

# The Hopf algebra of Fliess operators and its dual prelie algebra

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ABSTRACT. We study the Hopf algebra  $H$  of Fliess operators coming from Control Theory in the one-dimensional case. We prove that it admits a graded, finite-dimensional, connected gradation. Dually, the vector space  $\mathbb{R}\langle x_0, x_1 \rangle$  is both a prelie algebra for the prelie product dual to the coproduct of  $H$ , and an associative, commutative algebra for the shuffle product. These two structures admit a compatibility which makes  $\mathbb{R}\langle x_0, x_1 \rangle$  a Com-Prelie algebra. We give a presentation of this object as a Com-Prelie algebra and as a prelie algebra.

KEYWORDS. Fliess operators; prelie algebras; Hopf algebras.

AMS CLASSIFICATION. 16T05, 17B60, 93B25, 05C05.

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# Introduction

Right prelie algebras, or shortly prelie algebras [4, 1], are vector spaces with a bilinear product  $\bullet$  satisfying the following axiom:

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y).$$

Consequently, the antisymmetrization of  $\bullet$  is a Lie bracket. These objects are also called right-symmetric algebras or Vinberg algebra [12, 17]. If  $A$  is a prelie algebra, the symmetric algebra  $S(A)$  inherits a product  $\star$  making it a Hopf algebra, isomorphic to the enveloping algebra of the Lie algebra  $A$  [13, 14]. Whenever it is possible, we can consider the dual Hopf algebra  $S(A)^*$  and its group of characters  $G$ , which is the exponentiation, in some sense, of the Lie algebra  $A$ .

We here consider an inverse construction, departing from a group used in Control Theory, namely the group of Fliess operators [3, 5, 6]; this group is used to study the feedback product. We limit ourselves here to the one-dimensional case. This group is the set  $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$  of noncommutative formal series in two indeterminates, with a certain product generalizing the composition of formal series (definition 1). The Hopf algebra  $H$  of coordinates of this group is described in [5], where it is also proved that it is graded by the length of words; note that this gradation is not connected and not finite-dimensional. We first give a way to describe the composition in the group  $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$  and the coproduct of  $H$  by induction on the length of words (lemma 2 and proposition 3). We prove that  $H$  admits a second gradation, which is connected; the dimensions of this gradation are given by the Fibonacci sequence (proposition 8). As the product of  $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$  is left-linear,  $H$  is a commutative, right-sided combinatorial Hopf algebra [10], so, dually,  $\mathbb{R}\langle x_0, x_1 \rangle$  inherits a prelie product  $\bullet$ , which is inductively defined in proposition 11. We prove that the words  $x_1^n$ ,  $n \geq 0$ , form a minimal subset of generators of this prelie algebra (theorem 12).

The prelie algebra  $\mathbb{R}\langle x_0, x_1 \rangle$  has also an associative, commutative product, namely the shuffle product  $\sqcup$  [15]. We prove that the following axiom is satisfied (proposition 14):

$$(x \sqcup y) \bullet z = (x \bullet z) \sqcup y + x \sqcup (y \bullet z).$$

So  $\mathbb{R}\langle x_0, x_1 \rangle$  is a Com-Prelie algebra [11] (definition 15). We give a presentation of this Com-Prelie algebra in theorem 27. We use for this a description of free Com-Prelie algebras in terms of partitioned trees (definition 17), which generalizes the construction of prelie algebras in terms of rooted trees in [1]. We deduce a presentation of  $\mathbb{R}\langle x_0, x_1 \rangle$  as a prelie algebra in theorem 30. This presentation induces a new basis of  $\mathbb{R}\langle x_0, x_1 \rangle$  in terms of words with letters in  $\mathbb{N}^*$ , see corollary 31. The prelie product of two elements of this basis uses a dendriform structure [2, 9] on the algebra of words with letters in  $\mathbb{N}^*$  (theorem 34). The study of this dendriform structure is postponed to the appendix, as well as the enumeration of partitioned trees; we also prove that free Com-Prelie algebras are free as prelie algebras, using Livernet's rigidity theorem [7].

**Acknowledgment.** The research leading these results was partially supported by the French National Research Agency under the reference ANR-12-BS01-0017.

**Notation.** We denote by  $\mathbb{K}$  a commutative field of characteristic zero. All the objects (algebra, coalgebras, prelie algebras...) in this text will be taken over  $\mathbb{K}$ .

## 1 Construction of the Hopf algebra

### 1.1 Definition of the composition

Let us consider an alphabet of two letters  $x_0$  and  $x_1$ . We denote by  $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$  the completion of the free algebra generated by this alphabet, that is to say the set of noncommutative formal

series in  $x_0$  and  $x_1$ . Note that  $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$  is an algebra for the concatenation product and for the shuffle product, which we denote by  $\sqcup$ .

**Examples.** If  $a, b, c, d \in \{x_0, x_1\}$ :

$$\begin{aligned} abc \sqcup d &= abcd + abdc + adbc + dabc, \\ ab \sqcup cd &= abcd + acbd + cabd + acdb + cadb + cdab, \\ a \sqcup bcd &= abcd + bacd + bcad + bcda. \end{aligned}$$

The unit for both these products is the empty word, which we denote by  $\emptyset$ . The algebra  $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$  is given its usual ultrametric topology.

**Definition 1** [5].

1. For any  $d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ , we define a continuous algebra map  $\varphi_d$  from  $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$  to  $\text{End}(\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle)$  in the following way: for all  $X \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ ,

$$\varphi_d(x_0)(X) = x_0X, \quad \varphi_d(x_1)(X) = x_1X + x_0(d \sqcup X).$$

2. We define a composition  $\circ$  on  $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$  in the following way: for all  $c, d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ ,  $c \circ d = \varphi_d(c)(\emptyset) + d$ .

It is proved in [5] that this composition is associative.

**Notation.** For all  $c, d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ , we put  $c \tilde{\circ} d = c \circ d - d = \varphi_d(c)(\emptyset)$ .

**Remark.** If  $c_1, c_2, d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ ,  $\lambda \in \mathbb{K}$ :

$$(c_1 + \lambda c_2) \tilde{\circ} d = \varphi_d(c_1 + \lambda c_2)(\emptyset) = (\varphi_d(c_1) + \lambda \varphi_d(c_2))(\emptyset) = \varphi_d(c_1)(\emptyset) + \lambda \varphi_d(c_2)(\emptyset) = c_1 \tilde{\circ} d + \lambda c_2 \tilde{\circ} d.$$

So the composition  $\tilde{\circ}$  is linear on the left. As  $\varphi_d$  is continuous, the map  $c \rightarrow c \tilde{\circ} d$  is continuous for any  $d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ . Hence, it is enough to know how to compute  $c \tilde{\circ} d$  for any word  $c$ , which is made possible by the next lemma, using an induction on the length:

**Lemma 2** For any word  $c$ , for any  $d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ :

1.  $\emptyset \tilde{\circ} d = \emptyset$ .
2.  $(x_0c) \tilde{\circ} d = x_0(c \tilde{\circ} d)$ .
3.  $(x_1c) \tilde{\circ} d = x_1(c \tilde{\circ} d) + x_0(d \sqcup (c \tilde{\circ} d))$ .

**Proof.** 1.  $\emptyset \tilde{\circ} d = \varphi_d(\emptyset)(\emptyset) = Id(\emptyset) = \emptyset$ .

2.  $(x_0c) \tilde{\circ} d = \varphi_d(x_0c)(\emptyset) = \varphi_d(x_0) \circ \varphi_d(c)(\emptyset) = \varphi_d(x_0)(c \tilde{\circ} d) = x_0(c \tilde{\circ} d)$ .

3.  $(x_1c) \tilde{\circ} d = \varphi_d(x_1c)(\emptyset) = \varphi_d(x_1) \circ \varphi_d(c)(\emptyset) = \varphi_d(x_1)(c \tilde{\circ} d) = x_1(c \tilde{\circ} d) + x_0(d \sqcup (c \tilde{\circ} d))$ .  $\square$

## 1.2 Dual Hopf algebra

We here give a recursive description of the Hopf algebra of the coordinates of the group  $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$  of [5].

For any word  $c$ , let us consider the map  $X_c \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle^*$ , such that for any  $d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ ,  $X_c(d)$  is the coefficient of  $c$  in  $d$ . We denote by  $V$  the subspace of  $A^*$  generated by these maps.

Let  $H = S(V)$ , or equivalently the free associative, commutative algebra generated by the  $X_c$ 's. The elements of  $H$  are seen as polynomial functions on  $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ ; the elements of  $H \otimes H$  are seen as polynomial functions on  $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle \times \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ . Then  $H$  is given a multiplicative coproduct defined in the following way: for any word  $c$ , for any  $f, g \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ ,

$$\Delta(X_c)(f, g) = X_c(f \circ g).$$

As  $\circ$  is associative,  $\Delta$  is coassociative, so  $H$  is a bialgebra.

### Notations.

1. The space of words is a commutative algebra for the shuffle product  $\mathbb{W}$ . Dually, the space  $V$  inherits a coassociative, cocommutative coproduct, denoted by  $\Delta_{\mathbb{W}}$ . For example, if  $a, b, c \in \{x_0, x_1\}$ :

$$\begin{aligned} \Delta_{\mathbb{W}}(X_{\emptyset}) &= X_{\emptyset} \otimes X_{\emptyset}, \\ \Delta_{\mathbb{W}}(X_a) &= X_a \otimes X_{\emptyset} + X_{\emptyset} \otimes X_a, \\ \Delta_{\mathbb{W}}(X_{ab}) &= X_{ab} \otimes X_{\emptyset} + X_a \otimes X_b + X_b \otimes X_a + X_{\emptyset} \otimes X_{ab}, \\ \Delta_{\mathbb{W}}(X_{abc}) &= X_{abc} \otimes X_{\emptyset} + X_a \otimes X_{bc} + X_b \otimes X_{ac} + X_c \otimes X_{ab} \\ &\quad + X_{ab} \otimes X_c + X_{ac} \otimes X_b + X_{bc} \otimes X_a + X_{\emptyset} \otimes X_{abc}. \end{aligned}$$

2. We define two linear endomorphisms  $\theta_0, \theta_1$  of  $V$  by  $\theta_i(X_c) = X_{x_i c}$  for any word  $c$ .

The following proposition allows to compute  $\Delta(X_c)$  for any word  $c$  by induction on the length.

**Proposition 3** For all  $x \in V$ , we put  $\tilde{\Delta}(x) = \Delta(x) - 1 \otimes x$ .

1.  $\tilde{\Delta}(X_{\emptyset}) = X_{\emptyset} \otimes 1$ .
2.  $\tilde{\Delta} \circ \theta_0 = (\theta_0 \otimes Id) \circ \tilde{\Delta} + (\theta_1 \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}$ .
3.  $\tilde{\Delta} \circ \theta_1 = (\theta_1 \otimes Id) \circ \tilde{\Delta}$ .

**Proof.** For any word  $c$ , for any  $f, g \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ :

$$\tilde{\Delta}(X_c)(f, g) = \Delta(X_c)(f, g) - (1 \otimes X_c)(f, g) = X_c(f \circ g) - X_c(g) = X_c(f \otimes g - g) = X_c(f \tilde{\circ} g).$$

As  $\tilde{\circ}$  is linear on the left,  $\tilde{\Delta}(X_c) \in V \otimes H$ , so formulas in points 2 and 3 make sense.

Let  $f \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ . It can be uniquely written as  $f = x_0 f_0 + x_1 f_1 + \lambda \emptyset$ , with  $f_0, f_1 \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ ,  $\lambda \in K$ . For all  $g \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ :

$$f \tilde{\circ} g = (x_0 f_0) \tilde{\circ} g + (x_1 f_1) \tilde{\circ} g + \lambda \emptyset \tilde{\circ} g = x_0(f_0 \tilde{\circ} g + g \mathbb{W}(f_1 \tilde{\circ} g)) + x_1(f_1 \tilde{\circ} g) + \lambda \emptyset.$$

1. We obtain:

$$\tilde{\Delta}(X_{\emptyset})(f, g) = X_{\emptyset}(x_0(f_0 \tilde{\circ} g + g \mathbb{W}(f_1 \tilde{\circ} g)) + x_1(f_1 \tilde{\circ} g) + \lambda \emptyset) = 0 + 0 + \lambda = (X_{\emptyset} \otimes 1)(f, g),$$

so  $\Delta(X_{\emptyset}) = X_{\emptyset} \otimes 1$ .

2. Let  $c$  be a word.

$$\begin{aligned} \tilde{\Delta} \circ \theta_0(X_c)(f, g) &= \tilde{\Delta}(X_{x_0 c})(f, g) \\ &= X_{x_0 c}(x_0(f_0 \tilde{\circ} g + g \mathbb{W}(f_1 \tilde{\circ} g)) + x_1(f_1 \tilde{\circ} g) + \lambda \emptyset) \\ &= X_c(f_0 \tilde{\circ} g + g \mathbb{W}(f_1 \tilde{\circ} g)) + 0 + 0 \\ &= X_c(f_0 \tilde{\circ} g + (f_1 \tilde{\circ} g) \mathbb{W} g) + 0 + 0 \\ &= \tilde{\Delta}(X_c)(f_0, g) + (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}(X_c)(f_1, g, g) \\ &= \tilde{\Delta}(X_c)(f_0, g) + (Id \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}(X_c)(f_1, g) \\ &= (\theta_0 \otimes Id) \circ \tilde{\Delta}(X_c)(f, g) + (\theta_1 \otimes Id) \circ (Id \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}(X_c)(f, g), \end{aligned}$$

so  $\tilde{\Delta} \circ \theta_0(X_c) = (\theta_0 \otimes Id) \circ \tilde{\Delta}(X_c) + (\theta_1 \otimes Id) \circ (Id \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}(X_c)$ .

3. Let  $c$  be a word.

$$\begin{aligned} \tilde{\Delta} \circ \theta_1(X_c)(f, g) &= \tilde{\Delta}(X_{x_0c})(f, g) \\ &= X_{x_1c}(x_0(f_0\tilde{\circ}g + g\mathbb{W}(f_1\tilde{\circ}g)) + x_1(f_1\tilde{\circ}g) + \lambda\emptyset) \\ &= 0 + X_c(f_1\tilde{\circ}g) + 0 \\ &= \tilde{\Delta}(X_c)(f_1, g) \\ &= (\theta_1 \otimes Id) \circ \tilde{\Delta}(X_c)(f, g), \end{aligned}$$

so  $\tilde{\Delta} \circ \theta_1(X_c) = (\theta_1 \otimes Id) \circ \tilde{\Delta}(X_c)$ . □

**Examples.**

$$\begin{aligned} \Delta(X_{x_0}) &= X_{x_0} \otimes 1 + 1 \otimes X_{x_0} + X_{x_1} \otimes X_{\emptyset}, \\ \Delta(X_{x_0^2}) &= X_{x_0^2} \otimes 1 + 1 \otimes X_{x_0^2} + X_{x_0x_1} \otimes X_{\emptyset} + X_{x_1x_0} \otimes X_{\emptyset} + X_{x_1x_1} \otimes X_{\emptyset}^2 + X_{x_1} \otimes X_{x_0}, \\ \Delta(X_{x_0x_1}) &= X_{x_0x_1} \otimes 1 + 1 \otimes X_{x_0x_1} + X_{x_1x_1} \otimes X_{\emptyset} + X_{x_1} \otimes X_{x_1}, \\ \Delta(X_{x_1x_0}) &= X_{x_1x_0} \otimes 1 + 1 \otimes X_{x_1x_0} + X_{x_1x_1} \otimes X_{\emptyset}. \end{aligned}$$

**Corollary 4** For all  $n \geq 1$ ,  $\tilde{\Delta}(X_{x_1^n}) = X_{x_1^n} \otimes 1$  and  $\Delta(X_{x_1^n}) = X_{x_1^n} \otimes 1 + 1 \otimes X_{x_1^n}$ .

