COMBINATORIAL RESULTS FOR CERTAIN SEMIGROUPS OF ORDER-PRESERVING FULL CONTRACTION MAPPINGS OF A FINITE CHAIN

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Abstract

Let $\mathcal{T}_n$ be the full symmetric semigroup on $X_n = \{1, 2, \ldots, n\}$ and let $\mathcal{OCT}_n$ and $\mathcal{ORCT}_n$ be its subsemigroups of order-preserving and order-preserving or order-reversing full contraction mappings of X_n, respectively. In this paper we investigate the cardinalities of some equivalences on $\mathcal{OCT}_n$ and $\mathcal{ORCT}_n$ which lead naturally to obtaining the orders of these subsemigroups. [1] [2]

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1 Introduction and Preliminaries

Let $X_n = \{1, 2, \ldots, n\}$. A (partial) transformation $\alpha : \operatorname{Dom}\alpha \subseteq X_n \to \operatorname{Im}\alpha \subseteq X_n$ is said to be *full* or *total* if $\operatorname{Dom}\alpha = X_n$; otherwise it is called *strictly* partial. The set of of full transformations of X_n, denoted by $\mathcal{T}_n$, more commonly known as the *full transformation semigroup* is also known as the *full symmetric semigroup or monoid* with *composition of mappings* as the semigroup operation. We shall write $x\alpha$ for the image of x under α instead of $\alpha(x)$. The *height* of α is denoted and defined by $h(\alpha) = \mid \operatorname{Im}\alpha \mid$, the *right [left] waist* of α is denoted and defined by $w^+(\alpha) = \max(\operatorname{Im}\alpha)\,[w^-(\alpha) = \min(\operatorname{Im}\alpha)]$, the *fix* of α is denoted and defined by $f(\alpha) = \mid F(\alpha) \mid$ where

$$F(\alpha) = \{x \in \operatorname{Dom}\alpha : x\alpha = x\}.$$

It is also well-known that a partial transformation ϵ is *idempotent* ($\epsilon^2 = \epsilon$) if and only if $\operatorname{Im}\epsilon = F(\epsilon)$. It is worth noting that to define the left (right) waist of a transformation the base set X_n must be totally ordered.
A transformation $\alpha \in \mathcal{T}_n$ is said to be *order-preserving (order-reversing)* if $(\forall x, y \in \operatorname{Dom}\alpha)\ x \leq y \implies x\alpha \leq y\alpha$ $(x\alpha \geq y\alpha)$ and, a *contraction mapping* (or simply a *contraction*) if $(\forall x, y \in \operatorname{Dom}\alpha) \mid x - y \mid \geq \mid x\alpha - y\alpha \mid$. We shall denote by $\mathcal{OCT}_n$ and $\mathcal{ORCT}_n$, the semigroups of order-preserving full contractions and of order-preserving or order-reversing full contractions of an $n-$chain, respectively.

Recently, Zhao and Yang [21] initiated the algebraic study of semigroups of order-preserving partial contractions of an n-chain, where they referred

[1]*Key Words*: height, right (left) waist and fix of a transformation, idempotents and nilpotents.

[2]This work was begun when the first named author was visiting Sultan Qaboos University for a 3-month research visit in Fall 2012.

to our contractions as *compressions*. This paper investigates the combinatorial properties of $\mathcal{OCT}_n$, $\mathcal{ORCT}_n$ and $\mathcal{ODCT}_n$, and of their subsets of idempotents.

In this section we introduce basic terminologies and prove some preliminary results. In Section 2 we obtain the cardinalities of various equivalence classes defined on $\mathcal{OCT}_n$ and in Sections 3 and 4 we obtain the analogues of the results in Section 2 for $\mathcal{ORCT}_n$ and $\mathcal{ODCT}_n$, respectively. These cardinalities lead to formulae for the orders of $\mathcal{OCT}_n$, $\mathcal{ORCT}_n$ and $\mathcal{ODCT}_n$ as well as new triangles of numbers that as a result of this work were recently recorded in [16].

