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Multivariable Tangent and Secant q-derivative Polynomials

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Abstract. The derivative polynomials introduced by Knuth and Buckholtz in their calculations of the tangent and secant numbers are extended to a multivariable q–environment. The n-th q-derivatives of the classical q-tangent and q-secant are each given two polynomial expressions. The first polynomial expression is indexed by triples of integers, the second by compositions of integers. The functional relation between those two classes is fully given by means of combinatorial techniques. Moreover, those polynomials are proved to be generating functions for so-called t-permutations by multivariable statistics. By giving special values to those polynomials we recover classical q-polynomials such as the Carlitz q-Eulerian polynomials and the (t, q)-tangent and -secant analogs recently introduced. They also provide q-analogs for the Springer numbers. Finally, the t-compositions used in this paper furnish a combinatorial interpretation to one of the Fibonacci triangles.

Summary

Key words and phrases. q-derivative polynomials, q-secant numbers, q-tangent numbers, t-permutations, t-compositions, s-compositions, tq-secant numbers, tq-tangent numbers, q-Springer numbers, alternating permutations, inversion number, inverse major index.

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1. Introduction

Back in 1967, Knuth and Buckholtz [KB67] devised a clever method for computing the tangent and secant numbers T_{2n+1} and E_{2n} for large values of the subscripts n. For that purpose they introduced two sequences of polynomials, referred to as *derivative polynomials*. A few years later, Hoffman [Ho95, Ho99] calculated the exponential generating functions for those polynomials, and found a combinatorial interpretation for their coefficients, in terms of so-called *snakes*. The goal of this paper is to obtain a *multivariable q-analog* of all those results. We first recall the contributions made by those authors, then, introduce the q-environment that makes it possible to derive a handy algebra for these new q-derivative polynomials.

1.1. *The derivative polynomials.* Recall that *tangent* and *secant numbers* T_{2n+1} and E_{2n} occur as coefficients in the Taylor expansions of $\tan u$ and $\sec u$:

$$\tan u = \sum_{n\ge 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1} \tag{1.1}$$
$$= \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \cdots$$

$$\sec u = \frac{1}{\cos u} = \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n} \tag{1.2}$$
$$= 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots$$

See, e.g., [Ni23, p. 177-178], [Co74, p. 258-259].

Let $(A_n(x))$ $(n \ge 1)$ be the sequence of polynomials defined by

$$A_0(x) = x, \quad A_{n+1} = (1+x^2)\, DA_n(x),$$

with D being the differential operator. When writing $A_n(x) = \sum_{m\ge 0} a(n,m)x^m$, the coefficients $(a(n,m))$ satisfy the recurrence

$$a(0,m) = \delta_{1,m}, \quad a(n+1,m) = (m-1)a(n,m-1) + (m+1)a(n,m+1). \tag{1.3}$$

The $a(n,m)$'s form a triangle of integral numbers (see Table 1 in Section 11, where the numbers $a(n,m)$ are reproduced in boldface), now registered as the sequence A101343 in Sloane's Encyclopedia of Integer Sequences [Sl06] with an abundant bibliography. Knuth and Buckholtz [KB67] showed that the n-th derivative $D^n \tan u$ was equal to the polynomial

$$D^n \tan u = \sum_{m\ge 0} a(n,m) \tan^m u. \tag{1.4}$$

The same two authors also introduced the sequence $(b(n,m))$ by

$$b_{0,m} = \delta_{0,m}, \quad b(n+1,m) = mb(n,m-1) + (m+1)b(n,m+1).$$

Again, Knuth and Buckholtz [KB67] showed that the n-th derivative of $\sec u$ could be expressed as

$$D^n \sec u := \sum_{m\ge 0} b(n,m) \tan^m u \sec u. \tag{1.5}$$

The triangle of numbers $(b(n,m))$ also appears in Sloane's Enclycopedia [Sl06] with reference A008294. The first values of the numbers $b(n,m)$ are reproduced in Table 1 in plain type (not bold). From their very definitions the $a(n,m)$'s and $b(n,m)$'s can be imbricated in the same table, as done in Table 1. The meanings of the entries to the right of the table will be further explained.

The exponential generating functions for the polynomials

$$A_n(x) := \sum_{m\ge 0} a(n,m)x^m \quad \text{and} \quad B_n(x) := \sum_{m\ge 0} b(n,m)x^m,$$

called *derivative polynomials*, have been derived by Hoffman [Ho95] in the form

$$\sum_{n\ge 0} A_n(x)\frac{u^n}{n!} = \frac{x+\tan u}{1-x\,\tan u}; \tag{1.6}$$

$$\sum_{n\ge 0} B_n(x)\frac{u^n}{n!} = \frac{1}{\cos u - x\sin u}. \tag{1.7}$$

Those two exponential generating functions and recurrences for the $a(n,m)$'s and $b(n,m)$'s have also been obtained by other people in different contexts, in particular, by Carlitz and Scoville [CS72], Françon [Fr78].

By plugging $x=1$ in (1.6) the right-hand side becomes $\tan 2u + \sec 2u$, so that the sum $\sum_m a(n,m)$ is equal to $2^n E_n$, if n is even, and to $2^n T_n$ if n is odd. Likewise, (1.7) yields $\sum_n B_n(1)u^n/n! = 1/(\cos u - \sin u)$, which is the exponential generating function for the so-called *Springer numbers* (1, 1, 3, 11, 57, 361, 2763, ...) (see [Sp71], [Du95]) originally considered by Glaisher [Gl98, Gl99, Gl14], as noted in Sloane's Encyclopedia, under reference A001586. In Table 1 we have indicated the values of the row sums $\sum_m a(n,m)$ and $\sum_m b(n,m)$ to the right.

Finally, the combinatorial interpretations of the $a(n,m)$'s and $b(n,m)$'s are due to Hoffman in a later paper [Ho99]. An equivalent interpretation is also due to Josuat-Vergès [Jo11]. Both authors use the word *snake* of length n, a notion made popular by Arnold [Ar92, Ar92a] in the study of morsification of singularities, to designate each word $w = x_1x_2\cdots x_n$, whose letters are integers, positive or negative, with the further property that $x_1 > x_2$, $x_2 < x_3$, $x_3 > x_4$, ... in an alternating way and $|x_1|\,|x_2|\,\cdots\,|x_n|$ is a permutation of $1\,2\,\cdots\,n$. Note that Josuat-Vergès, Novelli and Thibon [JNT12] have recently developed an algebraic combinatorics of snakes from the Hopf algebra point of view. In [Ho99] and [Jo11] Hoffman and Josuat-Vergès show that $A_n(x)$ is the generating polynomial for the set of all snakes of length n by the number of sign changes; they also prove an analogous result for $B_n(x)$.

1.2. *Towards a multivariable q-analog.* In parallel with (1.4) and (1.5) the *q-derivative operator* D_q (see [GR90, p. 22]), as well as the *q-analogs* of tangent and secant (see [St76], [AG78], [AF80], [Fo81], [St97, p. 148-149], [St10]) are to be introduced. The first problem is to see whether the n-th q-derivatives of those q-analogs can be expressed as polynomials in those functions, and they can! But, contrary to formulas (1.4) and (1.5), those n-th q-derivatives have *several* polynomial forms. As will be seen, *two* such polynomial forms are derived in this paper for each q-analog of tangent and secant. The second problem is to work out an appropriate algebra for the polynomials involved that must appear as natural *multivariable* q-analogs of the entries $a(n,m)$ and $b(n,m)$.

Before stating the first results of this paper we recall a few basic notions on q-Calculus. The *t-ascending factorial* in a variable t is traditionally defined by

$$(t;q)_n := \begin{cases} 1, & \text{if } n=0; \\ (1-t)(1-tq)\cdots(1-tq^{n-1}), & \text{if } n\ge 1; \end{cases}$$

in its finite version and

$$(t;q)_\infty := \lim_n (t;q)_n = \prod_{n\ge 0}(1-tq^n);$$

in its infinite version. By q-series it is meant a series of the form $f(u) = \sum_{n\ge 0} f(n;q)u^n/(q;q)_n$, whose coefficients $f(n;q)$ belong to some ring [AAR00, chap. 10]. The *q-derivative operator* for fixed q used below is defined by

$$D_q f(u) := \frac{f(u)-f(qu)}{u},$$

instead of the traditional $(f(u)-f(qu))/(u(1-q))$.

At this stage we just have to note (see Section 2) that there is *only one q-tangent* attached to the classical Jackson definitions [Ja04] of the q-sine and q-cosine, namely,

$$\tan_q(u) := \frac{\sum\limits_{n\ge 0}(-1)^n u^{2n+1}/(q;q)_{2n+1}}{\sum\limits_{n\ge 0}(-1)^n u^{2n}/(q;q)_{2n}},$$

but, as it has been rarely noticed, there are *two q-secants*

$$\sec_q(u) := \frac{1}{\sum\limits_{n\ge 0}(-1)^n u^{2n}/(q;q)_{2n}};$$

$$\mathrm{Sec}_q(u) := \frac{1}{\sum\limits_{n\ge 0}(-1)^n q^{n(2n-1)} u^{2n}/(q;q)_{2n}}.$$

1.3. *The numerical and combinatorial background.* Following Andrews [An76, chap. 4] a *composition* of a positive integer n is defined to be a sequence $\mathbf{c} = (c_0, c_1, \ldots, c_m)$ of *nonnegative* integers such that $c_0 + c_1 + \cdots + c_m = n$, with the restriction that only c_0 and c_m can be zero; the c_i's are the *parts* of the composition and $m+1$ is the *number of parts*, denoted by $\mu\,\mathbf{c}+1$. A composition $\mathbf{c} = (c_0, c_1, \ldots, c_m)$ of n is said to be a *t-composition*, if the following two conditions hold:

(1) either $m = 0$, so that $\mathbf{c} = (c_0)$ and $c_0 = n$ is an *odd* integer, or $m \geq 1$ and both c_0, c_m are *even*;

(2) if $m \geq 2$, then all the parts $c_1, c_2, \ldots, c_{m-1}$ are *odd*.

For each $n \geq 1$ the set of all t-compositions of n is denoted by Θ_n. It is further assumed that Θ_0 consists of a unique empty composition denoted by $(0, 0)$.

The first t-compositions are the following:

Θ_0: $(0,0)$;
Θ_1: (1), $(0,1,0)$;
Θ_2: $(0,2)$, $(2,0)$, $(0,1,1,0)$;
Θ_3: (3), $(0,1,2)$, $(2,1,0)$, $(0,3,0)$, $(0,1,1,1,0)$;
Θ_4: $(0,4)$, $(2,2)$, $(4,0)$, $(0,3,1,0)$, $(0,1,3,0)$, $(0,1,1,2)$, $(2,1,1,0)$, $(0,1,1,1,1,0)$.

A t-composition $\mathbf{c} = (c_0, c_1, \ldots, c_{m-1}, c_m)$ from Θ_n such that $c_m = 0$ is called an *s-composition*. We write $\mathbf{c}^- = (c_0, c_1, \ldots, c_{m-1})$. The subset of Θ_n of all s-compositions is denoted by Θ_n^-.

For each composition $\mathbf{c} = (c_0, c_1, \ldots, c_m)$ of $n \geq 1$ let $\tan_q(q^{\mathbf{c}}u) := 1$ if $m = 0$ and for $m \geq 1$, whatever $n \geq 1$, let

$$\tan_q(q^{\mathbf{c}}u) := \tan_q(q^{c_0}u)\tan_q(q^{c_0+c_1}u)\cdots\tan_q(q^{c_0+c_1+\cdots+c_{m-1}}u).$$

Note that the rightmost part c_m does not occur in the previous expression. Also let $\rho\,\mathbf{c} := (c_m, \ldots, c_1, c_0)$ denote the mirror-image of $\mathbf{c}$.

A word $w = y_1 y_2 \cdots y_m$, whose letters are positive integers, is said to be *falling alternating*, or simply *alternating* (resp. *rising alternating*), if $y_1 > y_2$, $y_2 < y_3$, $y_3 > y_4$, … (resp. if $y_1 < y_2$, $y_2 > y_3$, $y_3 < y_4$, …) in an alternating manner. The notion goes back to Désiré André [An79, An81], who showed that the number of falling (resp. rising) alternating permutations $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $1\,2\ldots n$ is equal to T_n when n is odd, and to E_n when n is even. The length of each word w is denoted by λw and the *empty word* is the word of length 0, denoted by ϵ.

Definition. A *t-permutation* of order n is defined to be a nonempty sequence $w = (w_0, w_1, \ldots, w_m)$ of words having the properties:

(i) the juxtaposition product $w_0 w_1 \cdots w_m$ is a permutation of $12\cdots n$;

(ii) either $m = 0$ and w_0 is rising alternating of *odd* length, or $m \geq 1$ and then w_0 is rising alternating of *even* length and w_m is (falling) alternating of *even* length;

(iii) if $m \geq 2$, then all the components $w_1, w_2, \ldots, w_{m-1}$ are (falling) alternating of *odd* length.

For each $n \geq 0$ the set of all t-permutations of order n will be denoted by $\mathcal{T}_n$. The first t-permutations are the following:

$\mathcal{T}_0$: (ϵ, ϵ) ;
$\mathcal{T}_1$: (1); $(\epsilon, 1, \epsilon)$;
$\mathcal{T}_2$: $(\epsilon, 21)$, $(12, \epsilon)$; $(\epsilon, 2, 1, \epsilon)$, $(\epsilon, 1, 2, \epsilon)$;
$\mathcal{T}_3$: (132), (231);
$(\epsilon, 3, 21)$, $(\epsilon, 312, \epsilon)$, $(\epsilon, 213, \epsilon)$, $(12, 3, \epsilon)$, $(\epsilon, 1, 32)$, $(23, 1, \epsilon)$, $(13, 2, \epsilon)$, $(\epsilon, 2, 31)$;
$(\epsilon, 1, 2, 3, \epsilon)$, $(\epsilon, 1, 3, 2, \epsilon)$, $(\epsilon, 2, 1, 3, \epsilon)$, $(\epsilon, 2, 3, 1, \epsilon)$, $(\epsilon, 3, 1, 2, \epsilon)$, $(\epsilon, 3, 2, 1, \epsilon)$.

1.4. *The underlying statistics.* The number of factors in each t-permutation $w = (w_0, w_1, \ldots, w_m)$, minus one, is denoted by $\mu\, w = m$. If w is of order n, the sequence $(\lambda w_0, \lambda w_1, \ldots, \lambda w_m)$ is a t-composition of n, denoted by Λw. For each $n \geq 0$ the set of all t-permutations w from $\mathcal{T}_n$, such that $\mu\, w = m$, with $0 \leq m \leq n+1$ and $m \equiv n+1 \bmod 2$, will be denoted by $\mathcal{T}_{n,m}$.

Recall that the *ligne of route* of a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$, denoted by $\operatorname{Ligne}\sigma$, is defined to be the set of all i such that $1 \leq i \leq n-1$ and $\sigma(i) > \sigma(i+1)$; also, the *inverse ligne of route*, $\operatorname{Iligne}\sigma$, to be the set of all $\sigma(i)$ such that $\sigma(j) = \sigma(i)+1$ for some $j \leq i-1$. In an equivalent manner, $\operatorname{Iligne}\sigma = \operatorname{Ligne}\sigma^{-1}$. Next, let

$$\operatorname{ides}\sigma = \#\operatorname{Iligne}\sigma;$$
$$\operatorname{imaj}\sigma = \sum_i \sigma(i) \quad (\sigma(i) \in \operatorname{Iligne}\sigma);$$

and let $\operatorname{inv}\sigma$ be the *number of inversions* of σ, as being the number of pairs (i,j) such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$.

The *inverse ligne of route*, $\operatorname{Iligne} w$, of a t-permutation $w = (w_0, w_1, \ldots, w_m)$ is then defined by

$$\operatorname{Iligne} w := \operatorname{Iligne}(w_0 w_1 \cdots w_m);$$

and the *number of inversions*, $\operatorname{inv} w$, by

$$\operatorname{inv} w := \operatorname{inv}(w_0 w_1 \cdots w_m).$$

This makes sense, as the latter juxtaposition product is a permutation, say, σ of $12\cdots n$ if $w \in \mathcal{T}_n$. Finally, let $\min w := a$ if 1 is a letter in w_a.

For example, with the t-permutation $w = (4\,5,\, 11\,1\,\mathbf{3},\, \mathbf{10}\,7\,\mathbf{9},\, \mathbf{6},\, \mathbf{8}\,\mathbf{2})$, the elements of $\operatorname{Iligne} w$ being reproduced in boldface, we have: $\operatorname{ides} w = 6$, $\operatorname{imaj} w = \mathbf{3} + \mathbf{10} + \mathbf{9} + \mathbf{6} + \mathbf{8} + \mathbf{2} = \mathbf{38}$, $\operatorname{inv} w = 27$ and $\min w = 1$.

1.5. *The main results.* For each triple (k, a, b) let $\mathcal{T}_{n,k,a,b}$ denote the set of all t-permutations w from $\mathcal{T}_n$ such that $\operatorname{ides} w = k$, $\min w = a$ and $a + b = \mu\, w$.

