

# FROM STATE INTEGRALS TO $q$ -SERIES

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ABSTRACT. It is well-known to the experts that multi-dimensional state integrals of products of Faddeev's quantum dilogarithm which arise in Quantum Topology can be written as finite sums of products of basic hypergeometric series in  $q = e^{2\pi i\tau}$  and  $\tilde{q} = e^{-2\pi i/\tau}$ . We illustrate this fact by giving a detailed proof for a family of one-dimensional integrals which includes state-integral invariants of  $4_1$  and  $5_2$  knots.

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## 1. INTRODUCTION

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**1.1. State-integrals and their  $q$ -series.** Multi-dimensional state integrals of products of Faddeev's quantum dilogarithm appear in abundance in Quantum Topology, and were studied among others by Hikami [Hik01], Dimofte–Gukov–Lennels–Zagier [DGLZ09], Andersen–Kashaev [AK], and Kashaev–Luo–Vartanov [KLV12]. It is well-known to the experts that such state-integrals can be written as finite sums of products of pairs of  $q$ -series and  $\tilde{q}$ -series. The reason for this is a factorized structure of Faddeev's quantum dilogarithm, the structure of the set of its poles, and the specific form of exponential factors of the integrand of the state-integrals, while its derivation is based on an application of the residue theorem. Instead of formulating a general theorem for multi-dimensional integrals which obscures the principle, we will give a detailed proof for the case of a family of 1-dimensional integrals and illustrate it with some concrete examples taken from [AK, KLV12].

To state our results, recall that *Faddeev's quantum dilogarithm function*  $\Phi_b(x)$  is given by [Fad95]

$$(1) \quad \Phi_b(x) = \frac{(e^{2\pi b(x+c_b)}; q)_\infty}{(e^{2\pi b^{-1}(x-c_b)}; \tilde{q})_\infty},$$

where

$$q = e^{2\pi i b^2}, \quad \tilde{q} = e^{-2\pi i b^{-2}}, \quad c_b = \frac{i}{2}(b + b^{-1}), \quad \Im(b^2) > 0.$$

Remarkably, this function admits an extension to all values of  $b$  with  $b^2 \notin \mathbb{R}_{\leq 0}$ .  $\Phi_b(x)$  is a meromorphic function of  $x$  with

$$\text{poles: } c_b + i\mathbb{N}b + i\mathbb{N}b^{-1}, \quad \text{zeros: } -c_b - i\mathbb{N}b - i\mathbb{N}b^{-1}.$$

The functional equation

$$\Phi_b(x)\Phi_b(-x) = e^{\pi i x^2} \Phi_b(0)^2, \quad \Phi_b(0) = q^{\frac{1}{24}} \tilde{q}^{-\frac{1}{24}}$$

allows us to move  $\Phi_b(x)$  from the denominator to the numerator of the integrand of a state-integral.

For natural numbers  $A, B$  with  $B > A > 0$ , we consider the absolutely convergent integral

$$\mathcal{I}_{A,B}(b) = \int_{\mathbb{R}+i\epsilon} \Phi_b(x)^B e^{-A\pi i x^2} dx$$

with small positive  $\epsilon$ . The condition  $B > A > 0$  ensures not only the convergence of  $\mathcal{I}_{A,B}(b)$  for  $\Im(b^2) > 0$ , but also the convergence of the  $q$ -series and the  $\tilde{q}$ -series (for  $|q|, |\tilde{q}| < 1$ ) that appear in Theorem 1.1 below.

To express the above state-integral in terms of series, consider the generating series

$$(2) \quad F_{A,B}(q, x) = \sum_{m=0}^{\infty} \frac{(-1)^{Am} q^{A \frac{m(m+1)}{2}}}{(q)_m^B} x^m, \quad \tilde{F}_{A,B}(q, x) = F_{B-A,B}(q, x).$$

Consider the operators  $\delta$  and  $\delta_k$  (for  $k$  a positive natural number) which act on the space of functions of  $x$  as follows

$$(3) \quad (\delta F)(x) = x \partial_x F(x), \quad (\delta_k F)(x) = \sum_{s=1}^{\infty} \frac{s^{k-1} q^s}{1 - q^s} F(q^s x).$$

Likewise, there are operators  $\tilde{\delta}$  and  $\tilde{\delta}_k$  which act on the space of functions of  $\tilde{x}$  and with  $q$  replaced by  $\tilde{q}$ . It is easy to see that any two of the operators  $\delta, \delta_k, \tilde{\delta}, \tilde{\delta}_k$  commute and they freely generate over  $\mathbb{Q}$  a commutative ring  $\mathcal{D} \otimes \tilde{\mathcal{D}}$ , where

$$\mathcal{D} = \mathbb{Q}[\delta, \delta_1, \delta_2, \dots], \quad \tilde{\mathcal{D}} = \mathbb{Q}[\tilde{\delta}, \tilde{\delta}_1, \tilde{\delta}_2, \dots].$$