For standard concepts in semigroup and transformation semigroup theory, see for example [8, 4]. Let

(1) $$\mathcal{OR}_n = \{\alpha \in \mathcal{T}_n : (\forall x, y \in X_n)\, x \leq y \implies x\alpha \leq y\alpha \text{ or } x\alpha \geq y\alpha\}$$

be the subsemigroup of $\mathcal{T}_n$ consisting of all order-preserving or order-reversing full transformations of X_n, and let

(2) $$\mathcal{O}_n = \{\alpha \in \mathcal{T}_n : (\forall x, y \in X_n) x \leq y \implies x\alpha \leq y\alpha\}$$

be the subsemigroup of $\mathcal{T}_n$ consisting of all order-preserving full transformations of X_n. Also let

(3) $$\mathcal{CT}_n = \{\alpha \in \mathcal{T}_n : (\forall x, y \in X_n)\ \mid x - y \mid \geq \mid x\alpha - y\alpha \mid\}$$

be the subsemigroup of $\mathcal{T}_n$ consisting of all full contractions of X_n, and let

(4) $$\mathcal{D}_n = \{\alpha \in \mathcal{T}_n : (\forall x \in X_n)\, x\alpha \leq x\}$$

be the subsemigroup of $\mathcal{T}_n$ consisting of all *order-decreasing* full transformations of X_n.
We have the following results

Lemma 1.1 *Let $\alpha \in \mathcal{CT}_n$ be such that $f(\alpha) = m$. Then $F(\alpha) = \{i, i+1, \ldots, i+m-1\}$. Equivalently, $F(\alpha)$ is convex.*

Proof. Observe that it is sufficient to show that any point between two fixed points (of $\alpha \in \mathcal{CT}_n$) must also be a fixed point. Let $x, y \in F(\alpha)$. Then $x\alpha = x$ and $y\alpha = y$. Suppose also without loss of generality $x < x' < y$ for some $x' \in X_n$. Note that if $x' = x'\alpha$, there is nothing to prove. Thus we consider two cases: (i) $x' > x'\alpha$; (ii) $x' < x'\alpha$. In the former, we have

$$\mid y - x' \mid = \mid y\alpha - x' \mid = y\alpha - x' < y\alpha - x'\alpha = \mid y\alpha - x'\alpha \mid$$

which implies that α is not a contraction. Hence we get a contradiction. In the latter, we have

$$\mid x' - x \mid = \mid x' - x\alpha \mid = x' - x\alpha < x'\alpha - x\alpha = \mid x'\alpha - x\alpha \mid$$

which implies that α is not a contraction. Hence we get a contradiction. Thus, the proof is complete. □

Lemma 1.2 *Let $\alpha \in \mathcal{CT}_n$ be such that $h(\alpha) = p$. Then* $\operatorname{Im}\alpha = \{i, i+1, \ldots, i+p-1\}$. *Equivalently,* $\operatorname{Im}\alpha$ *is convex.*

Proof. (By contradiction) Suppose that $\operatorname{Im}\alpha$ is not convex. Then there exist $x, z \in \operatorname{Im}\alpha$ with $x < y < z$ for some $y \in X_n \setminus \operatorname{Im}\alpha$. Let $(y-1]$ and $[y+1)$ be the lower and upper saturations of $y-1$ and $y+1$, respectively. Notice that $x \in (y-1]$ and $z \in [y+1)$. Moreover, $(y-1]\alpha^{-1} \neq X_n \neq [y+1)\alpha^{-1}$ but $(y-1]\alpha^{-1} \cup [y+1)\alpha^{-1} = X_n$. If $(y-1]\alpha^{-1}$ is convex then since $(y-1]\alpha^{-1} \neq X_n$ there exist either (i) $t \in (y-1]\alpha^{-1}$ and $t+1 \in [y+1)\alpha^{-1}$; or (ii) $t \in (y-1]\alpha^{-1}$ and $t-1 \in [y+1)\alpha^{-1}$.

Case (i): it is clear that $t\alpha \leq y-1$ and $(t+1)\alpha \geq y+1$ so that

$$2 \leq (t+1)\alpha - t\alpha = \mid (t+1)\alpha - t\alpha \mid \leq \mid (t+1) - t \mid = 1,$$

which is a contradiction.

Case (ii): it is clear that $t\alpha \leq y-1$ and $(t-1)\alpha \geq y+1$ so that

$$2 \leq (t-1)\alpha - t\alpha = \mid (t-1)\alpha - t\alpha \mid \leq \mid (t-1) - t \mid = 1,$$

which is another contradiction.

