ENUMERATING MAXIMAL TATAMI MAT COVERINGS OF SQUARE GRIDS WITH v VERTICAL DOMINOES

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Abstract. We enumerate a certain class of monomino-domino coverings of square grids, which conform to the *tatami* restriction; no four tiles meet.

Let $\mathbf{T}_n$ be the set of monomino-domino tatami coverings of the $n \times n$ grid with the maximum number, n, of monominoes, oriented so that they have a monomino in each of the top left and top right corners. We give an algorithm for exhaustively generating the coverings in $\mathbf{T}_n$ with exactly v vertical dominoes in constant amortized time, and an explicit formula for counting them. The polynomial that generates these counts has the factorisation

$$P_n(z) \prod_{j \geq 1} S_{\lfloor \frac{n-2}{2^j} \rfloor}(z),$$

where $S_n(z) = \prod_{i=1}^{n}(1 + z^i)$, and $P_n(z)$ is an irreducible polynomial, at least for $1 < n < 200$. We present some compelling properties and conjectures about $P_n(z)$. For example $P_n(1) = n2^{\nu(n-2)-1}$ for all $n \geq 2$, where $\nu(n)$ is the number of 1s in the binary representation of n and $\deg(P_n(z)) = \sum_{k=1}^{n-2} Od(k)$, where $Od(k)$ is the largest odd divisor of k.

1. Introduction

The counting of domino coverings, together with its extension to counting perfect matchings in (planar) graphs, is a classic area of enumerative combinatorics and theoretical computer science. Less attention has been paid, however, to problems where the local interactions of the dominoes are restricted in some fashion. Perhaps the most natural such restriction is the "tatami" condition, defined below. The tatami condition is quite restrictive: for example, the 10×13 grid cannot be covered with dominoes and also satisfy the tatami condition. In this paper we restrict our attention to square grids, and explore in some detail the enumeration of certain extremal configurations.

Tatami mats are a traditional Japanese floor covering whose dimensions are approximately 1m $\times$ 1m or 1m $\times$ 2m. In certain arrangements, no four tatami mats may meet. Such an arrangement has a preferable structure which is discussed in [1] and [2].

A *tatami covering* is an arrangement of 1×1 monominoes, 1×2 horizontal dominoes, and 2×1 vertical dominoes, in which no four tiles meet. The present discussion is about tatami coverings of the $n \times n$ grid with exactly n monominoes and v vertical dominoes. On the basis of some computer investigations, Don Knuth discovered that the generating polynomial for small tatami coverings of this type, with respect to the number of vertical dominoes they contain, is a product of cyclotomic polynomials and a mainly mysterious, irreducible polynomial (private communication, December 2010). Knuth's discovery and our own observations motivated Conjecture 4 in [1], which is presented here as Equation (2). In this paper we generalize and prove Knuth's cyclotomic factors, and determine some important properties of the mysterious polynomial.

We prove in [1] that all of the $n \times n$ coverings with n monominoes can be rotated so that monominoes appear in each of the top two corners of the grid, so we let $\mathbf{T}_n$ be the set of these. Let $H(n,k)$ be the number of coverings in $\mathbf{T}_n$ with exactly k horizontal dominoes, and let $V(n,k)$ be the number with exactly k vertical dominoes. Let $S_n(z) = \prod_{i=1}^{n}(1+z^i)$. We prove that the polynomial

$$\text{(1)} \qquad V\!\!H_n(z) := 2 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{n-i-2}(z) S_{i-1}(z) z^{n-i-1} + \left(S_{\lfloor \frac{n-2}{2} \rfloor}(z) \right)^2,$$

is equal to $\sum_{k \geq 0} H(n,k) z^k$ for odd n and is equal to $\sum_{k \geq 0} V(n,k) z^k$ for even n. Knuth's observation generalizes to

$$\text{(2)} \qquad V\!\!H_n(z) = P_n(z) \prod_{j \geq 1} S_{\lfloor \frac{n-2}{2^j} \rfloor}(z),$$

where $P_n(z)$ is the "mysterious" polynomial. We prove Equation (2) in Theorem 3.

The remaining factors of Equation (2) are of the form $S_k(z)$, where k is a binary right shift of $n-2$, and the complete factorisation of these is known in general. The *ith cyclotomic polynomial*, $\Phi_i(z)$, is defined as $\prod_{\omega \in \Omega}(z - \omega)$, where Ω is the set of ith primitive roots of unity. Lemma 5 in [1] shows that $S_k(z)$ is a certain product of cyclotomic polynomials, which are known to be irreducible, and thus $V\!\!H_n(z)$ can apparently be factored completely as

$$\text{(3)} \qquad V\!\!H_n(z) = P_n(z) \prod_{j \geq 1} \Phi_{2j}(z)^{\lfloor \frac{n-2}{2j} \rfloor}.$$

We have verified the irreducibility of $P_n(z)$ for $1 < n < 200$ (the degree of $P_{199}(z)$ is 13022 and its largest coefficient has 55 digits), and thus we hope that Equation (3) is the complete factorisation of $V\!H_n(z)$ for all $n \geq 2$.

The class $P_n(z)$ of polynomials has some compelling properties, some of which are theory, others empirical. For example, we observe in Conjecture 2 that the alternating sums of $P_n(z)$ are the coefficients of the ordinary generating function

$$\sum_{n\geq 2} P_n(-1)z^{n-2} = \frac{(1+z)(1-2z)}{(1-2z^2)\sqrt{1-4z^2}},$$

for $1 < n < 200$. If the conjecture is true, then $P_{2(n+1)}(-1) = \binom{2n}{n}$. Furthermore $P_n(-1)$ is equal to the sum of the absolute values of the coefficients of $P_n(z)$, only for $n \geq 20$. This second fact is surprising, considering the way $P_n(z)$ is derived – why $n \geq 20$?

The complex roots of $P_n(z)$ appear to cluster neatly around the unit circle, and form convergent sequences as $n \longrightarrow \infty$. They are plotted in Fig. 7 for odd n; for even n, the plot has a similar look.

Theoretical progress on $P_n(z)$ comprises Theorem 4 and Theorem 5. The former states that $\deg(P_n(z)) = \sum_{k=1}^{n-2} Od(k)$, where $Od(n)$ is the largest odd divisor of n. We prove in Theorem 5 that for all $n \geq 2$, the sum of the coefficients of $P_n(z)$ is equal to $n2^{\nu(n-2)-1}$, where $\nu(n)$ is the number of 1-bits in the binary representation of n.

Our technique for finding $V\!H_n(z)$ employs an operation which preserves the tatami condition, called the *diagonal flip*, defined in [2]. The added observation that a diagonal flip changes the orientation of some dominoes, enables us to further exploit it. The crux of the argument uses the partition of $\mathbf{T}_n$, from Theorem 2 of [2], which reveals diagonal flips each with $1, 2, \ldots, k$ dominoes, respectively, that can be flipped independently. We use this to express $V\!H_n(z)$ in terms of $S_k(z)$, the generating polynomial for the number of subsets of $\{1, 2, \ldots, k\}$ whose elements sum to i.

The formula for the vth coefficient of $V\!H_n(z)$ translates into an algorithm for generating all possible $n \times n$ tatami coverings with v vertical (or horizontal) dominoes, given that we have one for generating all k-sum subsets of the n set, for $k, n \geq 0$. We employ an algorithm from [7] to generate our coverings in constant amortized time.

1.1. **Overview.** The structure of square tatami coverings with the maximum number of monominoes is summarized in Section 2 — see reference [1] for a complete proof. In Section 3 we describe a representation for the coverings in $\mathbf{T}_n$ using strings over a ternary alphabet.

Each symbol represents a monomino which can either be flipped in exactly one of two diagonals, or unflipped. This representation is essentially the same as the one in [2], but our revision of the indices greatly simplifies the counting of vertical and horizontal dominoes.