Theorem 1.1 (Multivariable q-analog of (1.4))**.** *Let*

$$A_{n,k,a,b}(q) = \sum_{w \in \mathcal{T}_{n,k,a,b}} q^{\operatorname{imaj} w} \tag{1.8}$$

be the generating polynomial for the set $\mathcal{T}_{n,k,a,b}$ by the statistic "imaj." *Then*

$$D_q^n \tan_q(u) = \sum_{k,a,b} A_{n,k,a,b}(q)\,(\tan_q(q^{k+1}u))^b\,(\tan_q(q^k u))^a; \tag{1.9}$$

where $0 \le k \le n-1$ and $0 \le a+b \le n+1$.

In the same manner, for each (k,a,b) let $\mathcal{T}^-_{n,k,a,b+1}$ denote the set of all t-permutations $w = (w_0, w_1, \ldots, w_m, w_{m+1})$ in $\mathcal{T}_{n,k,a,b+1}$ such that $w_{m+1} = \epsilon$.

Theorem 1.2 (Multivariable q-analog of (1.5))**.** *Let*

$$B_{n,k,a,b}(q) = \sum_{w \in \mathcal{T}^-_{n,k,a,b+1}} q^{\operatorname{imaj} w} \tag{1.10}$$

be the generating polynomial for the set $\mathcal{T}^-_{n,k,a,b+1}$ by "imaj." *Then*

$$D_q^n \sec_q(u) = \sum_{k,a,b} B_{n,k,a,b}(q)\big(\tan_q(q^{k+1}u)\big)^b \sec_q(q^{k+1}u)\,\big(\tan_q(q^k u)\big)^a; \tag{1.11}$$

$$D_q^n \operatorname{Sec}_q(u) = \sum_{k,a,b} q^{n(n-1)/2} B_{n,n-1-k,b,a}(q^{-1}) \times \big(\tan_q(q^{k+1}u)\big)^b \operatorname{Sec}_q(q^k u)\,\big(\tan_q(q^k u)\big)^a. \tag{1.12}$$

where $0 \le k \le n-1$ and $0 \le a+b \le n$.

Theorem 1.3 (Composition q-analogs of (1.4) and (1.5))**.** *For each $n \ge 1$ and each t-composition $\mathbf{c}$ of n let $A_{n,\mathbf{c}}(q)$ be the polynomial*

$$A_{n,\mathbf{c}}(q) = \sum_{w \in \mathcal{T}_n, \Lambda w = \mathbf{c}} q^{\operatorname{inv} w}. \tag{1.13}$$

Then,

$$D_q^n \tan_q(u) = \sum_{\mathbf{c} \in \Theta_n} A_{n,\mathbf{c}}(q)\,\tan_q(q^{\mathbf{c}} u); \tag{1.14}$$

$$D_q^n \sec_q(u) = \sum_{\mathbf{c} \in \Theta_n^-} A_{n,\mathbf{c}}(q)\,\tan_q(q^{\mathbf{c}^-} u)\,\sec_q(q^n u); \tag{1.15}$$

$$D_q^n \operatorname{Sec}_q(u) = \sum_{\mathbf{c} \in \Theta_n^-} q^{n(n-1)/2} A_{n,\mathbf{c}}(q^{-1})\,\tan_q(q^{\rho\mathbf{c}^-} u)\,\operatorname{Sec}_q(u). \tag{1.16}$$

Remark. The polynomials $A_{n,k,a,b}(q)$ are not uniquely defined by identity (1.9). On the contrary, both (1.11) and (1.12) uniquely define the polynomials $B_{n,k,a,b}(q)$ and $A_{n,\mathbf{c}}(q)$.

Polynomials indexed by triples (k,a,b) and those by compositions $\mathbf{c}$ are related to each other, as indicated in the next theorem.

Theorem 1.4. *We have*

$$\sum_{k\ge 0,\, a+b=m} A_{n,k,a,b}(q) = \sum_{\mathbf{c}\in\Theta_{n,m}} A_{n,\mathbf{c}}(q); \tag{1.17}$$

$$\sum_{k\ge 0,\, a+b=m} B_{n,k,a,b}(q) = \sum_{\mathbf{c}\in\Theta^-_{n,m+1}} A_{n,\mathbf{c}}(q). \tag{1.18}$$

Now, form the generating polynomials

$$A_n(x,q) := \sum_{m\ge 0} x^m \sum_{k\ge 0,\, a+b=m} A_{n,k,a,b}(q) = \sum_{m\ge 0} x^m \sum_{\mathbf{c}\in\Theta_{n,m}} A_{n,\mathbf{c}}(q);$$
$$B_n(x,q) := \sum_{m\ge 0} x^m \sum_{k\ge 0,\, a+b=m} B_{n,k,a,b}(q) = \sum_{m\ge 0} x^m \sum_{\mathbf{c}\in\Theta^-_{n,m+1}} A_{n,\mathbf{c}}(q).$$

Theorem 1.5. *The factorial generating functions for the polynomials $A_n(x,q)$ and $B_n(x,q)$ are given by:*

$$\sum_{n\ge 0} A_n(x,q)\frac{u^n}{(q;q)_n} = \tan_q(u) + \sec_q(u)(1-x\tan_q(u))^{-1} x\,\mathrm{Sec}_q(u); \tag{1.19}$$

$$\sum_{n\ge 0} B_n(x,q)\frac{u^n}{(q;q)_n} = \sec_q(u)(1-x\tan_q(u))^{-1}. \tag{1.20}$$

Those two formulas, derived in Section 9, are true q-analogs of Hoffman's identities (1.6) and (1.7), as the latter ones can be rewritten as:

$$\sum_{n\ge 0} A_n(x)\frac{u^n}{n!} = \tan(u) + \sec(u)(1-x\tan(u))^{-1} x\sec(u),$$

$$\sum_{n\ge 0} B_n(x)\frac{u^n}{n!} = \sec(u)(1-x\tan(u))^{-1}. \tag{1.21}$$

The four-variable polynomials $A_n(t,x,y,q) = \sum\limits_{k,a,b} A_{n,k,a,b}(q)t^k x^a y^b$ and $B_n(t,x,y,q) := \sum\limits_{k,a,b} B_{n,k,a,b}(q)t^k x^a y^b$, that may be considered as multivariable q-analogs of the entries $a(n,m)$ and $b(n,m)$, have several interesting *specializations* studied in Section 10. First, $tA_n(t,0,0,q)$ and $tB_n(t,0,0,q)$ are shown to be the (t,q)-analogs $T_n(t,q)$ and $E_n(t,q)$ of tangent and secant numbers, only defined so far [FH11] by their factorial generating functions $\sum_n T_{2n+1}(t,q)u^{2n+1}/(t;q)_{2n+2}$ and $\sum_n E_{2n}(t,q)u^{2n}/(t;q)_{2n+1}$. The recurrences of the polynomials $A_{n,k,a,b}(q)$ and $B_{n,k,a,b}(q)$ provide a handy method for calculating them. We also prove that the polynomial $A_n(t,x,q) := \sum_{k,a} A_{n,k,a,n+1-a}(q)t^k x^a$ is a *refinement* of the Carlitz q-analog [Ca54] $A_n(t,q)$

of the *Eulerian polynomial*, as $A_n(t,1,q)=A_n(t,q)$, with an explicit combinatorial interpretation. Referring to Tables 2 and 3 of the polynomials $A_{n,k,a,b}(q)$ and $B_{n,k,a,b}(q)$ displayed at the end of the paper, it is shown that the sum of the polynomials occurring in each box along the two top diagonals can be explicitly evaluated. Furthermore, the Springer numbers are given two q-analogs. Finally, generating functions and recurrence relations are provided for both t- and s-compositions.

2. A detour to the theory of q-trigonometric functions

By means of the q-binomial theorem ([GR90, § 1.3]; [AAR00, § 10.2]) we can express the *first q-exponential* $e_q(u)$ and the *second q-exponential* $E_q(u)$, either as an infinite q-series, or an infinite product:

$$e_q(u)=\sum_{n\ge 0}\frac{u^n}{(q;q)_n}=\frac{1}{(u;q)_\infty}; \tag{2.1}$$

$$E_q(u)=\sum_{n\ge 0}q^{n(n-1)/2}\frac{u^n}{(q;q)_n}=(-u;q)_\infty, \tag{2.2}$$

two results that go back to Euler [Eu48]. As already done by Jackson [Ja04] (also see [GR90, p. 23]), they both serve to define the q-trigonometric functions q-sine and q-cosine:

$$\sin_q(u):=\frac{e_q(iu)-e_q(-iu)}{2i}=\sum_{n\ge 0}(-1)^n\frac{u^{2n+1}}{(q;q)_{2n+1}};$$

$$\cos_q(u):=\frac{e_q(iu)+e_q(-iu)}{2}=\sum_{n\ge 0}(-1)^n\frac{u^{2n}}{(q;q)_{2n}};$$

$$\mathrm{Sin}_q(u):=\frac{E_q(iu)-E_q(-iu)}{2i}\qquad\text{and}\qquad \mathrm{Cos}_q(u):=\frac{E_q(iu)+E_q(-iu)}{2}.$$

We can also define: $\tan_q(u):=\sin_q(u)/\cos_q(u)$ and $\mathrm{Tan}_q(u):=\mathrm{Sin}_q(u)/\mathrm{Cos}_q(u)$; but, as $\sin_q(u)\,\mathrm{Cos}_q(u)-\mathrm{Sin}_q(u)\cos_q(u)=0$, we have:

$$\tan_q(u)=\frac{\sin_q(u)}{\cos_q(u)}=\frac{\mathrm{Sin}_q(u)}{\mathrm{Cos}_q(u)}=\mathrm{Tan}_q(u),$$

so that there is *only one* q-tangent. However, there are two q-secants:

$$\sec_q(u):=\frac{1}{\cos_q(u)}=\frac{1}{\sum\limits_{n\ge 0}(-1)^n u^{2n}/(q;q)_{2n}};$$

$$\mathrm{Sec}_q(u):=\frac{1}{\mathrm{Cos}_q(u)}=\frac{1}{\sum\limits_{n\ge 0}(-1)^n q^{n(2n-1)}u^{2n}/(q;q)_{2n}}.$$

Theorem 2.1. *The q-derivatives of the series* $\tan_q(u)$, $\sec_q(u)$, $\mathrm{Sec}_q(u)$ *can be evaluated as follows:*

$$D_q \tan_q(u) = 1 + \tan_q(u)\tan_q(qu); \tag{2.3}$$

$$D_q \sec_q(u) = \sec_q(qu)\ \tan_q(u); \tag{2.4}$$

$$D_q \mathrm{Sec}_q(u) = \mathrm{Sec}_q(u)\tan_q(qu). \tag{2.5}$$

Proof. These three identities can be proved, either by working with $e_q(u)$ and $E_q(u)$, when expressed as infinite q-series, or by using the infinite products appearing in (2.1) and (2.2). We choose the former way, because of its closeness to the traditional trigonometric calculus.

First,

$$e_q(\alpha u) - e_q(\alpha q u) = \sum_{n\ge1} \frac{(\alpha u)^n(1-q^n)}{(q;q)_n} = \alpha u \sum_{n\ge1} \frac{(\alpha u)^{n-1}}{(q;q)_{n-1}} = \alpha u\, e_q(\alpha u);$$

$$\begin{aligned} E_q(\alpha u) - E_q(\alpha q u) &= \sum_{n\ge1} \frac{(\alpha u)^n q^{n(n-1)/2}(1-q^n)}{(q;q)_n} \\ &= \alpha u \sum_{n\ge1} \frac{(\alpha u q)^{n-1} q^{(n-1)(n-2)/2}}{(q;q)_{n-1}} = \alpha u\, E_q(\alpha q u). \end{aligned}$$

Hence,

$$D_q\, e_q(\alpha u) = \alpha\, e_q(\alpha u); \qquad D_q\, E_q(\alpha u) = \alpha\, E_q(\alpha q u).$$

Next, by applying D_q to the familiar identities: $e_q(iu) = \cos_q(u) + i\sin_q(u)$ and $E_q(iu) = \mathrm{Cos}_q(u) + i\,\mathrm{Sin}_q(u)$, we get

$$D_q \cos_q(u) = -\sin_q(u); \quad D_q \sin_q(u) = \cos_q(u);$$
$$D_q \mathrm{Cos}_q(u) = -\mathrm{Sin}_q(u); \quad D_q \mathrm{Sin}_q(u) = \mathrm{Cos}_q(u).$$

Finally, we take advantage of the next formula that yields an expression for the q-derivative of a ratio $f(u)/g(u)$ of two q-series

$$D_q \frac{f(u)}{g(u)} = \frac{g(qu)\, D_q f(u) - f(qu)\, D_q g(u)}{g(u)g(qu)},$$

and use it for the ratios $\sin_q(u)/\cos_q(u)$, $1/\cos_q(u)$, $1/\mathrm{Cos}_q(u)$ to get:

$$D_q \tan_q(u) = \frac{\cos_q(qu)\cos_q(u) - \sin_q(qu)(-\sin_q(u))}{\cos_q(u)\cos_q(qu)} = 1 + \tan_q(u)\tan_q(qu);$$

$$D_q \sec_q(u) = \frac{\sin_q(u)}{\cos_q(u)\cos_q(qu)} = \sec_q(qu)\tan_q(u);$$

$$D_q \mathrm{Sec}_q(u) = \frac{\mathrm{Sin}_q(qu)}{\mathrm{Cos}_q(u)\,\mathrm{Cos}_q(qu)} = \mathrm{Sec}_q(u)\tan_q(qu). \quad \square$$

Identities (2.4) and (2.5) above show a certain duality between the q-derivatives of $\sec u$ and $\operatorname{Sec} u$, which is to be explored by using the base q^{-1} instead of q. First, for each q-series $f(u)$ let $Qf(u) := f(qu)$ and $Uf(u) := (1/u)f(u)$. We have the identities:

$$UQ = q\,QU, \qquad Q^{-1}U = q\,UQ^{-1}.$$

Next, the q-difference operators D_q and $D_{q^{-1}}$, the latter being defined by $D_{q^{-1}}f(u) := (1/u)(f(u) - f(q^{-1}u))$, also read

$$D_q = U(I-Q); \qquad D_{q^{-1}} = U(I-Q^{-1}). \tag{2.6}$$

Hence,

$$D_q = -D_{q^{-1}}Q. \tag{2.7}$$

Next, we have the relations:

$$e_{q^{-1}}(u) = QE_q(-u), \qquad E_{q^{-1}}(u) = Qe_q(-u). \tag{2.8}$$

$$\sin_{q^{-1}}(u) = -Q\operatorname{Sin}_q(u); \quad \cos_{q^{-1}}(u) = Q\operatorname{Cos}_q(u); \tag{2.9}$$

$$\tan_{q^{-1}}(u) = -Q\tan_q(u); \qquad \sec_{q^{-1}}(u) = Q\operatorname{Sec}_q(u). \tag{2.10}$$

We end this Section with three technical lemmas that will be used in Section 5 (resp. Section 6) to show how identity (1.12) (resp. (1.16)) in Theorem 1.2 (resp. in Theorem 1.3) can be obtained from (1.11) (resp. from (1.15)).

Lemma 2.2. *We have $D_{q^{-1}}Q = q\,QD_{q^{-1}}$, so that for each $n \ge 1$*

$$(D_{q^{-1}}Q)^n = q^{(n+1)n/2}Q^nD^n_{q^{-1}}. \tag{2.11}$$

Proof. By (2.6) we have: $D_{q^{-1}}Q = U(I-Q^{-1})Q = UQ - U = q\,QU - q\,QUQ = q\,QU(I-Q^{-1}) = q\,QD_{q^{-1}}$. Identity (2.11) is then a simple consequence, as there are $(n+1)n/2$ transpositions $QD_{q^{-1}} \leftrightarrow D_{q^{-1}}Q$ to be made to go from $(QD_{q^{-1}})^n$ to $Q^nD^n_{q^{-1}}$. ∎

Lemma 2.3. *For each composition $\mathbf{c} = (c_0, c_1, \ldots, c_m)$ of an integer $n \ge 1$, we have*

$$Q^n \tan_q((q^{-1})^{\rho\,\mathbf{c}}u) = \tan_q(q^{\mathbf{c}}u). \tag{2.12}$$

Proof. A simple verification:

$$\tan_q((q^{-1})^{\rho\,\mathbf{c}}u) = \tan_q((q^{-1})^{c_m}u)\tan_q((q^{-1})^{c_m+c_{m-1}}u) \times\cdots\times \tan_q((q^{-1})^{c_m+\cdots+c_1}u).$$

Hence, the left-hand side of (2.12) is equal to

$$\begin{aligned}
&Q^n \tan_q((q^{-1})^{\rho\,\mathbf{c}}u)\\
&\qquad = \tan_q(q^{n-c_m}u)\tan_q(q^{n-c_m-c_{m-1}}u)\times\cdots\times\tan_q(q^{n-c_m-\cdots-c_1}u)\\
&\qquad = \tan_q(q^{c_0+\cdots+c_{m-1}}u)\tan_q(q^{c_0+\cdots+c_{m-2}}u)\times\cdots\times\tan_q(q^{c_0}u)\\
&\qquad = \tan_q(q^{\mathbf{c}}u). \quad \square
\end{aligned}$$

Lemma 2.4. *We have:*

$$D_q^n \operatorname{Sec}_q(u) = (-1)^n q^{n(n-1)/2} Q^{n-1} D_{q^{-1}}^n \sec_{q^{-1}}(u).$$

Proof. Just write

$$\begin{aligned}
D_q^n \operatorname{Sec}_q(u) &= (-D_{q^{-1}}Q)^n\, Q^{-1}\sec_{q^{-1}}(u) &&\text{[by (2.7) and (2.10)]}\\
&= (-1)^n (D_{q^{-1}}Q)^{n-1} D_{q^{-1}}\sec_{q^{-1}}(u)\\
&= (-1)^n q^{n(n-1)/2} Q^{n-1} D_{q^{-1}}^n \sec_{q^{-1}}(u). \quad \square &&\text{[by (2.11)]}
\end{aligned}$$

3. Transformations on t-permutations

Say that a t-permutation $w=(w_0,w_1,\ldots,w_m)$ is of the *first* (resp. *second*) *kind*, if 1 appears (resp. does not appear) as a one-letter factor among the w_i's. Each set $\mathcal{T}_{n,m}$ can be partitioned into two subsets $\mathcal{T}^*_{n,m}$ and ${}^*\mathcal{T}_{n,m}$, the former one consisting of all permutations from $\mathcal{T}_{n,m}$ of the first kind, the latter one of those of the second kind. Let $[m]:=\{1,2,\ldots,m\}$ and for each integer y let $y^+:=y+1$ and $v^+:=y_1^+y_2^+\ldots y_m^+$ for each word $v=y_1y_2\ldots y_m$, whose letters are integers.