On the other hand if $(y-1]\alpha^{-1}$ is not convex then there exists $t \in (y-1]\alpha^{-1}$ and either $t+1 \in [y+1)\alpha^{-1}$ or $t-1 \in [y+1)\alpha^{-1}$. In the former, we see that $t\alpha \leq y-1$ and $(t+1)\alpha \geq y+1$. Thus,

$$2 \leq (t+1)\alpha - t\alpha = \mid (t+1)\alpha - t\alpha \mid \leq \mid (t+1) - t \mid = 1,$$

which is a contradiction. In the latter, we see that $t\alpha \leq y-1$ and $(t-1)\alpha \geq y+1$. Thus,

$$2 \leq (t-1)\alpha - t\alpha = \mid (t-1)\alpha - t\alpha \mid \leq \mid (t-1) - t \mid = 1,$$

which is another contradiction. Hence, the proof is complete. □

Next we state two important identities that will be needed later.

Lemma 1.3 *(Vandemonde's Convolution Identity, [15, (3a), p.8]). For all natural numbers m, n and p we have*

$$\sum_{k=0}^{n} \binom{n}{m-k}\binom{p}{k} = \binom{n+p}{m}.$$

Lemma 1.4 *[12, Lemma 1.3] For all natural numbers j and a we have*

$$\sum_{i=0}^{j-a} \binom{j-i}{a} = \binom{j+1}{a+1}.$$

Lemma 1.5 *([15, (3b), p.8]). For all natural numbers m, n and p we have*

$$\sum_{k=0}^{n} \binom{n-k}{n-m}\binom{p+k-1}{k} = \binom{n+p}{m}.$$

2 Order-preserving Full Contractions

Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations. Many numbers and triangle of numbers regarded as combinatorial gems like the Stirling numbers [7, pp. 42 & 96], the factorial [17], the Fibonacci number [6], Catalan numbers [4], Lah numbers [4], etc., have all featured in these enumeration problems. For a nice survey article concerning combinatorial problems in the partial transformation semigroup and some of its subsemigroups we refer the reader to Umar [19]. These enumeration problems lead to many numbers in Sloane's encyclopaedia of integer sequences [16] but there are also others that are not yet or have just been recorded in [16].
Now recall the definitions of *height*, *(right) waist* and *fix* of $\alpha \in \mathcal{T}_n$ stated earlier. From Umar [19], we quote this result.

Lemma 2.1 *Let $X_n = \{1, 2, \ldots, n\}$ and $P = \{p, m, k\}$, where for a given $\alpha \in \mathcal{T}_n$, we set $p = h(\alpha), m = f(\alpha)$ and $k = w^+(\alpha)$. Then we have the following:*

1. $n \geq k \geq p \geq m \geq 1$;
2. $k = 1 \implies p = 1 \implies m \leq 1$.

As in Umar [19] let S be a set full transformations on X_n and consider the combinatorial functions:

$$F(n; p, m, k) =\mid \{\alpha \in S : \wedge(h(\alpha) = p, f(\alpha) = m, w^+(\alpha) = k)\} \mid,$$
$$F(n; p, m) =\mid \{\alpha \in S : \wedge(h(\alpha) = p, f(\alpha) = m\} \mid,$$
$$F(n; p, k) =\mid \{\alpha \in S : \wedge(h(\alpha) = p, w^+(\alpha) = k)\} \mid,$$
$$F(n; m, k) =\mid \{\alpha \in S : \wedge(f(\alpha) = m, w^+(\alpha) = k)\} \mid,$$
$$F(n; p) =\mid \{\alpha \in S : h(\alpha) = p\} \mid,$$
$$F(n; m) =\mid \{\alpha \in S : f(\alpha) = m\} \mid,$$
$$F(n; k) =\mid \{\alpha \in S : w^+(\alpha) = k\} \mid .$$

Observe that

$$F(n; a_1, a_2) = \sum_{a_3} F(n; a_1, a_2, a_3),\ F(n; a_1) = \sum_{a_2} F(n; a_1, a_2),$$

and

$$|S| = \sum_{a_1} F(n; a_1)$$

where $\{a_1, a_2, a_3\} = \{p, m, k\}$.

The following lemma is crucial to our investigation.

Lemma 2.2 *Let*

$$\begin{aligned} a_{n,p} &= \mid \{\alpha \in \mathcal{OCT}_n : (F(\alpha) = \{1\}) \wedge (h(\alpha) = p)\} \mid \\ &= \mid \{\alpha \in \mathcal{OCT}_n : (F(\alpha) = \{n\}) \wedge (h(\alpha) = p)\} \mid . \end{aligned}$$

Then $a_{n,p} = \binom{n-2}{p-1}$.