**Proof.** It comes from an easy induction on  $n$ . □

### 1.3 gradation

It is proved in [5] that the Hopf algebra  $H$  is graded by the length of words, but this gradation is not connected, that is to say that the homogeneous component of degree 0 is not  $(0)$ , as it contains  $X_{\emptyset}$ . Moreover, the homogeneous components of  $H$  are not finite-dimensional, as for example  $X_{\emptyset}^n X_{x_0}^k$  is homogeneous of degree  $k$  for all  $n \geq 0$ . We now define another gradation on  $H$ , which is connected and finite-dimensional.

**Definition 5** 1. Let  $c = c_1 \dots c_n$  be a word. We put:

$$\deg(c) = n + 1 + \#\{i \in \{1, \dots, n\} \mid c_i = x_0\}.$$

2. For all  $k \geq 1$ , we put:

$$V_k = \text{Vect}(X_c \mid \deg(x) = k).$$

This define a connected gradation of  $V$ , that is to say:

$$V = \bigoplus_{k \geq 1} V_k.$$

3. This gradation induces a connected gradation of the algebra  $H$ :

$$H = \bigoplus_{k \geq 0} H_k, \text{ and } H_0 = \mathbb{K}.$$

**Lemma 6** If  $c$  is a word of degree  $n$ , then:

$$\tilde{\Delta}(X_c) \in \bigoplus_{i+j=n} V_i \otimes H_j.$$

**Proof.** Let us start by the following observations:

1. Let  $c$  be a word of degree  $k$ . Then  $x_0c$  is a word of degree  $k+2$ . Hence,  $\theta_0$  is homogeneous of degree 2 on  $V$ .
2. Let  $c$  be a word of degree  $k$ . Then  $x_1c$  is a word of degree  $k+1$ . Hence,  $\theta_1$  is homogeneous of degree 1 on  $V$ .
3. Let  $c$  and  $d$  be two words of respective degrees  $k$  and  $l$ . Then any word obtained by shuffling  $c$  and  $d$  is of degree  $k+l-1$ : its length is the sum of the length of  $c$  and  $d$ , and the number of  $x_0$  in it is the sum of the numbers of  $x_0$  in  $c$  and  $d$ . Hence, dually, the coproduct  $\Delta_{\mathbf{u}}$  is homogeneous of degree 1 from  $V$  to  $V \otimes V$ .

Let us prove the result by induction on the length  $k$  of  $c$ . If  $k=0$ , then  $c=\emptyset$  so  $n=1$ , and  $\tilde{\Delta}(X_c) = X_c \otimes 1 \in V_1 \otimes H_0$ . Let us assume the result for all words of length  $< k-1$ . Two cases can occur.

1. If  $c = x_0d$ , then  $\deg(d) = n-2$ . we put  $\Delta_{\mathbf{u}}(X_d) = \sum x'_i \otimes x''_i$ . By the preceding third observation, we can assume that for all  $i$ ,  $x'_i, x''_i$  are homogeneous elements of  $V$ , with  $\deg(x'_i) + \deg(x''_i) = n-2+1 = n-1$ . Then:

$$\tilde{\Delta}(X_c) = (\theta_0 \otimes Id) \circ \tilde{\Delta}(X_d) + \sum_i (\theta_1 \otimes m) \circ (\tilde{\Delta}(x'_i) \otimes x''_i).$$

By the induction hypothesis,  $\tilde{\Delta}(X_d) \in (V \otimes H)_{n-1}$ . By the second observation,  $(\theta_0 \otimes Id) \circ \tilde{\Delta}(X_d) \in (V \otimes H)_n$ . By the induction hypothesis applied to  $x'_i$ , for all  $i$ ,  $(\tilde{\Delta}(x'_i) \otimes x''_i) \in (V \otimes H \otimes V)_{n-1}$ , so by the first observation,  $(\theta_1 \otimes m) \circ (\tilde{\Delta}(x'_i) \otimes x''_i) \in (V \otimes H)_{n-1+1} \subseteq (V \otimes H)_n$ . So  $\Delta(X_c) \in (V \otimes H)_n$ .

2.  $c = x_1d$ , then  $\deg(d) = n-1$ . Moreover,  $\tilde{\Delta}(X_c) = (\theta_1 \otimes Id) \circ \tilde{\Delta}(X_d)$ . By the induction hypothesis,  $\tilde{\Delta}(X_d) \in (V \otimes H)_{n-1}$ . By the second observation,  $\tilde{\Delta}(X_c) \in (V \otimes H)_n$ .

So the result holds for any word  $c$ . □

**Proposition 7** *With this gradation,  $H$  is a graded, connected Hopf algebra.*

**Proof.** We have to prove that for all  $n \geq 0$ :

$$\Delta(H_n) \subseteq \bigoplus_{i+j=n} H_i \otimes H_j.$$

This comes from the multiplicativity of  $\Delta$ . □

Let us now study the formal series of  $V$  and  $H$ .

**Proposition 8** *1. For all  $k$ , let us put  $p_k = \dim(V_k)$  and  $F_V = \sum_{k=1}^{\infty} p_k X^k$ . Then:*

$$F_V = \frac{X}{1 - X - X^2},$$

and for all  $k \geq 1$ :

$$p_k = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^k - \left( \frac{1 - \sqrt{5}}{2} \right)^k \right).$$

*This is the Fibonacci sequence (A000045 in [16]).*

2. We put  $F_H = \sum_{k=0}^{\infty} \dim(H_k)X^k$ . Then:

$$F_H = \prod_{k=1}^{\infty} \frac{1}{(1 - X^k)^{p_k}}.$$

**Proof.** Let us consider the formal series:

$$F(X_0, X_1) = \sum_{i,j \geq 0} \#\{\text{words in } x_0, x_1 \text{ with } i \text{ } x_0 \text{ and } j \text{ } x_1\} X_0^i X_1^j.$$

Then  $F(X_0, X_1) = \frac{1}{1 - X_0 - X_1}$ . Moreover, by definition of the degree of a word:

$$F_V = XF(X^2, X) = \frac{X}{1 - X - X^2}.$$

As  $H$  is the symmetric algebra generated by  $V$ , its formal series is given by the second point.  $\square$

**Examples.** We obtain:

$k$	0	1	2	3	4	5	6	7	8	9	10
$\dim(V_k)$	0	1	1	2	3	5	8	13	21	34	55
$\dim(H_k)$	1	1	2	4	8	15	30	56	108	203	384

The third row is sequence A166861 of [16].

**Remark.** Consequently, the space  $V$  inherits a bigradation:

$$V_{k,n} = \text{Vect}(X_c \mid \text{deg}(c) = k \text{ and } \text{lg}(c) = n).$$

If  $c$  is a word of length  $n$  and of degree  $k$ , denoting by  $a$  the number of its letters equal to  $x_0$  and by  $b$  the number of its letters equal to  $x_1$ , then:

$$\begin{cases} a + b & = n, \\ 2a + b + 1 & = k, \end{cases}$$

so  $a = k - n - 1$ . Hence:

$$\dim(V_{k,n}) = \binom{n}{k - n - 1},$$

and the formal series of this bigradation is:

$$\sum_{k,n \geq 0} \dim(V_{k,n}) X^k Y^n = \frac{X}{1 - XY - X^2 Y}.$$

## 2 Prelie structure on $\mathbb{K}\langle x_0, x_1 \rangle$

### 2.1 Prelie coproduct on $V$

As the composition  $\circ$  is linear on the left, the dual coproduct satisfies  $\tilde{\Delta}(V) \subseteq V \otimes H$ , so  $H$  is a commutative right-sided Hopf algebra in the sense of [10], and  $V$  inherits a right prelie coproduct: if  $\pi$  is the canonical projection from  $H = S(V)$  onto  $V$ ,

$$\delta = (\pi \otimes \pi) \circ \Delta = (Id \otimes \pi) \circ \tilde{\Delta}.$$

It satisfies the right prelie coalgebra axiom:

$$(23).((\delta \otimes Id) \circ \delta - (Id \otimes \delta) \circ \delta) = 0.$$

The following proposition allows to compute  $\delta(X_c)$  by induction on the length of  $c$ .

**Proposition 9** 1.  $\delta(X_\emptyset) = 0$ .

2.  $\delta \circ \theta_0 = (\theta_0 \otimes Id) \circ \delta + (\theta_1 \otimes Id) \circ \Delta_{\mathbf{w}}$ .

3.  $\delta \circ \theta_1 = (\theta_1 \otimes Id) \circ \delta$ .

**Proof.** The first point comes from  $\Delta(X_\emptyset) = X_\emptyset \otimes 1 + 1 \otimes X_\emptyset$ . Let  $x \in V$ . We put  $\Delta_{\mathbf{w}}(x) = x' \otimes x'' \in V \otimes V$ . For any  $y \in V$ , we put  $\tilde{\Delta}(y) - y \otimes 1 = y^{(1)} \otimes y^{(2)} \in V \otimes H_+$ . Then:

$$\begin{aligned} (\theta_1 \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbf{w}}(x) &= (\theta_1 \otimes m)(x' \otimes 1 \otimes x'' + x'^{(1)} \otimes x'^{(2)} \otimes x'') \\ &= \theta_1(x') \otimes \underbrace{x''}_{\in V} + x'^{(1)} \otimes \underbrace{x'^{(2)} x''}_{\in Ker(\pi)}. \end{aligned}$$

Applying  $Id \otimes \pi$ , it remains:

$$(Id \otimes \pi) \circ (\theta_1 \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbf{w}}(x) = (\theta_1 \otimes Id) \circ \Delta_{\mathbf{w}}(x).$$

Let  $i = 0$  or  $1$ . Then:

$$(Id \otimes \pi) \circ (\theta_i \otimes Id) \circ \tilde{\Delta} = (\theta_i \otimes Id) \circ (Id \otimes \pi) \circ \tilde{\Delta} = (\theta_i \otimes Id) \circ \delta.$$

The result is induced by these remarks, combined with proposition 3. □

**Examples.**

$$\begin{aligned} \delta(X_{x_0}) &= X_{x_1} \otimes X_\emptyset, \\ \delta(X_{x_0^2}) &= X_{x_0 x_1} \otimes X_\emptyset + X_{x_1 x_0} \otimes X_\emptyset + X_{x_1} \otimes X_{x_0}, \\ \delta(X_{x_0 x_1}) &= X_{x_1 x_1} \otimes X_\emptyset + X_{x_1} \otimes X_{x_1}, \\ \delta(X_{x_1 x_0}) &= X_{x_1 x_1} \otimes X_\emptyset. \end{aligned}$$

**Proposition 10**  $Ker(\delta) = Vect(X_{x_1^n}, n \geq 0)$ .

**Proof.** The inclusion  $\supseteq$  is trivial by corollary 4. Let us prove the other inclusion.

*First step.* Let us prove the following property: if  $x \in V_k$  is such that

$$\delta(x) = \lambda \sum_{i+j=k-2} \frac{(k-2)!}{i!j!} X_{x_1^i} \otimes X_{x_1^j},$$

then there exists  $\mu \in \mathbb{K}$  such that  $x = \mu x_1^{k-1}$ . It is obvious if  $k = 1$ , as then  $x = \mu \emptyset$ . Let us assume the result at all ranks  $< k$ . We put  $x = x_1^\alpha (x_0 f_0 + x_1 f_1)$ , where  $\alpha \geq 0$ ,  $f_0$  is homogeneous of degree  $k - 2 - \alpha$  and  $f_1$  is homogeneous of degree  $k - 1 - \alpha$ .

$$\delta(x) = (\theta_1^\alpha \otimes Id) ((\theta_0 \otimes Id) \circ \delta(f_0) + (\theta_1 \otimes Id) \circ \delta(f_1) + (\theta_1 \otimes Id) \circ \Delta_{\mathbf{w}}(f_0)).$$

Let us consider the terms in this expression of the form  $X_\emptyset \otimes X_c$ , with  $c$  a word. This gives:

$$\lambda X_\emptyset \otimes X_{x_1^{k-2}} = 0,$$

so  $\lambda = 0$  and  $\delta(x) = 0$ . Let us now consider the terms of the form  $X_{x_1^\alpha x_0 c} \otimes X_d$ , with  $c, d$  words. We obtain:

$$(\theta_1^\alpha \circ \theta_0 \otimes Id) \circ \delta(f_0) = 0.$$

As both  $\theta_0$  and  $\theta_1$  are injective, we obtain  $\delta(f_0) = 0$ . By the induction hypothesis,  $f_0 = \nu X_1 x_1^l$ , with  $l = k - 2 - \alpha < k$ . Hence:

$$\Delta_{\mathbf{w}}(f_0) = \nu \sum_{i+j=l} \frac{l!}{i!j!} X_{x_1^i} \otimes X_{x_1^j},$$



and:

$$(\theta_1^{\alpha+1} \otimes Id) \left( \delta(f_1) + \nu \sum_{i+j=l} \frac{l!}{i!j!} X_{x_1^i} \otimes X_{x_1^j} \right) = 0.$$

As  $\theta_1$  is injective, we obtain with the induction hypothesis that  $f_1 = \mu X_{x_1^{k-2-\alpha}}$ , so:

$$x = \mu X_{x_1^{k-1}} + \nu X_{x_1^\alpha x_0 x_1^{k-\alpha-2}}.$$

This gives:

$$\begin{aligned} \delta(x) &= \nu(\theta_1^{\alpha+1} \otimes Id) \left( \sum_{i+j=k-\alpha-2} \frac{(k-\alpha-2)!}{i!j!} X_{x_1^i} \otimes X_{x_1^j} \right) \\ &= \nu \sum_{i+j=k-\alpha-2} \frac{(k-\alpha-2)!}{i!j!} X_{x_1^{i+\alpha}} \otimes X_{x_1^j} \\ &= 0, \end{aligned}$$

so necessarily  $\nu = 0$  and  $x = \mu X_{x_1^{k-1}}$ .

*Second step.* Let  $x \in Ker(\delta)$ . As  $\delta$  is homogeneous of degree 0, the homogeneous components of  $x$  are in  $Ker(\delta)$ . By the first step, with  $\lambda = 0$ , these homogeneous components, hence  $x$ , belong to  $Vect(X_{x_1^k}, k \geq 0)$ .  $\square$

## 2.2 Dual prelie algebra

As  $V$  is a graded prelie coalgebra, its graded dual is a prelie algebra. We identify this graded dual with  $\mathbb{K}\langle x_0, x_1 \rangle \subseteq \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ ; for any words  $c, d$ ,  $X_c(d) = \delta_{c,d}$ . The prelie product of  $\mathbb{K}\langle x_0, x_1 \rangle$  is denoted by  $\bullet$ . Dualizing proposition 9, we obtain:

- Proposition 11**
1. For all word  $c$ ,  $\emptyset \bullet c = 0$ .
  2. For all words  $c, d$ ,  $(x_0 c) \bullet d = x_0(c \bullet d)$ .
  3. For all words  $c, d$ ,  $(x_1 c) \bullet d = x_1(c \bullet d) + x_0(c \sqcup d)$ .