Our main result is Theorem 1, the generating polynomial $V\!H_n(z)$, but the technical part is proving the formula for its coefficients, which is presented in Lemma 1. We discuss the coefficients of $V\!H_n(z)$ and derive the self-reciprocal generating polynomial required to partition the coverings in $\mathbf{T}_n$ and their four distinct rotations according to their numbers of vertical dominos.

In Section 5 we apply the proof of Lemma 1 to generate $\mathbf{T}_n$ in constant amortized time by adapting an algorithm given in [7].

In Section 6 we depart from the geometrical interpretation of $V\!H_n(z)$ and prove Theorem 3, the factorisation in Equation (2), and the remainder of the section focuses on properties of the factor $P_n(z)$, including Theorem 4 and Theorem 5, introduced above.

2. Structure of a tatami covering

An *edge* refers to the edge of a tile, while a *boundary* refers to the outer boundary of a covering.

The structure of coverings in $\mathbf{T}_n$ is characterized in Lemma 3 and Corollary 2 of [1]. Corollary 2 states that an $n \times n$ covering with n monominoes has monominoes in exactly two of its corners, which must share a boundary. This ensures that each covering of $\mathbf{T}_n$ has distinct rotations through $0, \pi/2, \pi$, and $3\pi/2$ radians, and that rotations of distinct coverings in $\mathbf{T}_n$ are distinct from each other.

If $T \in \mathbf{T}_n$, then a *diagonal*, D, of T is a contiguous sequence of like-aligned dominoes whose centers lie on a line with slope 1 or -1. The sequence must begin with a domino with its long edge on the boundary; the final domino will share an edge with a monomino which is also considered to be part of the diagonal. In Fig. 1(a) there are 12 diagonals, only one of which contains horizontal dominoes.

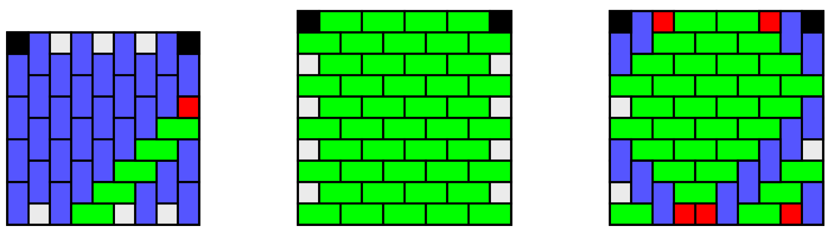
A *diagonal flip* of D consists of removing it from T, reflecting horizontally, rotating by $\frac{\pi}{2}$ radians, and placing it back onto the grid squares that were vacated.

There are three things to note about the diagonal flip:

- a flipped diagonal is a diagonal;
- the operation preserves the tatami restriction; and,
- it changes the orientation of the dominoes that it contains, and maps the monomino to the other extreme of the diagonal.

The *running bond*, or simply *bond*, is a rotation of the basic brick laying pattern, in which all dominoes have the same orientation. The restriction that coverings in $\mathbf{T}_n$ have monominoes in their upper corners, implies that exactly one bond pattern is possible for each n. When n is even, the bond consists of horizontal dominoes, with monominoes along the left and right boundaries, and when n is odd, the bond consists of vertical dominoes, with monominoes along the top and bottom boundaries (see Fig. 1(b)).

Lemma 3 in [1] shows that every covering in $\mathbf{T}_n$ can be produced from the running bond, via a finite sequence of diagonal flips in which each monomino is moved at most once, and the top corner monominoes are not moved at all. In addition, any such sequence of flips results in an element of $\mathbf{T}_n$ (see Fig. 1(c)).



(a) (b) (c)

Figure 1. Examples of coverings in $\mathbf{T}_n$. *(a)* A diagonal flip in a 9×9 covering. *(b)* The horizontal running bond for $n = 10$. *(c)* A sequence of diagonal flips results in a covering in $\mathbf{T}_{10}$. Flipped monominoes are coloured red.

From this perspective, we can look upon the original position of a monomino as its place in the running bond, and then describe it as *flipped* in a given direction if it has moved from its original position in T; otherwise it is *unflipped*.

3. Representing coverings as a ternary string

We describe a ternary string representation for $n \times n$ coverings with n monominoes. Recall that each monomino, besides the two corner monominoes, is in exactly two diagonals in the running bond, and in a given covering a monomino is flipped in one of these diagonals, or it is unflipped. A ternary symbol for each monomino indicates which of the three possible states it assumes. Each covering is described by a unique string of these ternary symbols, represented in the same order as the following indexed labelling (see caption at Fig. 2(c)).

Monominoes and their diagonals are labelled as shown in Fig. 2, such that the index, i, of a monomino is equal to the length of one of its diagonals, and $n-i-1$ is the length of the other. This relationship between diagonal length and index is helpful in Lemma 1.

The ternary string representing the 10×10 covering in Fig. 2(c) is $s = (0, 1, -1, 0, 0, 1, -1)$, where $s_i = 1$ if the i^{th} monomino is flipped upward, $s_i = -1$ if it is flipped downward, and $s_i = 0$ if it is unflipped.

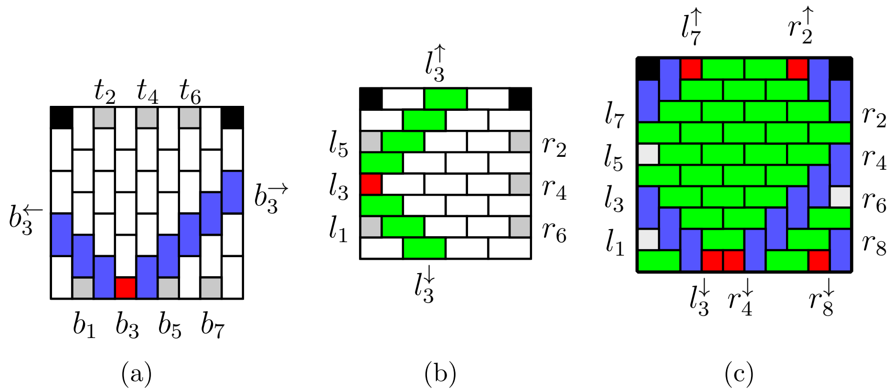


FIGURE 2. Labelling for $\mathbf{T}_n$. *(a)* For odd n, monominoes are labelled t_i and b_i. The distances from t_i and b_i to the left boundary are both i. *(b)* For even n, monominoes are labelled l_i and r_i. The distances from l_i to the bottom boundary, and from r_i to the top boundary, are both i. *(c)* The covering, $(0, 1, -1, 0, 0, 1, -1)$.

We use $l_i^{\uparrow}, l_i^{\downarrow}, r_i^{\uparrow}, r_i^{\downarrow}, t_i^{\rightarrow}, t_i^{\leftarrow}, b_i^{\rightarrow}, b_i^{\leftarrow}$ to denote the diagonals that the monominoes l_i, r_i, t_i, b_i can be flipped on. Naturally, l_i and r_i can only be (diagonally) flipped up or down, whilst t_i and b_i can only be flipped left or right.

Let $d_n(a)$ be the number of dominoes in the diagonal a, also called the length or size of the diagonal. It is a function of the index and direction of a:

$$d_n(a) = \begin{cases} i, & \text{if } a \in \{l_i^{\downarrow}, r_i^{\uparrow}, t_i^{\leftarrow}, b_i^{\leftarrow}\}; \\ n-i-1, & \text{if } a \in \{l_i^{\uparrow}, r_i^{\downarrow}, t_i^{\rightarrow}, b_i^{\rightarrow}\}. \end{cases}$$

Flipped diagonals which intersect are called *conflicting*, and can occur as one of two types (see Fig. 3).