Proof. Let $\alpha \in \mathcal{OCT}_n$ be such that $F(\alpha) = \{1\}$ and $h(\alpha) = p$. First observe that $\operatorname{Im}\alpha = \{1, 2, \ldots, p\}$, by Lemma 1.2. Next, recall that the *blocks* ($y\alpha^{-1}$ for $y \in \operatorname{Im}\alpha$) are convex by order-preservedness and so the number of partitions of X_n into p convex blocks is the number of ways of inserting $p-1$ bars between the $n-1$ spaces (between the points of the n-chain). However, notice that in this case the space between 1 and 2 is not available so that 2 does not become a fixed point since $F(\alpha) = \{1\}$. Thus, we have $n-2$ available places to insert $p-1$ bars, which can be done in $\binom{n-2}{p-1}$ ways, without new points being introduced. The result when $F(\alpha) = \{n\}$, follows by symmetry. □

The main result in this paper can now be stated.

Proposition 2.3 *Let* $S = \mathcal{OCT}_n = \mathcal{O}_n \cap \mathcal{CT}_n$. *Then* $F(n; p, m, k) = \binom{n-m-1}{n-p-1}$.

Proof. Let $\alpha \in \mathcal{OCT}_n$ be such that $h(\alpha) = p$, $f(\alpha) = m$ and $w^+(\alpha) = k$. Then by Lemmas 1.1 & 1.2 we see that

$F(\alpha) = \{i, i+1 \ldots, i+m-1\} \subseteq \{k-p+1, k-p+2 \ldots, k\} = \operatorname{Im}\alpha$, where $k-p+1 \le i \le k-m+1$. Next, observe that by order-preservedness we may decompose α into $\beta = \alpha|_{\{1,2,\ldots,i\}}$, $id_{\{i+1,i+2,\ldots,i+m-2\}}$ and $\beta' = \alpha|_{\{i+m-1,i+m,\ldots,n\}}$, where $F(\beta) = \{i\}$, $\operatorname{Im}\beta = \{k-p+1, \ldots, i\}$, $F(\beta') = \{i+m-1\}$, and $\operatorname{Im}\beta' = \{i+m-1, \ldots, k\}$. Now it is not difficult to see that for β there are $a_{i,i-k+p}$ possible maps, and for β' there are $a_{n-i-m+2,k-i-m+2}$ possible maps. Thus, multiplying the two numbers and taking the sum of the product over i from $k-p+1$ to $k-m+1$ and using Lemmas 1.5 & 2.2 gives

$$\begin{aligned} F(n; p, k, m) &= \sum_{i=k-p+1}^{k-m+1} a_{i,i-k+p} a_{n-i-m+2,k-i-m+2} \\ &= \sum_{i=k-p+1}^{k-m+1} \binom{i-2}{i-k+p-1} \binom{n-i-m}{k-i-m+1} \\ &= \sum_{i=k-p+1}^{k-m+1} \binom{i-2}{i-k+p-1} \binom{n-i-m}{n-k-1} \\ &= \binom{n-m-1}{p-m} = \binom{n-m-1}{n-p-1}, \end{aligned}$$

using the substitution $i-k+p-1 = j$, for the last step. □

Corollary 2.4 *Let $S = \mathcal{OCT}_n$. Then $F(n;p,m) = (n-p+1)\binom{n-m-1}{n-p-1}$, for $p \geq m \geq 1$.*

Corollary 2.5 *Let $S = \mathcal{OCT}_n$. Then $F(n;p,k) = \binom{n-1}{p-1}$, for $k \geq p \geq 1$.*

Proof. It follows directly from Proposition 2.3 and Lemma 1.4. $\square$

Corollary 2.6 *$S = \mathcal{OCT}_n$. Then $F(n;m,k) = \sum_{p=m}^{k} \binom{n-m-1}{n-p-1}$, for $k \geq m \geq 1$.*

Proof. It follows directly from Proposition 2.3 and Lemma 1.4. $\square$

Corollary 2.7 *Let $S = \mathcal{OCT}_n$. Then $F(n;p) = (n-p+1)\binom{n-1}{p-1}$, for $p \geq 1$.*

Proof. It follows from either of the Corollaries 2.4 & 2.5. $\square$

Corollary 2.8 *Let $S = \mathcal{OCT}_n$. Then $F(n;k) = \sum_{p=1}^{k} \binom{n-1}{p-1}$, for $k \geq 1$.*

Proof. It follows from either of the Corollaries 2.5 & 2.6. $\square$

For $0 \leq i \leq n$, let

$$F(n;m_i) = \mid \{\alpha \in S : f(\alpha) = i\} \mid .$$

Corollary 2.9 *Let $S = \mathcal{OCT}_n$. Then $F(n;m_n) = 1$ and $F(n;m) = (n-m+3)2^{n-m-2}$, for $n \geq 2$ and $n > m \geq 1$.*