Let

$$\mathcal{D}_b = \mathcal{D}[(2\pi i)^{-1}, b^{\pm 1}, e_2, e_4, e_6, \dots], \quad \tilde{\mathcal{D}}_b = \tilde{\mathcal{D}}[(2\pi i)^{-1}, b^{\pm 1}, e_2, e_4, e_6, \dots],$$

where  $e_l = e_l(\tilde{q}) = \tilde{\delta}_l(1) \in \mathbb{Z}[[\tilde{q}]]$ . Consider the following *operator valued polynomial*:

$$(4) \quad P_{A,B} = \text{Res}_{w=0} \left( e^{\frac{1}{4\pi i} w^2 + Aw(b(\delta + \frac{1}{2}) + b^{-1}(\tilde{\delta} + \frac{1}{2}))} \right)^A \left( \frac{\phi(bw, \delta_\bullet) \tilde{\phi}(b^{-1}w, \tilde{\delta}_\bullet)}{b(1 - e^{b^{-1}w})} \right)^B \in \mathcal{D}_b \otimes \tilde{\mathcal{D}}_b,$$

where

$$(5a) \quad \phi(w, \delta_\bullet) = \exp \left( - \sum_{l=1}^{\infty} \frac{\delta_l}{l!} w^l \right)$$

$$(5b) \quad \tilde{\phi}(w, \tilde{\delta}_\bullet) = \exp(-\tilde{\delta}w) \exp \left( 2 \sum_{l=\text{even}>0} e_l(\tilde{q}) \frac{w^l}{l!} \right) \exp \left( - \sum_{l=1}^{\infty} \frac{\tilde{\delta}_l}{l!} (-w)^l \right).$$

For a series  $F(x, \tilde{x})$ , we define:

$$(6) \quad \langle F(x, \tilde{x}) \rangle = F(1, 1).$$

**Theorem 1.1.** *We have:*

$$(7) \quad \mathcal{I}_{A,B}(b) = \left( \frac{\tilde{q}}{q} \right)^{\frac{B-3A}{24}} e^{\pi i \frac{B+2(A+1)}{4}} \left\langle P_{A,B} \left( F_{A,B}(q, x) \tilde{F}_{A,B}(\tilde{q}, \tilde{x}) \right) \right\rangle.$$

**Corollary 1.2.** Writing  $P_{A,B} = \sum_k p_k P_k$  (a finite sum), for  $p_k \in \mathcal{D}_b$  and  $P_k \in \tilde{\mathcal{D}}_b$ , it follows that

$$(8) \quad \mathcal{I}_{A,B}(b) = \left( \frac{\tilde{q}}{q} \right)^{\frac{B-3A}{24}} e^{\pi i \frac{B+2(A+1)}{4}} \sum_k g_k(q) G_k(\tilde{q})$$

where

$$(9) \quad g_k(q) = \langle p_k F_{A,B} \rangle, \quad G_k(\tilde{q}) = \langle P_k \tilde{F}_{A,B} \rangle.$$

**Remark 1.3.** The left hand side of Equation (8) has analytic continuation to the cut plane  $\mathbb{C} \setminus \{b^2 \mid b^2 < 0\}$  whereas each of the series  $g_k$  and  $G_k$  is only well-defined in the upper-half plane  $\{b^2 \mid \Im(b^2) > 0\}$ .

**Remark 1.4.**  $P_{A,B}$ , as a polynomial in the variables  $e_2, e_4, \dots$  has degree  $B - 1$ , where the degree of  $e_l$  is  $l$ .  $P_{A,B}$  as a Laurent polynomial in  $b$  has  $b$ -monomials of degrees in  $\{-B + 1, -B + 3, \dots, B - 3, B - 1\}$ .

**1.2.  $q$ -difference equations.** Next we describe a linear  $q$ -difference equation of  $F_{A,B}(q, x)$ . Consider the operators  $\hat{x}$  and  $\hat{E}$  which act on  $f(x) \in \mathbb{Q}(q)[[x]]$  by:

$$(\hat{E}f)(x) = f(qx), \quad (\hat{x}f)(x) = xf(x).$$

Observe that  $\hat{E}\hat{x} = q\hat{x}\hat{E}$ .

**Lemma 1.5.** (a) We have:

$$(10) \quad F_{A,B}(q^{-1}, x) = \tilde{F}_{A,B}(q, x).$$

(b)  $F_{A,B}$  satisfies the linear  $q$ -difference equation

$$(11) \quad \left( (1 - \hat{E})^B - (-1)^A q^A x \hat{E}^A \right) F_{A,B}(q, x) = 0.$$

**Corollary 1.6.** (a) If we define  $\omega(q, x) = F_{A,B}(q, qx)/F_{A,B}(q, x)$  and  $\omega(q, x)_n = \prod_{j=1}^n \omega(q, q^j x)$ , then  $\omega$  satisfies the nonlinear equation

$$\sum_{j=0}^B (-1)^j \binom{B}{j} \omega(q, x)_j - (-1)^A q^A x \omega(q, x)_A = 0.$$

(b)  $F$  is an admissible power series in the sense of Kontsevich-Soibelman [KS11, Sec.6], the limit  $\lim_{q \rightarrow 1} \omega(q, x) = \omega(x) \in \overline{\mathbb{Q}}[[x]]$  exists and satisfies the algebraic equation (also known as the Nahm equation or the gluing equation)

$$(12) \quad (1 - \omega(x))^B = (-1)^A x \omega(x)^A.$$

The Nahm equation has been studied by several authors including [Zag07, Sec.3], [Vla, VZ11], [RV, Sec.4].