For each pair (m,n) such that $0\le m\le n+1$ and $m\equiv n+1 \bmod 2$, we construct two bijections

$$\Delta^*:[m]\times\mathcal{T}_{n,m}\to\mathcal{T}^*_{n+1,m+1};\qquad {}^*\!\Delta:[m]\times\mathcal{T}_{n,m}\to{}^*\mathcal{T}_{n+1,m-1}; \tag{3.1}$$

Let $w=(w_0,w_1,w_2,\ldots,w_m)$ belong to $\mathcal{T}_{n,m}$ and $1\le i\le m$. The sequence

$$(w_0^+,\ldots,w_{i-2}^+,w_{i-1}^+,\,1\,,w_i^+,w_{i+1}^+,\ldots,w_m^+) \tag{3.2}$$

is obviously from $\mathcal{T}^*_{n+1,m+1}$; denote it by $\Delta^*(i,w)$. Next, the sequence

$$(w_0^+,\ldots,w_{i-2}^+,(w_{i-1}^+\,1\,w_i^+),w_{i+1}^+,\ldots,w_m^+) \tag{3.3}$$

is then from ${}^*\mathcal{T}_{n+1,m-1}$, and will be denoted by ${}^*\!\Delta(i,w)$; its i-th factor appears as $w_{i-1}^+\,1\,w_i^+$, as indicated in the inner parentheses in (3.3). By the above two bijections, recurrence (1.3) holds with the interpretation $a(n,m)=\#\mathcal{T}_{n,m}$.

The next step is to study the actions of the transformations ${}^*\!\Delta$ and Δ^* on the statistics "ides,", "imaj," "min," "inv" introduced in Subsection 1.4. Taking again the example used in that Subsection, namely the t-permutation $w=(4\,5,\ 11\,1\,\mathbf{3},\ \mathbf{10}\,7\,\mathbf{9},\ \mathbf{6},\ \mathbf{8}\,\mathbf{2})$, whose underlying statistics are $\operatorname{ides} w=6$, $\operatorname{imaj} w=38$, $\operatorname{inv} w=27$ and $\min w=1$, we get:

	ides	imaj	min	inv
$w = (4\,5,\ 11\,1\,\mathbf{3},\ \mathbf{10}\,7\,\mathbf{9},\ \mathbf{6},\ \mathbf{8}\,\mathbf{2})$	6	38	1	27
$\Delta^*(1,w) = (5\,6,\ 1,\ 12\,2\,\mathbf{4},\ \mathbf{11}\,8\,\mathbf{10},\ \mathbf{7},\ \mathbf{9}\,\mathbf{3})$	6	44	1	29
$\Delta^*(2,w) = (5\,6,\ 12\,2\,\mathbf{4},\ \mathbf{1},\ \mathbf{11}\,8\,\mathbf{10},\ \mathbf{7},\ \mathbf{9}\,\mathbf{3})$	7	45	2	32
$\Delta^*(3,w) = (5\,6,\ 12\,2\,\mathbf{4},\ \mathbf{11}\,8\,\mathbf{10},\ \mathbf{1},\ \mathbf{7},\ \mathbf{9}\,\mathbf{3})$	7	45	3	35
$\Delta^*(4,w) = (5\,6,\ 12\,2\,\mathbf{4},\ \mathbf{11}\,8\,\mathbf{10},\ \mathbf{7},\ \mathbf{1},\ \mathbf{9}\,\mathbf{3})$	7	45	4	36
${}^*\Delta(1,w) = (5\,6\,1\,12\,2\,\mathbf{4},\ \mathbf{11}\,8\,\mathbf{10},\ \mathbf{7},\ \mathbf{9}\,\mathbf{3})$	6	44	0	29
${}^*\Delta(2,w) = (5\,6,\ 12\,2\,\mathbf{4}\ \mathbf{1}\,\mathbf{11}\,8\,\mathbf{10},\ \mathbf{7},\ \mathbf{9}\,\mathbf{3})$	7	45	1	32
${}^*\Delta(3,w) = (5\,6,\ 12\,2\,\mathbf{4},\ \mathbf{11}\,8\,\mathbf{10}\,\mathbf{1}\,\mathbf{7},\ \mathbf{9}\,\mathbf{3})$	7	45	2	35
${}^*\Delta(4,w) = (5\,6,\ 12\,2\,\mathbf{4},\ \mathbf{11}\,8\,\mathbf{10},\ \mathbf{7}\,\mathbf{1}\,\mathbf{9}\,\mathbf{3})$	7	45	3	36

The proof of the next theorem is a simple verification and will not be reproduced here.

Theorem 3.1. *Let w be a t-permutation and let* $\operatorname{Iligne} w := \{j_1 < j_2 < \cdots < j_r\}$. *Furthermore, let $\Delta^*(i,w)$ and ${}^*\Delta(i,w)$ be defined as in* (3.1) - (3.3). *Then,*

$$\begin{aligned}&\operatorname{Iligne}\Delta^*(i,w)\\ &\quad= \operatorname{Iligne}{}^*\Delta(i,w) = \begin{cases}\{(j_1+1),(j_2+1),\dots,(j_r+1)\}, & \text{if } i \le \min w;\\ \{1,(j_1+1),(j_2+1),\dots,(j_r+1)\}, & \text{if } \min w < i.\end{cases}\end{aligned}$$

Hence,

$$\operatorname{ides}\Delta^*(i,w) = \operatorname{ides}{}^*\Delta(i,w) = \begin{cases}\operatorname{ides} w, & \text{if } i \le \min w;\\ 1+\operatorname{ides} w, & \text{if } \min w < i.\end{cases}$$

$$\operatorname{imaj}\Delta^*(i,w) = \operatorname{imaj}{}^*\Delta(i,w) = \begin{cases}\operatorname{ides} w + \operatorname{imaj} w, & \text{if } i \le \min w;\\ 1+\operatorname{ides} w + \operatorname{imaj} w, & \text{if } \min w < i;\end{cases}$$

Furthermore,

$$\operatorname{inv}\Delta^*(i,w) = \operatorname{inv}{}^*\Delta(i,w) = \operatorname{inv} w + \lambda(w_0 w_1 \cdots w_{i-1}),$$
$$\min(\Delta^*(i,w)) = i, \quad \min({}^*\Delta(i,w)) = i-1, \text{ for } i \ge 1.$$

4. Proof of Theorem 1.1

Two steps are needed to achieve the proof: first, the derivation of a *recurrence relation* for the polynomials $A_{n,k,a,b}(q)$, presented in the next theorem, then, an explicit *algorithm* for calculating them, described in Lemma 4.2.

Theorem 4.1 (Recurrence for the polynomials $A_{n,k,a,b}(q)$)**.** *With $m := a+b$ and $m' := a'+b'$ we have:*

$$\begin{aligned}(4.1)\quad A_{n+1,k',a',b'}(q) = q^{k'}\Bigl(&\sum_{0\le a\le a'-1\le m'-2} A_{n,k'-1,a,m'-1-a}(q) + \sum_{1\le a'\le a\le m'-1} A_{n,k',a,m'-1-a}(q)\\ &+ \sum_{0\le a\le a'} A_{n,k'-1,a,m'+1-a}(q) + \sum_{a'+1\le a\le m'+1} A_{n,k',a,m'+1-a}(q)\Bigr),\end{aligned}$$

valid for $n \ge 0$ with the initial condition: $A_{0,k,a,b}(q) = \delta_{k,0}\delta_{a,1}\delta_{b,0}$.

Proof. Let $w' = (w'_0, w'_1, \ldots, w'_{m'}) \in \mathcal{T}_{n+1,k',a',b'}$. When w' is of the first kind, the inverse ${\Delta^*}^{-1}(w')$ is obtained by deleting the unique $w'_{a'}$ equal to 1 and subtracting 1 from all the other letters of the components w'_j $(j = 0, \ldots, a'-1, a'+1, \ldots, m')$. When w' is of the second kind, the inverse ${}^*\Delta^{-1}(w')$ is obtained by deleting 1 from the component $w'_{a'} = u'_{a'}1v'_{a'}$ and subtracting 1 from all the letters of the components $w'_0, \ldots, w'_{a'-1}, u'_{a'}, v'_{a'}, w'_{a'+1}, \ldots, w'_{m'}$. The t-permutations w' from $\mathcal{T}_{n+1,k',a',b'}$ fall into four categories:

(1) w' is of the first kind and 2 is *to the left of* 1 in the product $w'_0 w'_1 \cdots w'_{m'}$; then $1 \le a' \le m'-1$, because, when $n+1 \ge 2$, the components w'_0 and $w'_{m'}$ are, either empty, or their lengths are at least equal to 2. Hence, ${\Delta^*}^{-1}(w') = (a, w)$ with $w \in \mathcal{T}_{n,k'-1,a,m'-1-a}$ and $0 \le a \le a'-1 \le m'-2$, as $\operatorname{ides} w = \operatorname{ides} w' - 1$. Also, $\operatorname{imaj} w' = 1 + \operatorname{ides} w + \operatorname{imaj} w = k' + \operatorname{imaj} w$. [By Theorem 3.1]

(2) w' is of the first kind and 2 is *to the right of* 1 in $w'_0 w'_1 \cdots w'_{m'}$; for an analogous reason as in case (1) we have: ${\Delta^*}^{-1}(w') = (a, w)$ with $w \in \mathcal{T}_{n,k',a,m'-1-a}$ and $a' \le a \le m'-1$. In this case $\operatorname{imaj} w' = \operatorname{ides} w + \operatorname{imaj} w = k' + \operatorname{imaj} w$.

(3) w' is of the second kind and 2 is *to the left of* 1 in $w'_0 w'_1 \cdots w'_{m'}$, so that 2 can still belong to $w'_{a'}$, or to any one of the components $w'_0, \ldots, w'_{a'-1}$. Hence, ${\Delta^*}^{-1}(w') = (a, w)$ with $w \in \mathcal{T}_{n,k'-1,a,m'+1-a}$ and $0 \le a \le a'$, for the number of components has increased by 1. Again, $\operatorname{imaj} w' = k' + \operatorname{imaj} w$.

(4) w' is of the second kind and 2 is *to the right of* 1 in $w'_0 w'_1 \cdots w'_{m'}$, so that 2 can still belong to $w'_{a'}$, or to any one of the components $w'_{a'+1}, \ldots, w'_{m'}$. Hence, ${\Delta^*}^{-1}(w') = (a, w)$ with $w \in \mathcal{T}_{n,k',a,m'+1-a}$ and $a'+1 \le a \le m'+1'$. Also, $\operatorname{imaj} w' = k' + \operatorname{imaj} w$.

Thus, identity (4.1) holds. $\square$

A consequence of the combinatorial interpretation is the symmetry property

$$q^{n(n-1)/2} A_{n,n-1-k,b,a}(q^{-1}) = A_{n,k,a,b}(q),$$

whose proof is easy and will be omitted.

Lemma 4.2. *Let* $[k, a, b] := (\tan_q(q^{k+1}u))^b \, (\tan_q(q^k u))^a$. *Then,*

$$\begin{aligned} D_q[k,a,b] &= q^k \sum_{0 \le i \le a-1} [k, i, a+b-1-i] + q^{k+1} \sum_{0 \le i \le b-1} [k+1, a+i, b-1-i] \\ &\quad + q^k \sum_{1 \le i \le a} [k, i, a+b+1-i] + q^{k+1} \sum_{1 \le i \le b} [k+1, a+i, b+1-i]. \end{aligned}$$

Proof. For taking the q-derivative of a product $f_1(u) f_2(u) \cdots f_n(u)$ of q-series we use the formula:

$$D_q \prod_{1 \le i \le n} f_i(u) = \sum_{1 \le i \le n} f_1(u) \cdots f_{i-1}(u) \, (D_q f_i(u)) \, f_{i+1}(qu) \cdots f_n(qu). \tag{4.2}$$

In particular,

$$
\begin{aligned}
&D_q(\tan_q(q^{k+1}u))^b\,(\tan_q(q^k u))^a \\
&\quad= \sum_{0\le i\le a-1} (\tan_q(q^{k+1}u))^b\,(\tan_q(q^k u))^i (D_q \tan_q(q^k u))(\tan_q(q^{k+1}u))^{a-1-i} \\
&\qquad+ \sum_{0\le j\le b-1} (\tan_q(q^{k+1}u))^j (D_q \tan_q(q^{k+1}u))(\tan_q(q^{k+2}u))^{b-1-j}(\tan_q(q^{k+1}u))^a \\
&\quad= q^k \sum_{0\le i\le a-1} (\tan_q(q^{k+1}u))^{a+b-1-i}\,(\tan_q(q^k u))^i \\
&\qquad+ q^k \sum_{0\le i\le a-1} (\tan_q(q^{k+1}u))^{a+b-i}\,(\tan_q(q^k u))^{i+1} \\
&\qquad+ q^{k+1} \sum_{0\le j\le b-1} (\tan_q(q^{k+2}u))^{b-1-j}(\tan_q(q^{k+1}u))^{a+j} \\
&\qquad+ q^{k+1} \sum_{0\le j\le b-1} (\tan_q(q^{k+2}u))^{b-j}(\tan_q(q^{k+1}u))^{a+j+1}. \quad \square
\end{aligned}
$$

With the notation $[k, a, b]$ we have $[0, 1, 0] = \tan_q(u)$, and identity (1.9) can be rewritten

$$D_q^n \tan_q(u) = \sum_{k,a,b} A_{n,k,a,b}(q)\,[k, a, b],$$

so that Lemma 4.2 can be used to calculate the polynomials $A_{n,k,a,b}(q)$ by iteration :

$$
\begin{aligned}
&D_q[0, 1, 0] \ (= D_q \tan_q(u)) = [0, 0, 0] + [0, 1, 1], \\
&\qquad \text{so that } A_{1,0,0,0}(q) = A_{1,0,1,1}(q) = 1; \\
&D_q^2[0, 1, 0] \ (= D_q^2 \tan_q(u)) = D_q[0, 1, 1] = [0, 0, 1] + q[1, 1, 0] + [0, 1, 2] + q[1, 2, 1], \\
&\qquad \text{so that } A_{2,0,0,1}(q) = 1,\ A_{2,1,1,0}(q) = q,\ A_{2,0,1,2}(q) = 1,\ A_{2,1,2,1}(q) = q; \\
&D_q^3[0, 1, 0] \ (= D_q^3 \tan_q(u)) = (q + q^2)[1, 0, 0] \\
&\qquad\qquad + [0, 0, 2] + q^2[1, 0, 2] + (2q + 2q^2)[1, 1, 1] + q[1, 2, 0] + q^3[2, 2, 0] \\
&\qquad\qquad + [0, 1, 3] + q^2[1, 1, 3] + (q + q^2)[1, 2, 2] + q[1, 3, 1] + q^3[2, 3, 1], \\
&\qquad \text{so that } A_{3,1,0,0}(q) = q + q^2,\ A_{3,0,0,2}(q) = 1, \text{ etc.}
\end{aligned}
$$

The polynomials $A_{n,k,a,b}(q)$ in Table 2 (Section 11) have been obtained using the previous calculation.