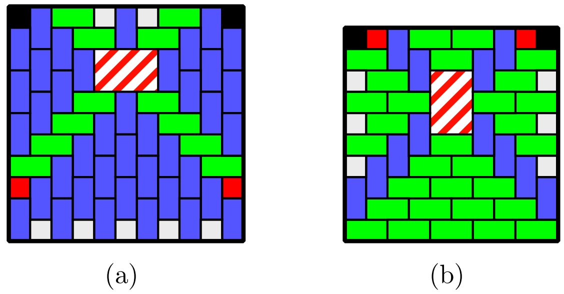


FIGURE 3. Example of, *(a)*, Type 1 conflict, and, *(b)*, Type 2 conflict.

Pair	Type 2 $\iff$
$l_i^{\downarrow}, r_j^{\downarrow}$	$j \leq i - 1$
$l_i^{\uparrow}, r_j^{\uparrow}$	$i \leq j - 1$
$t_i^{\leftarrow}, b_j^{\leftarrow}$	$n \leq j + i$
$t_i^{\rightarrow}, b_j^{\rightarrow}$	$i + j \leq n - 2$

TABLE 1. Conditions for Type 2 conflicts.

Type 1: A pair of diagonals with monominoes originating on the same boundary are flipped toward one another (e.g. $(t_i^{\rightarrow}, t_j^{\leftarrow})$ for some $i < j$).

Type 2: A pair of diagonals with monominoes originating on opposite boundaries are flipped in the same direction (e.g. $(l_i^{\uparrow}, r_j^{\uparrow})$) and their combined length is at least n (see Table 1).

Lastly, if a is a diagonal containing a given monomino, let $\bar{a}$ be the monomino's other diagonal.

3.1. **A partition of $\mathbf{T}_n$.** Let $\mathbf{T}_n(a) \subseteq \mathbf{T}_n$, where a is a diagonal such that $d_n(a) \geq d_n(\bar{a})$, be defined as the collection of coverings in $\mathbf{T}_n$ in which a is the *longest flipped diagonal*; for each flipped diagonal b, distinct from a, we have $d_n(b) < d_n(a)$.

Let $\mathbf{T}_n(\varnothing)$ be the set of coverings in which no monomino is flipped on its longest diagonal. Note the distinction between a monomino flipped on its longest diagonal, and the longest flipped diagonal in the whole covering.

The sets $\mathbf{T}_n(\varnothing)$ and $\mathbf{T}_n(a)$, for each diagonal a defined above, are a partition of $\mathbf{T}_n$, and the allowable diagonal flips of each subset can be applied independently of the other flips, by Theorem 2 in [2].

4. Enumeration

Let $S(s,k)$ be the number of subsets of $\{1,2,\ldots,s\}$ whose sum is k. The number of coverings with k vertical (or horizontal) dominoes is expressible in terms of this function by making independent flips of diagonals whose lengths are some subset $\{1,2,\ldots,s\}$. We identify these sets of diagonals in the proof of Lemma 1.

Lemma 1. *Let $V(n,k)$ and $H(n,k)$ be the number of coverings in $\mathbf{T}_n$ with exactly k vertical and horizontal dominoes, respectively. If n is even, then $V(n,k)$ is equal to*

$$\text{(4a)} \qquad V\!\!H(n,k) := 2\sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(\sum_{\substack{k_1+k_2= \\ k-(n-i-1)}} S(n-i-2,k_1)S(i-1,k_2) \right)$$

$$\text{(4b)} \qquad + \sum_{k_1+k_2=k} S\left(\left\lfloor \frac{n-2}{2} \right\rfloor, k_1 \right) S\left(\left\lfloor \frac{n-2}{2} \right\rfloor, k_2 \right).$$

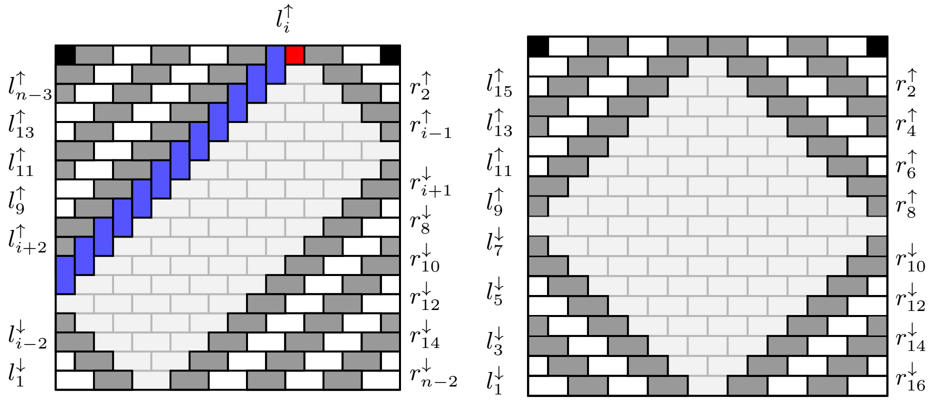
When n is odd, $V\!\!H(n,k)$ is equal to $H(n,k)$.

Proof. Each outer sum term of (4a) adds the coverings for $\mathbf{T}_n(a)$, for some diagonal a, and the term (4b) counts those in $\mathbf{T}_n(\varnothing)$.

Case n even: The trivial covering in $\mathbf{T}_n$ consists only of horizontal dominoes, and flipping the diagonal a contributes $d_n(a)$ vertical dominoes. Diagonals $l_i^{\uparrow}$ and $r_i^{\uparrow}$ have even length, for all i, while $l_i^{\downarrow}$ and $r_i^{\downarrow}$ have odd length. We use this fact to find sets of diagonals which have lengths $1,2,\ldots,s$, for some $s \in \mathbb{N}$, by combining allowable diagonals in opposite corners, for each $\mathbf{T}_n(a)$. Table 2 shows the lengths of the longest allowable diagonals in each corner for each $\mathbf{T}_n(a)$, and from this we can find the required sets of diagonals. For example, the allowable diagonals in $\mathbf{T}_n(l_i^{\uparrow})$ are shown in Fig. 4(a) (for $(n,i)=(18,5)$) and their respective lengths are

$$\begin{array}{ll} l_1^{\downarrow}, l_3^{\downarrow}, \ldots, l_{i-2}^{\downarrow} & 1,3,\ldots,i-2, \\ l_{i+2}^{\uparrow}, l_{i+4}^{\uparrow}, \ldots, l_{n-3}^{\uparrow} & n-i-3, n-i-5, \ldots, 2, \\ r_{i+1}^{\downarrow}, r_{i+3}^{\downarrow}, \ldots, r_{n-2}^{\downarrow} & n-i-2, n-i-4, \ldots, 1, \\ r_2^{\uparrow}, r_4^{\uparrow}, \ldots, r_{i-1}^{\uparrow} & 2,4,\ldots,i-1. \end{array}$$

We have $d_n(l_i^{\uparrow}) = n-i-1$, so we are interested in the number of combinations of the above independently flippable diagonals with



(a) (b)

FIGURE 4. Allowable diagonals shown in alternating grey and white, *(a)*, for $\mathbf{T}_n(l_i)$, where $(n, i) = (18, 5)$, and *(b)*, for $\mathbf{T}_{18}(\varnothing)$.

exactly $k - (n - i - 1)$ vertical dominoes. That number is

$$\sum_{\substack{k_1+k_2= \\ k-(n-i-1)}} S(n-i-2, k_1)S(i-1, k_2).$$

The indices of the diagonals $l_i^{\uparrow}$ for which $d_n(l_i^{\uparrow}) \geq d_n(l_i^{\downarrow})$ and $r_i^{\downarrow}$ for which $d_n(r_i^{\downarrow}) \geq d_n(r_i^{\uparrow})$, range from 1 to $\left\lfloor \frac{n-1}{2} \right\rfloor$, as required for (4a).