**1.3. The case of the  $4_1$  knot.** We now specialize Corollary 1.2 to the invariant of the  $4_1$  and  $5_2$  knots is given by [KLV12, AK]

$$\mathcal{I}_{1,2} = \mathcal{I}_{4_1} \quad \mathcal{I}_{2,3} = \mathcal{I}_{5_2}.$$

In this section, let

$$(13) \quad F(q, x) = F_{1,2}(q, x) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2} x^n.$$

**Corollary 1.7.** (a) We have:

$$(14) \quad \mathcal{I}_{4_1}(b) = -\frac{i}{2} \left( \frac{q}{\tilde{q}} \right)^{\frac{1}{24}} (bG(q)g(\tilde{q}) - b^{-1}G(\tilde{q})g(q))$$

where

$$(15a) \quad g(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2}$$

$$(15b) \quad G(q) = \sum_{m=0}^{\infty} \left( 1 + 2m - 4 \sum_{s=1}^{\infty} \frac{q^{s(m+1)}}{1 - q^s} \right) (-1)^m \frac{q^{\frac{1}{2}m(m+1)}}{(q)_m^2}$$

(b) The series  $g(q)$  and  $G(q)$  are given in terms of  $F(q, x)$  by:

$$(16a) \quad g(q) = \langle F \rangle$$

$$(16b) \quad G(q) = \langle (2 + 2\delta - 4\delta_1)F \rangle$$

(c)  $F$  satisfies the linear  $q$ -difference equation

$$(17) \quad F(q, q^{-1}x) + F(q, qx) = (2 - x)F(q, x)$$

The series  $g(q)$  that appears in Theorem 1.7 was known to the first author and Zagier to be closely related to the  $4_1$  knot. For a detailed discussion of experimental facts below, see [GZ]. Empirically, it appears that

- the pair  $(g(q), G(q))$  is related to the 3D index of the  $4_1$  knot,
- the radial asymptotics of the pair  $(g(q), G(q))$  are related to the asymptotics of the Kashaev invariant of the  $4_1$  knot,
- the above observations for  $4_1$  also hold for the case of  $5_2$  knot discussed below.

Recall that the index of an ideal triangulation was introduced in [DGGb, DGGa], necessary and sufficient conditions for its convergence was established in [Gar] and its topological invariance was proven in [GHRs]. For a detailed discussion of the above experimental facts, see [GZ].

1.4. **The case of the  $5_2$  knot.** In this section, let

$$F(q, x) = F_{2,3}(q, x) = \sum_{m=0}^{\infty} t_m(q)x^m, \quad \tilde{F}(q, \tilde{x}) = F_{1,3}(q, \tilde{x}) = \sum_{m=0}^{\infty} T_m(q)\tilde{x}^m$$

where

$$t_m(q) = \frac{q^{m(m+1)}}{(q)_m^3}, \quad T_n(q) = (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^3} = t_n(q^{-1}).$$

Let

$$\begin{aligned} R_{m,n}(q, \tilde{q}) &= -\frac{b^2}{2} \left( 1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_1^{(m)2}(q) - 3E_2^{(m)}(q) \right) \\ &\quad - \frac{1}{2\pi i} + \frac{1}{2} \left( 1 + 2m - 3E_1^{(m)}(q) \right) \left( 1 + 2n - 6E_1^{(n)}(\tilde{q}) \right) \\ &\quad + \frac{b^{-2}}{2} \left( -n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) - 9E_1^{(n)2}(\tilde{q}) + 3E_2^{(n)}(\tilde{q}) \right), \end{aligned}$$

where  $E_i^{(m)}(q)$  are defined in Equation (29a). For  $k = 1, \dots, 4$  let

$$(18) \quad g_k(q) = \sum_{m=0}^{\infty} p_k(m)t_m(q), \quad G_k(\tilde{q}) = \sum_{n=0}^{\infty} P_k(n)T_n(\tilde{q}),$$

where

$$(19a) \quad p_{1,m}(q) = 1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_1^{(m)2}(q) - 3E_2^{(m)}(q)$$

$$(19b) \quad p_{2,m}(q) = 1 + 2m - 3E_1^{(m)}(q)$$

$$(19c) \quad p_{3,m}(q) = 1$$

and

$$(20a) \quad P_{1,m}(q) = 1$$

$$(20b) \quad P_{2,m}(q) = 1 + 2n - 6E_1^{(n)}(\tilde{q})$$

$$(20c) \quad P_{3,m}(q) = -n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) - 9E_1^{(n)2}(\tilde{q}) + 3E_2^{(n)}(\tilde{q}).$$