Proof of Theorem 1.1. By induction. Assume that (1.9) is true for n.

$$D_q^{n+1}[0, 1, 0] = D_q D_q^n[0, 1, 0]$$

$$= D_q \sum_{k,a,b} A_{n,k,a,b}(q)[k,a,b] = \sum_{k,a,b} A_{n,k,a,b}(q) D_q[k,a,b]$$

$$= \sum_{k,a,b} A_{n,k,a,b}(q)\Big(q^k \sum_{0\le i\le a-1}[k,i,a+b-1-i]$$

$$+ q^{k+1}\sum_{0\le i\le b-1}[k+1,a+i,b-1-i]$$

$$+ q^k \sum_{1\le i\le a}[k,i,a+b+1-i] + q^{k+1}\sum_{1\le i\le b}[k+1,a+i,b+1-i]\Big).$$

We calculate the contribution of each sum to the triple $[k',a',b']$. For the first sum we have $[k,i,a+b-1-i]=[k',a',b']$ $(0\le i\le a-1)$ if and only if $k=k'$, $i=a'$, $a+b-1=a'+b'$, $a'+1\le a\le a'+b'+1$, so that the contribution is

$$q^{k'} \sum_{a'+1\le a\le a'+b'+1} A_{n,k',a,a'+b'+1-a}(q).$$

For the second sum we have $[k+1,a+i,b-1-i]=[k',a',b']$ $(0\le i\le b-1)$ if and only if $k=k'-1$, $a+i=a'$, $b-1-i=b'$, $0\le a'-a\le b-1$, that is, $k=k'-1$, $i=a'-a$, $a+b=a'+b'+1$, $0\le a\le a'$, so that the contribution is

$$q^{k'} \sum_{0\le a\le a'} A_{n,k'-1,a,a'+b'+1-a}(q).$$

with the convention that $A_{n,-1,a,a'+b'+1-a}(q)=0$. For the third sum we have $[k,i,a+b+1-i]=[k',a',b']$ $(1\le i\le a)$ if and only if $k=k'$, $i=a'$, $a+b+1=a'+b'$, $1\le a'\le a\le a+b=a'+b'-1$, so that the contribution is

$$q^{k'} \sum_{1\le a'\le a\le a'+b'-1} A_{n,k',a,a'+b'-1-a}(q).$$

For the fourth sum we have $[k+1,a+i,b+1-i]=[k',a',b']$ $(1\le i\le b)$ if and only if $k=k'-1$, $a+i=a'$, $b+1-i=b'$, $1\le i\le b$, that is, $k=k'-1$, $i=a'-a$, $a+b+1=a'+b'$, $0\le a\le a'-1=a+i-1\le a+b-1\le a'+b'-2$.

The contribution is then

$$q^{k'} \sum_{0\le a\le a'-1\le a'+b'-2} A_{n,k'-1,a,a'+b'-1-a}(q).$$

By Theorem 4.1 we have

$$D_q^{n+1}[0,1,0] = \sum_{k',a',b'} A_{n+1,k',a',b'}[k',a',b']. \quad \square$$

5. Proof of Theorem 1.2

As for Theorem 1.1, two steps are used to complete the proof of Theorem 1.2: first, the derivation of a recurrence relation, then, the construction of an explicit algorithm.

Theorem 5.1 (Recurrence relation for the polynomials $B_{n,k,a,b}(q)$). *With $m := a+b$ and $m' := a'+b'$ we have:*

$$B_{n+1,k',a',b'}(q) = q^{k'}\Big(\sum_{0\le a\le a'-1} B_{n,k'-1,a,m'-1-a}(q) + \sum_{1\le a'\le a\le m'-1} B_{n,k',a,m'-1-a}(q) + \sum_{0\le a\le a'} B_{n,k'-1,a,m'+1-a}(q) + \sum_{a'+1\le a\le m'+1} B_{n,k',a,m'+1-a}(q)\Big), \tag{5.1}$$

valid for $n\ge 0$ with the initial condition: $B_{0,k,a,b}(q) = \delta_{k,-1}\delta_{a,0}\delta_{b,0}$.

Proof. Let $w' = (w'_0, w'_1, \ldots, w'_{m'}, \epsilon) \in \mathcal{T}'_{n+1,k',a',b'+1}$. What has been said for recurrence (4.1) can be reproduced, except for the first sum, when w' is of the first kind and 2 is *to the left of* 1 in the product $w'_0 w'_1 \cdots w'_{m'}$; now, $1\le a'\le m'$ and not $m'-1$. Hence, $\Delta^{*-1}(w') = (a,w)$ with $w\in \mathcal{T}'_{n,k'-1,a,m'-a}$ and $0\le a\le a'-1\le m'-1$. Thus, identity (5.1) holds. ∎

Lemma 5.2. *Let $\langle k,a,b\rangle := (\tan_q(q^{k+1}u))^b \sec_q(q^{k+1}u)(\tan_q(q^k u))^a$. Then*

$$D_q\langle k,a,b\rangle := q^k \sum_{0\le i\le a-1}\langle k,i,a+b-1-i\rangle + q^k\sum_{1\le i\le a}\langle k,i,a+b+1-i\rangle + q^{k+1}\sum_{0\le i\le b-1}\langle k+1,a+i,b-1-i\rangle + q^{k+1}\sum_{1\le i\le b+1}\langle k+1,a+i,b+1-i\rangle. \tag{5.2}$$

Proof. By using (4.2) we derive

$$\begin{aligned}
&D_q(\tan_q(q^{k+1}u))^b \sec_q(q^{k+1}u)\,(\tan_q(q^k u))^a\\
&= \sum_{0\le i\le a-1}(\tan_q(q^{k+1}u))^{a+b-1-i}\sec_q(q^{k+1}u)(\tan_q(q^k u))^i(D_q\tan_q(q^k u))\\
&\quad+\sum_{0\le j\le b-1}(\tan_q(q^{k+1}u))^{a+j}(D_q\tan_q(q^{k+1}u))(\tan_q(q^{k+2}u))^{b-1-j}\sec_q(q^{k+2}u)\\
&\quad+(\tan_q(q^{k+1}u))^b\,D_q(\sec_q(q^{k+1}u))\,(\tan_q(q^k u))^a\\
&= q^k\sum_{0\le i\le a-1}(\tan_q(q^{k+1}u))^{a+b-1-i}\sec_q(q^{k+1}u)(\tan_q(q^k u))^i\\
&\quad+q^k\sum_{0\le i\le a-1}(\tan_q(q^{k+1}u))^{a+b-i}\sec_q(q^{k+1}u)(\tan_q(q^k u))^{i+1}\\
&\quad+q^{k+1}\sum_{0\le j\le b-1}(\tan_q(q^{k+2}u))^{b-1-j}\sec_q(q^{k+2}u)(\tan_q(q^{k+1}u))^{a+j}\\
&\quad+q^{k+1}\sum_{0\le j\le b-1}(\tan_q(q^{k+2}u))^{b-j}\sec_q(q^{k+2}u)(\tan_q(q^{k+1}u))^{a+j+1}\\
&\quad+q^{k+1}(\tan_q(q^{k+1}u))^{a+b+1}\sec_q(q^{k+2}u). \qquad ∎
\end{aligned}$$

Lemma 5.2 provides a way to calculate the polynomials $B_{n,k,a,b}(q)$. As $\langle -1,0,0\rangle = \sec_q(u)$, we get

$D_q\langle -1,0,0\rangle = \langle 0,1,0\rangle$, so that $B_{1,0,1,0}(q)=1$;

$D_q^2\langle -1,0,0\rangle = D_q\langle 0,1,0\rangle = \langle 0,0,0\rangle + \langle 0,1,1\rangle + q\langle 1,2,0\rangle$,

so that $B_{2,0,0,0}(q)=B_{2,0,1,1}(q)=1$, $B_{2,1,2,0}(q)=q$;

$$
\begin{aligned}
D_q^3\langle -1,0,0\rangle &= D_q\bigl(\langle 0,0,0\rangle + \langle 0,1,1\rangle + q\langle 1,2,0\rangle\bigr)\\
&= q\langle 1,1,0\rangle\\
&\quad + \bigl(\langle 0,0,1\rangle + \langle 0,1,2\rangle + q\langle 1,1,0\rangle + q\langle 1,2,1\rangle + q\langle 1,3,0\rangle\bigr)\\
&\quad + \bigl(q^2\langle 1,0,1\rangle + q^2\langle 1,1,0\rangle + q^2\langle 1,1,2\rangle + q^2\langle 1,2,1\rangle + q^3\langle 2,3,0\rangle\bigr)\\
&= \langle 0,0,1\rangle + q^2\langle 1,0,1\rangle + (2q+q^2)\langle 1,1,0\rangle\\
&\quad + \langle 0,1,2\rangle + q^2\langle 1,1,2\rangle + (q+q^2)\langle 1,2,1\rangle + q\langle 1,3,0\rangle + q^3\langle 2,3,0\rangle,
\end{aligned}
$$

so that $B_{3,0,0,1}(q)=1$, $B_{3,1,0,1}(q)=q^2$, $B_{3,1,1,0}(q)=2q+q^2$, $B_{3,0,1,2}(q)=1$, $B_{3,1,1,2}(q)=q^2$, $B_{3,1,2,1}(q)=q+q^2$, $B_{3,1,3,0}(q)=q$, $B_{3,2,3,0}(q)=q^3$.

The polynomials $B_{n,k,a,b}(q)$ in Table 2 have been calculated by means of the previous algorithm.

Proof of (1.11). By induction. Assume that (1.11) is true for n. Then,

$$
\begin{aligned}
D_q^{n+1}\langle -1,0,0\rangle &= D_q D_q^n\langle -1,0,0\rangle\\
&= D_q\sum_{k,a,b} B_{n,k,a,b}(q)\langle k,a,b\rangle = \sum_{k,a,b} B_{n,k,a,b}(q) D_q\langle k,a,b\rangle\\
&= \sum_{k,a,b} B_{n,k,a,b}(q)\Bigl(q^k \sum_{0\le i\le a-1}\langle k,i,a+b-1-i\rangle\\
&\quad + q^{k+1}\sum_{0\le i\le b-1}\langle k+1,a+i,b-1-i\rangle\\
&\quad + q^k\sum_{1\le i\le a}\langle k,i,a+b+1-i\rangle + q^{k+1}\sum_{1\le i\le b+1}\langle k+1,a+i,b+1-i\rangle\Bigr).
\end{aligned}
$$

We calculate the contribution of each sum to the triple $\langle k',a',b'\rangle$. For the first three sums we can simply reproduce the arguments developed in the proof of Theorem 1.1. Only the fourth sum is to be checked. This time, the double inequality $1\le i\le b+1$ prevails, instead of $1\le i\le b$. This leads to the sequence $0\le a\le a'-1 = a+i-1 \le a+(b+1)-1 \le a+b \le a'+b'-1$. Hence, the contribution is

$$
q^{k'}\sum_{0\le a\le a'-1\le a'+b'-1} B_{n,k'-1,a,a'+b'-1-a}(q).
$$

By Theorem 5.1 we have

$$D_q^{n+1}\langle -1,0,0\rangle = \sum_{k',a',b'} B_{n+1,k',a',b'}(q)\langle k',a',b'\rangle. \qquad \square$$

Proof of (1.12). Rewrite identity (1.11) taking (2.10) into account:

$$\begin{aligned}
&D^n_{q^{-1}} \sec_{q^{-1}}(u)\\
&\quad = \sum_{k,a,b} B_{n,k,a,b}(q^{-1})(\tan_{q^{-1}}(q^{-k-1}u))^b \sec_{q^{-1}}(q^{-k-1}u)(\tan_{q^{-1}}(q^{-k}u))^a\\
&\quad = \sum_{k,a,b} B_{n,k,a,b}(q^{-1})(-1)^b(\tan_q(q^{-k}u))^b \operatorname{Sec}_q(q^{-k}u)(-1)^a(\tan_q(q^{-k+1}u))^a.
\end{aligned}$$

Hence, as $0 \le k \le n-1$,

$$\begin{aligned}
&Q^{n-1}D^n_{q^{-1}} \sec_{q^{-1}}(u)\\
&\quad = \sum_{k,a,b} B_{n,k,a,b}(q^{-1})(-1)^{a+b}(\tan_q(q^{n-1-k}u))^b \operatorname{Sec}_q(q^{n-1-k}u)(\tan_q(q^{n-k}u))^a\\
&\quad = \sum_{k,a,b} B_{n,n-1-k,a,b}(q^{-1})(-1)^{a+b}(\tan_q(q^{k}u))^b \operatorname{Sec}_q(q^{k}u)(\tan_q(q^{k+1}u))^a.
\end{aligned}$$

Finally, as n and $a+b$ are of the same parity, we have by Lemma 2.4

$$\begin{aligned}
&D^n_q \operatorname{Sec}_q(u) = (-1)^n q^{n(n-1)/2} Q^{n-1} D^n_{q^{-1}} \sec_{q^{-1}}(u),\\
&\quad = \sum_{k,a,b} q^{n(n-1)/2} B_{n,n-1-k,a,b}(q^{-1})(\tan_q(q^{k}u))^b \operatorname{Sec}_q(q^{k}u)(\tan_q(q^{k+1}u))^a.
\end{aligned}$$

This establishes identity (1.12). $\square$

6. Proof of Theorem 1.3

Let $\mathbf{c}' = (c'_0, c'_1, \ldots, c'_{m'}) \in \Theta_{n+1}$ and $0 \le i \le m'$. If $c'_0 \ge 2$, and if $1 \le 2j+1 \le c'_0$, let

$$\mathbf{c}'[0, 2j+1] := (2j, c'_0 - (2j+1), c'_1, \ldots, c'_{m'}). \tag{6.1}$$

If $1 \le i \le m'$ and $2 \le 2j \le c'_i$, let

$$\mathbf{c}'[i, 2j] := (c'_0, \ldots, c'_{i-1}, 2j-1, c'_i - 2j, c'_{i+1}, \ldots, c'_{m'}). \tag{6.2}$$

Finally, if $1 \le i \le m'-1$ and $c'_i = 1$, let

$$\mathbf{c}'[i, 1] := (c'_0, \ldots, c'_{i-1}, c'_{i+1}, \ldots, c'_{m'}). \tag{6.3}$$

Theorem 6.1 (Recurrence relation for the polynomial $A_{n,\mathbf{c}}(q)$). *With* $\mathbf{c}'=(c'_0,c'_1,\ldots,c'_{m'})$ *we have:*

$$A_{n+1,\mathbf{c}'}(q)=\sum_{0\le 2j+1\le c'_0} q^{2j}A_{n,\mathbf{c}'[0,2j+1]}(q) + \sum_{1\le i\le m'} q^{c'_0+\cdots+c'_{i-1}}\sum_{2\le 2j\le c'_i} q^{2j-1}A_{n,\mathbf{c}'[i,2j]}(q) + \sum_{1\le i\le m'-1}\chi(c'_i=1)q^{c'_0+\cdots+c'_{i-1}}A_{n,\mathbf{c}'[i,1]}(q), \tag{6.4}$$

where $\chi(c'_i=1)$ *is equal to 1 if* $c'_i=1$ *holds and* 0 *otherwise.*

Proof. Let $w'=(w'_0,w'_1,\ldots,w'_{m'})$ be a t-permutation from $\mathcal{T}_{n+1}$ such that $\Lambda w'=\mathbf{c}'=(c'_0,c'_1,\ldots,c'_{m'})$. Two cases are to consider: (i) 1 belongs to the component w_i $(0\le i\le m')$ having at least two letters: write $w\in\mathcal{T}_{n+1,\mathbf{c}',2}$; (ii) the component w_i is equal to the one-letter 1: write $w\in\mathcal{T}_{n+1,\mathbf{c}',1}$. For each word v, whose letters are integers, let v^- designate the word obtained from v by subtracting 1 from each of its letters. By convention, $\epsilon^-:=\epsilon$.

Case (i): we may write: $w'_i=u'_i1v'_i$, where at least one of the factors u'_i, v'_i is nonempty. To w' there corresponds a unique triple $(w,i,\lambda(u'_i1))$ where $w:=(w_0'^-,\ldots,w_{i-1}'^-,u_i'^-,v_i'^-,w_{i+1}'^-,w_{m'}'^-)$. As w'_0 is rising alternating (when nonempty) and the other components falling alternating, the length $\lambda(u'_i1)$ of the word u'_i1 is odd, say, $2j+1$, when $i=0$ and even, say, $2j$, when $i\ge 1$.

Thus, the mapping $w\mapsto(w,i,\lambda(u'_i1))$ is a bijection of $\mathcal{T}_{n+1,\mathbf{c}',2}$ to the set of triples (w,i,j), where $0\le i\le m'$; $w\in\mathcal{T}_n$, Λw is equal to $\mathbf{c}'[0,2j+1]$ when $i=0$ and to $\mathbf{c}'[i,2j]$ when $1\le i\le m'$; $0\le 2j+1\le c'_0$ when $i=0$ and $2\le 2j\le c'_i$ when $1\le i\le m'$. Moreover,

$$\operatorname{inv} w'=\begin{cases}2j+\operatorname{inv} w, & \text{when } i=0;\\ c'_0+\cdots+c'_{i-1}+2j-1+\operatorname{inv} w, & \text{when } 1\le i\le m'.\end{cases}$$

Case (ii): the mapping $w'\mapsto(w,i)$ is a bijection of $\mathcal{T}_{n+1,\mathbf{c}',1}$ to the set of the pairs (w,i), where $w:=(w_0'^-,\ldots,w_{i-1}'^-,w_{i+1}'^-,w_{m'}'^-)$ and $1\le i\le m'-1$. Thus, $\Lambda w=\mathbf{c}'[i,1]$. Moreover, $\operatorname{inv} w'=c'_0+\cdots+c'_{i-1}+\operatorname{inv} w$.