Proof. It follows from either of the Corollaries 2.4 & 2.6. $\square$

Corollary 2.10 *Let $S = \mathcal{OCT}_n$. Then $\mid S \mid = \mid \mathcal{OCT}_n \mid = (n+1)2^{n-2}$, for $n \geq 1$.*

Proof. It follows from any one of the Corollaries 2.7, 2.8 & 2.9. $\square$

Corollary 2.11 *Let $S = E(\mathcal{OCT}_n)$. Then $F(n;p,k) = F(n;m,k) = 1$, for $k \geq p = m \geq 1$.*

Proof. Since $F(\alpha) = \operatorname{Im}\alpha$ for idempotents, it follows that $p = m$. Hence the result follows from Proposition 2.3. $\square$

Corollary 2.12 *Let $S = E(\mathcal{OCT}_n)$. Then $F(n;p) = F(n;m) = n-p+1 = n-m+1$, for $p = m \geq 1$.*

Corollary 2.13 *Let $S = E(\mathcal{OCT}_n)$. Then $F(n;k) = k$, for $k \geq 1$.*

Corollary 2.14 *$\mid E(\mathcal{OCT}_n) \mid = n(n+1)/2 = \binom{n+1}{2}$, for $n \geq 1$.*

Remark 2.15 *The triangle of numbers $F(n;m)$ has as a result of this work been recorded in [16]. However, $F(n;p)$, $F(n;k)$, $F(n;m_1)$ and $\mid \mathcal{OCT}_n \mid$ were recorded (in [16]) as A104698, A008949, A045623 and A001792, respectively.*

3 Order-preserving or Order-reversing Full Contractions

Remark 3.1 *For $h(\alpha) = p = 1$ the concepts of order-preserving and order-reversing coincide but distinct otherwise. However, the map $\alpha \longmapsto \alpha h$, where $xh = n - x + 1$, for all x in X_n is a bijection between the two sets for $p \geq 2$, see [3, page 2, last paragraph].*

Remark 3.2 *Every idempotent is necessarily order-preserving. Thus, there are no additional idempotents from reversing the order.*

The main result of this section is

Proposition 3.3 *Let $S = \mathcal{ORCT}_n = \mathcal{OR}_n \cap \mathcal{CT}_n$. Then*

$$F(n;p,k) = \begin{cases} 2\binom{n-1}{p-1}, & p > 1; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. It follows from Corollary 2.5 and Remark 3.1. □

Corollary 3.4 *Let $S = \mathcal{ORCT}_n$. Then*

$$F(n;p) = \begin{cases} 2(n-p+1)\binom{n-1}{p-1}, & p > 1; \\ n, & \text{otherwise.} \end{cases}$$

Proof. It follows from Proposition 3.3 and the fact that $p \leq k \leq n$. □

Corollary 3.5 *Let $S = \mathcal{ORCT}_n$. Then $F(n;k) = 2\sum_{p=1}^{k}\binom{n-1}{p-1} - 1$, for $k \geq 1$.*

Proof. It follows from Proposition 3.3 and the fact that $1 \leq p \leq k$. □

Corollary 3.6 *Let $S = \mathcal{ORCT}_n$. Then $\mid S \mid = \mid \mathcal{ORCT}_n \mid = (n+1)2^{n-1} - n$, for $n \geq 1$.*

Proof. It follows from either of the Corollaries 3.4 & 3.5. □

Let us denote the set of all order-reversing full contraction of X_n by $\mathcal{ORCT}_n^*$ and let

$$b(n,p) = \mid \{\alpha \in \mathcal{ORCT}_n^* : h(\alpha) = p \text{ and } f(\alpha) = 1\} \mid,$$

be the number of order-reversing full contractions of height p and with exactly one fixed point. Then we have

Lemma 3.7 *For $n \geq p \geq 1$, we have $b(n,p) = (n-p+1)\sum_{i \geq 1}\binom{n-2i}{p-2i}$.*

Lemma 3.8 *For $n \geq p \geq 2$, we have*

(a) $b(n,p) = (n-p+1)\binom{n-2}{p-1} + b(n-2,p-2),\ \ b(n,1) = n, b(2,2) = 0,$

(b) $(n-p)b(n,p) = (n-p)b(n-1,p-1) + (n-p+1)b(n-2,p-2),$
$b(2r+1,2r+1) = 1, b(2r,2r) = 0.$