**Proof.** Let  $u, v, w$  be words. Then  $X_w(u \bullet v) = \delta(X_w)(u \otimes v)$ . Hence, if  $d$  is a word:

$$\begin{aligned} X_\emptyset(u \bullet v) &= 0, \\ X_{x_0 d}(u \bullet v) &= (\theta_0 \otimes Id) \circ \delta(X_d)(u \otimes v) + (\theta_1 \otimes Id) \circ \Delta_\sqcup(X_d)(u \otimes v) \\ &= X_d(\theta_0^*(u) \bullet v + \theta_1^*(u) \sqcup v), \\ X_{x_1 d}(u \bullet v) &= (\theta_1 \otimes Id) \circ \delta(X_d)(u \otimes v) \\ &= X_d(\theta_1^*(u) \bullet v). \end{aligned}$$

Moreover, for all word  $c$ :

$$\begin{array}{lll} \theta_0^*(\emptyset) = 0, & \theta_0^*(x_0 c) = c, & \theta_0^*(x_1 c) = 0, \\ \theta_1^*(\emptyset) = 0, & \theta_1^*(x_0 c) = 0, & \theta_1^*(x_1 c) = c. \end{array}$$

Hence, for any words  $c, d$ :

$$\begin{aligned} X_{x_0 d}(x_0 c \bullet v) &= X_d(c \bullet v) & X_{x_0 d}(x_1 c \bullet v) &= X_d(c \sqcup v) \\ &= X_{x_0 d}(x_0(x \bullet v)), & &= X_{x_0 d}(x_1(c \bullet v) + x_0(c \sqcup v)), \\ \\ X_{x_1 d}(x_0 c \bullet v) &= 0 & X_{x_1 d}(x_1 c \bullet v) &= X_d(c \bullet v) \\ &= X_{x_1 d}(x_0(x \bullet v)), & &= X_{x_1 d}(x_1(c \bullet v) + x_0(c \sqcup v)). \end{aligned}$$

Hence, for any  $w$ ,  $X_w(x_0c \bullet v) = X_w(x_0(x \bullet v))$  and  $X_w(x_1c \bullet v) = X_w((x_1(c \bullet v) + x_0(c \Downarrow v)))$ .  $\square$

### Examples.

$$\begin{array}{lll} x_0 \bullet x_0 = 0 & x_0 \bullet x_0x_0 = 0 & x_1 \bullet x_0x_0 = x_0x_0x_0 \\ x_0 \bullet x_1 = 0 & x_0 \bullet x_0x_1 = 0 & x_1 \bullet x_0x_1 = x_0x_0x_1 \\ x_1 \bullet x_0 = x_0x_0 & x_0 \bullet x_1x_0 = 0 & x_1 \bullet x_1x_0 = x_0x_1x_0 \\ x_1 \bullet x_1 = x_0x_1 & x_0 \bullet x_1x_1 = 0 & x_1 \bullet x_1x_1 = x_0x_1x_1 \end{array}$$

$$\begin{array}{ll} x_0x_0 \bullet x_0 = 0 & x_0x_0 \bullet x_1 = 0 \\ x_0x_1 \bullet x_0 = x_0x_0x_0 & x_0x_1 \bullet x_1 = x_0x_0x_1 \\ x_1x_0 \bullet x_0 = 2x_0x_0x_0 & x_1x_0 \bullet x_1 = x_0x_0x_1 + x_0x_1x_0 \\ x_1x_1 \bullet x_0 = x_1x_0x_0 + x_0x_1x_0 + x_0x_0x_1 & x_1x_1 \bullet x_1 = x_1x_0x_1 + 2x_0x_1x_1 \end{array}$$

Dualizing proposition 10:

**Theorem 12**  $\mathbb{K}\langle x_0, x_1 \rangle = Vect(x_1^n, n \geq 0) \oplus (\mathbb{K}\langle x_0, x_1 \rangle \bullet \mathbb{K}\langle x_0, x_1 \rangle)$ . Hence,  $(x_1^n)_{n \geq 0}$  is a minimal system of generators of the prelie algebra  $\mathbb{K}\langle x_0, x_1 \rangle$ .

**Proof.** As  $\bullet = \delta^*$ ,  $Im(\bullet) = Ker(\delta)^\perp = Vect(X_{x_1^n}, n \geq 0)^\perp$ . The first assertion is then immediate. As  $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$  is a graded, connected prelie coalgebra,  $\mathbb{K}\langle x_0, x_1 \rangle$  is a graded, connected prelie algebra. The result then comes from the next lemma.  $\square$

**Lemma 13** Let  $A$  be a graded, connected prelie algebra, and  $V$  be a graded subspace of  $A$ .

1.  $V$  generates  $A$  if, and only if,  $A = V + A \bullet A$ .
2.  $V$  is a minimal subspace of generators of  $A$  if, and only if,  $A = V \oplus A \bullet A$ .

**Proof.** 1.  $\implies$ . Let  $x \in A$ . Then it can be written as an element of the prelie subalgebra generated by  $v$ , so as the sum of an element of  $V$  and of iterated prelie products of elements of  $V$ . Hence,  $x \in V + A \bullet A$ . Note that we did not use the gradation of  $A$  to prove this point.

1.  $\impliedby$ . Let  $B$  be the prelie subalgebra generated by  $V$ . Let  $x \in A_n$ , let us prove that  $x \in B$  by induction on  $n$ . As  $A_0 = (0)$ , it is obvious if  $n = 0$ . Let us assume the result at all ranks  $< n$ . We obtain, by the gradation:

$$A_n = V_n \oplus \sum_{i=1}^{n-1} A_i \bullet A_{n-i}.$$

So we can write  $x = \lambda x_1^{n-1} + \sum x_i \bullet y_i$ , where  $x_i, y_i$  are homogeneous of degree  $< n$ . By the induction hypothesis, these elements belong to  $B$ , so  $x \in B$ .

2.  $\implies$ . By 1.  $\implies$ ,  $A = V + A \bullet A$ . If  $V \cap A \bullet A \neq (0)$ , we can choose a graded subspace  $W \subsetneq V$ , such that  $A = W \oplus A \bullet A$ . By 1.  $\impliedby$ ,  $W$  generates  $A$ , so  $V$  is not a minimal system of generators of  $A$ : contradiction. So  $A = V \oplus A \bullet A$ .

2.  $\impliedby$ . By 1.  $\impliedby$ ,  $V$  is a space of generators of  $A$ . If  $W \subsetneq V$ , then  $W \oplus A \bullet A \subsetneq A$ . By 1.  $\implies$ ,  $W$  does not generate  $V$ . So  $V$  is a minimal subspace of generators.  $\square$

**Proposition 14** For all  $x, y, z \in \mathbb{K}\langle x_0, x_1 \rangle$ ,  $(x \Downarrow y) \bullet z = (x \bullet z) \Downarrow y + x \Downarrow (y \bullet z)$ .

**Proof.** We prove it if  $x, y, z$  are words. If  $x = \emptyset$ , then:

$$(\emptyset \sqcup y) \bullet z = y \bullet z = (\emptyset \bullet z) \sqcup y + \emptyset \sqcup (y \bullet z).$$

If  $y = \emptyset$ , the result is also true, using the commutativity of  $\sqcup$ . We can now consider that  $x, y$  are nonempty words.

Let us proceed by induction on  $k = lg(x) + lg(y)$ . If  $k = 0$  or  $1$ , there is nothing to prove. Let us assume the result at all rank  $< k$ . Four cases can occur.

*First case.*  $x = x_0a$  and  $y = x_0b$ . Then:

$$\begin{aligned} (x \sqcup y) \bullet z &= (x_0(a \sqcup x_0b)) \bullet z + (x_0(x_0a \sqcup b)) \bullet z \\ &= x_0((a \sqcup x_0b) \bullet z) + x_0((x_0a \sqcup b) \bullet z) \\ &= x_0((a \bullet z) \sqcup x_0b) + x_0(a \sqcup ((x_0b) \bullet z)) + x_0(((x_0a) \bullet z) \sqcup b) + x_0(x_0a \sqcup (b \bullet z)) \\ &= x_0((a \bullet z) \sqcup x_0b) + x_0(a \sqcup (x_0(b \bullet z))) + x_0((x_0(a \bullet z)) \sqcup b) + x_0(x_0a \sqcup (b \bullet z)) \\ &= x_0(a \bullet z) \sqcup x_0b + x_0a \sqcup x_0(b \bullet z) \\ &= (x \bullet z) \sqcup y + x \sqcup (y \bullet z). \end{aligned}$$

*Second case.*  $x = x_1a$  and  $y = x_0b$ . This gives:

$$\begin{aligned} (x \sqcup y) \bullet z &= (x_1(a \sqcup x_0b)) \bullet z + (x_0(x_1a \sqcup b)) \bullet z \\ &= x_1((a \bullet z) \sqcup x_0b) + x_1(a \sqcup x_0(b \bullet z)) \\ &\quad + x_0(a \sqcup x_0b \sqcup z) + x_0((x_1a) \bullet z) \sqcup b + x_0(x_1a \sqcup (b \bullet z)) \\ &= x_1((a \bullet z) \sqcup x_0b) + x_1(a \sqcup x_0(b \bullet z)) \\ &\quad + x_0(a \sqcup x_0b \sqcup z) + x_0((x_1(a \bullet z)) \sqcup b) + x_0((x_0(a \sqcup z)) \sqcup b) + x_0(x_1a \sqcup (b \bullet z)), \end{aligned}$$

$$\begin{aligned} (x \bullet z) \sqcup y &= (x_1(a \bullet z)) \sqcup x_0b + (x_0(a \sqcup z)) \sqcup (x_0b) \\ &= x_1((a \bullet z) \sqcup (x_0b)) + x_0(x_1(a \bullet z) \sqcup b) \\ &\quad + x_0(a \sqcup z \sqcup x_0b) + x_0((x_0(a \sqcup z)) \sqcup b), \end{aligned}$$

$$\begin{aligned} x \sqcup (y \bullet z) &= x_1a \sqcup x_0(b \bullet z) \\ &= x_1(a \sqcup x_0(b \bullet z)) + x_0(x_1a \sqcup (b \bullet z)). \end{aligned}$$

These computations imply the required equality.

*Third case.*  $x = x_0a$  and  $y = x_1b$ . This is a consequence of the second case, using the commutativity of  $\sqcup$ .

*Last case.*  $x = x_1a$  and  $y = x_1b$ . Similar computations give:

$$\begin{aligned} (x \sqcup y) \bullet z &= x_1((a \bullet z) \sqcup x_1b) + x_1(a \sqcup x_1(b \bullet w)) + x_1(a \sqcup x_0(b \sqcup z)) + x_0(a \sqcup x_1b \sqcup z) \\ &\quad + x_1(x_1a \sqcup (b \bullet z)) + x_1((x_1(a \bullet z)) \sqcup b) + x_1((x_0(a \sqcup z)) \sqcup b) + x_0(a \sqcup x_1b \sqcup z), \end{aligned}$$

$$(x \bullet z) \sqcup y = x_1((a \bullet z) \sqcup x_1b) + x_1((x_1(a \bullet z)) \sqcup b) + x_0(a \sqcup x_1b \sqcup z) + x_1((x_0(a \sqcup z)) \sqcup b),$$

$$x \sqcup (y \bullet z) = x_1(a \sqcup x_1(b \bullet w)) + x_1(a \sqcup x_0(b \sqcup z)) + x_1(x_1a \sqcup (b \bullet z)) + x_0(a \sqcup x_1b \sqcup z).$$

So the result holds in all cases.  $\square$

### 3 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a Com-Prelie algebra

Proposition 14 motivates the following definition:

**Definition 15** [11] *A Com-Prelie algebra is a triple  $(V, \bullet, \sqcup)$ , such that:*

1.  $(V, \bullet)$  is a prelie algebra.
2.  $(V, \sqcup)$  is a commutative, associative algebra (non necessarily unitary).
3. For all  $a, b, c \in V$ ,  $(a \sqcup b) \bullet c = (a \bullet c) \sqcup b + a \sqcup (b \bullet c)$ .

For example,  $\mathbb{K}\langle x_0, x_1 \rangle$  is a Com-Prelie algebra. See [11] for an example of Com-Prelie algebra based on rooted trees.

#### 3.1 Free Com-Prelie algebras

**Definition 16** 1. *A partitioned forest is a pair  $(F, I)$  such that:*

- (a)  $F$  is a rooted forest (the edges of  $F$  being oriented from the leaves to the roots).
  - (b)  $I$  is a partition of the vertices of  $F$  with the following condition: if  $x, y$  are two vertices of  $F$  which are in the same part of  $I$ , then either they are both roots, or they have the same direct descendant.
2. We shall say that a partitioned forest is a partitioned tree if all the roots are in the same part of the partition.
  3. Let  $\mathcal{D}$  be a set. A partitioned tree decorated by  $\mathcal{D}$  is a pair  $(t, d)$ , where  $t$  is a partitioned tree and  $d$  is a map from the set of vertices of  $t$  into  $\mathcal{D}$ . For any vertex  $x$  of  $t$ ,  $d(x)$  is called the decoration of  $x$ .
  4. The set of isoclasses of partitioned trees will be denoted by  $\mathcal{PT}$ . For any set  $\mathcal{D}$ , the set of isoclasses of partitioned trees decorated by  $\mathcal{D}$  will be denoted by  $\mathcal{PT}(\mathcal{D})$ .

**Examples.** We represent partitioned trees by the Hasse graph of the underlying rooted forest, the partition being represented by horizontal edges, of different colors. Here are all the partitioned trees with  $\leq 4$  vertices:

$$\begin{aligned} \cdot; \uparrow, \text{---}; \nabla, \nabla, \uparrow, \uparrow = \downarrow, \text{---}; \nabla, \nabla = \nabla, \nabla, \downarrow = \downarrow, \downarrow = \downarrow, \nabla, \nabla, \uparrow, \\ \nabla = \nabla, \uparrow, \downarrow = \downarrow, \nabla = \nabla, \downarrow, \downarrow = \downarrow = \downarrow, \text{---}. \end{aligned}$$

**Definition 17** Let  $t = (t, I)$  and  $t' = (t', J) \in \mathcal{PT}$ .

1. Let  $s$  be a vertex of  $t'$ . The partitioned tree  $t \bullet_s t'$  is defined as follows:
  - (a) As a rooted forest,  $t \bullet_s t'$  is obtained by grafting all the roots of  $t'$  on the vertex  $s$  of  $t$ .
  - (b) We put  $I = \{I_1, \dots, I_k\}$  and  $J = \{J_1, \dots, J_l\}$ . The partition of the vertices of this rooted forest is  $I \sqcup J = \{I_1, \dots, I_k, J_1, \dots, J_l\}$ .
2. The partitioned tree  $t \sqcup t'$  is defined as follows:
  - (a) As a rooted forest,  $t \sqcup t'$  is  $tt'$ .
  - (b) We put  $I = \{I_1, \dots, I_k\}$  and  $J = \{J_1, \dots, J_l\}$  and we assume that the set of roots of  $t$  is  $I_1$  and the set of roots of  $t'$  is  $J_1$ . The partition of the vertices of  $t \sqcup t'$  is  $\{I_1 \sqcup J_1, I_2, \dots, I_k, J_2, \dots, J_l\}$ .

### Examples.

1. Here are the three possible graftings  $\nabla \bullet_s \cdot$ :  $\nabla$ ,  $\nabla$  and  $\nabla$ .

2. Here are the two possible graftings  $\mathbf{!} \bullet_s \cdot$ :  $\nabla$  and  $\nabla$ .

These operations can also be defined for decorated partitioned trees.

