and O. Patashnik. *Concrete Mathematics.* Addison Wesley, Boston, 2nd edition, 1994.

[7] A. Hamdi and J. Zeng. Orthogonal polynomials and operator orderings. *J. Math. Phys.*, 51:043506, 2010.

[8] T. Koorwinder. Meixner-Pollaczek polynomials and the Heisenberg algebra. *J. Math. Phys.*, 30:767–769, 1989.

[9] A. M. Navon. Combinatorics and Fermion algebra. *Riv. Nuovo Cimento B*, 16:324–330, 1973.

[10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[11] M. Petkovšek, H. Wilf, and D. Zeilberger. *A=B.* A. K. Peters, Ltd., 1st edition, 1996.

[12] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series*, volume 3: More special functions. Gordon and Breach Science Publishers, 1998.
[13] R. Stanley. Differential posets. *Journal of AMS*, 1:919–961, 1988.
[14] A. Varvak. Rook numbers and the normal ordering problem. *Journal of Comb. Theory Series A*, 112:292–307, 2005.

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