Accordingly, (6.4) holds. ∎

Let $\mathbf{c}=(c_0,c_1,\ldots,c_m)$ be a composition of a nonnegative integer n. If $1\le i\le m$, let

$$^{(i)}\mathbf{c}:=(c_0,\ldots,c_{i-2},c_{i-1}+1+c_i,c_{i+1},\ldots,c_m); \tag{6.5}$$

$$\mathbf{c}^{(i)}:=(c_0,\ldots,c_{i-1},1,c_i,\cdots,c_m). \tag{6.6}$$

With the notations of (6.1)–(6.3) and (6.5)–(6.6) we then have:

$$^{(1)}(\mathbf{c}'[0,2j+1])=\mathbf{c}'. \tag{6.7}$$

$$^{(i+1)}(\mathbf{c}'[i,2j])=\mathbf{c}'. \tag{6.8}$$

$$(\mathbf{c}'[i,1])^{(i)}=\mathbf{c}'. \tag{6.9}$$

Lemma 6.2. *Let* $\mathbf{c} = (c_0, c_1, \dots, c_m) \in \Theta_n$. *Then,*

$$D_q \tan_q(q^{\mathbf{c}} u) = \sum_{1 \le i \le m} q^{c_0 + \cdots + c_{i-1}} \big(\tan_q(q^{{}^{(i)}\mathbf{c}} u) + \tan_q(q^{\mathbf{c}^{(i)}} u) \big).$$

Proof.

$$\begin{aligned} D_q \tan_q(q^{\mathbf{c}} u) &= D_q \prod_{1 \le i \le m} \tan_q(q^{c_0 + \cdots + c_{i-1}} u) \\ &= \sum_{1 \le i \le m} \tan_q(q^{c_0} u) \cdots \tan_q(q^{c_0 + \cdots + c_{i-2}} u) \\ &\quad \times q^{c_0 + \cdots + c_{i-1}} \big(1 + \tan_q(q^{c_0 + \cdots + c_{i-1}} u) \tan_q(q^{c_0 + \cdots + c_{i-1} + 1} u) \big) \\ &\quad \times \tan_q(q^{c_0 + \cdots + c_{i-1} + 1 + c_i} u) \cdots \tan_q(q^{c_0 + \cdots + c_{i-1} + 1 + c_i + \cdots + c_{m-1}} u) \\ &= \sum_{1 \le i \le m} q^{c_0 + \cdots + c_{i-1}} \big(\tan_q(q^{{}^{(i)}\mathbf{c}} u) + \tan_q(q^{\mathbf{c}^{(i)}} u) \big). \quad \square \end{aligned}$$

In the next example, let $\mathbf{c} := \tan_q(q^{\mathbf{c}} u)$ by convention, so that

$$D_q \mathbf{c} = \sum_{1 \le i \le m} q^{c_0 + \cdots + c_{i-1}} \big({}^{(i)}\mathbf{c} + \mathbf{c}^{(i)} \big).$$

We get

$$\begin{aligned} D_q((0,0)) &:= (1) + (0,1,0) \\ D_q^2((0,0)) &:= D_q\big((1) + (0,1,0)\big) \\ &= 0 + \big((2,0) + q(0,2) + (0,1,1,0) + q(0,1,1,0)\big) \\ &= (2,0) + q(0,2) + (1+q)(0,1,1,0); \\ D_q^3((0,0)) &:= D_q\big((2,0) + q(0,2) + (1+q)(0,1,1,0)\big) \\ &= \big(q^2(3) + q^2(2,1,0)\big) + q\big((3) + (0,1,2)\big) \\ &\qquad + (1+q)((2,1,0) + q(0,3,0) + q^2(0,1,2) \\ &\qquad\qquad + (0,1,1,1,0) + q(0,1,1,1,1,0) + q^2(0,1,1,1,0)\big) \\ &= (q+q^2)(3) + (1+q+q^2)(2,1,0) + (q+q^2+q^3)(0,1,2) \\ &\qquad\qquad + (q+q^2)(0,3,0) + (1+q)(1+q+q^2)(0,1,1,1,0). \end{aligned}$$

We can then calculate the polynomials (see also Table 3): $A_{1,(1)}(q) = A_{1,(0,1,0)}(q) = 1$;
$A_{2,(2,0)}(q) = 1$, $A_{2,(0,2)}(q) = q$, $A_{2,(0,1,1,0)}(q) = 1$, $A_{2,(0,1,1,0)}(q) = q$;
$A_{3,(3)}(q) = q + q^2$, $A_{3,(2,1,0)}(q) = 1 + q + q^2$, $A_{3,(0,1,2)}(q) = q + q^2 + q^3$, $A_{3,(0,3,0)}(q) = q + q^2$, $A_{3,(0,1,1,1,0)}(q) = (1+q)(1+q+q^2)$.

Proof of (1.14). Let $\mathbf{c}'=(c'_0,c'_1,\ldots,c'_{m'})$. By (6.1) and (6.7) we have ${}^{(1)}\mathbf{c}=\mathbf{c}'$ when $c'_0\ge 2$, if and only if $\mathbf{c}=\mathbf{c}'[0,2j+1]$ for some j such that $1\le 2j+1\le c'_0$. By (6.2) and (6.8) the relation ${}^{(i)}\mathbf{c}=\mathbf{c}'$ holds for i such that $1\le i\le m'$ and $c'_i\ge 2$, if and only if $\mathbf{c}=\mathbf{c}'[i,2j]$ for some j such that $2\le 2j\le c'_i$. Finally, $\mathbf{c}^{(i)}=\mathbf{c}'$ holds for i such that $1\le i\le m'-1$ and $c'_i=1$, if and only if $\mathbf{c}=\mathbf{c}'[1,i]$.

Finally, (1.14) is proved by induction. Assume it is rue for n. Then,

$$\begin{aligned}
D_q^{n+1}\tan_q(u)&=D_qD_q^n\tan_q(u)=D_q\Bigl(\sum_{\mathbf{c}\in\Theta_n}A_{n,\mathbf{c}}(q)\,\tan_q(q^{\mathbf{c}}u)\Bigr)\\
&=\sum_{\mathbf{c}\in\Theta_n}A_{n,\mathbf{c}}(q)\,D_q\tan_q(q^{\mathbf{c}}u)\\
&=\sum_{\mathbf{c}\in\Theta_n}A_{n,\mathbf{c}}(q)\sum_{1\le i\le m}q^{c_0+\cdots+c_{i-1}}\bigl(\tan_q(q^{{}^{(i)}\mathbf{c}}u)+\tan_q(q^{\mathbf{c}^{(i)}}u)\bigr).\\
&=\sum_{\mathbf{c}'\in\Theta_{n+1}}\Bigl(\sum_{0\le 2j+1\le c'_0}q^{2j}A_{n,\mathbf{c}'[0,2j+1]}(q)\\
&\qquad+\sum_{1\le i\le m'}q^{c'_0+\cdots+c'_{i-1}}\sum_{2\le 2j\le c'_i}q^{2j-1}A_{n,\mathbf{c}'[i,2j]}(q)\\
&\qquad+\sum_{1\le i\le m'-1}\chi(c'_i=1)q^{c'_0+\cdots+c'_{i-1}}A_{n,\mathbf{c}'[i,1]}(q)\Bigr)\,\tan_q(q^{\mathbf{c}'}u)\\
&=\sum_{\mathbf{c}'\in\Theta_{n+1}}\Bigl(A_{n+1,\mathbf{c}'}(q)\Bigr)\,\tan_q(q^{\mathbf{c}'}u).\quad\text{[By Theorem 6.1]}\quad\square
\end{aligned}$$

For each $\mathbf{c}=(c_0,c_1,\ldots,c_m,0)\in\Theta_n^-$, let $\Phi(\mathbf{c}):=\tan_q(q^{\mathbf{c}^-}u)\sec_q(q^nu)$.

Lemma 6.3. *Let* $\mathbf{c}=(c_0,c_1,\ldots,c_m,0)\in\Theta_n^-$. *Then,*

$$D_q\Phi(\mathbf{c})=\sum_{1\le i\le m}\Bigl(q^{c_0+\cdots+c_{i-1}}(\Phi({}^{(i)}\mathbf{c})+\Phi(\mathbf{c}^{(i)})\Bigr)+q^n\Phi\mathbf{c}^{(m+1)}.$$

Proof.

$$\begin{aligned}
D_q\Phi(\mathbf{c})&=D_q\tan_q(q^{\mathbf{c}^-}u)\sec_q(q^nu)\\
&=\sum_{1\le i\le m}\tan_q(q^{c_0}u)\cdots\tan_q(q^{c_0+\cdots+c_{i-2}}u)\\
&\qquad\times q^{c_0+\cdots+c_{i-1}}\bigl(1+\tan_q(q^{c_0+\cdots+c_{i-1}}u)\tan_q(q^{c_0+\cdots+c_{i-1}+1}u)\bigr)\\
&\qquad\times\tan_q(q^{c_0+\cdots+c_{i-1}+1+c_i}u)\cdots\tan_q(q^{c_0+\cdots+c_{i-1}+1+c_i+\cdots+c_{m-1}}u)\\
&\qquad\times\sec_q(q^{n+1}u)\\
&\quad+q^n\tan_q(q^{\mathbf{c}^-}u)\tan_q(q^nu)\sec_q(q^{n+1}u)\\
&=\sum_{1\le i\le m}\Bigl(q^{c_0+\cdots+c_{i-1}}(\Phi({}^{(i)}\mathbf{c})+\Phi(\mathbf{c}^{(i)})\Bigr)+q^n\Phi\mathbf{c}^{(m+1)}.\quad\square
\end{aligned}$$

Proof of (1.15). Again (1.15) is proved by induction. Assume that (1.15) holds for n. Then,

$$\begin{aligned}
D_q^{n+1}\sec_q(u) &= D_q D_q^n \sec_q(u) = D_q\Big(\sum_{\mathbf{c}\in\Theta_n^-} A_{n,\mathbf{c}}(q)\,\Phi(\mathbf{c})\Big)\\
&= \sum_{\mathbf{c}\in\Theta_n^-} A_{n,\mathbf{c}}(q)\, D_q\Phi(\mathbf{c})\\
&= \sum_{\mathbf{c}\in\Theta_n^-} A_{n,\mathbf{c}}(q) \sum_{1\le i\le m}\Big(q^{c_0+\cdots+c_{i-1}}(\Phi({}^{(i)}\mathbf{c})+\Phi(\mathbf{c}^{(i)})\Big)+q^n\Phi\mathbf{c}^{(m+1)}\\
&= \sum_{\mathbf{c}'\in\Theta_{n+1}^-}\Big(\sum_{0\le 2j+1\le c'_0} q^{2j}A_{n,\mathbf{c}'[0,2j+1]}(q)\\
&\qquad + \sum_{1\le i\le m'} q^{c'_0+\cdots+c'_{i-1}} \sum_{2\le 2j\le c'_i} q^{2j-1}A_{n,\mathbf{c}'[i,2j]}(q)\\
&\qquad + \sum_{1\le i\le m'-1}\chi(c'_i=1)q^{c'_0+\cdots+c'_{i-1}}A_{n,\mathbf{c}'[i,1]}(q)\Big)\,\Phi(\mathbf{c}')\\
&= \sum_{\mathbf{c}'\in\Theta_{n+1}^-} A_{n+1,\mathbf{c}'}(q)\,\Phi(\mathbf{c}'). \quad \text{[By Theorem 6.1]} \quad \square
\end{aligned}$$

Proof of (1.16). In our proof of (1.16) given next we again make a full use of the duality derived in Section 2 between the q-series $\mathrm{Sec}_q(u)$, $\tan_q(u)$ and their analogs $\sec_{q^{-1}}(u)$, $\tan_{q^{-1}}(u)$. Since $\tan_{q^{-1}}((q^{-1})^{\mathbf{c}^-}u) = (-1)^{\mu\mathbf{c}}\tan_q((q^{-1})^{\mathbf{c}^-}qu)$, identity (1.15) can be rewritten

$$\begin{aligned}
D_{q^{-1}}^n\sec_{q^{-1}}(u) &= \sum_{\mathbf{c}\in\Theta_n^-} A_{n,\mathbf{c}}(q^{-1})\ \tan_{q^{-1}}((q^{-1})^{\mathbf{c}^-}u)\ \sec_{q^{-1}}((q^{-1})^n u)\\
&= (-1)^{\mu\mathbf{c}}\sum_{\mathbf{c}\in\Theta_n^-} A_{n,\mathbf{c}}(q^{-1})\ \tan_q((q^{-1})^{\mathbf{c}^-}qu)\ \mathrm{Sec}_q((q^{-1})^n qu).
\end{aligned}$$

By Lemma 2.3,

$$Q^{n-1}D_{q^{-1}}^n\sec_{q^{-1}}(u) = (-1)^{\mu\mathbf{c}}\sum_{\mathbf{c}\in\Theta_n^-} A_{n,\mathbf{c}}(q^{-1})\tan_q(q^{\rho\mathbf{c}^-}u)\,\mathrm{Sec}_q(u).$$

By Lemma 2.4, and since n and $\mu\mathbf{c}$ are of the same parity, we get

$$\begin{aligned}
D_q^n\,\mathrm{Sec}_q(u) &= (-1)^n q^{n(n-1)/2}Q^{n-1}D_{q^{-1}}^n\sec_{q^{-1}}(u)\\
&= \sum_{\mathbf{c}\in\Theta_n^-} q^{n(n-1)/2}A_{n,\mathbf{c}}(q^{-1})\tan_q(q^{\rho\mathbf{c}^-}u)\,\mathrm{Sec}_q(u).
\end{aligned}$$

This proves identity (1.16). $\square$

7. More on q-trigonometric functions

Some parts of this section are of semi-expository nature, although, to our knowledge, the combinatorial properties of "$\mathrm{Sec}_q(u)$" have not been explicitly written down. Let

$$\tan_q(u) = \sum_{n\ge 0} A_{2n+1}(q)\frac{u^{2n+1}}{(q;q)_{2n+1}}; \tag{7.1}$$

$$\sec_q(u) = \sum_{n\ge 0} A_{2n}(q)\frac{u^{2n}}{(q;q)_{2n}}; \tag{7.2}$$

$$\mathrm{Sec}_q(u) = \sum_{n\ge 0} A_{2n}^{\mathrm{Sec}}(q)\frac{u^{2n}}{(q;q)_{2n}}; \tag{7.3}$$

be the q-expansions of the three series $\tan_q(u)$, $\sec_q(u)$, $\mathrm{Sec}_q(u)$, respectively.

Let $\left[{N \atop M}\right]_q := (q;q)_N/((q;q)_M\,(q;q)_{N-M})$ $(0\le M\le N)$ be the Gaussian polynomial. Identities (2.3)-(2.5) yield

$$A_{2n+1}(q) = \sum_{0\le k\le n-1} \begin{bmatrix} 2n \\ 2k+1\end{bmatrix} q^{2k+1} A_{2k+1}(q)\, A_{2n-2k-1}(q), \tag{7.4}$$

$$A_{2n}(q) = \sum_{0\le k\le n-1} \begin{bmatrix} 2n-1 \\ 2k\end{bmatrix}_q q^{2k} A_{2k}(q)\, A_{2n-2k-1}(q), \tag{7.5}$$

$$A_{2n}^{\mathrm{Sec}}(q) = \sum_{0\le k\le n-1} \begin{bmatrix} 2n-1 \\ 2k\end{bmatrix}_q A_{2k}^{\mathrm{Sec}}(q)\, q^{2n-2k-1} A_{2n-2k-1}(q), \tag{7.6}$$

for $n\ge 1$ with the initial conditions $A_1(q)=1$, $A_0(q)=1$ and $A_0^{\mathrm{Sec}}(q)=1$. Hence, the coefficients $A_n(q)$, $A_{2n}^{\mathrm{Sec}}(q)$ $(n\ge 0)$ occurring in the q-expansions of $\tan_q(u)$, $\sec_q(u)$, $\mathrm{Sec}_q(u)$ in (7.1)–(7.2) are polynomials with positive integral coefficients.