**Proposition 18** *Let  $\mathcal{D}$  be a set. We denote by  $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$  the vector space generated by  $\mathcal{PT}(\mathcal{D})$ . We extend  $\sqcup$  by bilinearity on  $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$  and we define a second product  $\bullet$  on  $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$  in the following way: if  $t, t' \in \mathcal{PT}(\mathcal{D})$ ,*

$$t \bullet t' = \sum_{s \in V(t)} t \bullet_s t'.$$

*Then  $(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}, \bullet, \sqcup)$  is a Com-Prelie algebra.*

**Proof.** Let  $t, t', t''$  be three partitioned trees.

If  $s', s''$  are two vertices of  $t$ , we define by  $t \bullet_{s, s'}(t', t'')$  the partitioned trees obtained by grafting the roots of  $t'$  on  $s'$ , the roots of  $t''$  on  $s''$ , the partition of the vertices of the obtained rootes forest being the union of the partitions of  $t, t'$  and  $t''$ . Then:

$$\begin{aligned} (t \bullet t') \bullet t'' &= \sum_{s' \in V(t)} (t \bullet_{s'} t') \bullet t'' \\ &= \sum_{s', s'' \in V(t)} (t \bullet_{s'} t') \bullet_{s''} t'' + \sum_{s' \in V(t), s'' \in V(t')} (t \bullet_{s'} t') \bullet_{s''} t'' \\ &= \sum_{s', s'' \in V(t)} t \bullet_{s' s''}(t', t'') + \sum_{s' \in V(t), s'' \in V(t')} t \bullet_{s'}(t' \bullet_{s''} t'') \\ &= \sum_{s', s'' \in V(t)} t \bullet_{s' s''}(t', t'') + t \bullet(t' \bullet t''). \end{aligned}$$

So  $(t \bullet t') \bullet t'' - t \bullet(t' \bullet t'')$  is clearly symmetric in  $t$  and  $t'$ , and  $\bullet$  is prelie.

Moreover,  $(t \sqcup t') \sqcup t'' = t \sqcup(t' \sqcup t'')$  is the rooted forest  $tt't''$ , the partition being  $\{I_1 \sqcup J_1 \sqcup K_1, I_2, \dots, I_k, J_2, \dots, J_l, K_2, \dots, K_m\}$ , with immediate notations;  $t \sqcup t' = t' \sqcup t$  is the rooted forest  $t't$ , the partition being  $\{I_1 \sqcup J_1, I_2, \dots, I_k, J_2, \dots, J_l\}$ . So  $\sqcup$  is an associative, commutative product.

Finally:

$$\begin{aligned} (t \sqcup t') \bullet t'' &= \sum_{s \in V(t)} (t \sqcup t') \bullet_s t'' + \sum_{s' \in V(t')} (t \sqcup t') \bullet_{s'} t'' \\ &= \sum_{s \in V(t)} (t \bullet_s t'') \sqcup t' + \sum_{s' \in V(t')} t \sqcup (t' \bullet_{s'} t'') \\ &= (t \bullet t') \sqcup t'' + t \sqcup (t' \bullet t''). \end{aligned}$$

So  $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$  is Com-Prelie. □

In particular,  $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$  is prelie. Let us use the extension of the prelie product  $\bullet$  to  $S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$  defined by Oudom and Guin [13, 14]:

1. If  $t_1, \dots, t_k \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ ,  $t_1 \dots t_k \bullet 1 = t_1 \dots t_k$ .
2. If  $t, t_1, \dots, t_k \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ ,  $t \bullet t_1 \dots t_k = (t \bullet t_1 \dots t_{k-1}) \bullet t_k - t \bullet (t_1 \dots t_{k-1} \bullet t_k)$ .

3. If  $a, b, c \in S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$ ,  $ab \bullet c = (a \bullet c^{(1)})(b \bullet c^{(2)})$ , where  $\Delta(c) = c^{(1)} \otimes c^{(2)}$  is the usual coproduct of  $S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$ . In particular, if  $t_1, \dots, t_k, t \in \mathcal{PT}(\mathcal{D})$ :

$$t_1 \dots t_k \bullet t = \sum_{i=1}^k t_1 \dots (t_i \bullet t) \dots t_k.$$

**Lemma 19** *Let  $t = (t, I)$ ,  $t_1 = (t_1, I^{(1)})$ ,  $\dots$ ,  $t_k = (t_k, I^{(k)})$  be partitioned trees ( $k \geq 1$ ). Let  $s_1, \dots, s_k \in V(t)$ . The partitioned tree  $t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k)$  is obtained by grafting the roots of  $t_i$  on  $s_i$  for all  $i$ , the partition being  $I \sqcup I^{(1)} \sqcup \dots \sqcup I^{(k)}$ . Then:*

$$t \bullet t_1 \dots t_k = \sum_{s_1, \dots, s_k \in V(t)} t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k).$$

**Proof.** By induction on  $k$ . This is obvious if  $k = 1$ . Let us assume the result at rank  $k$ .

$$\begin{aligned} t \bullet t_1 \dots t_{k+1} &= (t \bullet t_1 \dots t_k) \bullet t_{k+1} - \sum_{i=1}^k t \bullet (t_1 \dots (t_i \bullet t_{k+1}) \dots t_k) \\ &= \sum_{s_1, \dots, s_k \in V(t)} (t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k)) \bullet t_{k+1} - \sum_{i=1}^k \sum_{s \in V(t_i)} t \bullet (t_1 \dots (t_i \bullet_s t_{k+1}) \dots t_i) \\ &= \sum_{s_1, \dots, s_{k+1} \in V(t)} (t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k)) \bullet_{s_{k+1}} t_{k+1} \\ &+ \sum_{i=1}^k \sum_{s \in V(t_i)} (t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k)) \bullet_s t_{k+1} \\ &- \sum_{i=1}^k \sum_{s_1, \dots, s_k \in V(t)} \sum_{s \in V(t_i)} t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_i \bullet_s t_{k+1}, \dots, t_i) \\ &= \sum_{s_1, \dots, s_{k+1} \in V(t)} t \bullet_{s_1, \dots, s_{k+1}} (t_1, \dots, t_{k+1}). \end{aligned}$$

Hence, the result holds for all  $k$ . □

**Theorem 20** *Let  $\mathcal{D}$  be a set, let  $A$  be a Com-Prelie algebra, and let  $a_d \in A$  for all  $d \in \mathcal{D}$ . There exists a unique morphism of Com-Prelie algebra  $\phi : \mathfrak{g}_{\mathcal{PT}(\mathcal{D})} \rightarrow A$ , such that  $\phi(\bullet_d) = a_d$  for all  $d \in \mathcal{D}$ . In other words,  $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$  is the free Com-Prelie algebra generated by  $\mathcal{D}$ .*

**Proof.** *Unicity.* Let  $t \in \mathcal{T}^d$ . We denote by  $r_1, \dots, r_n$  its roots. For all  $1 \leq i \leq n$ , let  $t_{i,1}, \dots, t_{i,k_i}$  be the partitioned trees born from  $r_i$  and let  $d_i$  be the decoration of  $r_i$ . Then:

$$t = (\bullet_{d_1} \bullet t_{1,1} \dots t_{1,k_1}) \sqcup \dots \sqcup (\bullet_{d_n} \bullet t_{n,1} \dots t_{n,k_n}).$$

So  $\phi$  is inductively defined by:

$$\phi(t) = (a_{d_1} \bullet \phi(t_{1,1}) \dots \phi(t_{1,k_1})) \sqcup \dots \sqcup (a_{d_n} \bullet \phi(t_{n,1}) \dots \phi(t_{n,k_n})). \quad (1)$$

*Existence.* As the product  $\sqcup$  of  $A$  is commutative and associative, (1) defines inductively a morphism  $\phi$  from  $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$  to  $A$ . By definition, it is compatible with the product  $\sqcup$ . Let us prove the compatibility with the product  $\bullet$ . Let  $t, t'$  be two partitioned trees, let us prove that  $\phi(t \bullet t') = \phi(t) \bullet \phi(t')$  by induction on the number  $N$  of vertices of  $t$ . If  $N = 1$ , then  $t = \bullet_d$  and:

$$\phi(t \bullet t') = a_d \bullet \phi(t') = \phi(t) \bullet \phi(t'),$$

by definition of  $t'$ . If  $N > 1$ , two cases are possible.

*First case.* If  $t$  has only one root, then  $t = \cdot_d \bullet t_1 \dots t_k$ , and:

$$t \bullet t' = \cdot_d \bullet t_1 \dots t_k t' + \sum_{i=1}^k \cdot_d \bullet t_1 \dots t_i \circ t' \bullet t_k.$$

Using the induction hypothesis on  $t_1, \dots, t_k$ :

$$\begin{aligned} \phi(t \bullet t') &= a_d \bullet \phi(t_1) \dots \phi(t_k) \phi(t') + \sum_{i=1}^k a_d \bullet \phi(t_1) \dots \phi(t_i \circ t') \dots \phi(t_k) \\ &= a_d \bullet \phi(t_1) \dots \phi(t_k) \phi(t') + \sum_{i=1}^k a_d \bullet (\phi(t_1) \dots \phi(t_i) \circ \phi(t') \dots \phi(t_k)) \\ &= (a_d \bullet \phi(t_1) \dots \phi(t_k)) \bullet \phi(t') \\ &= \phi(t) \bullet \phi(t'). \end{aligned}$$

*Second case.* If  $t$  has  $k > 1$  roots, we put  $t = t_1 \sqcup \dots \sqcup t_k$ . The induction hypothesis holds for  $t_1, \dots, t_k$ , so:

$$\begin{aligned} \phi(t \circ t') &= \sum_{i=1}^k \phi(t_1 \sqcup t_i \bullet t' \sqcup \dots \sqcup t_k) \\ &= \sum_{i=1}^k \phi(t_1) \sqcup \phi(t_i \bullet t') \sqcup \dots \sqcup \phi(t_k) \\ &= \sum_{i=1}^k \phi(t_1) \sqcup \phi(t_i) \bullet \phi(t') \sqcup \dots \sqcup \phi(t_k) \\ &= (\phi(t_1) \sqcup \dots \sqcup \phi(t_k)) \bullet \phi(t') \\ &= \phi(t) \bullet \phi(t'). \end{aligned}$$

Hence,  $\phi$  is a morphism of Com-Prelie algebras. □

### 3.2 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a Com-Prelie algebra

**Proposition 21** *As a Com-Prelie algebra,  $\mathbb{K}\langle x_0, x_1 \rangle$  is generated by  $\emptyset$  and  $x_1$ .*

**Proof.** Let  $A$  be the Com-Prelie subalgebra of  $\mathbb{K}\langle x_0, x_1 \rangle$  generated by  $\emptyset$  and  $x_1$ . For all  $n \geq 1$ , it contains  $x_1^{\sqcup n} = n!x_1^n$ , so it contains  $x_1^n$  for all  $n \geq 0$ . As  $\mathbb{K}\langle x_0, x_1 \rangle$  is generated by these elements as a prelie algebra,  $A = \mathbb{K}\langle x_0, x_1 \rangle$ . □

We denote by  $\phi_{CPL} : \mathfrak{g}_{\mathcal{PT}(\{1,2\})} \longrightarrow \mathbb{K}\langle x_0, x_1 \rangle$  the unique morphism of Com-Prelie algebras which sends  $\cdot_1$  to  $\emptyset$  and  $\cdot_2$  to  $\cdot_2$ . By proposition 21, it is surjective.

**Lemma 22** *Let  $t_1, \dots, t_k \in \mathcal{PT}(\{1, 2\})$ .*

1.  $\phi_{CPL}(\cdot_1 \bullet t_1 \dots t_k) = 0$  if  $k \geq 1$ .
2.  $\phi_{CPL}(\cdot_2 \bullet t_1 \dots t_k) = 0$  if  $k \geq 2$ .
3. If  $t \in \mathcal{PT}(\{1, 2\})$ ,  $\phi_{CPL}(\cdot_2 \bullet t) = x_0 \phi_{CPL}(t)$ .

**Proof.** We prove 1.-3. by induction on  $k$ . If  $k = 1$ :

$$\begin{aligned} \phi_{CPL}(\cdot_1 \bullet t) &= \emptyset \bullet \phi_{CPL}(t) = 0, \\ \phi_{CPL}(\cdot_2 \bullet t) &= x_1 \bullet \phi_{CPL}(t) = x_0 \phi_{CPL}(t). \end{aligned}$$

Let us assume the results at rank  $k - 1 \geq 1$ . Then:

$$\begin{aligned}
\phi_{CPL}(\bullet_1 \bullet t_1 \dots t_k) &= \emptyset \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_k) \\
&= (\emptyset \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_{k-1})) \bullet \phi_{CPL}(t_k) \\
&\quad - \sum_{i=1}^k \emptyset \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_i \bullet t_k) \dots \phi_{CPL}(t_{k-1}) \\
&= 0, \\
\phi_{CPL}(\bullet_2 \bullet t_1 \dots t_k) &= x_1 \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_k) \\
&= (x_1 \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_{k-1})) \bullet \phi_{CPL}(t_k) \\
&\quad - \sum_{i=1}^k x_1 \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_i \bullet t_k) \dots \phi_{CPL}(t_{k-1}).
\end{aligned}$$

If  $k \geq 3$ , the induction hypothesis immediately allows to conclude that  $\phi_{CPL}(\bullet_2 \bullet t_1 \dots t_k) = 0 - 0 = 0$ . If  $k = 2$ , this gives:

$$\begin{aligned}
\phi_{CPL}(\bullet_2 \bullet t_1 t_2) &= (x_1 \bullet \phi_{CPL}(t_1)) \bullet \phi_{CPL}(t_2) - x_1 \bullet \phi_{CPL}(t_1 \bullet t_2) \\
&= (x_0 \phi_{CPL}(t_1)) \bullet \phi_{CPL}(t_2) - x_0 \phi_{CPL}(t_1 \bullet t_2) \\
&= x_0 (\phi_{CPL}(t_1) \bullet \phi_{CPL}(t_2)) \phi_{CPL}(t_1 \bullet t_2) \\
&= 0.
\end{aligned}$$

Hence, the result holds for all  $k \geq 1$ . □

**Lemma 23** *For all  $t \in \mathcal{PT}(\{1, 2\})$ ,  $\phi_{CPL}(t)$  is a linear span of words of length the number of vertices of  $t$  decorated by 2.*

**Proof.** By induction on the number of vertices  $N$  of  $t$ . If  $N = 1$ , then  $t = \bullet_1$  or  $\bullet_2$  and the result is obvious. Let us assume the result at all rank  $< N$ .

*First case.* If  $t$  has only one root, we put  $t = \bullet_i \bullet t_1 \dots t_k$ . By the preceding lemma, we can assume that  $i = 2$  and  $k = 1$ . Then  $\phi_{CPL}(t) = x_0 \phi_{CPL}(t_1)$  and the result is obvious.

*Second case.* If  $t$  has  $k > 1$  roots, we put  $t = t_1 \sqcup \dots \sqcup t_k$ . Then  $\phi_{CPL}(t_1)$  is equal to  $\phi_{CPL}(t_1) \sqcup \dots \sqcup \phi_{CPL}(t_k)$  and the result is immediate. □

**Lemma 24** *We define inductively a family  $F$  of elements of  $\mathcal{PT}(\{1, 2\})$  by:*

1.  $F(1) = \{\bullet_1, \bullet_2\}$ .
2.  $F(n+1) = (\bullet_2 \bullet F(n)) \cup \bigcup_{i=1}^n (F(i) \sqcup F(n+1-i))$ .
3.  $F = \bigcup_{n \geq 1} F(n)$ .

*Let  $t \in \mathcal{PT}(\{1, 2\})$ . Then  $\phi_{CPL}(t) \neq 0$  if, and only if,  $t \in F$ .*

**Proof.**  $\implies$ . We proceed by induction on the number  $N$  of vertices of  $t$ . This is obvious if  $N = 1$ . Let us assume the result at all rank  $< N$ .

*First case.* If  $N$  has only one root, we put  $N = \bullet_i \bullet t_1 \dots t_k$ . By lemma 22,  $i = 2$  and  $k = 1$ . Then  $\phi_{CPL}(t) = x_0 \phi_{CPL}(t_1)$ . By the induction hypothesis,  $t_1 \in F$ , so  $t \in F$ .



*Second case.* If  $N$  has  $k > N$  roots, we put  $t = t_1 \sqcup \dots \sqcup t_k$ . Then:

$$\phi_{CPL}(t) = \phi_{CPL}(t_1) \sqcup \phi_{CPL}(t_2) \sqcup \dots \sqcup \phi_{CPL}(t_k) \neq 0,$$

so by the induction hypothesis,  $t_1$  and  $t_2 \sqcup \dots \sqcup t_k \in F$ , and  $t \in F$ .