Define a sequence $\{a_n\}$ by

$$na_n = (n+2)a_{n-1} + 2(n+1)a_{n-2},\ \ a_1 = 1, a_2 = 2.$$

Then we have

Lemma 3.9 *Let* $S = \mathcal{ORCT}_n^*$. *Then*

(a) $F(n;m_0) = a_{n-1},\ n \geq 2;$

(b) $F(n,m_1) = a_n,\ n \geq 1;$

(c) $F(n;m) = 0,\ m \geq 2.$

Lemma 3.10 *Let* $S = \mathcal{ORCT}_n$. *Then* $F(1;m_0) = 0,\ F(n;m_n) = 1$ *and*

(a) $F(n;m_0) = a_{n-1},\ n \geq 2;$

(b) $F(n,m_1) = a_n + (n+2)2^{n-3} - n,\ n \geq 1;$

(c) $F(n;m) = (n-m+3)2^{n-m-2},\ n > m \geq 2.$

Remark 3.11 *The triangles of numbers* $F(n;p)$, $F(n;k)$ *and* $F(n;m)$; *and the sequences* $F(n;m_0)$, $F(n;m_1)$ *and* $\mid \mathcal{ORCT}_n \mid$ *have as a result of this work just been recorded in [16].*

4 Order-preserving and Order-decreasing Full Contractions

The main result of this section is

Proposition 4.1 *Let* $S = \mathcal{ODCT}_n = \mathcal{D}_n \cap \mathcal{OCT}_n$. *Then*

$$F(n;p,k,m) = \begin{cases} \binom{n-m-1}{p-m}, & p = k; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let $\alpha \in \mathcal{ODCT}_n$ be such that $h(\alpha) = p$, $f(\alpha) = m$ and $w^+(\alpha) = k$. Then by Lemmas 1.1 & 1.2 we see that

$$F(\alpha) = \{1, 2\ldots, m\} \subseteq \{1, 2\ldots, p\} = \operatorname{Im}\alpha.$$

Moreover, $p = k$. Next, observe that by order-preservedness we may decompose α into $id_{\{1,2,\ldots,m-1\}}$ and $\beta' = \alpha|_{\{m,m+1,\ldots,n\}}$, where $F(\beta') = \{m\}$, and $\operatorname{Im}\beta' = \{m, m+1\ldots, p\}$. Now it is not difficult to see that there are $a_{n-m+1,p-m+1}$ possible β's, by Lemma 2.2. Thus,

$$F(n;p,k,m) = \begin{cases} \binom{n-m-1}{p-m}, & p = k; \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

Corollary 4.2 *Let* $S = \mathcal{ODCT}_n$. *Then* $F(n; p, m) = \binom{n-m-1}{p-m}$, *for* $p \geq m \geq 1$.

Corollary 4.3 $S = \mathcal{ODCT}_n$. *Then* $F(n; m, k) = \binom{n-m-1}{k-m}$, *for* $k \geq m \geq 1$.

Corollary 4.4 *Let* $S = \mathcal{ODCT}_n$. *Then* $F(n; p) = \binom{n-1}{p-1}$, *for* $p \geq 1$.

Corollary 4.5 *Let* $S = \mathcal{ODCT}_n$. *Then* $F(n; k) = \binom{n-1}{k-1}$, *for* $k \geq 1$.

Corollary 4.6 *Let* $S = \mathcal{ODCT}_n$. *Then* $F(n; m_n) = 1$ *and* $F(n; m) = 2^{n-m-1}$, *for* $n > m \geq 1$.

Corollary 4.7 *Let* $S = \mathcal{ODCT}_n$. *Then* $\mid S \mid = \mid \mathcal{ODCT}_n \mid = 2^{n-1}$, *for* $n \geq 1$.

Corollary 4.8 *Let* $S = E(\mathcal{ODCT}_n)$. *Then*

$$F(n; p, m) = \begin{cases} 1, & p = m; \\ 0, & otherwise. \end{cases}$$

Proof. Since $F(\alpha) = \operatorname{Im} \alpha$ for idempotents, it follows that $p = m$. Hence the result follows from Proposition 2.3. □

Corollary 4.9 *Let* $S = E(\mathcal{ODCT}_n)$. *Then* $F(n; p) = F(n; m) = F(n; k) = 1$, *for* $p, k, m \geq 1$.

Corollary 4.10 $\mid E(\mathcal{ODCT}_n) \mid = n$, *for* $n \geq 1$.

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