The first values of the polynomials $A_n(q)$ and $A_{2n}^{\mathrm{Sec}}(q)$ $(n\ge 0)$ can be calculated by means of (7.4)—(7.6):

$A_1(q)=1$; $A_3(q)=q+q^2$; $A_5(q)=q^2+2q^3+3q^4+4q^5+3q^6+2q^7+q^8$;
$A_0(q)=A_2(q)=1$; $A_4(q)=q+2q^2+q^3+q^4$;
$A_6(q)=q^2+3q^3+5q^4+8q^5+10q^6+10q^7+9q^8+7q^9+5q^{10}+2q^{11}+q^{12}$;
$A_0^{\mathrm{Sec}}(q)=A_2^{\mathrm{Sec}}(q)=q$; $A_4^{\mathrm{Sec}}(q)=q^2+q^3+2q^4+q^5$;
$A_6^{\mathrm{Sec}}(q)=q^3+2q^4+5q^5+7q^6+9q^7+10q^8+10q^9+8q^{10}+5q^{11}+3q^{12}+q^{13}$;

We note that the two identities (7.5) and (7.6) can be combined into the following single formula, valid for $n\ge 1$ with $A_0(q)=1$:

$$A_n(q) = \sum_{0\le k\le \lfloor n/2\rfloor -1} \begin{bmatrix} n-1 \\ 2k+1\end{bmatrix}_q q^{n-2k-2} A_{2k+1}(q) A_{n-2k-2}(q).$$

The polynomials $A_{2n+1}(q)$ (resp. $A_{2n}(q)$) defined by (7.1) and (7.2) are usually called the *q-tangent numbers* and *q-secant numbers*, respectively. No traditional name exists for the polynomials $A_{2n}^{\rm Sec}(q)$, as they are intimately related to the $A_{2n+1}(q)$'s by identity (7.12).

For each $n \geq 0$ the set of all rising (resp. falling) alternating permutations $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $1\,2\,\ldots\,n$ is denoted by $\mathcal{RA}_n$ (resp. $\mathcal{FA}_n$). As already mentioned in Section 7, the old result by Désiré André [An81] asserts that both $\#\mathcal{RA}_{2n+1}$ and $\#\mathcal{FA}_{2n+1}$ (resp. $\#\mathcal{RA}_{2n}$ and $\#\mathcal{FA}_{2n}$) are equal to the tangent number T_{2n+1} (resp. secant number E_{2n}) occurring in (1.1) and (1.2)

For the q-analog of this result we keep the same combinatorial set-up, namely, $\mathcal{RA}_n$ and $\mathcal{FA}_n$ ($n \geq 0$), but, as there are two different q-secants, $\sec_q(u)$ and $\mathrm{Sec}_q(u)$, the coefficients in their q-expansions will have different combinatorial interpretations. For each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$ (not necesarily an alternating permutation), let $\mathrm{inv}\,\sigma$ denote the traditional *number of inversions* of σ.

Theorem 7.1. *Let $A_{2n+1}(q)$ (resp. $A_{2n}(q)$, resp. $A_{2n}^{\rm Sec}(q)$) be the coefficients in the q-expansion of $\tan_q(u)$ (resp. of $\sec_q(u)$, resp. of $\mathrm{Sec}_q(u)$), as defined in* (7.1)–(7.2). *Then,*

$$A_{2n+1}(q) = \sum_{\sigma\in\mathcal{RA}_{2n+1}} q^{\mathrm{inv}\,\sigma} = \sum_{\sigma\in\mathcal{FA}_{2n+1}} q^{\mathrm{inv}\,\sigma}; \tag{7.7}$$

$$A_{2n}(q) = \sum_{\sigma\in\mathcal{RA}_{2n}} q^{\mathrm{inv}\,\sigma}; \tag{7.8}$$

$$A_{2n}^{\rm Sec}(q) = \sum_{\sigma\in\mathcal{FA}_{2n}} q^{\mathrm{inv}\,\sigma}. \tag{7.9}$$

The proofs of (7.7) – (7.9) are not reproduced here. It suffices to q-mimick Desiré André's [An81] classical proof.

The statistics "Ligne" and "imaj" have been defined in Section 8. By means of the so-called "second fundamental transformation" (see, e.g., [Lo83, § 10.6], [Fo68], [FS78]) we can construct a bijection Φ of the group of all permutations onto itself with the property that

$$\mathrm{Ligne}\,\sigma = \mathrm{Ligne}\,\Phi(\sigma) \quad \text{and} \quad \mathrm{inv}\,\sigma = \mathrm{imaj}\,\Phi(\sigma). \tag{7.10}$$

Saying that a permutation σ is falling (resp. rising) alternating is equivalent to saying that $\mathrm{Ligne}\,\sigma = \{1,3,5,\ldots\}$ (resp. $= \{2,4,6,\ldots\}$). Accordingly, we also have

$$A_{2n+1}(q) = \sum_{\sigma\in\mathcal{RA}_{2n+1}} q^{\mathrm{imaj}\,\sigma} = \sum_{\sigma\in\mathcal{FA}_{2n+1}} q^{\mathrm{imaj}\,\sigma};$$

$$A_{2n}(q)=\sum_{\sigma\in\mathcal{RA}_{2n}} q^{\operatorname{imaj}\sigma};$$
$$A_{2n}^{\rm Sec}(q)=\sum_{\sigma\in\mathcal{FA}_{2n}} q^{\operatorname{imaj}\sigma}.$$

For each permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ let ρ (the *mirror-image*) and γ (the *complement*) be defined by

$$\gamma\,\sigma(i):=n+1-\sigma(i);\quad \rho\,\sigma(i):=\sigma(n+1-i)\quad (1\le i\le n).$$

The transformation $\rho\gamma$ is a bijection of $\mathcal{RA}_{2n+1}$ onto $\mathcal{FA}_{2n+1}$ preserving the number of inversions. This makes up a combinatorial proof of the second identity in (7.7).

Proposition 10.1. *We have*

$$q^{(2n+1)(2n)/2}A_{2n+1}(q^{-1})=A_{2n+1}(q); \tag{7.11}$$
$$q^{(2n)(2n-1)/2}A_{2n}(q^{-1})=A_{2n}^{\rm Sec}(q). \tag{7.12}$$

Proof. The transformation ρ is a bijection of $\mathcal{RA}_{2n+1}$ onto $\mathcal{RA}_{2n+1}$ with the property that: $\operatorname{inv}\sigma+\operatorname{inv}\rho\,\sigma=(2n)(2n+1)/2$. By (7.7) we have $A_{2n+1}(q)=\sum_{\sigma\in\mathcal{FA}_{2n+1}} q^{\operatorname{inv}\sigma}=\sum_{\sigma\in\mathcal{FA}_{2n+1}} q^{\operatorname{inv}\rho\,\sigma}=q^{2n(2n+1)/2}\sum_{\sigma\in\mathcal{FA}_{2n+1}} q^{-\operatorname{inv}\sigma}=q^{2n(2n+1)/2}A_{2n+1}(q^{-1})$. In the same manner, the transformation ρ is a bijection of $\mathcal{RA}_{2n}$ onto $\mathcal{FA}_{2n}$ with the property that: $\operatorname{inv}\sigma+\operatorname{inv}\rho\,\sigma=(2n)(2n-1)/2$. Hence, $A_{2n}^{\rm Sec}(q)=\sum_{\sigma\in\mathcal{FA}_{2n}} q^{\operatorname{inv}\sigma}=\sum_{\sigma\in\mathcal{RA}_{2n}} q^{\operatorname{inv}\rho\,\sigma}=q^{2n(2n-1)/2}\sum_{\sigma\in\mathcal{RA}_{2n}} q^{-\operatorname{inv}\sigma}=q^{2n(2n-1)/2}A_{2n}(q^{-1})$ □

8. Proof of Theorem 1.4

By means of the second fundamental transformation Φ, already mentioned in (7.10), and the bijection $\mathbf{i}$ that maps each permutation σ from the symmetric group $\mathfrak{S}_n$ onto its inverse σ^{-1}, we can form $\psi:=\mathbf{i}\,\Phi\,\mathbf{i}$. The latter bijection has the following properties:

$$\operatorname{Ligne}\psi(\sigma)=\operatorname{Ligne}\sigma;\qquad \operatorname{inv}\psi(\sigma)=\operatorname{imaj}\sigma.$$

If $\mathbf{c}=(c_0,c_1,\ldots,c_m)$ is a composition of n and $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ a permutation from $\mathfrak{S}_n$, let

$$(\operatorname{Ligne}\setminus\mathbf{c})\sigma:=\operatorname{Ligne}\sigma\setminus\{c_0,c_0+c_1,\ldots,c_0+c_1+\cdots+c_{m-1}\},$$

so that for every composition $\mathbf{c}$ of n

$$(\operatorname{Ligne}\setminus\mathbf{c})\psi(\sigma)=(\operatorname{Ligne}\setminus\mathbf{c})\sigma;\qquad \operatorname{inv}\psi(\sigma)=\operatorname{imaj}\sigma. \tag{8.1}$$

Now, consider a t-permutation $w = (w_0, w_1, \ldots, w_m)$ of order n such that $\Lambda w = \mathbf{c} = (c_0, c_1, \ldots, c_m)$. The transformation ψ maps the *permutation* $\sigma := w_0 w_1 \cdots w_m$ to another permutation $\psi(\sigma)$. Let $(w'_0, w'_1, \ldots, w'_m)$ be the factorization of $\psi(\sigma)$, written as a word of n letters, defined by: $\lambda w'_0 = c_0$, $\lambda w'_1 = c_1, \ldots, \lambda w'_m = c_m$. Property (8.1) implies that the mapping

$$\psi : w = (w_0, w_1, \ldots, w_m) \mapsto w' := (w'_0, w'_1, \ldots, w'_m)$$

is a bijection of $\mathcal{T}_n$ onto itself having the properties:

$$\Lambda w' = \Lambda w = \mathbf{c}, \quad \operatorname{inv} w' = \operatorname{imaj} w. \tag{8.2}$$

For instance, (see [FS78, p. 147] where the same numerical example is here reproduced) we have: $\Phi(7\,4\,9\,2\,6\,1\,5\,8\,3) = 4\,7\,2\,6\,1\,9\,5\,8\,3$. Hence,

$$w = 6\,4\,9\,2\,7\,5\,1\,8\,3 \xrightarrow{\mathbf{i}} 7\,4\,9\,2\,6\,1\,5\,8\,3 \xrightarrow{\Phi} 4\,7\,2\,6\,1\,9\,5\,8\,3 \xrightarrow{\mathbf{i}} 5\,3\,9\,1\,7\,4\,2\,8\,6 = w'.$$

The t-permutation $w = (\epsilon, 6, 4, 927, 5183)$ is then mapped under ψ onto $w' = (\epsilon, 5, 3, 917, 4286)$. Moreover, $\operatorname{imaj} w = 1+3+5+8 = 17 = 1+3+1+3+5+1+3 = \operatorname{inv} w'$.

From Theorem 1.3 and (8.2) it then follows that

$$A_{n,\mathbf{c}}(q) = \sum_{w \in \mathcal{T}_n,\, \Lambda w = \mathbf{c}} q^{\operatorname{inv} w} = \sum_{w \in \mathcal{T}_n,\, \Lambda w = \mathbf{c}} q^{\operatorname{imaj} w}.$$

Now, sum the previous identity over all t-compositions of n. We get:

$$\begin{aligned}\sum_{\mu\, \mathbf{c} = m} A_{n,\mathbf{c}}(q) &= \sum_{\mu\, \mathbf{c} = m} \sum_{w \in \mathcal{T}_n,\, \Lambda w = \mathbf{c}} q^{\operatorname{inv} w} = \sum_{\mu\, \mathbf{c} = m} \sum_{w \in \mathcal{T}_n,\, \Lambda w = \mathbf{c}} q^{\operatorname{imaj} w} \\ &= \sum_{\substack{k \geq 0,\, a+b=m \\ w \in \mathcal{T}_{n,k,a,b}}} q^{\operatorname{imaj} w} = \sum_{\substack{k \geq 0 \\ a+b=m}} A_{n,k,a,b}(q).\end{aligned}$$

This proves (1.17).

For instance, from the tables 2 and 3 we can verify that (1.17) holds for $n = 3$ and for $m = 2$: $A_{3,0,0,2}(q) + A_{3,1,0,2}(q) + A_{3,1,1,1}(q) + A_{3,1,2,0}(q) + A_{3,2,2,0}(q) = 1 + q^2 + (2q + 2q^2) + q + q^3 = (q + q^2) + (1 + q + q^2) + (q + q^2 + q^3) = A_{3,(3)}(q) + A_{3,(2,1,0)}(q) + A_{3,(0,1,2)}(q)$.

The same technique of proof can be used for the s-permutations. We get:

$$\begin{aligned}\sum_{\mathbf{c} \in \Theta^-_{n,m+1}} A_{n,\mathbf{c}}(q) &= \sum_{\mathbf{c} \in \Theta^-_{n,m+1}} \sum_{w \in \mathcal{T}^-_n,\, \Lambda w = \mathbf{c}} q^{\operatorname{inv} w} = \sum_{\mathbf{c} \in \Theta^-_{n,m+1}} \sum_{w \in \mathcal{T}^-_n,\, \Lambda w = \mathbf{c}} q^{\operatorname{imaj} w} \\ &= \sum_{\substack{k \geq 0,\, a+b=m \\ w \in \mathcal{T}^-_{n,k,a,b+1}}} q^{\operatorname{imaj} w} = \sum_{\substack{k \geq 0 \\ a+b=m}} B_{n,k,a,b}(q).\end{aligned}$$

This proves (1.18).

9. Proof of Theorem 1.5

If $\mathcal{J}=(J_0,J_1,\ldots,J_{m-1},J_m)$ is a sequence of disjoint subsets of the interval $[\,n\,]:=\{1,2,\ldots,n\}$ of union $[\,n\,]$ with $m\ge 1$, then $(\#J_0,\#J_1,\ldots,\#J_m)$ is a composition $\mathbf{c}=(c_0,c_1,\ldots,c_m)$ of n. We then write $\#\,\mathcal{J}:=\mathbf{c}$. Also, let $\operatorname{inv}\mathcal{J}$ denote the number of ordered pairs (x,y) where $x\in J_k$, $y\in J_l$, $k<l$ and $x>y$. A classical result that goes back to MacMahon (see, e.g. [An76, § 3.4]) makes it possible to write for each composition $\mathbf{c}$ of n

$$\sum_{\mathcal{J},\,\#\mathcal{J}=\mathbf{c}} q^{\operatorname{inv}\mathcal{J}} = \begin{bmatrix} n \\ c_0,c_1,\ldots,c_m\end{bmatrix}_q,$$

where the right-hand side is the *q-multinomial coefficient* equal to

$$\frac{(q;q)_n}{(q;q)_{c_0}\,(q;q)_{c_1}\,\cdots\,(q;q)_{c_m}}.$$

Theorem 9.1. *For each t-composition* $\mathbf{c}=(c_0,c_1,\ldots,c_{m-1},c_m)$ *of* n $(n\ge 1)$ *we have:*

$$A_{n,\mathbf{c}}(q)=\begin{bmatrix} n \\ c_0,c_1,\ldots,c_{m-1},c_m\end{bmatrix}_q A_{c_0}(q)A_{c_1}(q)\cdots A_{c_{m-1}}(q)A^{\mathrm{Sec}}_{c_m}(q).$$

Proof. Each t-permutation w from $\mathcal{T}_n$ such that $\Lambda w=\mathbf{c}$ and $\mu\,\mathbf{c}=m\ge 1$ is completely characterized by a sequence

$$\bigl((I_0,\sigma_0),(I_1,\sigma_1),\ldots,(I_m,\sigma_m)\bigr),$$

having the following properties:

(i) the sequence $\mathcal{I}(w):=(I_0,I_1,\ldots,I_m)$ consists of disjoint subsets of the interval $[\,n\,]:=\{1,2,\ldots,n\}$ of union $[\,n\,]$; moreover, $\#\mathcal{I}(w):=\mathbf{c}$;

(ii) $\sigma_0\in\mathcal{RA}_{c_0}$, $\sigma_1\in\mathcal{FA}_{c_1}$, … , $\sigma_m\in\mathcal{FA}_{c_m}$.

If $\mathcal{I}(w)=\mathcal{J}$, then $\operatorname{inv}w=\operatorname{inv}\mathcal{J}+\operatorname{inv}\sigma_0+\operatorname{inv}\sigma_1+\cdots+\operatorname{inv}\sigma_{m-1}+\operatorname{inv}\sigma_m$. Hence,

$$\begin{aligned}
A_{n,\mathbf{c}}(q)&=\sum_{w\in\mathcal{T}_n,\Lambda w=\mathbf{c}} q^{\operatorname{inv}w}=\sum_{w\in\mathcal{T}_n,\,\#\mathcal{I}(w)=\mathbf{c}} q^{\operatorname{inv}w}\\
&=\sum_{\mathcal{J},\,\#\mathcal{J}=\mathbf{c}}\ \sum_{w,\,\mathcal{I}(w)=\mathcal{J}} q^{\operatorname{inv}w}\\
&=\sum_{\mathcal{J},\,\#\mathcal{J}=\mathbf{c}} q^{\operatorname{inv}\mathcal{J}}\sum_{\sigma_0\in\mathcal{RA}_{c_0}} q^{\operatorname{inv}\sigma_0}\sum_{\sigma_1\in\mathcal{FA}_{c_1}} q^{\operatorname{inv}\sigma_1}\times\cdots\times\sum_{\sigma_m\in\mathcal{FA}_{c_m}} q^{\operatorname{inv}\sigma_m}\\
&=\begin{bmatrix} n \\ c_0,c_1,\ldots,c_m\end{bmatrix}_q A_{c_0}(q)A_{c_1}(q)\cdots A^{\mathrm{Sec}}_{c_m}(q).\qquad\Box
\end{aligned}$$

The factorial generating functions for the polynomials

$$A_n(x,q) := \sum_{\mathbf{c}\in\Theta_n} x^{\mu\,\mathbf{c}} A_{n,\mathbf{c}}(q), \qquad B_n(x,q) := \sum_{\mathbf{c}\in\Theta_n^-} x^{\mu\,\mathbf{c}-1} B_{n,\mathbf{c}}(q),$$

can be derived from Theorem 9.1.