**Corollary 1.8.** (a) We have:

(21)

$$\begin{aligned} \mathcal{I}_{2,3}(q) &= -e^{\frac{3\pi i}{4}} \left( \frac{q}{\tilde{q}} \right)^{\frac{1}{8}} \sum_{m,n=0}^{\infty} R_{m,n}(q, \tilde{q}) t_m(q) T_n(\tilde{q}) \\ &= -e^{\frac{3\pi i}{4}} \left( \frac{q}{\tilde{q}} \right)^{\frac{1}{8}} \left( -\frac{b^2}{2} g_1(q) G_1(\tilde{q}) - \frac{1}{2\pi i} g_3(q) G_1(\tilde{q}) + \frac{1}{2} g_2(q) G_2(\tilde{q}) + \frac{b^{-2}}{2} g_3(q) G_3(\tilde{q}) \right) \end{aligned}$$

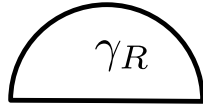
(b)  $F$  and  $\tilde{F}$  satisfy the linear  $q$ -difference equations

$$\begin{aligned} F(q, q^3x) - (3 - q^2x)F(q, q^2x) + 3F(q, qx) - F(q, x) &= 0 \\ \tilde{F}(q, q^3x) - 3\tilde{F}(q, q^2x) + (3 - q^2x)\tilde{F}(q, qx) - \tilde{F}(q, x) &= 0. \end{aligned}$$

**Remark 1.9.** A computation gives that  $P(A, B) = P(B - A, B)$  for  $(A, B) = (1, 2)$  and  $(A, B) = (2, 3)$  corresponding to the invariants of the  $4_1$  and  $5_2$  knots. In all other cases that we tried, we found that  $P(A, B)$  is not equal to  $P(B - A, B)$ .

## 2. PROOFS

**2.1. A residue computation.** To relate the state-integral  $\mathcal{I}_{A,B}$  to a sum, we will apply the residue theorem on a semicircle  $\gamma_R$  with center 0 and radius  $R$ , oriented counterclockwise in the upper half-plane:



Then, we will take the limit  $R \rightarrow \infty$ . To compute the residue of the integrand, we need to expand  $\Phi_b(x)$  near the pole

$$x_{m,n} = c_b + ibm + ib^{-1}n$$

for natural numbers  $m$  and  $n$ . Let

$$(23) \quad \phi_m(x) = \frac{(q^{m+1}e^x; q)_\infty}{(q^{m+1}; q)_\infty}$$

$$(24) \quad \tilde{\phi}_n(x) = \frac{(\tilde{q}; \tilde{q})_\infty (\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}e^x; \tilde{q})_\infty (\tilde{q}^{-1}e^x; \tilde{q}^{-1})_n}$$

**Lemma 2.1.** We have:

$$(25) \quad \Phi_b(x + x_{m,n}) = \frac{(q; q)_\infty}{(\tilde{q}; \tilde{q})_\infty} \frac{1}{(q; q)_m} \frac{1}{(\tilde{q}^{-1}; \tilde{q}^{-1})_n} \frac{\phi_m(2\pi bx) \tilde{\phi}_n(2\pi b^{-1}x)}{1 - e^{2\pi b^{-1}x}}.$$

*Proof.* Equation (1) implies the functional equations

$$\frac{\Phi_b(x + c_b + ib)}{\Phi_b(x + c_b)} = \frac{1}{1 - qe^{2\pi bx}}$$

$$\frac{\Phi_b(x + c_b + ib^{-1})}{\Phi_b(x + c_b)} = \frac{1}{1 - \tilde{q}^{-1}e^{2\pi b^{-1}x}}$$

which give

$$\Phi_b(x + x_{m,n}) = \Phi_b(x + c_b) \frac{1}{(qe^{2\pi bx}; q)_m} \frac{1}{(\tilde{q}^{-1}e^{2\pi b^{-1}x}; \tilde{q}^{-1})_n}$$

$$\Phi_b(x + c_b) = \frac{1}{1 - e^{2\pi b^{-1}x}} \frac{(qe^{2\pi bx}; q)_\infty}{(\tilde{q}e^{2\pi b^{-1}x}; \tilde{q})_\infty}.$$

Thus,

$$\begin{aligned} \Phi_b(x + x_{m,n}) &= \frac{(q; q)_\infty}{(\tilde{q}; \tilde{q})_\infty} \frac{1}{(q; q)_m} \frac{1}{(\tilde{q}^{-1}; \tilde{q}^{-1})_n} \\ &= \frac{1}{1 - e^{2\pi b^{-1}x}} \frac{(qe^{2\pi bx}; q)_\infty}{(q; q)_\infty} \frac{(\tilde{q}; \tilde{q})_\infty}{(\tilde{q}e^{2\pi b^{-1}x}; \tilde{q})_\infty} \frac{(q; q)_m}{(qe^{2\pi bx}; q)_m} \frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^{2\pi b^{-1}x}; \tilde{q}^{-1})_n} \\ &= \frac{(q; q)_\infty}{(\tilde{q}; \tilde{q})_\infty} \frac{1}{(q; q)_m} \frac{1}{(\tilde{q}^{-1}; \tilde{q}^{-1})_n} \\ &= \frac{1}{1 - e^{2\pi b^{-1}x}} \frac{(q^{m+1}e^{2\pi bx}; q)_\infty}{(q^{m+1}; q)_\infty} \frac{(\tilde{q}; \tilde{q})_\infty}{(\tilde{q}e^{2\pi b^{-1}x}; \tilde{q})_\infty} \frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^{2\pi b^{-1}x}; \tilde{q}^{-1})_n} \end{aligned}$$