$\Leftarrow$ . Let  $t \in T(n)$ . We proceed by induction on  $n$ . It  $n = 1$ , this is obvious. If  $n > 1$  then  $t = \bullet_2 \bullet t'$ , with  $t' \in F(n-1)$ , or  $t = t' \sqcup t''$ , with  $t' \in F(i)$ ,  $t'' \in F(n-i)$ . In the first case, by the induction hypothesis,  $\phi_{CPL}(t') \neq 0$  and  $\phi_{CPL}(t) = x_0 \phi_{CPL}(t') \neq 0$ . In the second case,  $\phi_{CPL}(t'), \phi_{CPL}(t'') \neq 0$  by the induction hypothesis, so  $\phi_{CPL}(t) = \phi_{CPL}(t') \sqcup \phi_{CPL}(t'') \neq 0$ .  $\square$

**Examples.**

$$\begin{aligned} F(1) &= \{\bullet_1, \bullet_2\}, \\ F(2) &= \{\mathbf{1}_2^1, \mathbf{1}_2^2, \mathbf{1}_{\rightarrow 1}, \mathbf{1}_{\rightarrow 2}, \mathbf{2}_{\rightarrow 2}\}, \\ F(3) &= \left\{ \mathbf{1}_2^1, \mathbf{1}_2^2, \mathbf{1}_{\nabla 2}^1, \mathbf{1}_{\nabla 2}^2, \mathbf{2}_{\nabla 2}^2, \mathbf{1}_{\rightarrow 1}, \mathbf{2}_{\rightarrow 1}, \mathbf{2}_{\rightarrow 2}, \mathbf{1}_{\rightarrow 1}, \mathbf{1}_{\rightarrow 2}, \mathbf{1}_{\rightarrow 2}, \mathbf{2}_{\rightarrow 2} \right\}. \end{aligned}$$

We define a second family of elements of  $\mathcal{PT}(\{1, 2\})$  in the following way:

1.  $F'(1) = \{\bullet_1, \bullet_2\}$ .
2.  $F'(2) = \{\mathbf{1}_2^2, \mathbf{1}_{\rightarrow 2}\}$ .
3.  $F'(n+1) = (\bullet_2 \bullet F'(n)) \cup \bigcup_{i=2}^{n-1} (F'(i) \sqcup F'(n+1-i)) \cup (\bullet_2 \sqcup F'(n))$  if  $n \geq 2$ .
4.  $F' = \bigcup_{n \geq 1} F'(n)$ .

For example:

$$\begin{aligned} F'(3) &= \left\{ \mathbf{1}_2^1, \mathbf{1}_2^2, \mathbf{2}_{\nabla 2}^2, \mathbf{1}_{\rightarrow 2}, \mathbf{2}_{\rightarrow 2}, \mathbf{2}_{\rightarrow 2} \right\}, \\ F'(4) &= \left\{ \mathbf{1}_2^1, \mathbf{1}_2^2, \mathbf{2}_{\nabla 2}^2, \mathbf{2}_{\nabla 2}^1, \mathbf{2}_{\nabla 2}^2, \mathbf{2}_{\nabla 2}^2, \mathbf{2}_{\nabla 2}^2, \mathbf{2}_{\nabla 2}^2, \mathbf{1}_{\rightarrow 2}, \mathbf{2}_{\rightarrow 2}, \mathbf{1}_{\rightarrow 2}, \mathbf{1}_{\rightarrow 2}, \mathbf{1}_{\rightarrow 2}, \mathbf{2}_{\rightarrow 2}, \mathbf{2}_{\rightarrow 2}, \mathbf{2}_{\rightarrow 2} \right\}. \end{aligned}$$

We define a map  $\pi$  from  $F$  to  $\mathcal{PT}(\{1, 2\})$  in the following way:

1.  $\pi(\bullet_i) = \bullet_i$  if  $i = 1, 2$ .
2.  $\pi(\bullet_1 \sqcup \dots \sqcup \bullet_1) = \bullet_1$ .
3. If  $t = \bullet_1 \sqcup \dots \sqcup \bullet_1 \sqcup t_1 \sqcup \dots \sqcup t_k$ ,  $k \geq 1$ , with  $t_1, \dots, t_k \neq \bullet_1$ , then  $\pi(t) = \pi(t_1) \sqcup \dots \sqcup \pi(t_k)$ .
4. If  $t = \bullet_2 \bullet t_1 \dots t_k$ , then  $\pi(t) = \bullet_2 \bullet \pi(t_1) \dots \pi(t_k)$ .

**Lemma 25**  $\pi$  is a projection on  $F'$  and  $\phi_{CPL} \circ \pi = \phi_{CPL}|_F$ .

**Proof.** Let  $t \in F$ . Let us prove by induction on the number  $N$  of vertices of  $t$  that:

1.  $\pi(t) \in F'$ .
2. If  $t \in F'$ ,  $\pi(t) = t$ .
3.  $\phi_{CPL} \circ \pi(t) = \phi_{CPL}(t)$ .

4. If  $\pi(t) = \bullet_1$ , then  $t = \bullet_1 \sqcup^N$ .

All these points are immediate if  $N = 1$ . Let us assume the result at all ranks  $< N$ ,  $N \geq 2$ . We put  $t = \bullet_1 \sqcup \dots \sqcup \bullet_1 \sqcup t_1 \sqcup \dots \sqcup t_k$ ,  $k \geq 0$ , with  $t_1, \dots, t_k \neq \bullet_1$ .

*First case.* If  $k \geq 2$ , then  $\pi(t) = \pi(t_1) \sqcup \dots \sqcup \pi(t_k)$ . Following the induction hypothesis,  $\pi(t_1), \dots, \pi(t_k) \in F'$  and are not equal to  $\bullet_1$ , so  $\pi(t) \in F'$ : moreover,  $\pi(t_1) \neq \bullet_1$ , so  $\pi(t) \neq \bullet_1$ .

$$\begin{aligned} \phi_{CPL}(t) &= \phi_{CPL}(\bullet_1) \sqcup \dots \sqcup \phi_{CPL}(\bullet_1) \sqcup \phi_{CPL}(t_1) \sqcup \dots \sqcup \phi_{CPL}(t_k) \\ &= \emptyset \sqcup \dots \sqcup \emptyset \sqcup \phi_{CPL} \circ \pi(t_1) \sqcup \dots \sqcup \phi_{CPL} \circ \pi(t_k) \\ &= \phi_{CPL}(\pi(t_1) \sqcup \dots \sqcup \pi(t_k)) \\ &= \phi_{CPL} \circ \pi(t). \end{aligned}$$

If  $t \in F'$ , necessarily  $t = t_1 \sqcup \dots \sqcup t_k$ , and  $t_1, \dots, t_k \in F'$ . By the induction hypothesis,  $\pi(t_1) = t_1, \dots, \pi(t_k) = t_k$ , so  $\pi(t) = t$ .

*Second case.* If  $k = 1$ , as  $t_1 \in F$ , we put  $t_1 = \bullet_2 \bullet s$ . Then  $\pi(t) = \bullet_2 \bullet \pi(s)$ . By the induction hypothesis,  $\pi(s) \in F'$ , so  $\pi(t) \in F'$ . Moreover:

$$\begin{aligned} \phi_{CPL}(t) &= \phi_{CPL}(\bullet_1) \sqcup \dots \sqcup \phi_{CPL}(\bullet_1) \sqcup (\phi_{CPL}(\bullet_2) \bullet \phi_{CPL}(s)) \\ &= \emptyset \sqcup \dots \sqcup \emptyset \sqcup (\phi_{CPL}(\bullet_2) \bullet \phi_{CPL}(s)) \\ &= \phi_{CPL} \circ \pi(\bullet_2) \bullet \phi_{CPL} \circ \pi(s) \\ &= \phi_{CPL} \circ \pi(t). \end{aligned}$$

If  $t' \in F'$ , then  $s \in F'$ , and  $t = \bullet_2 \bullet s$ . Then  $\pi(t) = \bullet_2 \bullet \pi(s) = \bullet_2 \bullet s = t$ .

*Last case.* If  $k = 0$ , all the results are obvious. □

**Lemma 26** *Let  $t, t' \in \mathcal{PT}(\{1, 2\})$ . Then:*

$$\phi_{CPL}((\bullet_2 \bullet t) \sqcup (\bullet_2 \bullet t')) = \phi_{CPL}(\bullet_2 \bullet ((\bullet_2 \bullet t) \sqcup t' + t \sqcup (\bullet_2 \bullet t'))).$$

**Proof.** Indeed, putting  $w = \phi_{CPL}(t)$  and  $w' = \phi_{CPL}(t')$ :

$$\begin{aligned} \phi_{CPL}((\bullet_2 \bullet t) \sqcup (\bullet_2 \bullet t')) &= x_0 w \sqcup x_0 w' \\ &= x_0(w \sqcup x_0 w') + x_0(x_0 w \sqcup w') \\ &= \phi_{CPL}(\bullet_2 \bullet ((\bullet_2 \bullet t) \sqcup t' + t \sqcup (\bullet_2 \bullet t'))). \end{aligned}$$

We used lemma 22 for the first and third equalities. □

**Theorem 27** *The kernel of  $\phi_{CPL}$  is the Com-Prelie ideal generated by the elements:*

1.  $\bullet_1 \bullet t_1 \dots t_k$ , where  $k \geq 1$ ,  $t_1, \dots, t_k \in \mathcal{PT}(\{1, 2\})$ .
2.  $\bullet_2 \bullet t_1 \dots t_k$ , where  $k \geq 2$ ,  $t_1, \dots, t_k \in \mathcal{PT}(\{1, 2\})$ .
3.  $\bullet_1 \sqcup t - t$ , where  $t \in \mathcal{PT}(\{1, 2\})$ .
4.  $(\bullet_2 \bullet t) \sqcup (\bullet_2 \bullet t') - \bullet_2 \bullet ((\bullet_2 \bullet t) \sqcup t' - t \sqcup (\bullet_2 \bullet t'))$ , where  $t, t' \in \mathcal{PT}(\{1, 2\})$ .

**Proof.** Let  $I$  be the ideal generated by these elements. Lemmas 22 and 26 prove that the elements 1, 2 and 4 belong to  $\text{Ker}(\phi_{CPL})$ . Moreover, for all  $t \in \mathcal{PT}(\{1, 2\})$ ,  $\pi(\bullet_1 \sqcup t) = \pi(t)$ . For all  $t \in \mathcal{PT}(\{1, 2\})$ :

$$\phi_{CPL}(\bullet_1 \sqcup t) = \emptyset \sqcup \phi_{CPL}(t) = \phi_{CPL}(t),$$

so elements 3. also belong to  $Ker(\phi_{CPL})$ . Hence,  $I \subseteq Ker(\phi_{CPL})$ .

Let  $h = \mathfrak{gPT}_{\{1,2\}}/I$ . As the elements 1 and 2 belong to  $I$ ,  $h$  is linearly spanned by the elements  $\bar{t}$ ,  $t \in F$ . As the elements 3 belong to  $I$ , for all  $t \in F$ ,  $\overline{\pi(t)} = \bar{t}$ . As  $\pi$  is a projection on  $F'$ ,  $h$  is linearly spanned by the elements  $\bar{t}$ ,  $t \in F'$ .

We now define inductively two families of partitionned trees in the following way:

1.  $T''(1) = \{\bullet_2\}$  and  $F''(1) = \{\bullet_1, \bullet_2\}$ .
2.  $T''(n+1) = \bullet_2 \bullet F''(n)$ .
3.  $F''(n+1) = \bigcup_{i=1}^{n+1} T''(i) \sqcup \bullet_2 \sqcup^{(n+1-i)}$ .
4.  $F'' = \bigcup_{n \geq 1} F''(n)$ .

For example:

$$F''(3) = \left\{ \begin{array}{c} \mathbb{1}_{\frac{1}{2}}^1, \mathbb{1}_{\frac{2}{2}}^2, \mathbb{2}\nabla_2^2, \mathbb{1}_{\frac{1}{2}}^1, \mathbb{2}\mathbb{1}_{\frac{1}{2}}^1, \mathbb{2}\mathbb{1}_{\frac{2}{2}}^2, \mathbb{2}\mathbb{2}_{\frac{2}{2}}^2 \end{array} \right\},$$

$$F''(4) = \left\{ \begin{array}{c} \mathbb{1}_{\frac{1}{2}}^1, \mathbb{1}_{\frac{2}{2}}^2, \mathbb{2}\nabla_2^2, \mathbb{2}\nabla_2^1, \mathbb{2}\nabla_2^2, \mathbb{2}\nabla_2^2, \mathbb{2}\nabla_2^2, \mathbb{2}\nabla_2^2, \mathbb{2}\mathbb{1}_{\frac{1}{2}}^1, \mathbb{2}\mathbb{1}_{\frac{2}{2}}^2, \mathbb{2}\mathbb{1}_{\frac{2}{2}}^2, \mathbb{2}\mathbb{2}_{\frac{2}{2}}^2 \end{array} \right\}.$$

Let us prove that for all  $t \in F'$ , there exists  $t' \in Vect(F'')$  such that  $\bar{t} = \overline{t'}$ . We proceed by induction on the number  $N$  of vertices of  $t$ . If  $N = 1$ , then  $t = \bullet_1$  or  $\bullet_2$  and we take  $t' = t$ . Let us assume the result at all rank  $< N$ . We put  $t = t_1 \sqcup \dots \sqcup t_k \sqcup \bullet_2 \sqcup \dots \sqcup \bullet_2$ , with  $t_i = \bullet_2 \bullet s_i$ ,  $s_i \neq 1$ , for all  $1 \leq i \leq k$ . We proceed by induction on  $k$ . If  $k = 0$ , we take  $t' = t = \bullet_2 \sqcup \dots \sqcup \bullet_2$ . If  $k = 1$ , then, by the induction hypothesis on  $N$  applied to  $s_1$ :

$$\bar{t} = (\overline{\bullet_2 \bullet s_1}) \sqcup \overline{\bullet_2} \sqcup \dots \sqcup \overline{\bullet_2} = (\overline{\bullet_2 \bullet s'_1}) \sqcup \overline{\bullet_2} \sqcup \dots \sqcup \overline{\bullet_2} = \overline{(\bullet_2 \bullet s'_1) \sqcup \bullet_2 \sqcup \dots \sqcup \bullet_2}.$$

We take  $t' = (\bullet_2 \bullet s'_1) \sqcup \bullet_2 \sqcup \dots \sqcup \bullet_2$ , which clearly belongs to  $Vect(F'')$ , as  $s'_1 \in Vect(F'')$ . Let us assume the result at all rank  $< k$ . Then, as the elements 4 belong to  $I$ :

$$\overline{t_1 \sqcup t_2} = \underbrace{\overline{\bullet_2 \bullet (t_1 \sqcup s_2)}}_{t'_1} + \underbrace{\overline{\bullet_2 \bullet (s_1 \bullet t_2)}}_{t''_1},$$

so:

$$\bar{t} = \overline{t'_1 \sqcup t_3 \sqcup \dots \sqcup t_k \sqcup \bullet_2 \sqcup \dots \sqcup \bullet_2} + \overline{t''_1 \sqcup t_3 \sqcup \dots \sqcup t_k \sqcup \bullet_2 \sqcup \dots \sqcup \bullet_2}.$$

By the induction hypothesis on  $k$  applied to these two partitionned trees, there exists  $x'_1$  and  $x''_1 \in Vect(F'')$ , such that  $\bar{t} = \overline{x'_1} + \overline{x''_1}$ . We take  $t' = x'_1 + x''_1$ . Consequently, the elements  $\bar{t}$ ,  $t \in F''$ , linearly span  $h$ .