Proof of Theorem 1.5. For $m \ge 1$ we have:

$$\begin{aligned}
&\sum_{n\ge0} \frac{u^n}{(q;q)_n} \sum_{\substack{w\in\Theta_n\\ \mu\,\mathbf{c}=m}} A_{n,\mathbf{c}}(q)\\
&\quad= \sum_{n\ge0} \frac{u^n}{(q;q)_n} \sum_{\substack{c_0+c_1+\cdots\\ +c_m=n}} \begin{bmatrix} n \\ c_0,c_1,\dots,c_m\end{bmatrix}_q A_{c_0}(q)A_{c_1}(q)\cdots A^{\mathrm{Sec}}_{c_m}(q)\\
&\quad= \sum_{n\ge0} u^n \sum_{\substack{c_0+c_1+\cdots\\ +c_m=n}} \frac{A_{c_0}(q)}{(q;q)_{c_0}} \frac{A_{c_1}(q)}{(q;q)_{c_1}} \cdots \frac{A_{c_{m-1}}(q)}{(q;q)_{c_{m-1}+1}} \frac{A^{\mathrm{Sec}}_{c_m}(q)}{(q;q)_{c_m}}.
\end{aligned}$$

When $m \ge 1$, the integers c_0 and c_m are even; if, furthermore, $m \ge 2$, then c_1, $\dots$, c_{m-1} are odd. Hence, the previous identity may be rewritten as

$$\begin{aligned}
&\sum_{n\ge0} \frac{u^n}{(q;q)_n} \sum_{\substack{w\in\Theta_n\\ \mu\,\mathbf{c}=m}} A_{n,\mathbf{c}}(q)\\
&\quad= \sum_{j_0\ge0} A_{2j_0}(q)\frac{u^{2j_0}}{(q;q)_{2j_0}} \Bigl(\sum_{j\ge0} A_{2j+1}(q)\frac{u^{2j+1}}{(q;q)_{2j+1}}\Bigr)^{m-1} \sum_{j_m\ge0} A^{\mathrm{Sec}}_{2j_m}(q)\frac{u^{2j_m}}{(q;q)_{2j_m}}\\
&\quad= \sec_q(u)\,\bigl(\tan_q(u)\bigr)^{m-1}\ \mathrm{Sec}_q(u).
\end{aligned}$$

When $m=0$, then n is *odd*. Hence, $A_{n,\mathbf{c}}(q) = A_n(q)$ (n odd) and

$$\sum_{n\ge0} \frac{u^n}{(q;q)_n} \sum_{\substack{w\in\Theta_n\\ \mu\,\mathbf{c}=0}} A_{n,\mathbf{c}}(q) = \tan_q(u).$$

Thus,

$$\begin{aligned}
\sum_{n\ge0} A_n(x,q)\frac{u^n}{(q;q)_n} &= \sum_{n\ge0} \frac{u^n}{(q;q)_n} \sum_{m\ge0} x^m \sum_{\substack{w\in\Theta_n\\ \mu\,\mathbf{c}=m}} A_{n,\mathbf{c}}(q)\\
&= \sum_{n\ge0} \frac{u^n}{(q;q)_n} \sum_{\substack{w\in\Theta_n\\ \mu\,\mathbf{c}=0}} A_{n,\mathbf{c}}(q) + \sum_{m\ge1} x^m \sum_{n\ge0} \frac{u^n}{(q;q)_n} \sum_{\substack{w\in\Theta_n\\ \mu\,\mathbf{c}=m}} A_{n,\mathbf{c}}(q)\\
&= \tan_q(u) + \sum_{m\ge1} \sec_q(u)\,\bigl(x\tan_q(u)\bigr)^{m-1}\, x\,\mathrm{Sec}_q(u)\\
&= \tan_q(u) + \sec_q(u)(1-x\tan_q(u))^{-1} x\,\mathrm{Sec}_q(u),
\end{aligned}$$

which proves (1.19).

The proof of (1.20) is quite similar. The only difference is the fact that $\mathbf{c} = (c_0, c_1, \ldots, c_m, 0)$ is now an s-composition, so that, c_0 is even and if $m \ge 1$ all the other c_i's are odd. We then get:

$$\sum_{n\ge 0} \frac{u^n}{(q;q)_n} \sum_{\substack{w\in\Theta_n^- \\ \mu\,\mathbf{c}=m+1}} A_{n,\mathbf{c}}(q) = \sum_{j_0\ge 0} A_{2j_0}(q)\frac{u^{2j_0}}{(q;q)_{2j_0}} \Bigl(\sum_{j\ge 0} A_{2j+1}(q)\frac{u^{2j+1}}{(q;q)_{2j+1}}\Bigr)^m$$
$$= \sec_q(u)\,\bigl(\tan_q(u)\bigr)^m.$$

Then

$$\sum_{n\ge 0} B_n(x,q)\frac{u^n}{(q;q)_n} = \sum_{m\ge 0} x^m \sum_{n\ge 0} \frac{u^n}{(q;q)_n} \sum_{\substack{w\in\Theta_n^- \\ \mu\,\mathbf{c}=m+1}} A_{n,\mathbf{c}}(q)$$
$$= \sum_{m\ge 0} \sec_q(u)\,\bigl(x\tan_q(u)\bigr)^m$$
$$= \sec_q(u)(1 - x\tan_q(u))^{-1},$$

which proves (1.20). □

10. Specializations

Our three families of polynomials $(A_{n,k,a,b}(q))$, $(B_{n,k,a,b}(q))$, and $(A_{n,\mathbf{c}}(q))$ involve specializations that relate to other classes of generating polynomials or classical numbers that have been studied in previous works. Those polynomials are displayed in Tables 2–4 at the end of the paper. Each table appears as a matrix, whose (n,m)-cell contains several polynomials. In the (n,m)-cell of Table 2 (resp. Table 3) are reproduced all the polynomials $A_{n,k,a,b}(q)$ (or $B_{n,k,a,b}(q)$) such that $a+b=m$ (resp. $A_{n,\mathbf{c}}(q)$ (or $B_{n,\mathbf{c}}(q)$) such that $\mu\,\mathbf{c} = m$. The specializations we deal with refer to rows, columns or diagonals of those tables. Others are obtained by summing the above polynomials with respect to certain subscripts.

To this end we use the following notations:

$$A_{n,k,a+b=m}(q) := \sum_{a+b=m} A_{n,k,a,b}(q); \tag{10.1}$$

$$A_{n,a+b=m}(q) := \sum_{\substack{k\ge 0 \\ a+b=m}} A_{n,k,a,b}(q); \tag{10.2}$$

By Theorem 1.4 we also have:

$$A_{n,a+b=m}(q) := \sum_{\mu\,\mathbf{c}=m} A_{n,\mathbf{c}}(q).$$

Analogous definitions are made for the polynomials $B_{n,k,a,b}(q)$.

10.1. *The first column of* $(A_{n,k,a,b}(q))$, $(B_{n,k,a,b}(q))$. The (t,q)-analogs of tangent and secant have been introduced in our previous paper [FH11]. For each $r \geq 0$ form the q-series:

$$\begin{aligned}
\sin_q^{(r)}(u) &:= \sum_{n\geq 0} (-1)^n \frac{(q^r;q)_{2n+1}}{(q;q)_{2n+1}} u^{2n+1};\\
\cos_q^{(r)}(u) &:= \sum_{n\geq 0} (-1)^n \frac{(q^r;q)_{2n}}{(q;q)_{2n}} u^{2n};\\
\tan_q^{(r)}(u) &:= \frac{\sin_q^{(r)}(u)}{\cos_q^{(r)}(u)};\\
\sec_q^{(r)}(u) &:= \frac{1}{\cos_q^{(r)}(u)}.
\end{aligned}$$

The (t,q)-*analogs of the tangent and secant numbers* have been defined as the coefficients $T_{2n+1}(t,q)$ and $E_{2n}(t,q)$, respectively, in the following two series:

$$\sum_{r\geq 0} t^r \tan_q^{(r)}(u) = \sum_{n\geq 0} \frac{u^{2n+1}}{(t;q)_{2n+2}} T_{2n+1}(t,q); \tag{10.3}$$

$$\sum_{r\geq 0} t^r \sec_q^{(r)}(u) = \sum_{n\geq 0} \frac{u^{2n}}{(t;q)_{2n+1}} E_{2n}(t,q). \tag{10.4}$$

It was then proved that $T_{2n+1}(t,q)$ and $E_{2n}(t,q)$ have the following combinatorial interpretations:

$$\begin{aligned}
T_{2n+1}(t,q) &= \sum_{\sigma\in\mathcal{RA}_{2n+1}} t^{1+\operatorname{ides}\sigma} q^{\operatorname{imaj}\sigma};\\
E_{2n}(t,q) &= \sum_{\sigma\in\mathcal{RA}_{2n}} t^{1+\operatorname{ides}\sigma} q^{\operatorname{imaj}\sigma}.
\end{aligned}$$

Now, the set $\mathcal{T}_{2n+1,k,0,0}$ is the set of all t-permutations $w = (w_0)$ of order $(2n+1)$, where w_0 is simply an element of $\mathcal{RA}_{2n+1}$, that is, a rising alternating permutation of order $(2n+1)$. Hence,

$$T_{2n+1}(t,q) = \sum_{k\geq 1} t^{k+1} A_{2n+1,k,0,0}(q).$$

Accordingly, Theorem 4.1 provides a method for calculating the polynomials $T_{2n+1}(t,q)$, only defined so far by their generating function (10.3). In an equivalent manner, we can also say that the factorial generating function for the first column of the matrix $(A_{n,k,a,b}(q))$ is given by

$$\sum_{r\geq 0} t^r \tan_q^{(r)}(u) = \sum_{n\geq 0} \frac{u^{2n+1}}{(t;q)_{2n+2}} \sum_{k\geq 1} t^{k+1} A_{2n+1,k,0,0}(q).$$

In the same way, we get:

$$E_{2n}(t,q) = \sum_{k\geq 1} t^{k+1} B_{2n,k,0,0}(q),$$

which also provides, either a way of calculating the (t,q)-analogs $E_{2n}(t,q)$ of the secant numbers, or writing the factorial generating function for the first column of the matrix $(B_{n,k,a,b}(q))$.

10.2. *The super-diagonal* $(n, n+1)$ *of the matrix* $(A_{n,k,a,b}(q))$. As will be shown, the polynomials $A_{n,k,a,b}(q)$ $(a+b=n+1)$ of that super-diagonal provide a *refinement* of the *Carlitz q-analogs of the Eulerian polynomials* [Ca54]. Twenty-one years later [Ca75] Carlitz also showed that they were generating polynomials for the symmetric groups by the pair "des" (number of descents) and "maj" (major index). Let $(A_n(t,q))$ be the sequence of those polynomials, written as $A_n(t,q) = \sum_{j\geq 0} A_{n,j}(q)$ $(n \geq 0)$. The recurrence

$$A_{n,j}(q) = (1+q+\cdots+q^j)\, A_{n-1,j}(q) + (q^j+q^{j+1}+\cdots+q^{n-1})\, A_{n-1,j-1}(q),$$

with the initial conditions $A_{0,j}(q) = A_{1,j}(q) = \delta_{0,j}$, provides a method for calculating them.

Their first values are reproduced in the following table:

$A_0(t,q) = A_1(t,q) = 1$; $A_2(t,q) = 1 + tq$; $A_3(t,q) = 1 + 2tq(q+1) + t^2q^3$;
$A_4(t,q) = 1 + tq(3q^2+5q+3) + t^2q^3(3q^2+5q+3) + t^3q^6$;
$A_5(t,q) = 1 + tq(4q^3+9q^2+9q+4) + t^2q^3(6q^4+16q^3+22q^2+16q+6) + t^3q^6(4q^3+9q^2+9q+4) + t^4q^{10}$.

Now, go back to the recurrence for the polynmials $A_{n,k,a,b}(q)$ shown in (4.1) and rewrite it when $a'+b' = m' = n+2$. The coefficients $A_{n,k'-1,a,m'+1-a}(q) = A_{n,k'-1,a,n+3-a}(q)$ and $A_{n,k',a,m'+1-a}(q) = A_{n,k',a,n+3-a}(q)$ vanish, because $A_{n,k,a,b}(q) = 0$ when $a+b \geq n+2$. Hence,

$$A_{n+1,k',a',n+2-a'}(q) = q^{k'}\Bigl(\sum_{0\leq a\leq a'-1\leq n} A_{n,k'-1,a,n+1-a}(q) + \sum_{1\leq a'\leq a\leq n+1} A_{n,k',a,n+1-a}(q)\Bigr),$$

valid for $n \geq 0$ with the initial condition: $A_{0,k,a,b}(q) = \delta_{k,0}\delta_{a,1}\delta_{b,0}$. For $n \geq 0$ let $A_{n,k,a}(q) := A_{n,k,a,n+1-a}(q)$, so that the previous recurrence can be written in the form

$$A_{n+1,k',a'}(q) = q^{k'}\Big(\sum_{0\le a\le a'-1\le n} A_{n,k'-1,a}(q) + \sum_{1\le a'\le a\le n+1} A_{n,k',a}(q)\Big),$$

with $0 \leq a' \leq n+2$ and makes it possible the calculations of those polynomials:

$A_{1,0,1}(q) = 1$;
$A_{2,0,1}(q) = 1$, $A_{2,1,2}(q) = q$;
$A_{3,0,1}(q) = 1$, $A_{3,1,1}(q) = q^2$, $A_{3,1,2}(q) = q + q^2$, $A_{3,1,3}(q) = q$, $A_{3,2,3}(q) = q^3$;
$A_{4,0,1}(q) = 1$, $A_{4,1,1}(q) = 2q^2+2q^3$, $A_{4,1,2}(q) = q+2q^2+q^3$, $A_{4,1,3}(q) = q+q^2$, $A_{4,1,4}(q) = q$, $A_{4,2,1}(q) = q^5$, $A_{4,2,2}(q) = q^4 + q^5$, $A_{4,2,3}(q) = q^3 + 2q^4 + q^5$, $A_{4,2,4}(q) = 2q^3 + 2q^4$, $A_{4,3,4}(q) = q^6$.

Theorem 10.1. *For each $n \geq 0$ and each $j \geq 0$ the coefficient $A_{n,j}(q)$ of t^j in the Carlitz q-Eulerian polynomial $A_n(t,q)$ admits the following refinement:*

$$A_{n,j}(q) = \sum_{a\ge 0} A_{n,j,a}(q). \tag{10.5}$$

For instance, $A_{4,1}(q) = 3q+5q^2+3q^3 = (2q^2+2q^3)+(q+2q^2+q^3)+(q+q^2)+q = A_{4,1,1}(q)+A_{4,1,2}(q)+A_{4,1,3}(q)+A_{4,1,4}(q)$.

Proof. Each polynomial $A_n(t,q)$ is the generating polynomial for $\mathfrak{S}_n$ by the pair ("number of descents","major index"), as established by Carlitz [Ca75], or, in an equivalent manner, by the pair (ides, imaj). This can also be expressed by

$$A_{n,j}(q) = \sum_{\sigma\in\mathfrak{S}_n,\,\mathrm{ides}\,\sigma=j} q^{\mathrm{imaj}\,\sigma}, \tag{10.6}$$

By Theorem 9.1 $A_{n,j,a}(q) = A_{n,j,a,n+1-a}(q)$ is the generating polynomial for the set of all t-permutations w of order n such that $\mu\, w = n+1$, $\mathrm{ides}\, w = j$ and $\min w = a$ by "imaj." Such t-permutations are of the form $w = (w_0, w_1, w_2, \ldots, w_{n+1})$, so that necessarily, $w_0 = w_{n+1} = \epsilon$, and the other components w_i are one-letter words. Accordingly, $A_{n,j,a}(q)$ is the generating polynomial for all (ordinary) *permutations* $\sigma = w_1w_2\cdots w_n$ of $12\cdots n$ such that $\mathrm{ides}\,\sigma = j$ and $w_a = 1$, that is,

$$A_{n,j,a}(q) = \sum_{\substack{\sigma\in\mathfrak{S}_n,\,\mathrm{ides}\,\sigma=j,\\ \min\sigma=a}} q^{\mathrm{imaj}\,\sigma}. \tag{10.7}$$

Thus (10.5) is a consequence of (10.6) and (10.7). $\square$

For each integer $n \geq 1$ let $[n]_q := (q;q)_n/(1-q)^n = 1+q+\cdots+q^{n-1}$. By summing the $A_{n,j,a}$'s over the pair (j,a) we get the polynomial $A_{n,a+b=n+1}(q)$ defined in (10.2), which is the generating polynomial for $\mathfrak{S}_n$ by "inv," well-known to be equal to

$$A_{n,a+b=n+1}(q) = [1]_q\,[2]_q\,\cdots\,[n]_q, \tag{10.8}$$

also equal (using the same combinatorial interpretation) to $B_{n,a+b=n}(q) = \sum_{k\geq 0, a+b=n} B_{n,k,a,b}(q)$.