The result follows.  $\square$

The decoupling of  $(m, n)$  in the quadratic form comes as follows: since  $A, m, n$  are integers,  $e^{A\pi imn} = 1$  and a computation gives

$$e^{-A\pi i(x+x_n, m)^2} = i^A \left( \frac{q}{\tilde{q}} \right)^{\frac{A}{8}} t_m^A(q) \tilde{t}_n^A(\tilde{q}) e^{-A\pi ix^2 + 2A\pi x(b(m+\frac{1}{2}) + b^{-1}(n+\frac{1}{2}))}$$

where

$$t_m^A(q) = (-1)^{Am} q^{A\frac{m(m+1)}{2}}, \quad \tilde{t}_n^A(\tilde{q}) = (-1)^{An} \tilde{q}^{-A\frac{n(n+1)}{2}}.$$

The Dedekind function  $\eta(\tau) = q^{1/24}(q; q)_\infty$  (with  $q = e^{2\pi i\tau}$ ) satisfies the modular equation  $\eta(-\tau^{-1}) = \sqrt{-i\tau}\eta(\tau)$  [And76]. It follows that

$$(26) \quad \frac{(q; q)_\infty}{(\tilde{q}; \tilde{q})_\infty} = e^{\frac{\pi i}{4}} \left( \frac{\tilde{q}}{q} \right)^{\frac{1}{24}} b^{-1}.$$

After we set  $w = x/(2\pi)$ , the above discussion implies that

$$(27) \quad \mathcal{I}_{A,B}(b) = \left( \frac{\tilde{q}}{q} \right)^{\frac{B-3A}{24}} e^{\pi i \frac{B+2(A+1)}{4}} \sum_{m,n=0}^{\infty} (\text{Res}_{w=0} F_{A,B,m,n}(w)) \frac{t_m^A(q)}{(q; q)_m^B} \frac{\tilde{t}_n^A(\tilde{q})}{(\tilde{q}^{-1}; \tilde{q}^{-1})_n^B},$$

where

$$(28) \quad F_{A,B,m,n}(w) = e^{\frac{A}{4\pi i}w^2 + Aw(b(m+\frac{1}{2})+b^{-1}(n+\frac{1}{2}))} \left( \frac{\phi_m(bw) \tilde{\phi}_n(b^{-1}w)}{b(1-e^{b^{-1}w})} \right)^B.$$

**2.2. The Taylor series of  $\phi_m(x)$  and  $\tilde{\phi}_n(x)$ .** In this section we express the Taylor series of  $\phi_m(x)$  and  $\tilde{\phi}_n(x)$  in terms of the  $q$ -series  $E_l^{(m)}(q)$  and  $\tilde{E}_l^{(m)}(\tilde{q})$  defined by:

$$(29a) \quad E_l^{(m)}(q) = \sum_{s=1}^{\infty} \frac{s^{l-1} q^{s(m+1)}}{1-q^s} = \langle \delta_l(x^m) \rangle$$

$$(29b) \quad \tilde{E}_l^{(n)}(\tilde{q}) = \begin{cases} -n + E_1^{(n)}(\tilde{q}) & \text{if } l = 1 \\ E_l^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is odd} \\ 2E_l^{(0)}(\tilde{q}) - E_l^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is even} \end{cases}$$

**Proposition 2.2.** We have:

$$(30a) \quad \phi_m(x) = \exp \left( - \sum_{l=1}^{\infty} \frac{1}{l!} E_l^{(m)}(q) x^l \right)$$

$$(30b) \quad \tilde{\phi}_n(x) = \exp \left( \sum_{l=1}^{\infty} \frac{1}{l!} \tilde{E}_l^{(m)}(\tilde{q}) x^l \right).$$

The proof of this proposition is given in Section 2.6. The first few terms in Equations (30a)-(30a) are given by:

$$(31a) \quad \phi_m(x) = \exp \left( -E_1^{(m)}x - \frac{1}{2}E_2^{(m)}x^2 - \frac{1}{6}E_3^{(m)}x^3 - \frac{1}{24}E_4^{(m)}x^4 - \dots \right)$$

$$(31b) \quad = 1 - E_1^{(m)}x + \frac{1}{2}(E_1^{(m)2} - E_2^{(m)})x^2 + \frac{1}{6}(-E_1^{(m)3} + 3E_1^{(m)}E_2^{(m)} - E_3^{(m)})x^3 + \frac{1}{24}(E_1^{(m)4} - 6E_1^{(m)2}E_2^{(m)} + 3E_2^{(m)2} + 4E_1^{(m)}E_3^{(m)} - E_4^{(m)})x^4 + \dots$$

$$(31c) \quad \tilde{\phi}_n(x) = \exp \left( \tilde{E}_1^{(n)}x + \frac{1}{2}\tilde{E}_2^{(n)}x^2 + \frac{1}{6}\tilde{E}_3^{(n)}x^3 + \frac{1}{24}\tilde{E}_4^{(n)}x^4 - \dots \right)$$