Let  $t \in F''(n)$ . Then it has  $n$  vertices, and at most one of them is decorated by 1. We denote by  $F''_1(n)$  the set of elements of  $F''(n)$  with one vertex decorated by 1, and we put  $F''_2(n) = F''(n) \setminus F''_1(n)$ . Let us prove that for all  $n \geq 1$ ,  $|F''_1(n+1)| \leq 2^{n-1}$  and  $|F''_2(n)| \leq 2^{n-1}$ . For  $n = 0$ , as  $F''_1(2) = \{\mathbb{1}_{\frac{1}{2}}^1\}$  and  $F''_2(1) = \{\bullet_2\}$ , this is immediate. Let us assume the result at all rank  $\leq n$ . Then:

$$F''_2(n+1) = \bigcup_{i=1}^{n+1} \bullet_2 \sqcup^{(n+1-i)} \sqcup T''(i) \cap F''_2(i) = \{\bullet_2 \sqcup^{(n+1)}\} \cup \bigcup_{i=1}^n \bullet_2 \sqcup^{(n+1-i)} \sqcup \bullet_2 \bullet F''_2(i).$$

Hence,  $|F_2''(n+1)| \leq 1 + 1 + 2 + \dots + 2^{n-1} = 2^n$ .

$$F_1''(n+2) = \bigcup_{i=1}^{n+2} \cdot_2 \mathfrak{W}^{(n+2-i)} \sqcup T''(i) \cap F_1''(i) = \bigcup_{i=2}^{n+2} \cdot_2 \mathfrak{W}^{(n+2-i)} \sqcup \cdot_2 \bullet F_1''(i-1).$$

Hence,  $|F_1''(n+2)| \leq +1 + 1 + \dots + 2^{n-1} = 2^n$ .

Let  $\bar{\phi}_{APL}$  be the linear map induced by  $\phi_{CPL}$  on  $h$ . If  $t \in F_1''(n)$ , by lemma 23,  $\bar{\phi}_{APL}(\bar{t})$  is a linear span of words of length  $n-1$ . If  $t \in F_2''(n)$ , by lemma 23,  $\bar{\phi}_{APL}(\bar{t})$  is a linear span of words of length  $n$ . Hence, for all  $n \geq 0$ :

$$\bar{\phi}_{APL}(Vect(F_2''(n)) + Vect(F_1''(n+1))) \subseteq Vect(\text{words of length } n).$$

As  $\phi_{CPL}$  is surjective, we obtain:

$$\bar{\phi}_{APL}(Vect(F_2''(n)) + Vect(F_1''(n+1))) = Vect(\text{words of length } n).$$

Moreover, as  $\dim(Vect(\text{words of length } n)) = 2^n$  and  $\dim(Vect(F_2''(n)) + Vect(F_1''(n+1))) \leq |F_2''(n)| + |F_1''(n)| \leq 2^{n-1} + 2^{n-1} = 2^n$ , the restriction of  $\bar{\phi}_{APL}$  to  $Vect(F_2''(n)) + Vect(F_1''(n+1))$  is injective. Finally,  $\bar{\phi}_{APL}$  is injective, so  $Ker(\phi_{CPL}) = I$ .  $\square$

## 4 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a prelie algebra

### 4.1 A surjective morphism

Let  $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$  be the free prelie algebra generated by  $\mathbb{N}^*$ , as described in [1]. It can be seen as the subspace of  $\mathfrak{g}_{\mathcal{PT}(\mathbb{N}^*)}$  generated by rooted trees (which are seen as partitioned trees such that any part of the partition is a singleton), with the restriction of the prelie product  $\bullet$  defined by graftings. For example, in  $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ , if  $a, b, c, d > 0$ :

$$\mathbf{1}_a^b \bullet \mathbf{1}_c^d = b \mathbf{V}_a^c \begin{matrix} d \\ \vdots \\ c \end{matrix} + \begin{matrix} d \\ \vdots \\ c \end{matrix} \mathbf{1}_a^b.$$

This prelie algebra is graded, the degree of a tree being the sum of its decorations.

By theorem 12, there exists a unique surjective map of prelie algebras  $\Phi_{PL} : \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} \longrightarrow \mathbb{K}\langle x_0, x_1 \rangle$ , sending  $\cdot_n$  to  $x_1^{n-1}$  for all  $n \geq 1$ . As  $x_1^{i-1}$  is homogeneous of degree  $i$  for all  $i$ , this morphism is homogeneous of degree 0.

**Notation.** If  $t_1 \dots t_k \in \mathcal{T}(\mathbb{N}^*)$  and  $n \in \mathbb{N}^*$ , we put:

$$B_n(t_1 \dots t_k) = \cdot_n \bullet t_1 \dots t_k.$$

This is the tree obtained by grafting  $t_1, \dots, t_k$  on a common root decorated by  $n$ .

**Proposition 28** *Let  $t = B_n(t_1 \dots t_k) \in \mathcal{T}(\mathbb{N}^*)$ . We put  $\phi_{PL}(t_i) = w_i$  for all  $1 \leq i \leq k$ . Then:*

$$\phi_{PL}(t) = \begin{cases} x_0 w_1 \sqcup \dots \sqcup x_0 w_k \sqcup x_1^{n-1-k} & \text{if } k < n, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** As  $\mathfrak{g}_{\mathcal{PT}(\{1,2\})}$  is prelie, there exists a unique morphism of prelie algebras:

$$\psi : \begin{cases} \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} & \longrightarrow \mathfrak{g}_{\mathcal{PT}(\{1,2\})} \\ \cdot_n & \longrightarrow \frac{1}{(n-1)!} \cdot_2 \mathfrak{W}^{(n-1)}. \end{cases}$$

Then  $\phi_{APL} \circ \psi$  is a prelie algebra morphism sending  $\cdot_n$  to  $\frac{1}{(n-1)!}x_1^{\mathbb{W}(n-1)} = x_1^{n-1}$  for all  $n \geq 1$ , so  $\phi_{APL} \circ \psi = \phi_{PL}$ . We obtain, by lemma 19:

$$\begin{aligned} \psi(\cdot_n \bullet t_1 \dots t_k) &= \frac{1}{(n-1)!} \cdot_2^{\mathbb{W}(n-1)} \bullet (\psi(t_1) \dots \psi(t_k)) \\ &= \frac{1}{(n-1)!} \sum_{I_1 \sqcup \dots \sqcup I_{n-1} = \{1, \dots, k\}} \cdot_2 \bullet \left( \prod_{i \in I_1} t_i \right) \mathbb{W} \dots \mathbb{W} \cdot_2 \bullet \left( \prod_{i \in I_{n-1}} t_i \right) \end{aligned}$$

Let us apply  $\phi_{APL}$  to this expression. If  $|I_j| \geq 2$ , by theorem 27:

$$\phi_{APL} \left( \cdot_2 \bullet \left( \prod_{i \in I_j} t_i \right) \right) = 0.$$

Consequently, if  $k \geq n$ , at least one of the  $I_j$  contains two elements, so  $\phi_{APL} \circ \psi(t) = \phi_{PL}(t) = 0$ . Let us assume that  $k < n$ . Hence, using the commutativity of  $\mathbb{W}$ :

$$\begin{aligned} \phi_{PL}(\cdot_n \bullet t_1 \dots t_k) &= \frac{1}{(n-1)!} \sum_{I_1 \sqcup \dots \sqcup I_{n-1} = \{1, \dots, k\}, |I_j| \leq 1} x_1 \bullet \left( \prod_{i \in I_1} w_i \right) \mathbb{W} \dots \mathbb{W} x_1 \bullet \left( \prod_{i \in I_k} w_i \right) \\ &= \frac{1}{(n-1)!} \sum_{\nu: \{1, \dots, k\} \rightarrow \{1, \dots, n-1\}, \text{ injective}} x_1 \bullet w_1 \mathbb{W} \dots x_1 \bullet w_k \mathbb{W} x_1^{\mathbb{W}(n-1-k)} \\ &= \frac{1}{(n-1)!} \sum_{\nu: \{1, \dots, k\} \rightarrow \{1, \dots, n-1\}, \text{ injective}} x_0 w_1 \mathbb{W} \dots x_0 w_k \mathbb{W} x_1^{\mathbb{W}(n-1-k)} \\ &= \frac{(n-1) \dots (n-k)}{(n-1)!} x_0 w_1 \mathbb{W} \dots x_0 w_k \mathbb{W} x_1^{\mathbb{W}(n-1-k)} \\ &= \frac{(n-1) \dots (n-k)(n-1-k)!}{(n-1)!} x_0 w_1 \mathbb{W} \dots x_0 w_k \mathbb{W} x_1^{n-1-k} \\ &= x_0 w_1 \mathbb{W} \dots x_0 w_k \mathbb{W} x_1^{n-1-k}, \end{aligned}$$

which is the announced result.  $\square$

**Corollary 29** *Let  $s_1, \dots, s_k, t_1, \dots, t_l \in \mathcal{T}(\{N^*\})$ ,  $k, l \geq 0$ . For all  $i, j, n \geq 1$ :*

$$\begin{aligned} \phi_{PL}(B_{n+1}((B_i(s_1 \dots s_k)B_j(t_1 \dots t_l))) \\ = \phi_{PL}(B_n(B_{i+1}(s_1 \dots s_k)B_j(t_1 \dots t_l))) + \phi_{PL}(B_n(B_{j+1}(B_i(s_1 \dots s_k)t_1 \dots t_l))). \end{aligned}$$

**Proof.** We note:

$$\begin{aligned} T_1 &= B_{n+1}((B_i(s_1 \dots s_k)B_j(t_1 \dots t_l))) = \cdot_{n+1} \bullet ((\cdot_i \bullet s_1 \dots s_k)(\cdot_j \bullet t_1 \dots t_l)), \\ T_2 &= B_n(B_{i+1}(s_1 \dots s_k)B_j(t_1 \dots t_l)) = \cdot_n \bullet (\cdot_{i+1} \bullet (s_1 \dots s_k)(\cdot_j \bullet t_1 \dots t_l)), \\ T_3 &= B_n(B_{j+1}(B_i(s_1 \dots s_k)t_1 \dots t_l)) = \cdot_n \bullet (\cdot_{j+1} \bullet ((\cdot_i \bullet s_1 \dots s_k)t_1 \dots t_l)). \end{aligned}$$

If  $k \geq i$ , or  $l \geq j$ , or  $n = 1$ , all these elements are sent to zero by  $\phi_{PL}$  by proposition 28. Let us assume now that  $k < i$ ,  $l < j$ ,  $n < 1$ . We put  $v_i = \phi_{PL}(s_i)$  and  $w_i = \phi_{PL}(t_i)$ . Then:

$$\begin{aligned} \phi_{PL}(T_1) &= x_0 \underbrace{(x_0 v_1 \mathbb{W} \dots \mathbb{W} x_0 v_k \mathbb{W} x_1^{i-1-k})}_X \underbrace{\mathbb{W} x_0 (x_0 w_1 \mathbb{W} \dots \mathbb{W} x_0 w_l \mathbb{W} x_1^{j-1-l})}_Y \mathbb{W} x_1^{n-2} \\ &= x_0 X \mathbb{W} x_0 Y \mathbb{W} x_1^{n-2}, \\ \phi_{PL}(T_2) &= x_0 (x_0 v_1 \mathbb{W} \dots \mathbb{W} x_0 (x_0 w_1 \mathbb{W} \dots \mathbb{W} x_0 w_l \mathbb{W} x_1^{j-1-l})) \mathbb{W} x_1^{i-1-k} \mathbb{W} x_1^{n-2} \\ &= x_0 (X \mathbb{W} x_0 Y) \mathbb{W} x_1^{n-2}, \\ \phi_{PL}(T_3) &= x_0 (x_0 (x_0 v_1 \mathbb{W} \dots \mathbb{W} x_0 v_k \mathbb{W} x_1^{i-1-k})) \mathbb{W} x_0 w_1 \mathbb{W} x_0 w_l \mathbb{W} x_1^{j-1-l} \mathbb{W} x_1^{n-2} \\ &= x_0 (x_0 X \mathbb{W} Y) \mathbb{W} x_1^{n-2}. \end{aligned}$$

As  $x_0 X \mathbb{W} x_0 Y = x_0 (X \mathbb{W} x_0 Y) + x_0 (x_0 X \mathbb{W} Y)$ , we obtain the result.  $\square$

**Theorem 30** *The kernel of  $\phi_{PL}$  is the prelie ideal generated by:*

1.  $B_1(t_1 \dots t_k)$ , where  $k \geq 1$ ,  $t_1, \dots, t_k \in \mathcal{T}(\mathbb{N}^*)$ .
2.  $B_{n+1}(B_i(s_1 \dots s_k)B_j(t_1 \dots t_l)) - B_n(B_{i+1}(s_1 \dots s_k)B_j(t_1 \dots t_l)) - B_{j+1}(B_i(s_1 \dots s_k)t_1 \dots t_l)$ , where  $k, l \geq 0$ ,  $s_1, \dots, s_k, t_1, \dots, t_l \in \mathcal{T}(\mathbb{N}^*)$ .

**Proof.** Let  $I$  be the ideal generated by these elements. By proposition 28 and corollary 29,  $I \subseteq \text{Ker}(\phi_{PL})$ . We put  $h = \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}/I$ . Applying repeatedly the relation given by elements of the second form, it is not difficult to prove that for any  $t \in \mathcal{T}(\mathbb{N}^*)$ , there exists a linear span of ladders  $t'$  such that  $\bar{t} = \bar{t}'$  in  $h$ . Moreover, by the relation given by elements 1., if one of the vertices of a ladder  $t$  which is not the leaf is decorated by 1, then  $\bar{t} = 0$ . Let us denote by  $L(n)$  the set of ladders decorated by  $\mathbb{N}^*$ , of weight  $n$ , such that all the vertices which are not the leaf are decorated by integer  $S > 1$ . It turns out that  $h$  is generated by the elements  $\bar{t}$ ,  $t \in L = \bigcup L(n)$ .

Let  $\bar{\phi}_{PL}$  be the morphism from  $h$  to  $\mathbb{K}\langle x_0, x_1 \rangle$  induced by  $\phi_{PL}$ . By homogeneity, as  $\phi_{PL}$  is surjective, for all  $n \geq 1$ :

$$\bar{\phi}_{PL}(\text{Vect}(L(n))) = \text{Vect}(\text{words of degree } n).$$

In order to prove that  $I = \text{Ker}(\phi_{PL})$ , it is enough to prove that  $\bar{\phi}_{PL}$  is injective. By homogeneity, it is enough to prove that  $\bar{\phi}|_{\text{Vect}(L(n))}$  is injective for all  $n \geq 1$ . Hence, it is enough to prove that for all  $n \geq 1$ ,

$$|L(n)| = \dim(\text{Vect}(\text{words of degree } n)) = p_n,$$

where the  $p_n$  are the integers defined in proposition 8. Let  $l_n = |L(n)|$  and  $q_n$  be the number of  $t \in L(n)$  with no vertex decorated by 1. Then for all  $n \geq 2$ ,  $l_n = q_n + q_{n-1}$ , and  $l_1 = 1$ . We put:

$$L = \sum_{n=1}^{\infty} l_n X^n, \quad Q = \sum_{n=1}^{\infty} q_n X^n.$$

We obtain  $P = X + Q + XQ$ . Moreover:

$$Q = \frac{1}{1 - \sum_{i \geq 2} X^i} - 1 = \frac{1}{1 - \frac{X^2}{1-X}} - 1 = \frac{X^2}{1 - X - X^2},$$

Finally:

$$L = \frac{X}{1 - X - X^2} = F.$$

So, for all  $n \geq 1$ ,  $|L(n)| = p_n$ . □

As an immediate corollary, a basis of  $h$  is given by the classes of the elements of  $L$ . Turning to  $\mathbb{K}\langle x_0, x_1 \rangle$ , we obtain:

**Corollary 31** *Let  $w = a_1 \dots a_k$  be a word with letters in  $\mathbb{N}^*$ .*

1. *We put:*

$$m_w = x_1^{a_1-1} \bullet (x_1^{a_1-1} \bullet (\dots (x_1^{a_{k-1}-1} \bullet x_1^{a_k}) \dots)).$$

2. *We shall say that  $w$  is admissible if  $a_1, \dots, a_{k-1} > 1$ . The set of admissible words is denoted by  $\text{Adm}$ .*

*Then  $(m_w)_{w \in \text{Adm}}$  is a basis of  $\mathbb{K}\langle x_0, x_1 \rangle$ .*

**Remark.** If  $w$  is not admissible, that is to say if there exists  $1 \leq i < k$ , such that  $a_i = 1$ , then  $m_w = 0$  by proposition 28.