10.3. *The subdiagonal $(n,n-1)$ of the matrix $(A_{n,k,a,b}(q))$.* Our purpose is to evaluate the polynomial $A_{n,a+b=n-1}(q)$ for each $n \geq 1$, which is the generating function for all t-permutations of order n such that $\mu\, w = n-1$ by "inv." Such t-permutations w have one of the three forms:

(1) $w = (x_1x_2, x_3, x_4, \ldots, x_n, \epsilon)$ with $x_1 < x_2$;

(2) $w = (\epsilon, x_1, x_2, \ldots, x_{n-2}, x_{n-1}x_n)$ with $x_{n-1} > x_n$;

(3) $(\epsilon, x_1, \ldots, x_{i-1}, x_ix_{i+1}x_{i+2}, x_{i+3}, \ldots, x_n, \epsilon)$ with $1 \leq i \leq n-2$, $x_i > x_{i+1}$, $x_{i+1} < x_{i+2}$.

The g.f. of the t-permutations of form (1) or (2) by "inv" is equal to

$$(1+q)\begin{bmatrix} n \\ 2, 1^{n-2} \end{bmatrix}_q = [2]_q\,[3]_q\,[4]_q\,\cdots\,[n]_q.$$

The g.f. of the t-permutations of form (3) by "inv" is equal to

$$\begin{aligned}\sum_{1\leq i\leq n-2} \begin{bmatrix} n \\ 1^{i-1}, 3, 1^{n-i-2} \end{bmatrix}_q \times q(q+1) &= (n-2)q(q+1)\begin{bmatrix} n \\ 3, 1^{n-3} \end{bmatrix}_q \\ &= (n-2)q\,[2]_q\,[4]_q\,\cdots\,[n]_q,\end{aligned}$$

so that the total g.f. is equal to

$$\begin{aligned}&[2]_q\,[4]_q\,\cdots\,[n]_q([3]_q + (n-2)q) \\ &\qquad = [2]_q\,[4]_q\,\cdots\,[n]_q(1+q+q^2+(n-2)q) \\ &\qquad = [1]_q\,[2]_q\big(1+(n-1)q+q^2\big)\,[4]_q\,\cdots\,[n]_q.\end{aligned}$$

Thus, $A_{n,a+b=n-1}(q) = [1]_q\,[2]_q\big(1+(n-1)q+q^2\big)\,[4]_q\,\cdots\,[n]_q$.

10.4. *The subdiagonal $(n,n-2)$ of the matrix $(B_{n,k,a,b}(q))$.* Using the same combinatorial technique as in 10.3, but this time operating with the s-permutations we get the following evaluation for each $n \geq 2$:

$$B_{n,a+b=n-2}(q) = (1+(n-1)q+(n-1)q^2)\,[4]_q\,\cdots\,[n]_q.$$

10.5. *Two q-analogs of the Springer numbers.* It was recalled in the Introduction ((1.7) and (1.20)) that $\sec(u)(1-x\tan(u))^{-1}$ for $x=1$ was the exponential generating function for the *Springer numbers.* Referring to (1.21) we then see that $\sec_q(u)(1-\tan_q(u))^{-1}$ is the factorial generating function for the q-analogs of the Springer numbers, which are simply the generating polynomials for the s-permutations by "imaj" or "inv."

Note that $\mathrm{Sec}_q(u)(1-\tan_q(u))^{-1}$ is also the factorial generating for such q-analogs, but this time for the S-permutations by "imaj" or "inv;"

10.6. *t-compositions and Fibonacci triangle.* Let $\alpha(n,m) := \#\Theta_{n,m}$ (resp. $\beta(n,m) := \#\Theta^{-}_{n,m+1}$) be the *number* of t-compositions (resp. s-compositons). From the previous lists of the Θ_i's made in Section 1 we have the next table, where the $\alpha(n,m)$'s (resp. $\beta(n,m)$'s) have been reproduced in bold face (resp. plain type). To the right are displayed the row sums of those entries, which will be proven to be the classical Fibonacci numbers.

$$
\begin{array}{r}
m= \\
n=0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
1 & \mathbf{1} & & & & & & \\
\mathbf{1} & 1 & \mathbf{1} & & & & & \\
1 & \mathbf{2} & 1 & \mathbf{1} & & & & \\
\mathbf{1} & 2 & \mathbf{3} & 1 & \mathbf{1} & & & \\
1 & \mathbf{3} & 3 & \mathbf{4} & 1 & \mathbf{1} & & \\
\mathbf{1} & 3 & \mathbf{6} & 4 & \mathbf{5} & 1 & \mathbf{1} & \\
1 & \mathbf{4} & 6 & \mathbf{10} & 5 & \mathbf{6} & 1 & \mathbf{1}
\end{pmatrix}
\end{array}
\begin{array}{cc}
 & \\
\mathbf{1} & 1 \\ \mathbf{2} & 1 \\ \mathbf{3} & 2 \\ \mathbf{5} & 3 \\ \mathbf{8} & 5 \\ \mathbf{13} & 8 \\ \mathbf{21} & 13
\end{array}
$$

Fig. 10.1. The coefficients $\alpha(\mathbf{n},\mathbf{m})$ and $\beta(n,m)$

The mapping $(c_0,c_1,\ldots,c_m,0)\mapsto(c_0,c_1,\ldots,c_m-1)$ is a bijection of $\Theta^{-}_{n,m+1}$ onto $\Theta_{n-1,m}$, because c_m is odd. Hence $\beta(n,m)=\alpha(n-1,m)\quad(n\ge 1)$. We now only study the numbers $\alpha(n,m)$.

Proposition 10.2. *With the initial values $\alpha(0,m)=\delta_{1,m}$, $\alpha(1,m)=\delta_{0,m}+\delta_{2,m}$, the entries $\alpha(n,m)$ are inductively given by*

$$\alpha(n,m)=\alpha(n-1,m-1)+\alpha(n-2,m)\quad(n\ge 2).$$

Proof. Let $\mathbf{c}=(c_0,c_1,\ldots,c_{m-1},c_m)\in\Theta_{n,m}$. If $c_m=0$ we define $\phi(\mathbf{c})=(c_0,c_1,\ldots,c_{m-1}-1)\in\Theta_{n-1,m-1}$. If $c_m\ge 1$, then $c_m\ge 2$ because c_m is even. We define $\phi(\mathbf{c})=(c_0,c_1,\ldots,c_{m-1},c_m-2)\in\Theta_{n-2,m}$. We verify that ϕ is a bijection between $\Theta_{n,m}$ and $\Theta_{n-1,m-1}+\Theta_{n-2,m}$. $\square$

For each $n\ge 0$ let $A_n(x)=\sum_{m\ge 0}\alpha(n,m)x^m$ be the generating polynomials of the coefficients $\alpha(n,m)$. From Proposition 10.23 it follows that $A_0(x)=x$, $A_1(x)=1+x^2$ and the recurrence formula

$$A_{n+1}(x)=xA_n(x)+A_{n-1}(x)\quad(n\ge 1).$$

Let $A(x;u) := \sum_{n\ge 0} A_n(x)u^n$. Then, $A(x;u) - x - u(1+x^2) = xu(A(x;u) - x) + u^2 A(x;u)$, so that $A(x;u)(1 - xu - u^2) = x + u$ and

$$A(x;u) = \sum_{n\ge 0} A_n(x)u^n = \sum_{\mathbf{c}\in\Theta} x^{\mu\,\mathbf{c}} u^{|\mathbf{c}|} = \frac{x+u}{1-u(x+u)}.$$

Let $x = 1$ we obtain the generating function for the row sums $A_n(1) = \sum \alpha(n,m)$ $(m \ge 0)$, which is equal to $(1+u)/(1-u-u^2)$. Thus, the row sums are the classical Fibonacci numbers.

The polynomials $A_n(x)$ are related to the polynomials $F_n(x)$ already introduced in Sloane's Integer Encyclopedia [Sl06] under reference A102426 by $A_n(x) = x^{n+1}F_{n+1}(x^{-2})$ for $n \ge 0$. Accordingly, the t-compositions provide a natural combinatorial interpretation for their coefficients.

10.7. *Further comment.* Dominique Dumont [Du12] has drawn our attention to the two papers by Carlitz-Scoville [CS72] and Françon [Fr78]. Instead of t-permutations or snakes, Carlitz and Scoville have dealt with "up-down sequences of length $n+m$ with m infinite elements." Such a sequence is a rising alternating permutation $x_1x_2\cdots x_{n+m}$ containing all the integers $1, 2, \ldots, n$ and m letters equal to $-\infty$. Note that replacing all the commas in each t-permutation $w = (w_0, w_1, \ldots, w_m)$ by $-\infty$ makes up a bijection of the set of all t-permutations onto the set of all Carlitz-Scoville sequences. In Françon [Fr78] can be found an unexpected combinatorial interpretation of the entries $b(n,m)$ in terms of computer file histories.

11. Tables

Four tables are being displayed, the first one containing the values of $a(n,m)$ and $b(n,m)$ for $0 \le n \le 6$, the second one containing the values of the polynomials $A_{n,k,a,b}(q)$ and $B_{n,k,a,b}(q)$ for $0 \le n \le 4$, the third one for the polynomials $A_{n,\mathbf{c}}(q)$ for $0 \le n \le 4$. The last one contains the values of the polynomials $A_{n,a+b=n}(q)$ and $B_{n,a+b=n}(q)$, whose definitions are given in (10.1)–(10.2).

$m =$	0	1	2	3	4	5	6	7		
$n = 0$	1	**1**							$\mathbf{1.2^0}$	1
1	**1**	1	**1**						$\mathbf{1.2^1}$	1
2	1	**2**	2	**2**					$\mathbf{1.2^2}$	3
3	**2**	5	**8**	6	**6**				$\mathbf{2.2^3}$	11
4	5	**16**	28	**40**	24	**24**			$\mathbf{5.2^4}$	57
5	**16**	61	**136**	180	**240**	120	**120**		$\mathbf{16.2^5}$	361
6	61	**272**	662	**1232**	1320	**1680**	720	**720**	$\mathbf{61.2^6}$	2763

Table 1. The coefficients $\mathbf{a(n,m)}$ and $b(n,m)$

$m=$	0	1	2	3
$n=0$	$B_{0,-1,0,0}=1$	$A_{0,0,1,0}=1$		
1	$A_{1,0,0,0}=1$	$B_{1,0,1,0}=1$	$A_{1,0,1,1}=1$	
2	$B_{2,0,0,0}=1$	$A_{2,0,0,1}=1$ $A_{2,1,1,0}=q$	$B_{2,0,1,1}=1$ $B_{2,1,2,0}=q$	$A_{2,0,1,2}=1$ $A_{2,1,2,1}=q$
3	$A_{3,1,0,0}=q+q^2$	$B_{3,0,0,1}=1$ $B_{3,1,0,1}=q^2$ $B_{3,1,1,0}=2q+q^2$	$A_{3,0,0,2}=1$ $A_{3,1,0,2}=q^2$ $A_{3,1,1,1}=2q+2q^2$ $A_{3,1,2,0}=q$ $A_{3,2,2,0}=q^3$	$B_{3,0,1,2}=1$ $B_{3,1,1,2}=q^2$ $B_{3,1,2,1}=q+q^2$ $B_{3,1,3,0}=q$ $B_{3,2,3,0}=q^3$
4	$B_{4,1,0,0}=q+2q^2+q^3$ $B_{4,2,0,0}=q^4$	$A_{4,1,0,1}=q+3q^2+2q^3$ $A_{4,1,1,0}=q+q^2$ $A_{4,2,0,1}=q^4+q^5$ $A_{4,2,1,0}=2q^3+3q^4+q^5$	$B_{4,0,0,2}=1$ $B_{4,1,0,2}=2q^2+2q^3$ $B_{4,1,1,1}=2q+4q^2+2q^3$ $B_{4,1,2,0}=2q+q^2$ $B_{4,2,0,2}=q^5$ $B_{4,2,1,1}=2q^4+q^5$ $B_{4,2,2,0}=3q^3+4q^4+q^5$	$A_{4,0,0,3}=1$ $A_{4,1,0,3}=2q^2+2q^3$ $A_{4,1,1,2}=2q+5q^2+3q^3$ $A_{4,1,2,1}=2q+2q^2$ $A_{4,1,3,0}=q$ $A_{4,2,0,3}= q^5$ $A_{4,2,1,2}=2q^4+2q^5$ $A_{4,2,2,1}=3q^3+5q^4+2q^5$ $A_{4,2,3,0}=2q^3+2q^4$ $A_{4,3,3,0}=q^6$

$m=$	4	5
$n=3$	$A_{3,0,1,3}=1$ $A_{3,1,1,3}=q^2$ $A_{3,1,2,2}=q+q^2$ $A_{3,1,3,1}=q$ $A_{3,2,3,1}=q^3$	
4	$B_{4,0,1,3}=1$ $B_{4,1,1,3}=2q^2+2q^3$ $B_{4,1,2,2}=q+2q^2+q^3$ $B_{4,1,3,1}=q+q^2$ $B_{4,1,4,0}=q$ $B_{4,2,1,3}=q^5$ $B_{4,2,2,2}=q^4+q^5$ $B_{4,2,3,1}=q^3+2q^4+q^5$ $B_{4,2,4,0}=2q^3+2q^4$ $B_{4,3,4,0}=q^6$	$A_{4,0,1,4}=1$ $A_{4,1,1,4}=2q^2+2q^3$ $A_{4,1,2,3}=q+2q^2+q^3$ $A_{4,1,3,2}=q+q^2$ $A_{4,1,4,1}=q$ $A_{4,2,1,4}=q^5$ $A_{4,2,2,3}=q^4+q^5$ $A_{4,2,3,2}=q^3+2q^4+q^5$ $A_{4,2,4,1}=2q^3+2q^4$ $A_{4,3,4,1}=q^6$

Table 2. Polynomials $A_{n,k,a,b}(q)$ and $B_{n,k,a,b}(q)$ for $0 \le n \le 4$ ($A_{n,k,a,b} := A_{n,k,a,b}(q)$ and $B_{n,k,a,b} := B_{n,k,a,b}(q)$)

$m=$	0	1	2	3
$n=0$		$A_{0(00)}=1$		
1	$A_{1(1)}=1$		$A_{1(010)}=1$	
2		$A_{2(20)}=1$ $A_{2(02)}=q$		$A_{2(010)}=1+q$
3	$A_{3(3)}=q+q^2$		$A_{3(210)}=1+q+q^2$ $A_{3(012)}=q+q^2+q^3$ $A_{3(030)}=q+q^2$	
4		$A_{4(04)}=q^2+q^3+2q^4+q^5$ $A_{4(22)}=(q+q^3)(1+q+q^2)$ $A_{4(40)}=q+2q^2+q^3+q^4$		$A_{4(0112)}=(1+q+q^2)(q+q^2+q^3+q^4)$ $A_{4(0130)}=(q+q^2)(1+q+q^2+q^3)$ $A_{4(0310)}=(q+q^2)(1+q+q^2+q^3)$ $A_{4(2110)}=(1+q+q^2)(1+q+q^2+q^3)$

$m=$	4	5
$n=3$	$A_{3(01110)}=(1+q)(1+q+q^2)$	
4		$A_{4(011110)}=(1+q)(1+q+q^2)(1+q+q^2+q^3)$

Table 3. Polynomials $A_{n,\mathbf{c}}(q)$ for $0 \le n \le 4$, ($A_{n(c_0\cdots c_m)} := A_{n,(c_0,\ldots,c_m)}(q)$)

$m=$	0	1	2	3	4	5
$n=0$	1	1				
1	1	1	1			
2	1	$1+q$	$1+q$	$1+q$		
3	$q+q^2$	$1+2q+2q^2$	$1+3q+3q^2+q^3$	$1+2q+2q^2+q^3$	$1+2q+2q^2+q^3$	
4	$q+2q^2+q^3+q^4$	$2q+4q^2+4q^3$ $+4q^4+2q^5$	$1+4q+7q^2+7q^3$ $+6q^4+3q^5$	$1+5q+9q^2+10q^3$ $+9q^4+5q^5+q^6$	$1+3q+5q^2+6q^3$ $+5q^4+3q^5+q^6$	$1+3q+5q^2+6q^3$ $+5q^4+3q^5+q^6$

Table 4. Polynomials $A_{n,a+b=m}(q)$ ($n \equiv m+1 \pmod 2$) and $B_{n,a+b=m}(q)$ ($n \equiv m \pmod 2$) for $0 \le n \le 4$

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