$$= 1 + \tilde{E}_1^{(n)}x + \frac{1}{2}(\tilde{E}_1^{(n)2} + \tilde{E}_2^{(n)})x^2 + \frac{1}{6}(\tilde{E}_1^{(n)3} + 3\tilde{E}_1^{(n)}\tilde{E}_2^{(n)} + \tilde{E}_3^{(n)})x^3 +$$

$$(31d) \quad \frac{1}{24}(\tilde{E}_1^{(n)4} + 6\tilde{E}_1^{(n)2}\tilde{E}_2^{(n)} + 3\tilde{E}_2^{(n)2} + 4\tilde{E}_1^{(n)}\tilde{E}_3^{(n)} + \tilde{E}_4^{(n)})x^4 + \dots$$

where  $E_l^{(m)} = E_l^{(m)}(q)$  and  $\tilde{E}_l^{(m)} = \tilde{E}_l^{(m)}(\tilde{q})$ .

**2.3. The connection with the differential operators  $\delta_l$  and  $\tilde{\delta}_l$ .** In this section we connect the series  $E_l^{(m)}(q)$  and  $\tilde{E}_l^{(m)}(\tilde{q})$  with the action of the differential operators  $\delta_l$  and  $\tilde{\delta}_l$  on a series  $F(x)$  and  $\tilde{F}(\tilde{x})$  respectively. Consider formal power series

$$F(x) = \sum_{m=0}^{\infty} t(m)x^m \quad \tilde{F}(\tilde{x}) = \sum_{m=0}^{\infty} \tilde{t}(m)\tilde{x}^m.$$



**Lemma 2.3.** We have:

$$(32) \quad \sum_{m=0}^{\infty} \left( \prod_{j=1}^r E_{l_j}^{(m)}(q) \right) t(m) = \left\langle \prod_{j=1}^r \delta_{l_j} F \right\rangle$$

$$(33) \quad \sum_{m=0}^{\infty} m^r t(m) = \langle \delta^r F \rangle$$

and

$$(34) \quad \sum_{n=0}^{\infty} \left( \prod_{j=1}^r \tilde{E}_{l_j}^{(n)}(\tilde{q}) \right) \tilde{t}(n) = \left\langle \prod_{j=1}^r \tilde{\delta}_{l_j} \tilde{F} \right\rangle$$

$$(35) \quad \sum_{n=0}^{\infty} n^r \tilde{t}(n) = \langle \tilde{d}^r \tilde{F} \rangle.$$

*Proof.* For a positive natural number  $l$  we have:

$$\sum_{m=0}^{\infty} E_l^{(m)}(q) t(m) = \sum_{m=0}^{\infty} \langle \delta_l(x^m) \rangle t(m) = \left\langle \delta_l \left( \sum_{m=0}^{\infty} t(m) x^m \right) \right\rangle = \langle \delta_l F \rangle.$$

Moreover, for positive natural numbers  $l, l'$  we have:

$$\begin{aligned} \sum_{m=0}^{\infty} E_l^{(m)}(q) E_{l'}^{(m)}(q) t(m) &= \sum_{m=0}^{\infty} \langle \delta_l(x^m) \rangle \langle \delta_{l'}(x^m) \rangle t(m) \\ &= \left\langle \delta_l \left( \sum_{m=0}^{\infty} \langle \delta_{l'}(x^m) \rangle t(m) x^m \right) \right\rangle. \end{aligned}$$

Now,

$$\langle \delta_{l'}(x^m) \rangle t(m) x^m = \sum_{s=1}^{\infty} \frac{s^{l'-1} q^s}{1 - q^s} q^{sm} t(m) x^m = \delta_{l'}(x^m) t(m)$$

and summing up over  $m$ , we obtain that

$$\sum_{m=0}^{\infty} \langle \delta_{l'}(x^m) \rangle t(m) x^m = \delta_{l'} F(q, x).$$

It follows that

$$\sum_{m=0}^{\infty} E_l^{(m)}(q) E_{l'}^{(m)}(q) t(m) = \langle \delta_l \delta_{l'} F \rangle.$$

The general case of Equation (32) follows by induction on  $r$ . Equation (33) is obvious.  $\square$

2.4. **Proof of Theorem 1.1.** Fix natural numbers  $A$  and  $B$  with  $B > A \geq 1$ , and let

$$t(m) = \frac{(-1)^{Am} q^{A \frac{m(m+1)}{2}}}{(q)_m^B}, \quad F(q, x) = \sum_{m=0}^{\infty} t(m) x^m$$

and

$$\tilde{t}(n) = \frac{(-1)^{(B-A)n} \tilde{q}^{(B-A) \frac{n(n+1)}{2}}}{(\tilde{q})_n^B}, \quad \tilde{F}(\tilde{q}, \tilde{x}) = \sum_{n=0}^{\infty} \tilde{t}(n) \tilde{x}^n.$$