Now suppose $a = \varnothing$. If i is the largest index such that $d_n(l_i^{\downarrow}) < d_n(l_i^{\uparrow})$ and j is the largest index such that $d_n(r_j^{\uparrow}) < d_n(r_j^{\downarrow})$, then $\max(i, j) = \left\lfloor \frac{n-2}{2} \right\rfloor$ and $|i - j| = 1$. The allowable diagonals in $\mathbf{T}_n(\varnothing)$ and their respective sizes are shown in the table below (see Fig. 4(b)).

$l_1^{\downarrow}, l_3^{\downarrow}, \ldots, l_i^{\downarrow}$	$1, 3, \ldots, i$
$l_{i+2}^{\uparrow}, l_{i+4}^{\uparrow}, \ldots, l_{n-3}^{\uparrow}$	$n-i-3, n-i-5, \ldots, 2$
$r_2^{\uparrow}, r_4^{\uparrow}, \ldots, r_j^{\uparrow}$	$2, 4, \ldots, j,$
$r_{j+2}^{\downarrow}, r_{j+4}^{\downarrow}, \ldots, r_{n-2}^{\downarrow}$	$n-j-3, n-j-5, \ldots, 1.$

Choosing subsets of the independently flippable diagonals with k vertical dominoes contributes the term

$$\sum_{k_1+k_2=k} S\left(\left\lfloor \frac{n-2}{2} \right\rfloor, k_1\right) S\left(n - \left(\left\lfloor \frac{n-2}{2} \right\rfloor - 1\right) - 3, k_2\right),$$

and since $n - (\lfloor (n-2)/2 \rfloor - 1) - 3 = \lfloor (n-2)/2 \rfloor$, this is equal to (4b) for even n.

$\mathbf{T}_n(a)$	Index and size of largest diagonal in this corner			
n even	$l_j^{\downarrow}$ (j odd)	$l_j^{\uparrow}$ (j odd)	$r_j^{\downarrow}$ (j even)	$r_j^{\uparrow}$ (j even)
$\mathbf{T}_n(l_i^{\uparrow})$	$j < i$	$j > i$	$j > i$	c. 2(a)
index j:	$i-2$	$i+2$	$i+1$	$i-1$
size:	$i-2$	$n-i-3$	$n-i-2$	$i-1$
$\mathbf{T}_n(r_i^{\downarrow})$	c. 2(a)	$i < j$	$j > i$	$j < i$
index j:	$i-1$	$i+1$	$i+2$	$i-2$
size:	$i-1$	$n-i-2$	$n-i-3$	$i-2$
$\mathbf{T}_n(l_i^{\downarrow})$		Symmetric with $\mathbf{T}_n(r_i^{\downarrow})$.		
$\mathbf{T}_n(r_i^{\uparrow})$		Symmetric with $\mathbf{T}_n(l_i^{\uparrow})$.		
n odd	$t_j^{\leftarrow}$ (j even)	$t_j^{\rightarrow}$ (j even)	$b_j^{\leftarrow}$ (j odd)	$b_j^{\rightarrow}$ (j odd)
$\mathbf{T}_n(t_i^{\rightarrow})$	$j < i$	$j > i$	$j < n-i-1$	c. 2(a)
index j:	$i-2$	$i+2$	$n-i-2$	$n-i$
size:	$i-2$	$n-i-3$	$n-i-2$	$i-1$
$\mathbf{T}_n(b_i^{\rightarrow})$	$j < n-i-1$	c. 2(a)	$j < i$	$j > i$
index j:	$n-i-2$	$n-i$	$i-2$	$i+2$
size:	$n-i-2$	$i-1$	$i-2$	$n-i-3$
$\mathbf{T}_n(t_i^{\leftarrow})$		Symmetric with $\mathbf{T}_n(t_i^{\rightarrow})$.		
$\mathbf{T}_n(b_i^{\leftarrow})$		Symmetric with $\mathbf{T}_n(b_i^{\rightarrow})$.		

TABLE 2. The longest allowable diagonals in each of four corners for each $\mathbf{T}_n(a)$. Entries are calculated using the parity of i and j, the avoidance of conflicts, and the requirement that a be the longest diagonal in $\mathbf{T}_n(a)$. Note that "c. 2(a)", above, refers to conflict 2(a) which occurs between diagonals a and b if $d_n(a) + d_n(b) \geq n$.

Case n odd: The trivial covering is a vertical bond with $\lfloor (n-2)/2 \rfloor$ monominoes at the top (besides the two that are fixed) and $\lceil (n-2)/2 \rceil$ non-fixed monominoes along the bottom boundary. When diagonal a is flipped, $d_n(a)$ horizontal dominoes are added to the covering, instead of vertical dominoes. Hence we argue for $H(n,k)$ rather than $V(n,k)$.

Now $t_j^{\leftarrow}$ and $t_j^{\rightarrow}$ have even length, and $b_j^{\leftarrow}$ and $b_j^{\rightarrow}$ have odd length (see Table 2). For example, the allowable diagonals in $\mathbf{T}_n(t_i^{\rightarrow})$ are shown in Fig. 5(a) (for $(n,i) = (17,6)$), and their respective lengths are

$$\begin{array}{ll}
t_1^{\leftarrow}, t_3^{\leftarrow}, \ldots, t_{i-2}^{\leftarrow} & 1, 3, \ldots, i-2, \\
t_{i+2}^{\rightarrow}, t_{i+4}^{\rightarrow}, \ldots, t_{n-3}^{\rightarrow} & n-i-3, n-i-5, \ldots, 2, \\
b_1^{\leftarrow}, b_3^{\leftarrow}, \ldots, b_{n-i-2}^{\leftarrow} & 1, 3, \ldots, n-i-2, \\
b_{n-i}^{\rightarrow}, b_{n-i+2}^{\rightarrow}, \ldots, b_{n-2}^{\rightarrow} & i-1, i-3, \ldots, 1.
\end{array}$$

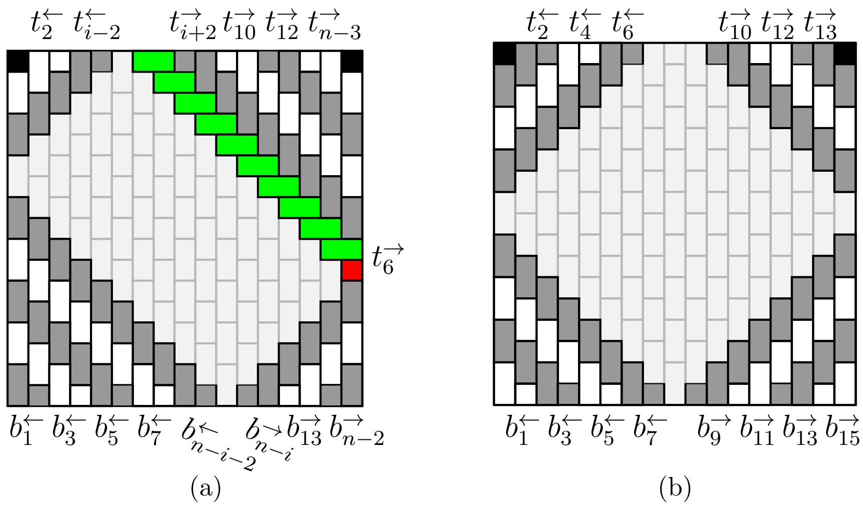


FIGURE 5. Allowable diagonals shown in alternating grey and white, *(a)*, for $\mathbf{T}_n(t_i)$, where $(n, i) = (17, 6)$, and *(b)*, for $\mathbf{T}_{17}(\varnothing)$.