We extend the map  $w \rightarrow m_w$  by linearity.

## 4.2 Prelie product in the basis of admissible words

**Notations.**

1. For all  $k, l$ , we denote by  $Sh(k, l)$  the set of  $(k, l)$ -shuffles, that is to say permutations  $\zeta \in \mathfrak{S}_{k+l}$  such that  $\zeta(1) < \dots < \zeta(k)$ ,  $\zeta(k+1) < \dots < \zeta(k+l)$ .
2. For all  $k, l$  we denote by  $Sh_{\prec}(k, l)$  the set of  $(k, l)$ -shuffles  $\zeta$  such that  $\zeta^{-1}(k+l) = k$ .
3. For all  $k, l$  we denote by  $Sh_{\succ}(k, l)$  the set of  $(k, l)$ -shuffles  $\zeta$  such that  $\zeta^{-1}(k+l) = k+l$ .
4. The symmetric group  $\mathfrak{S}_n$  acts on the set of words with letters in  $\mathbb{N}^*$  of length  $n$  by permutation of the letters:

$$\sigma.(a_1 \dots a_n) = a_{\sigma^{-1}(1)} \dots a_{\sigma^{-1}(n)}.$$

**Proposition 32** *Let  $\mathbb{K}\langle\mathbb{N}^*\rangle$  be the space generated by words with letters in  $\mathbb{N}^*$ . We define a dendriform structure on this space by:*

$$\begin{aligned} (a_1 \dots a_k) \prec (b_1 \dots b_l) &= \sum_{\zeta \in Sh_{\prec}(k,l)} \zeta.a_1 \dots a_k b_1 \dots b_{k-1} (b_k + 1) \\ (a_1 \dots a_k) \succ (b_1 \dots b_l) &= \sum_{\zeta \in Sh_{\succ}(k,l)} \zeta.a_1 \dots a_{k-1} (a_k + 1) b_1 \dots b_k. \end{aligned}$$

The associative product  $\prec + \succ$  is denoted by  $\star$ .

**Proof.** We denote by  $Sh(k, l, m)$  the set of  $k+l+m$ -permutations such that  $\zeta(1) < \dots < \zeta(k)$ ,  $\zeta(k+1) < \dots < \zeta(k+l)$ ,  $\zeta(k+l+1) < \dots < \zeta(k+l+m)$ . Then:

$$\begin{aligned} (a_1 \dots a_k \prec b_1 \dots b_l) \prec c_1 \dots c_m &= a_1 \dots a_k \prec (b_1 \dots b_l \star c_1 \dots c_m) \\ &= \sum_{\zeta \in Sh(k,l,m), \zeta^{-1}(k+l+m)=k} \zeta.a_1 \dots a_k b_1 \dots (b_l + 1) c_1 \dots (c_m + 1); \\ (a_1 \dots a_k \succ b_1 \dots b_l) \prec c_1 \dots c_m &= a_1 \dots a_k \succ (b_1 \dots b_l \prec c_1 \dots c_m) \\ &= \sum_{\zeta \in Sh(k,l,m), \zeta^{-1}(k+l+m)=k+l} \zeta.a_1 \dots (a_k + 1) b_1 \dots b_l c_1 \dots (c_m + 1); \\ (a_1 \dots a_k \star b_1 \dots b_l) \succ c_1 \dots c_m &= a_1 \dots a_k \succ (b_1 \dots b_l \succ c_1 \dots c_m) \\ &= \sum_{\zeta \in Sh(k,l,m), \zeta^{-1}(k+l+m)=k+l+m} \zeta.a_1 \dots (a_k + 1) b_1 \dots (b_l + 1) c_1 \dots c_m. \end{aligned}$$

So  $\mathbb{K}\langle\langle\mathbb{N}^*\rangle\rangle$  is a dendriform algebra. □

We postpone the study of this dendriform algebra to section 5.2.

**Notations.** For all  $a_1, \dots, a_k \in \mathbb{N}^*$ , we denote by  $l(a_1 \dots a_k) = B_{a_1} \circ \dots \circ B_{a_k}(1)$  the ladder decorated from the root to the leaf by  $a_1, \dots, a_k$ . Note that  $m_{a_1 \dots a_k} = \phi_{PL}(l(a_1 \dots a_k))$ .

**Lemma 33** *Let  $k, l \geq 1$  and let  $a_1, \dots, a_l, b_1, \dots, b_l \in \mathbb{N}^*$ . Then:*

$$\phi_{PL}(B_{a_1+1}(l(a_2 \dots a_k)l(b_1 \dots b_l))) + B_{b_1+1}(l(a_1 \dots a_k)l(b_2 \dots b_l)) = m_{a_1 \dots a_k \star b_1 \dots b_l}.$$

**Proof.** By induction on  $k+l$ . If  $k=l=1$ , then:

$$\phi_{PL}(\mathbf{1}_{a_1+1}^{b_1} + \mathbf{1}_{b_1+1}^{a_1}) = m_{(a_1+1)b_1+(b_1+1)a_1} = m_{a_1 \star b_1}.$$

Let us assume the result at all ranks  $< k + l$ . If  $k = 1$ , then:

$$\begin{aligned}
& \phi_{PL}(B_{a_1+1}(l(b_2 \dots b_l)) + B_{b_1+1}(l(a_1)l(b_2 \dots b_l))) \\
&= \phi_{PL}(\bullet_{a_1+1} \bullet l(b_2 \dots b_l) + \bullet_{b_1+1} \bullet (l(a_1)l(b_2 \dots b_l))) \\
&= \phi_{PL}(l((a_1 + 1)b_2 \dots b_l)) + \phi_{PL}(\bullet_{b_1} \bullet (l((a_1 + 1)b_2 \dots b_l) + \bullet_{b_2+1} \bullet (l(a_1)l(b_3 \dots b_l)))) \\
&= m_{(a_1+1)b_2 \dots b_l} + m_{b_1(a_1 \star b_2 \dots b_l)} \\
&= m_{(a_1+1)b_2 \dots b_l} + \sum_{i=1}^{l-1} m_{b_1 \dots b_i(a_1+1) \dots b_l} + m_{b_1 \dots (b_l+1)a_1} \\
&= m_{a_1 \star b_1 \dots b_l}.
\end{aligned}$$

If  $l = 1$ , a similar computation, permuting the  $a_i$ 's and the  $b_j$ 's, proves the result. If  $k, l > 1$ , then:

$$\begin{aligned}
& \phi_{PL}(B_{a_1+1}(l(a_2 \dots a_k)l(b_1 \dots b_l)) + B_{b_1+1}(l(a_1 \dots a_k)l(b_2 \dots b_l))) \\
&= \phi_{PL}(\bullet_{a_1} \bullet (\bullet_{a_2+1} \bullet l(a_3 \dots a_k)l(b_1 \dots b_l)) + \bullet_{b_1+1} \bullet l(a_1 \dots a_k)l(b_2 \dots b_l)) \\
&+ \phi_{PL}(\bullet_{b_1} \bullet (\bullet_{a_1+1} \bullet l(a_2 \dots a_k)l(b_2 \dots b_l)) + \bullet_{b_2+1} \bullet l(a_1 \dots a_k)l(b_3 \dots b_l)) \\
&= m_{a_1(a_2 \dots a_k \star b_1 \dots b_l) + b_1(a_1 \dots a_k \star b_2 \dots b_l)} \\
&= m_{a_1 \dots a_k \star b_1 \dots b_l}.
\end{aligned}$$

Hence, the result holds for all  $k, l \geq 1$ . □

**Theorem 34** For all  $a_1, \dots, a_k, b_1, \dots, b_l \in \mathbb{N}^*$ :

$$m_{a_1 \dots a_k} \bullet m_{b_1 \dots b_l} = \sum_{i=1}^{k-1} m_{a_1 \dots a_{i-1}(a_i-1)(a_{i+1} \dots a_k \star b_1 \dots b_l)} + m_{a_1 \dots a_k b_1 \dots b_l}.$$

**Proof.** By definition of  $m_{a_1 b_1 \dots b_l}$ , if  $k = 1$ ,  $m_{a_1} \bullet m_{b_1 \dots b_l} = m_{a_1 b_1 \dots b_l}$ . So the result holds if  $k = 1$ . Let us assume that  $k \geq 2$ . In  $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ , we have:

$$l(a_1 \dots a_k) \bullet l(b_1 \dots b_l) = \bullet_{a_1} \bullet (l(a_2 \dots a_k) \bullet l(b_1 \dots b_l)) + \bullet_{a_1} \bullet l(a_2 \dots a_k)l(b_1 \dots b_l).$$

Applying  $\phi_{PL}$ :

$$\begin{aligned}
m_{a_1 \dots a_k} \bullet m_{b_1 \dots b_l} &= m_{a_1(a_2 \dots a_k) \bullet (b_1 \dots b_l)} \\
&+ \phi_{PL}(\bullet_{a_1-1} \bullet (\bullet_{a_2+1} l(a_3 \dots a_k)l(b_1 \dots b_l)) + \bullet_{b_1+1} \bullet l(a_1 \dots a_k)l(b_2 \dots b_l)) \\
&= m_{a_1(a_2 \dots a_k) \bullet (b_1 \dots b_l)} + m_{(a_1-1)(a_2 \dots a_k \star b_1 \dots b_l)},
\end{aligned}$$

by the preceding lemma. The result follows from an easy induction. □

**Remark.** In particular,  $m_1 \circ m_{b_1 \dots b_l} = 0$ .

**Corollary 35** Let  $a_1 \dots a_k, b_1 \dots b_l$  be two words with letters in  $\mathbb{N}^*$ . Then  $m_{a_1 \dots a_k} \bullet m_{b_1 \dots b_l}$  is a span of  $m_w$ , where  $w$  is a word with  $k + l$  letters and of weight  $a_1 + \dots + a_k + b_1 + \dots + b_l$ .

Hence,  $\mathbb{K}\langle x_0, x_1 \rangle$  is a bigraded prelie algebra, with:

$$\mathbb{K}\langle x_0, x_1 \rangle_{n,k} = Vect(m_{a_1 \dots a_k} \mid a_1 + \dots + a_k = n).$$

We put:

$$G = \sum_{k,n \geq 0} dim(\mathbb{K}\langle x_0, x_1 \rangle_{n,k}) X^n Y^k.$$



**Proposition 36**  $G = \frac{XY}{1-X-X^2Y} = \sum_{k=1}^{\infty} \sum_{l=2k-1}^{\infty} \binom{l-k}{k-1} X^l Y^k.$

**Proof.** Note that  $\dim(\mathbb{K}\langle x_0, x_1 \rangle_{n,k})$  is the number of words  $a_1 \dots a_k$  of length  $k$ , such that  $a_1, \dots, a_{k-1} \geq 2$ , and  $a_1 + \dots + a_k = n$ . Hence:

$$G = \sum_{k=1}^{\infty} \left( \frac{X^2Y}{1-X} \right)^{k-1} \frac{XY}{1-X} = \frac{XY}{1-X} \frac{1}{1 - \frac{X^2Y}{1-X}} = \frac{XY}{1-X-X^2Y}.$$

An easy developement in formal series gives the second formula. □

### 4.3 An associative product on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$

We now define an associative product on  $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ , in such a way that  $\phi_{PL}$  becomes a morphism of Com-Prelie algebras.

**Proposition 37** *We define a product  $\sqcup$  on  $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$  by:*

$$B_p(s_1 \dots s_k) \sqcup B_q(t_1 \dots t_l) = \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l).$$

Then  $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$  is a Com-Prelie algebra and  $\phi_{PL}$  is a morphism of Com-Prelie algebras.

**Proof.** As  $\binom{p+q-k-l-2}{p-k-1} = \binom{p+q-k-l-2}{q-l-1}$ ,  $\sqcup$  is commutative. Let  $t = B_p(s_1 \dots s_k)$ ,  $t' = B_q(\bullet t_1 \dots t_l)$  and  $t'' = B_r(u_1 \dots u_m)$ . Then:

$$\begin{aligned} t \sqcup (t' \sqcup t'') &= \underbrace{\binom{q+r-l-m-2}{q-l-1} \binom{p+q+r-k-l-m-3}{q+r-l-m-2}}_A B_{p+q+r-2}(s_1 \dots s_k t_1 \dots t_l u_1 \dots u_m), \\ (t \sqcup t') \sqcup t'' &= \underbrace{\binom{p+q-k-l-2}{p-k-1} \binom{p+q+r-k-l-m-3}{p+q-k-l-2}}_B B_{p+q+r-2}(s_1 \dots s_k t_1 \dots t_l u_1 \dots u_m). \end{aligned}$$

If  $p \leq k$  or  $q \leq l$  or  $r \leq m$ , then  $A = B = 0$ . If  $p > k$  and  $q > l$  and  $r > m$ , then:

$$A = B = \frac{(p+q+r-k-l-m-3)!}{(p-k-1)!(q-l-1)!(r-m-1)!}.$$

So  $\sqcup$  is associative.

Let  $t_1 = B_p(s_1 \dots s_k)$ ,  $t_2 = B_q(t_1 \dots t_l)$  and  $t \in \mathcal{T}(\mathbb{N}^*)$ . Then:

$$\begin{aligned}
(t_1 \mathbb{W} t_2) \circ T &= \binom{p+q-k-l-2}{m-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t) \\
&+ \sum_{i=1}^k \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots (s_i \bullet t) \dots s_k t_1 \dots t_l) \\
&+ \sum_{j=1}^l \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots (t_j \bullet t) \dots t_l), \\
(t_1 \bullet t) \mathbb{W} t_2 &= \left( \sum_{i=1}^k B_p(s_1 \dots (s_i \bullet t) \dots s_k) + B_p(s_1 \dots s_k t) \right) \mathbb{W} t_2 \\
&= \sum_{i=1}^k \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots (s_i \bullet t) \dots s_k t_1 \dots t_l) \\
&+ \binom{p+q-k-l-3}{p-k-2} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t), \\
t_1 \mathbb{W} (t_2 \bullet t) &= t_1 \mathbb{W} \left( \sum_{j=1}^l B_q(t_1 \dots (t_j \bullet t) \dots t_l) + B_q(t_1 \dots t_l t) \right) \\
&= \sum_{j=1}^l \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots (t_j \bullet t) \dots t_l) \\
&+ \binom{p+q-k-l-3}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t).
\end{aligned}$$

As  $\binom{p+q-k-l-3}{p-k-2} + \binom{p+q-k-l-3}{p-k-1} = \binom{p+q-k-l-2}{p-k-1}$ , we obtain  $(t_1 \mathbb{W} t_2) \bullet t = (t_1 \bullet t) \mathbb{W} t_2 + t_1 \mathbb{W} (t_2 \bullet t)$ . So  $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$  is Com-Prelie.