Use Equations (27) and (28) and Proposition 2.2 to expand  $F_{A,B,m,n}(w)$  as a power series with coefficients polynomials in the variables  $m, E_l^{(m)}(q)$  and  $n, \tilde{E}_l^{(n)}(\tilde{q})$  and  $b^{\pm 1}$  and  $(2\pi i)^{-1}$ . Now apply Lemma 2.3 to convert the variables  $m, E_l^{(m)}(q), n, \tilde{E}_l^{(n)}(\tilde{q})$  in terms of the action of the operators  $\delta, \delta_l, \tilde{\delta}, \tilde{\delta}_l$  respectively. This concludes the proof of Theorem 1.1.  $\square$

2.5. **Some auxiliary power series.** Consider the auxiliary series

$$(36) \quad \frac{1}{ae^x - 1} = \sum_{n=0}^{\infty} p_n(a) x^n$$

where

$$p_n(a) = -\frac{a}{n!(1-a)^{n+1}} \sum_{m=0}^{n-1} A_{n,m} a^m \quad p_0(a) = -\frac{1}{1-a}$$

and  $A_{n,m}$  are *Euler triangular numbers* (sequence A008292 in the online encyclopedia of integer sequences [Slo]) that satisfy the recursion

$$A_{n,m} = (n-m)A_{n-1,m-1} + (m+1)A_{n-1,m}$$

and also given by the sum

$$A_{n,m} = \sum_{k=0}^m (-1)^k \binom{n+1}{k} (m+1-k)^n.$$

For a detailed discussion on this subject, see [FS70]. A table of the first few numbers  $A_{n,m}$  is given by

$n \setminus m$	0	1	2	3	4	5	6	7	8
1	1								
2	1	1							
3	1	4	1						
4	1	11	11	1					
5	1	26	66	26	1				
6	1	57	302	302	57	1			
7	1	120	1191	2416	1191	120	1		
8	1	247	4293	15619	15619	4293	247	1	
9	1	502	14608	88234	156190	88234	14608	502	1

**Lemma 2.4.** For  $l \geq 1$ , we have:

$$(37) \quad \frac{d^l}{dx^l} \log(1 - q^k e^{bx})|_{x=0} = b^l p_{l-1}(q^k) + b \delta_{l,1}$$

*Proof.* It follows from

$$\frac{d}{dx} \log(1 - q^k e^{bx}) = b \left( 1 + \frac{1}{q^k e^{bx} - 1} \right)$$

and Equation (36).  $\square$

For positive natural numbers  $l, r$  with  $l \geq r$  and  $m$  consider the  $q$ -series  $E_{l,r}^{(m)}(q)$  defined by

$$(38) \quad E_{l,r}^{(m)}(q) = \sum_{k=m+1}^{\infty} \frac{q^{kr}}{(1 - q^k)^l}$$

**Lemma 2.5.** (a) We have

$$(39) \quad E_{l,r}^{(m)}(q) = \sum_{s=r}^{\infty} a_{l,s} \frac{q^{s(m+1)}}{1 - q^s}$$

where

$$\frac{x^r}{(1 - x)^l} = \sum_{s=r}^{\infty} a_{l,s} x^s$$

(b) It follows that

$$(40) \quad \sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{(m)}(q) = E_l^{(m)}(q)$$

*Proof.* For (a), interchange  $k$  and  $s$  summation:

$$E_{l,r}^{(m)}(q) = \sum_{k=m+1}^{\infty} \sum_{s=r}^{\infty} a_{l,s} q^{sk} = \sum_{s=r}^{\infty} \sum_{k=m+1}^{\infty} a_{l,s} q^{sk} = \sum_{s=r}^{\infty} q^{(m+1)s} \sum_{k=0}^{\infty} a_{l,s} q^{sk} = \sum_{s=r}^{\infty} a_{l,s} \frac{q^{(m+1)s}}{1 - q^s}$$

(b) follows from (a) and the fact that

$$\frac{\sum_{r=0}^{l-1} A_{l-1,r} x^r}{(1 - x)^l} = \sum_{s=1}^{\infty} s^{l-1} x^s.$$

$\square$

**Lemma 2.6.** We have:

$$(41) \quad \phi_m(x) = \exp \left( - \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{(m)}(q) x^l \right)$$

*Proof.* It follows from Lemma 2.4 combined with

$$\log(\phi_m(x)) = \log \left( \frac{(q^{m+1} e^x; q)_{\infty}}{(q^{m+1}; q)_{\infty}} \right) = \sum_{l=m+1}^{\infty} (\log(1 - q^l e^x) - \log(1 - q^l))$$

$\square$

2.6. **Proof of Proposition 2.2.** Part (a) of Proposition 2.2 follows from Lemma 2.5 and Lemma 2.6. For part (b), we will use the series

$$E_l^{[m]}(q) = \sum_{s=1}^{\infty} \frac{s^{k-1} q^{s(m+1)}}{1 - q^s}$$

Using

$$\log(\tilde{\phi}_n(x)) = \log\left(\frac{(\tilde{q}; \tilde{q})_{\infty}}{(\tilde{q}e^x; \tilde{q})_{\infty}}\right) + \log\left(\frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^x; \tilde{q}^{-1})_n}\right)$$