Once again $d_n(t_i^{\rightarrow}) = n - i - 1$, so we are interested in the number of combinations of the above independently flippable diagonals with exactly $k - (n - i - 1)$ horizontal dominoes. As before, that number is

$$\sum_{\substack{k_1+k_2= \\ k-(n-i-1)}} S(n-i-2, k_1) S(i-1, k_2).$$

Now suppose $a = \varnothing$, then if i is the largest index such that $d_n(t_i^{\rightarrow}) < d_n(t_i^{\leftarrow})$ and j is the largest index such that $d_n(b_j^{\rightarrow}) < d_n(b_j^{\leftarrow})$ then $\max(i, j) = \left\lfloor \frac{n-2}{2} \right\rfloor$ and $|i - j| = 1$. The allowable leftward diagonals in $\mathbf{T}_n(\varnothing)$ and their respective sizes are given in the table below.

$$\begin{array}{ll} t_2^{\leftarrow}, t_4^{\leftarrow}, \ldots, t_j^{\leftarrow} & 2, 4, \ldots, j, \\ b_1^{\leftarrow}, b_3^{\leftarrow}, \ldots, b_i^{\leftarrow} & 1, 3, \ldots, i \end{array}$$

and by horizontal symmetry, the rightward diagonals have the same lengths. We conclude that the coverings with k horizontal dominoes of $\mathbf{T}_n(\varnothing)$ is also generated by (4b) when n is odd. $\square$

The terms $V\!H(n, k) z^k$ can be summed over k to obtain the generating polynomial $T(n, z)$ (same as $V\!H_n(z)$), mentioned in Conjecture 4 of [1].

Theorem 1. *The generating polynomial for (4a-4b) is*

$$VH_n(z) := 2 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{n-i-2}(z) S_{i-1}(z) z^{n-i-1} + \left(S_{\lfloor \frac{n-2}{2} \rfloor}(z) \right)^2, \tag{5}$$

where $S_n(z) = \sum_{k \in \mathbb{Z}} S(n,k) z^k$. This "generates" $V(n,k)$ for even n, and $H(n,k)$ for odd n.

Proof. This follows from Lemma 1. □

The degree of $VH_n(z)$ is $\frac{n^2-n}{2} - (n-1)$, because this is the largest number of vertical dominoes possible in a covering of $\mathbf{T}_n$, for even n (and horizontal dominoes for odd n). For example, the covering with all l_i flipped up and all r_i flipped down contains exactly $n-1$ horizontal dominoes.

The coefficients of $VH_n(z)$ are listed in Table 3 up to $n = 10$, and the following conjecture is true at least up to $n = 20$. If $Q(z)$ is a polynomial, then write $\langle z^k \rangle Q(z)$ to denote the coefficient of z^k.

Conjecture 1.

(a) For $k \leq n-2$, we have $\langle z^k \rangle VH_n(z) = \langle z^k \rangle \prod_{m \geq 0} (1+z^m)^2$, the number of partitions of k into distinct parts with two types of each part (see A022567 in [6]*).*

(b) For $0 \leq k < n-3$, we have

$$\langle z^{\deg(VH_n(z))-k} \rangle VH_n(z) = 2 \langle z^k \rangle \prod_{m \geq 0} (1+z^m),$$

twice the number of partitions of k into distinct parts (see A000009 in [6]*).*

Rotating a covering of $\mathbf{T}_n$ by $\pi/2$ radians interchanges vertical and horizontal dominoes, and this transformation can be applied to the generating polynomial $VH_n(z)$ to obtain the polynomial $VH_n(z^{-1}) z^{\frac{n^2-n}{2}}$. Thus we can easily derive the bivariate generating polynomial $R_n(x,y)$, whose coefficient of $x^v y^h$ is the number of tatami coverings with exactly v vertical dominoes and h horizontal dominoes.

Our remarks prove the following corollary.

Corollary 1. *Let $R_n(x,y)$ be as defined above. We have*

$$R_n(x,y) = 2VH_n(xy^{-1}) y^{\frac{n^2-n}{2}} + 2VH_n(x^{-1}y) x^{\frac{n^2-n}{2}}. \tag{6}$$

We list some basic properties of $R_n(x,y)$.

- The degree of $R_n(x,1)$ as well as the degree of every term in $R_n(x,y)$ is $\frac{n^2-2}{2}$;

$n \backslash z^k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1															
3	1	2														
4	1	2	3	2												
5	1	2	3	6	4	2	2									
6	1	2	3	6	9	8	7	6	2	2	2					
7	1	2	3	6	9	14	15	14	14	10	8	6	4	2	2	2
8	1	2	3	6	9	14	22	24	25	28	25	22	19	14	10	10
9	1	2	3	6	9	14	22	32	37	42	49	48	49	46	38	34
10	1	2	3	6	9	14	22	32	46	56	66	78	84	90	92	88
$n \backslash z^k$	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
8	8	4	4	2	2	2										
9	30	24	20	16	12	12	10	6	4	4	2	2	2			
10	81	76	69	58	51	44	38	34	28	22	20	16	14	12	8	6
$n \backslash z^k$	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
10	4	4	2	2	2											

TABLE 3. Table of coefficients of $VH_n(z)$ for $2 \leq n \leq 10$. The (n, k)th entry represents the number of coverings of $\mathbf{T}_n$ with k vertical dominoes when n is even, and k horizontal dominoes when n is odd.

- the polynomial $R_n(x, y)$ can be recovered from $R_n(x, 1)$, and the latter is the generating polynomial for the set of all $n \times n$ coverings with n monominoes with exactly v vertical dominoes (or h horizontal dominoes);
- the polynomial $R_n(x, 1)$ is self reciprocal because of interchangeability of vertical and horizontal dominoes; and finally,
- the polynomial $R_n(x, 1)$ has similar properties to those listed for $VH_n(z)$ in Conjecture 1, in the sense that for some increasing integer function f, we have $\langle x^k \rangle R_n(x, 1) = \langle x^k \rangle R_{n+1}(x, 1)$, whenever $k < f(n + 1)$.

If there is an even number of dominoes, which is the case when $(n^2 - n)/4$ is an integer, then $\langle x^k y^k \rangle R_n(x, y) = 4\langle z^k \rangle VH_n(z)$, where $k = (n^2 - n)/4$. Rotating the covering maps k vertical dominoes to k horizontal dominoes, and vice versa. The coverings counted by these coefficients are called *balanced tatami coverings*, appropriately named by Knuth (private communication), because the number of vertical and horizontal dominoes are equal. Here is $\langle z^k \rangle VH_n(z)$ for $2 \leq n \leq 56$: 0, 0, 2, 2, 0, 0, 10, 20, 0, 0, 114, 210, 0, 0, 1322, 2460, 0, 0, 16428, 31122, 0, 0, 214660, 410378, 0, 0, 2897424, 5575682, 0, 0, 40046134, 77445152, 0, 0, 563527294, 1093987598, 0, 0, 8042361426, 15660579168, 0, 0, 116083167058, 226608224226, 0, 0, 1691193906828, 3308255447206, 0, 0, 24830916046462, 48658330768786, 0, 0, 366990100477712, (see A182107 in [6]). Note that this is perhaps better viewed as four sequences, one for each $0 \leq j < 4$ such that $n \pmod 4 = j$.

5. Exhaustive generation of tatami coverings in constant amortized time

In this section, we present procedure `genVH`(n, k), which generates the coverings counted by $V\!H(n, k)$, while doing a constant amount of data structure change per covering that is produced (example output is shown in Fig. 6). Let $\mathbf{S}(n, k)$ denote the set of subsets of $\{1, 2, \dots, n\}$ whose elements sum to k; thus $|\mathbf{S}(n, k)| = S(n, k)$. The procedure follows naturally from the sums in Equation (4a-4b), since each term $S(a, i)S(b, j)$ counts some set of '$\diagup$' and '$\diagdown$'-oriented diagonals. The sets $\mathbf{S}(a, i) \times \mathbf{S}(b, j)$ are generated in constant amortized time (CAT) by a modification of `C4` from [7] (see Listing 1). Our modified algorithm, `modC`, is invoked for each sum term of Equation (4a-4b). Procedure `modC` is CAT for the same reasons that `C4` is CAT.