Let  $t_1 = B_p(s_1 \dots s_k)$  and  $t_2 = B_q(t_1 \dots t_l)$ . If  $k \geq p$ , then  $\binom{p+q-k-l-2}{p-k-1} = 0$ , so  $t_1 \mathbb{W} t_2 = 0$ . By proposition 28,  $\phi_{PL}(t_1) = 0$ , so  $\phi_{PL}(t_1 \mathbb{W} t_2) = \phi_{PL}(t_1) \mathbb{W} \phi_{PL}(t_2) = 0$ . Similarly, if  $l \geq q$ ,  $\phi_{PL}(t_1 \mathbb{W} t_2) = \phi_{PL}(t_1) \mathbb{W} \phi_{PL}(t_2) = 0$ . If  $k < p$  and  $l < q$ , we put  $w_i = \phi_{PL}(s_i)$  and  $w'_j = \phi_{PL}(t_j)$ . Then:

$$\begin{aligned}
\phi_{PL}(t_1) \mathbb{W} \phi_{PL}(t_2) &= x_0 w_1 \mathbb{W} \dots \mathbb{W} x_0 w_k \mathbb{W} x_1^{p-1-k} \mathbb{W} x_0 w'_1 \mathbb{W} \dots \mathbb{W} x_0 w'_l \mathbb{W} x_1^{q-1-l} \\
&= \binom{p+q-k-l-2}{p-k-1} x_0 w_1 \mathbb{W} \dots \mathbb{W} x_0 w'_l \mathbb{W} x_1^{p+q-k-l-2} \\
&= \binom{p+q-k-l-2}{p-k-1} \phi_{PL}(B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l)) \\
&= \phi_{PL}(t_1 \mathbb{W} t_2).
\end{aligned}$$

So  $\phi_{PL}$  is a Com-Prelie algebra morphism. □

**Remark.** By the proof of proposition 28, we have a commutative diagram of prelie algebra morphisms:

$$\begin{array}{ccc}
\mathfrak{g}_{\mathcal{T}(\{1,2\})} & \xrightarrow{\phi_{CPL}} & \mathbb{K}\langle x_0, x_1 \rangle \\
\psi \uparrow & \nearrow \phi_{PL} & \\
\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} & & 
\end{array}$$

Moreover,  $\phi_{CPL}$  is a morphism of Com-Prelie algebra. With the commutative, associative product previously defined on  $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ ,  $\phi_{PL}$  is now a morphism of Com-Prelie algebra. However,  $\psi$  is not compatible with  $\sqcup$ . Indeed,  $\psi(\mathbf{1}_2^1) = \psi(\cdot_2) \bullet \psi(\cdot_1) = \mathbf{1}_2^1$ , so:

$$\psi(\mathbf{1}_2^1) \sqcup \psi(\mathbf{1}_2^1) = \mathbf{1}_2^1 \sqcup \mathbf{1}_2^1 = \mathbf{1}_2^1 \mathbf{1}_2^1.$$

Moreover,  $\mathbf{1}_2^1 \sqcup \mathbf{1}_2^1 = \mathbf{1}V_3^1$ , so:

$$\psi(\mathbf{1}_2^1 \sqcup \mathbf{1}_2^1) = \psi(\cdot_3) \bullet \psi(\cdot_1) \psi(\cdot_1) = \frac{1}{2} \mathbf{2} \bullet \mathbf{1} \bullet \mathbf{1} = \mathbf{1}_2^1 \mathbf{1}_2^1 + \mathbf{1}_2^1 \mathbf{1}_2^1.$$

## 5 Appendix

### 5.1 Enumeration of partitioned trees

Let  $d \geq 1$ . For all  $n \geq 1$ , let  $f_n$  be the number of partitioned trees decorated by  $\{1, \dots, d\}$  with  $n$  vertices and let  $t_n$  be the number of partitioned trees decorated by  $\{1, \dots, d\}$  with  $n$  vertices and one root. By convention,  $f_0 = 1$ . We put:

$$T = \sum_{n=1}^{\infty} t_n X^n, \quad F = \sum_{n=0}^{\infty} f_n X^n.$$

Let  $V_T$  be the vector space generated by the set of partitioned trees decorated by  $\{1, \dots, d\}$  and  $V_F$  be the vector space generated by the set of partitioned trees decorated by  $\{1, \dots, d\}$  with only one root. There is a bijection:

$$\begin{cases} S(V_T) & \longrightarrow V_F \\ t_1 \dots t_k & \longrightarrow t_1 \sqcup \dots \sqcup t_k. \end{cases}$$

Hence:

$$F = \prod_{i=1}^{\infty} \frac{1}{(1 - X^i)^{t_i}}. \quad (2)$$

There is a bijection:

$$\begin{cases} \bigoplus_{i=1}^d S(V_F) & \longrightarrow V_T \\ (F_{1,1} \dots, F_{1,k_1}, \dots, F_{d,1} \dots F_{d,k_d}) & \longrightarrow \sum_{i=1}^d \cdot_i \bullet (F_{i,1} \dots F_{i,k_i}). \end{cases}$$

This gives:

$$T = dX \prod_{i=1}^{\infty} \frac{1}{(1 - X^i)^{f_{i-1}}}. \quad (3)$$

Formulas (2) and (3) allow to compute inductively  $f_k$  and  $t_k$  for all  $k \geq 1$ . This gives:

$$\begin{cases} f_1 & = d \\ f_2 & = \frac{d(3d+1)}{2} \\ f_3 & = \frac{d(19d^2+9d+2)}{6} \\ f_4 & = \frac{d(63d^2+34d^2+13d+2)}{8} \\ f_5 & = \frac{d(644d^4+400d^3+175d^2+35d+6)}{30} \end{cases}$$

Here are examples of  $f_n$  for  $d = 1$  or  $2$ :

$n$	1	2	3	4	5	6	7	8	9	10
$d = 1$	1	2	5	14	42	134	444	1518	5318	18989
$d = 2$	2	7	32	167	952	5759	36340	236498	1576156	10702333

The row  $d = 1$  is sequence A035052 of [16].

## 5.2 Study of the dendriform structure on admissible words

We here study the dendriform algebra  $\mathbb{K}\langle\mathbb{N}^*\rangle$  of proposition 32. It is clearly commutative, via the bijection from  $Sh_{\prec}(k, l)$  to  $Sh_{\succ}(l, k)$  given by the composition (on the left) by the permutation  $(l + 1 \dots l + k \ 1 \dots l)$ : in other terms, it is a Zinbiel algebra [8].

Let  $V$  be a vector space. The shuffle dendriform algebra  $Sh(V)$  is  $T_+(V)$ , with the products given by:

$$\begin{aligned} (a_1 \dots a_k) \prec (b_1 \dots b_l) &= \sum_{\zeta \in Sh_{\prec}(k, l)} \zeta.a_1 \dots a_k b_1 \dots b_{k-1} b_k \\ (a_1 \dots a_k) \succ (b_1 \dots b_l) &= \sum_{\zeta \in Sh_{\succ}(k, l)} \zeta.a_1 \dots a_{k-1} a_k b_1 \dots b_k. \end{aligned}$$

Moreover, this is the free commutative dendriform algebra generated by  $V$ , that is to say if  $A$  is a commutative dendriform algebra and  $f : V \rightarrow A$  is any linear map, there exists a morphism of dendriform algebras  $\phi : Sh(V) \rightarrow A$  such that  $\phi|_V = f$ . As  $a_1 \dots a_k \succ b = a_1 \dots a_k b$  in  $Sh(V)$  for all  $a_1, \dots, a_k, b \in V$ , this morphism  $\phi$  is defined by:

$$\phi(a_1 \dots a_k) = (\dots (a_1 \succ a_2) \succ a_3) \dots \succ a_k.$$

**Proposition 38** 1. *Let  $V$  be the space generated by the words  $1^k i$ ,  $k \in \mathbb{N}$ ,  $i \geq 1$ . Then  $K\langle\mathbb{N}^*\rangle$  is isomorphic, as a dendriform algebra, to  $Sh(V)$ .*

2. *Let  $A$  be the subspace of  $K\langle\mathbb{N}^*\rangle$  generated by admissible words. Then it is a dendriform subalgebra of  $K\langle\mathbb{N}^*\rangle$ . Moreover, if  $W$  is the space generated by the letters  $i$ ,  $i \geq 1$ , then  $A$  is isomorphic, as a dendriform algebra, to  $Sh(W)$ .*

**Proof.** Let  $w = a_1 \dots a_k$  be a word with letters in  $\mathbb{N}^*$ . We denote by  $o(w)$  the sequence of indices  $j \in \{1, \dots, k-1\}$  such that  $a_j \neq 1$ . This sequences are totally ordered in this way:  $(j_1, \dots, j_k) < (j'_1, \dots, j'_l)$  if there exists a  $p$  such that  $j_k = j'_l$ ,  $j_{k-1} = j'_{l-1}$ ,  $\dots$ ,  $j_{k-p+1} = j'_{l-p+1}$ ,  $j_{k-p} < j'_{l-p}$ , with the convention  $j_0 = j_{-1} = \dots = j'_0 = j'_{-1} = \dots = 0$ .

Let  $\phi : Sh(V) \rightarrow K\langle\mathbb{N}^*\rangle$  be the unique morphism of dendriform algebras which extends the identity of  $V$ . Then:

$$\begin{aligned} \phi((1^{k_1-1} a_1) \dots (1^{k_n-1} a_n)) &= 1^{k_1-1} (a_1 + 1) \dots 1^{k_n-1} (a_{n-1} + 1) 1^{k_n-1} a_n \\ &\quad + \text{words } w' \text{ such that } o(w') > (k_1, \dots, k_{n-1}). \end{aligned}$$

By triangularity,  $\phi$  is an isomorphism. Moreover, for all  $a_1, \dots, a_n \geq 1$ :

$$\phi(a_1 \dots a_n) = (a_1 + 1) \dots (a_{n-1} + 1) a_n.$$

Consequently,  $\phi(Sh(W)) = A$ , so  $A$  is a dendriform subalgebra of  $K\langle\mathbb{N}^*\rangle$  and is isomorphic to  $Sh(W)$ .  $\square$

### 5.3 Freeness of the pre-Lie algebra $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$

**Notations.** Let  $k \geq 1$ ,  $d_1, \dots, d_k \in \mathcal{D}$  and let  $F_1, \dots, F_k$  be decorated partitioned forests. We put:

$$B_{d_1, \dots, d_k}(F_1, \dots, F_k) = (\bullet_{d_1} \bullet F_1) \sqcup \dots \sqcup (\bullet_{d_k} \bullet F_k).$$

Note that any partitioned tree can be written under the form  $B_{d_1, \dots, d_k}(F_1, \dots, F_k)$ . This writing is unique up to a common permutation of the  $d_i$ 's and the  $F_i$ 's.

**Proposition 39** *We define a coproduct  $\delta$  on  $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$  in the following way: for any decorated partitioned tree  $t = B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{k,1} \dots t_{k,n_k})$ ,*

$$\delta(t) = \frac{1}{k} \sum_{i=1}^k \sum_{j=1}^{n_i} B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,j-1} t_{i,j+1} \dots t_{i,n_i}, \dots, t_{k,1} \dots t_{k,n_k}) \otimes t_{i,j}.$$

1. For all  $x \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ ,  $(\delta \otimes Id) \circ \delta(x) = (23)(\delta \otimes Id) \circ \delta(x)$ .
2. For all  $x, y \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ ,  $\delta(x \bullet y) = x \otimes y + \delta(x) \bullet y$ .

**Proof.** 1. Let  $t = B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{k,1} \dots t_{k,n_k})$ . For all  $i, j$ , we put:

$$t/t_{i,j} = B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,j-1} t_{i,j+1} \dots t_{i,n_i}, \dots, t_{k,1} \dots t_{k,n_k}).$$

Then:

$$\delta(t) = \frac{1}{k} \sum_{i,j} t/t_{i,j} \otimes t_{i,j}.$$

Hence:

$$(\delta \otimes Id) \circ \delta(t) = \sum_{(i,j) \neq (i',j')} (t/t_{i,j})/t_{i',j'} \otimes t_{i',j'} \otimes t_{i,j}$$

As  $(t/t_{i,j})/t_{i',j'}$  and  $(t/t_{i',j'})/t_{i,j}$  are both the partitioned tree obtained by cutting  $t_{i,j}$  and  $t_{i',j'}$  in  $t$ , they are equal, so  $(\delta \otimes Id) \circ \delta(t)$  is invariant under the action of (23).

2. Let  $t'$  be a decorated partitioned tree.

$$\begin{aligned} \delta(t \bullet t') &= \sum_{i=1}^k \delta(B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,n_i} t', \dots, t_{k,1} \dots t_{k,n_k})) \\ &\quad + \sum_{i,j} \delta(B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,j} \bullet t' \dots t_{i,n_i}, \dots, t_{k,1} \dots t_{k,n_k})) \\ &= \frac{1}{k} k t \otimes t' + \frac{1}{k} \sum_i \sum_{i',j'} B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,n_i} t', \dots, t_{k,1} \dots t_{k,n_k})/t_{i',j'} \otimes t_{i',j'} \\ &\quad + \frac{1}{k} \sum_{(i,j) \neq (i',j')} B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,j} \bullet t' \dots t_{i,n_i}, \dots, t_{k,1} \dots t_{k,n_k})/t_{i',j'} \otimes t_{i',j'} \\ &\quad + \frac{1}{k} \sum_{i,j} t/t_{i,j} \otimes t_{i,j} \bullet t' \\ &= t \otimes t' + \sum t^{(1)} \otimes t^{(2)} \bullet t' + \sum t^{(1)} \otimes t^{(2)} \bullet t'. \end{aligned}$$

So  $\delta(t \bullet t') = t \otimes t' + \delta(t) \bullet t'$ . □

By Livernet's pre-Lie rigidity theorem [7]:

**Corollary 40** *The pre-Lie algebra  $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$  is freely generated by  $\text{Ker}(\delta)$ .*

## Remarks.

1. It is not difficult to prove that for any  $x, y \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ :

$$\delta(x \sqcup y) = \sum x^{(1)} \otimes x^{(2)} \sqcup y + \sum y^{(1)} \otimes x \sqcup y^{(2)}.$$

Hence,  $Ker(\delta)$  is an algebra for the product  $\sqcup$ .

2. Here are elements of  $Ker(\delta)$  in the non decorated case. Let  $t_1, t_2, t_3, t_4$  be partitioned trees.

$$\begin{aligned} X &= B(t_1 t_2, 1) - B(t_1, t_2), \\ Y &= B(t_1 t_2 t_3, 1, 1) - B(t_1 t_2, t_3, 1) - B(t_1 t_3, t_2, 1) - B(t_2 t_3, t_1, 1) + 2B(t_1, t_2, t_3), \\ Z &= B(t_1 t_2 t_3 t_4, 1) - B(t_1 t_2 t_3, t_4) - B(t_1 t_2 t_4, t_3) - B(t_1 t_3 t_4, t_2) - B(t_2 t_3 t_4, t_1) \\ &\quad + B(t_1 t_2, t_3 t_4) + B(t_1 t_3, t_2 t_4) + b(t_1 t_4, t_2 t_3), \\ T &= B(t_1 t_2, t_3 t_4, 1, 1) + B(t_1 t_3, t_2 t_4, 1, 1) + B(t_1 t_4, t_2 t_3, 1, 1) - B(t_1 t_2, t_3, t_4, 1) \\ &\quad - B(t_1 t_3, t_2, t_4, 1) - B(t_1 t_4, t_2, t_3, 1) - B(t_2 t_3, t_1, t_4, 1) - B(t_2 t_4, t_1, t_3, 1) \\ &\quad - B(t_3 t_4, t_1, t_2, 1) + 3B(t_1, t_2, t_3, t_4). \end{aligned}$$

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