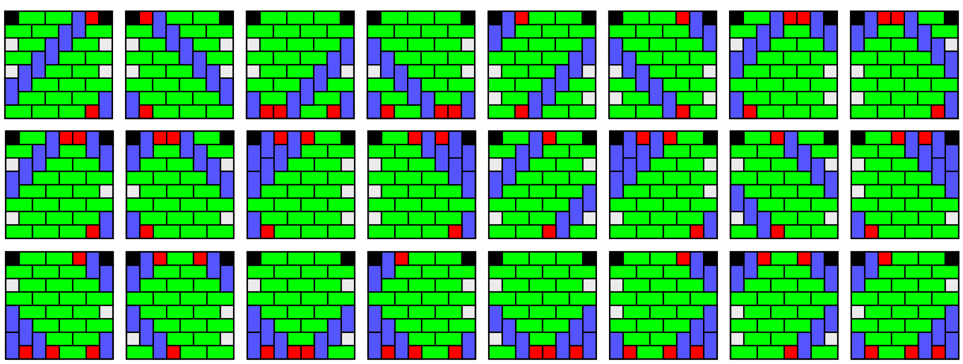


Figure 6. The coverings of $\mathbf{T}_8$ with exactly 7 vertical dominoes. This is the output of `genVH`$(8, 7)$ printed in the order the coverings are generated (as one would naturally read text).

There is one subtlety involved in exploiting the CATness of `C4`. Invoking `C4`(a, i) requires $\Omega(a)$ preprocessing steps if its input list is recreated for each call, but `C4`(a, i) may not produce so many combinations for small a and large i. The result is that we may make many calls to `modC` that require too much preprocessing, but this is dealt with, as follows: a top level call to `C4`(i, j) in [7] takes the list $[i{+}1, 2, 3, \dots, i{+}1]$, which requires $i{+}1$ steps to create, however, `C4`(i, j) also concludes with the same list (see Listing 1). Let A and B be the largest integers for which `modC` is called to compute $\mathbf{S}(a, i) \times \mathbf{S}(b, i)$. We set `aiSet`$= [1, 2, 3, \dots, A]$ and `biSet`$= [1, 2, 3, \dots, B]$, and by setting `aiSet`$[0] = a + 1$ and `biSet`$[0] = b + 1$, we initialize for each call to `modC` with exactly two operations.

LISTING 1. Python code for a modified version of C4 from [7] to compute $\mathbf{S}(a,i) \times \mathbf{S}(b,j)$. Global variables aiSet and bjSet are the lists representing $\mathbf{S}(a,i)$ and $\mathbf{S}(b,i)$, respectively

```
def modC(a,i,b,j,comp,isFirst):
    global aiSet,bjSet
    if( a == 0):
        if(isFirst):
            modC(b,j,0,0,False,False)
        else:
            Output(aiSet,bjSet)
    else:
        if(isFirst):
            L = aiSet
        else:
            L = bjSet
        if( i > a*(a+1)/2 ):
            i = a*(a+1)/2 - i; comp = not comp
        if( i<a ):
            if(comp):
                L[a] = L[0]; L[0] = i+1
                modC( i, i, b, j, comp, isFirst)
                L[0] = L[a]; L[a] = a+1
            else:
                modC( i, i, b, j, comp, isFirst)
        else:
            L[a] = L[0]; L[0] = a
            if(comp):
                modC( a-1, i, b, j, comp, isFirst)
                L[0] = L[a]; L[a] = a+1
                modC( a-1, i-a, b, j, comp, isFirst)
            else:
                modC( a-1, i-a, b, j, comp, isFirst)
                L[0] = L[a]; L[a] = a+1
                modC( a-1, i, b, j, comp, isFirst)
```

Theorem 2. *The coverings in $\mathbf{T}_n$ with exactly k vertical dominoes if n is even and horizontal dominoes if n is odd, can be exhaustively generated in constant amortized time.*

Proof. The outer procedure does a constant amount of work per call to `modC`. This subroutine is CAT, so the outer procedure is also CAT. □

6. A mysterious factor of $V\!H_n(z)$

In this section we prove that the generating polynomial $V\!H_n(z)$ has (very nearly) the factorisation conjectured in [1]. We use the following lemma.

Lemma 2. *For all $x \geq 0$,*

$$\lfloor x \rfloor = \sum_{k \geq 1} \left\lfloor \frac{x}{2^k} + \frac{1}{2} \right\rfloor . \tag{7}$$

Proof. Let $n = \lfloor x \rfloor$ and apply strong induction on n. Clearly Equation (7) holds for the base case, when $n = 0$. Suppose it holds for $0, 1, \ldots, n-1$, then

$$\begin{aligned} \sum_{k \geq 1} \left\lfloor \frac{x}{2^k} + \frac{1}{2} \right\rfloor &= \left\lfloor \frac{x}{2} + \frac{1}{2} \right\rfloor + \sum_{k \geq 1} \left\lfloor \frac{\frac{x}{2}}{2^k} + \frac{1}{2} \right\rfloor \\ &= \left\lfloor \frac{x}{2} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{x}{2} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{2} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{\lfloor x \rfloor}{2} \right\rfloor = \lfloor x \rfloor . \end{aligned}$$

Two applications of Equation (3.11) in [3] yield the penultimate equation and the final equation follows by considering the parity of $\lfloor x \rfloor$, or by using (3.26) in [3] with $m = 2$ and $x' = x/2$. □

Theorem 3. *The generating polynomial $V\!H_n(z)$ has the factorisation*

$$V\!H_n(z) = P_n(z) D_n(z)$$

where $P_n(z)$ is a polynomial and

$$D_n(z) = \prod_{j \geq 1} S_{\lfloor \frac{n-2}{2^j} \rfloor}(z). \tag{8}$$

Proof. We prove that $D_n(z)$ divides $V\!H_n(z)$ by using the factorisation of $S_n(z)$ into cyclotomic polynomials ([1], Lemma 5),

$$S_n(z) = \prod_{j \geq 1} \Phi_{2j}(z)^{\lfloor \frac{n+j}{2j} \rfloor}. \tag{9}$$

and showing that the power of $\Phi_i(z)$ is greater in each term of $V\!H_n(z)$ than it is in $D_n(z)$.

The power of $\Phi_{2j}(z)$ in $D_n(z)$ is obtained by substituting Equation (9) into Equation (8),

$$D_n(z) = \prod_{i\geq 1} S_{\lfloor \frac{n-2}{2^i} \rfloor}(z) = \prod_{i\geq 1, j\geq 1} \Phi_{2j}(z)^{\left\lfloor \frac{\lfloor \frac{n-2}{2^i} \rfloor + j}{2j} \right\rfloor}$$

$$= \prod_{j\geq 1} \Phi_{2j}(z)^{\sum_{i\geq 1} \left\lfloor \frac{\lfloor \frac{n-2}{2^i} \rfloor + j}{2j} \right\rfloor}.$$

We simplify $D_n(z)$ to

$$D_n(z) = \prod_{j\geq 1} \Phi_{2j}(z)^{\lfloor \frac{n-2}{2j} \rfloor}, \tag{10}$$

by applying Lemma 2 and with Equation (3.11) in [3].

Expanding the second term of $V\!H_n(z)$ gives

$$\left(S_{\frac{n-2}{2}}(z)\right)^2 = \prod_{j\geq 1} \Phi_{2j}(z)^{2\left\lfloor \frac{\frac{n-2}{2} + j}{2j} \right\rfloor},$$

which is divisible by $D_n(z)$, since

$$\left\lfloor \frac{n-2}{2j} \right\rfloor \leq 2 \left\lfloor \frac{\frac{n-2}{2} + j}{2j} \right\rfloor,$$

for all $j \geq 1$ and positive even integers n.

The other terms in $V_n(z)$ are of the form

$$S_{n-k-2}(z)S_{k-1}(z)z^d = \left(\prod_{j>0} \Phi_{2j}(z)^{\lfloor \frac{(n-k-2)+j}{2j} \rfloor}\right)\left(\prod_{j>0} \Phi_{2j}(z)^{\lfloor \frac{(k-1)+j}{2j} \rfloor}\right) z^d$$

for each $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ where d is the appropriate power of z. These terms are all divisible by $D_n(z)$ if the exponents in Equation (10) satisfy

$$\left\lfloor \frac{n-2}{2j} \right\rfloor \leq \left\lfloor \frac{k-1}{2j} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{n-k-2}{2j} + \frac{1}{2} \right\rfloor. \tag{11}$$

Let r_1 and r_2 be integers such that $0 \leq r_i < 2j$ and $\frac{k-1}{2j} = \left\lfloor \frac{k-1}{2j} \right\rfloor + \frac{r_1}{2j}$ and $\frac{n-2}{2j} = \left\lfloor \frac{n-2}{2j} \right\rfloor + \frac{r_2}{2j}$. We eliminate occurrences of $\left\lfloor \frac{k-1}{2j} \right\rfloor$ and $\left\lfloor \frac{n-2}{2j} \right\rfloor$ from Inequality (11), since they are integers and can be removed from floors, and rewrite the inequality as

$$0 \leq \left\lfloor \frac{r_1}{2j} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{r_2 - r_1 - 1}{2j} + \frac{1}{2} \right\rfloor. \tag{12}$$

It is straightforward to show that if the second term is -1, then the first term is equal to 1.

Therefore, $D_n(z)$ divides each and every term of $V\!H_n(z)$. □

$n\backslash z^k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1	2														
4	1	1	2													
5	1	1	2	4	0	2										
6	1	0	1	2	2	-2	2									
7	1	0	1	2	2	4	-2	4	0	2	-2	2				
8	1	0	1	1	2	3	4	-2	2	0	4	-2	2	-2	2	
9	1	0	1	1	2	3	4	6	-2	6	0	8	-2	4	-4	6
10	1	-1	1	0	1	1	1	2	2	-6	6	-2	6	-6	4	-4
11	1	-1	1	0	1	1	1	2	2	4	-8	10	-4	10	-8	8
$n\backslash z^k$	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
9	-2	4	-2	2	-2	2										
10	6	-6	6	-4	4	-4	2									
11	-8	10	-10	12	-8	10	-12	10	-6	6	-6	6	-4	4	-4	2

TABLE 4. Table of coefficients of $P_n(z)$ for $3 \le n \le 11$.

Our computer investigations show that $P_n(z)$ is irreducible for $1 < n < 200$, and we know the complete factorisation of $S_k(z)$, for each positive integer k. We suspect, therefore, that the complete factorisation is

$$V\!H_n(z) = P_n(z) \prod_{j \ge 1} \Phi_{2j}(z)^{\left\lfloor \frac{n-2}{2j} \right\rfloor}. \tag{13}$$

The factor $P_n(z)$ is somewhat more mysterious than $D_n(z)$; e.g., we have no formula to express it besides $V\!H_n(z)/D_n(z)$. Take $P_{11}(z)$ for example, which is equal to $1 - 1z^1 + 1z^2 + 0z^3 + 1z^4 + 1z^5 + 1z^6 + 2z^7 + 2z^8 + 4z^9 - 8z^{10} + 10z^{11} - 4z^{12} + 10z^{13} - 8z^{14} + 8z^{15} - 8z^{16} + 10z^{17} - 10z^{18} + 12z^{19} - 8z^{20} + 10z^{21} - 12z^{22} + 10z^{23} - 6z^{24} + 6z^{25} - 6z^{26} + 6z^{27} - 4z^{28} + 4z^{29} - 4z^{30} + 2z^{31}$. The coefficients are almost all non-zero, the central coefficients are larger than the ones at the tails, a great many of them are even, they alternate in sign for a long stretch, and the polynomial is irreducible.

The degree of $P_{11}(z)$ is $\deg(P_{11}(z)) = \deg(V\!H_{11}(z)) - \deg(D_{11}(z))$, both of which are easily calculated. In general $\deg(P_n(z))$ is equal to the sum of the sequence of largest odd divisors of the numbers $1, 2, \ldots, n-2$, which is a sequence with some nice properties (see A135013 in [6]).

Theorem 4. *For each $n \ge 2$,*

$$\deg(P_n(z)) = \sum_{k=1}^{n-2} Od(k),$$

where $Od(k)$ is the largest odd divisor of k.

Proof. Theorem 3 gives the degree of $D_n(z)$ so we can write

$$\deg(P_n(z)) = \binom{n-1}{2} - \sum_{k\geq 1} \binom{\left\lfloor \frac{n-2}{2^k} \right\rfloor + 1}{2}, \tag{14}$$

since $\deg(S_n(z)) = \binom{n+1}{2}$.

The proof that $\sum_{k=1}^{n} Od(k) = \deg(P_{n+2}(z))$ is by induction, and the base case, where $n = 0$, is easily verified. Let $p_n = \deg(P_n(z))$, for $n \geq 2$, to abbreviate the notation. It remains for us to show that $p_{n+3} - p_{n+2} = Od(n+1)$.

Let $n'01^\alpha$ be the binary representation of n, so that $n+1 = n'10^\alpha$, and let $[\![A]\!] = 1$ if the statement A is true, and $[\![A]\!] = 0$ otherwise. Observe that

$$\left\lfloor \frac{n+1}{2^k} \right\rfloor = [\![k \leq \alpha]\!] + \left\lfloor \frac{n}{2^k} \right\rfloor, \tag{15}$$

which we use to simplify

$$\sum_{k\geq 1} \left(\binom{\left\lfloor \frac{n+1}{2^k} \right\rfloor + 1}{2} - \binom{\left\lfloor \frac{n}{2^k} \right\rfloor + 1}{2} \right),$$

and write

$$p_{n+3} - p_{n+2} = (n+1) - \sum_{k=1}^{\alpha} \left(\left\lfloor \frac{n}{2^k} \right\rfloor + 1 \right).$$

Using Equation (15) and the fact that $(n+1)/2^k$ is an integer for $1 \leq k \leq \alpha$, we write

$$p_{n+3} - p_{n+2} = (n+1) - \sum_{k=1}^{\alpha} \left(\left\lfloor \frac{n+1}{2^k} + \frac{1}{2} \right\rfloor \right),$$

and then express this as the remaining sum terms in Equation (7)

$$\begin{aligned} p_{n+3} - p_{n+2} &= \sum_{k\geq \alpha+1} \left(\left\lfloor \frac{n+1}{2^k} + \frac{1}{2} \right\rfloor \right) \\ &= \sum_{k-\alpha\geq 1} \left(\left\lfloor \frac{\frac{n+1}{2^\alpha}}{2^{k-\alpha}} + \frac{1}{2} \right\rfloor \right). \end{aligned}$$

Applying Equation (7) again, we have $p_{n+3} - p_{n+2} = (n+1)/2^\alpha$, which is equal to $Od(n+1)$, as required. ☐

In addition to finding $\deg(P_{11}(z))$, we can evaluate at $z = 1$ with $P_{11}(1) = V\!H_{11}(1)/D_{11}(1) = 22$, a ratio which is also easy to calculate in general because $V\!H_n(1)$ and $D_n(1)$ have well understood combinatorial interpretations. It also leads to an interesting sequence, whose derivation for all n is given below.

Theorem 5. *The sum of the coefficients of $P_n(z)$ is equal to $n2^{\nu(n-2)-1}$, where $\nu(n)$ is the number of 1s in the binary representation of n*

Proof. The sum of the coefficients of $P_n(z)$ is equal to $P_n(1)$, which is expressible as $V\!H_n(1)/D_n(1)$. The numerator evaluates to $n2^{n-3}$, since this is the number of coverings in $\mathbf{T}_n$, and the denominator is evaluated as described below.

It is well known that $\Phi_k(1) = p$ if k is a non-zero power of a prime p and $\Phi_k(1) = 1$ if k is divisible by two distinct primes (see [4], p.74). We can evaluate $D_n(1)$ using Equation (10),

$$D_n(1) = \prod_{i\geq 1} \Phi_{2i}(1)^{\left\lfloor \frac{n-2}{2i} \right\rfloor} = 2^{\sum_{i\geq 1} \left\lfloor \frac{n-2}{2^i} \right\rfloor},$$

by ignoring the factors for which $2i$ is not a power of 2. Apply Equation (4.24) in [3] to obtain $D_n(1) = 2^{n-2-\nu(n-2)}$. Thus

$$P_n(1) = n2^{n-3-(n-2)+\nu(n-2)} = n2^{\nu(n-2)-1}.$$

□

We have verified that $P_n(z)$ is irreducible over the integers for $1 < n < 200$, but we do not understand its structure well enough to prove it for all n. We state below some of the observable structure which has also been verified for $1 < n < 200$, as Conjecture 2, and we plot some complex roots for odd n up to 67 in Fig. 7.

Conjecture 2.

(a) If $k \geq 1$ and $n \pmod{2^k} = 2$, then $\langle z^i\rangle P_n(z) = \langle z^i\rangle P_{n+j}(z)$ for $i \leq \frac{n-2}{2^{k-1}}$ and $j \leq 2^k$.

(b) When n is odd, $P_n(z)$ has exactly one real root α_n, with $-1 < \alpha_n \leq -0.5$, and $\{\alpha_n\}_{n \text{ odd}}$ is a monotonically decreasing sequence.

(c) When n is even, $P_n(z)$ has no real root.

(d) The polynomial $P_n(z)$ is irreducible over the integers for $n \geq 2$.

(e) The alternating sums of coefficients are given by the generating function

$$\sum_{n\geq 2} P_n(-1) z^{n-2} = \frac{(1+z)(1-2z)}{(1-2z^2)\sqrt{1-4z^2}}. \tag{16}$$

(f) For even n, the sum of the absolute values of coefficients of $P_n(z)$ is equal to $P_n(-1)$ when $n \geq 20$.

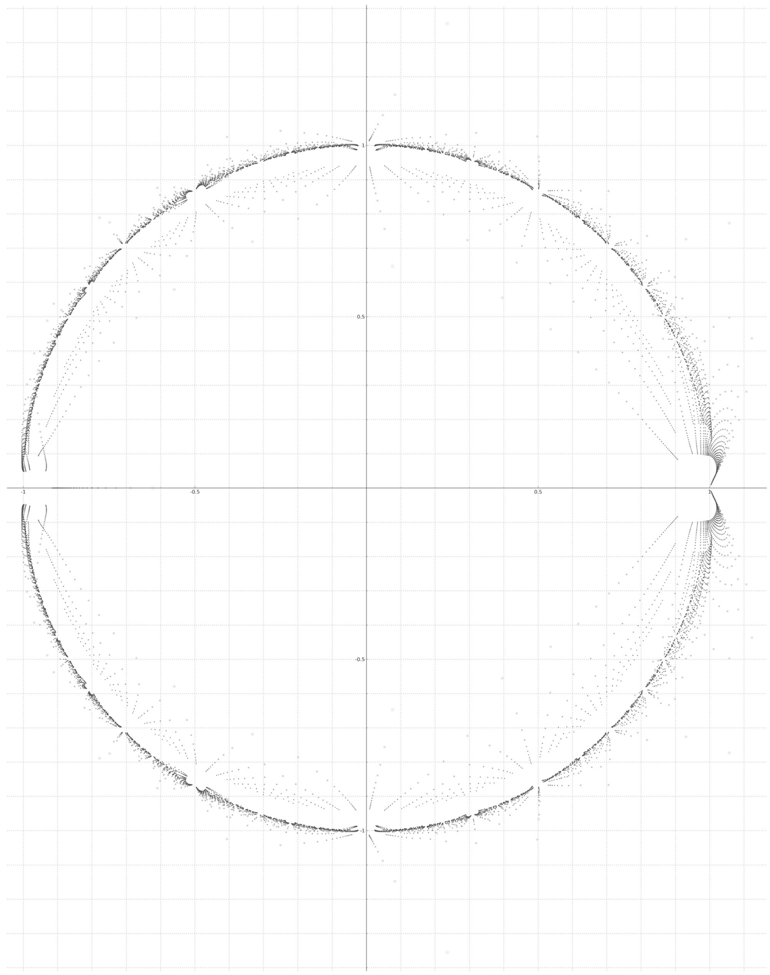


FIGURE 7. The complex zeros of $P_n(z)$ for odd n, where $3 \leq n \leq 67$. Darker and smaller points are used for larger n. Larger versions may be viewed at `http://webhome.cs.uvic.ca/~ruskey/Publications/Tatami/HoriVert.html`.

The right hand side of Equation (16) is the sum of two generating functions, with odd and even powered terms, respectively. The sequence of coefficients of the odd power terms is $-\sum_{i=0}^{k} 2^{k-i}\binom{2i}{i}$, for $k \geq 0$ (see A082590 in [6]), and that of the even power terms is $\binom{2k}{2}$, for $k \geq 1$ (see A000984 in [6]). The first few numbers $P_n(-1)$, starting with $n = 2$, are: 1, -1, 2, -4, 6, -14, 20, -48, 70, -166, 252, -584, 924, -2092, 3432, -7616, 12870, -28102, 48620, -104824, 184756, -394404, 705432, -1494240, 2704156, -5692636, 10400600.

Conjecture 2(f) compares the above sequence with the sum of the absolute values of the coefficients of $P_n(z)$. The first few of these are listed, also starting with $n = 2$: 1, 3, 4, 10, 10, 22, 28, 64, 76, 180, 260, 606, 932, 2124, 3440, 7666, 12872, 28178, 48620, 104946, 184756, 394638, 705432, 1494600, 2704156, 5693376, 10400600.

7. Conclusions and further research

The polynomials $P_n(z)$ exhibit numerous patterns in the signs of their coefficients, their plots and zeros (see `http://webhome.cs.uvic.ca/~ruskey/Publications/Tatami/HoriVert.html`), their degrees, and the values of $P_n(-1)$, but yet all we have seen is that they fall magically out of these coverings. What is the geometric interpretation, if there is one, for the factorisation of $V\!H_n(z)$, and how do we calculate the coefficients of $P_n(z)$ without dividing $D_n(z)$ into $V\!H_n(z)$?

In the present paper we deal with $n \times n$ coverings with n monominoes, and the techniques can be readily applied to $r \times c$ coverings with $r < c$ and a maximum number of monominoes. When m is not maximum, however, we may have to deal with other features of tatami coverings, called bidimers and vortices, described in [1]. The numbers of horizontal and vertical dominoes that any given bidimer or vortex will introduce is easily calculated, and they provide the advantage of sometimes isolating corners of the grid, which makes diagonal flips easier to count. We may not, however, have the good fortune of encountering a formula like the one in Lemma 1, which easily yields the generating polynomial of Theorem 1.

A generating function for fixed height balanced coverings, or perhaps some other relation between the numbers of vertical and horizontal tiles, would extend the results on fixed height coverings in [5] and [1].

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