and the proof of part (a) of Proposition 2.2, it follows that

$$\begin{aligned} \log(\tilde{\phi}_n(x)) &= \log\left(\frac{(\tilde{q}; \tilde{q})_{\infty}}{(\tilde{q}e^x; \tilde{q})_{\infty}}\right) + \log\left(\frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^x; \tilde{q}^{-1})_n}\right) \\ &= \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{(0)}(\tilde{q}) x^l + \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{[n]}(\tilde{q}^{-1}) x^l \\ &= \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} \left( E_{l,r+1}^{(0)}(\tilde{q}) + E_{l,r+1}^{[n]}(\tilde{q}^{-1}) \right) x^l \end{aligned}$$

where

$$(42) \quad E_{l,r}^{[n]}(q) = \sum_{k=1}^n \frac{q^{kr}}{(1 - q^k)^l}.$$

Let

$$(43) \quad \tilde{E}_{l,r}^{(n)}(\tilde{q}) = \begin{cases} -n + E_{1,1}^{(n)}(\tilde{q}) & \text{if } l = r = 1 \\ E_{l,r}^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is odd} \\ 2E_{l,r}^{(0)}(\tilde{q}) - E_{l,r}^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is even} \end{cases}$$

We claim that

$$(44) \quad E_{l,r}^{(0)}(\tilde{q}) + E_{l,l-r}^{[n]}(\tilde{q}^{-1}) = \tilde{E}_{l,r}^{(n)}(\tilde{q})$$

for  $l > r \geq 1$  and

$$(45) \quad E_{1,1}^{(0)}(\tilde{q}) + E_{1,1}^{[n]}(\tilde{q}^{-1}) = \tilde{E}_{1,1}^{(n)}(\tilde{q})$$

Assuming Equations (44) and (45), it follows that

$$\begin{aligned} \log(\tilde{\phi}_n(x)) &= \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} \tilde{E}_{l,r+1}^{(n)}(\tilde{q}) x^l \\ &= \sum_{l=1}^{\infty} \frac{1}{l!} \tilde{E}_l^{(n)}(\tilde{q}) x^l \end{aligned}$$

where the last step follows from part (b) of Lemma 2.5.

It remains to prove Equations (44) and (45). Equation (44) follows from the definition of  $\widetilde{E}_{1,1}^{(n)}(\tilde{q})$  and

$$\begin{aligned} E_{l,r}^{(0)}(\tilde{q}) + E_{l,l-r}^{[n]}(\tilde{q}^{-1}) &= \sum_{k=1}^{\infty} \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} + \sum_{k=1}^n \frac{\tilde{q}^{-k(l-r)}}{(1-\tilde{q}^{-k})^l} \\ &= \sum_{k=1}^{\infty} \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} + (-1)^l \sum_{k=1}^n \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} \\ &= (1 + (-1)^l) \sum_{k=1}^n \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} + \sum_{k=n+1}^{\infty} \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} \end{aligned}$$

Equation (45) follows from

$$\begin{aligned} E_{1,1}^{(0)}(\tilde{q}) + E_{1,1}^{[n]}(\tilde{q}^{-1}) &= \sum_{k=1}^{\infty} \frac{\tilde{q}^k}{1-\tilde{q}^k} + \sum_{k=1}^n \frac{\tilde{q}^{-k}}{1-\tilde{q}^{-k}} \\ &= \sum_{k=1}^{\infty} \frac{1-1+\tilde{q}^k}{1-\tilde{q}^k} - \sum_{k=1}^n \frac{1}{1-\tilde{q}^k} = -n + \sum_{k=n+1}^{\infty} \frac{\tilde{q}^k}{1-\tilde{q}^k} \end{aligned}$$

This completes the proof of Proposition 2.2.  $\square$

**2.7. Proof of Lemma 1.5.** Part (a) of Lemma 1.5 follows from the definition of  $F_{A,B}$  and  $\widetilde{F}_{A,B}$ .

Part (b) follows from an application of Zeilberger's creative telescoping [Zei91]. To apply the method, define

$$t(m, x) = \frac{(-1)^{Am} q^{A \frac{m(m+1)}{2}}}{(q)_m^B} x^m$$

Then, observe that  $t$  satisfies the recursions with respect to  $m$  and  $x$ :

$$(1 - q^{m+1})^B t(m+1, x) = (-1)^A q^{A(m+1)} t(m, x) \quad t(m, qx) = q^m t(m, x).$$

Now, we eliminate  $q^m$  from the above equations as follows. The second equation implies that  $t(m+1, q^j x) = q^{j(m+1)} t(m+1, x)$ . Expanding the first equation, it follows that

$$\sum_{j=0}^B (-1)^j \binom{B}{j} t(m+1, q^j x) = (-1)^A q^A x t(m, q^A x)$$

Summing for  $m \geq 0$  implies (b).  $\square$

*Proof.* (of Corollary 1.6) The admissibility of  $F$  in the sense of Kontsevich-Soibelman, follows from [KS11, Sec.6.1] and [KS11, Thm.9]. Given this, the Nahm Equation (12) for  $\omega$  follows easily from part (b) of Lemma 1.5.  $\square$

3. AN APPLICATION: STATE-INTEGRALS OF THE  $4_1$  AND  $5_2$  KNOTS

3.1. **Proof of Corollary 1.7.** Assume now that  $(A, B) = (1, 2)$ . Then,

$$\begin{aligned} \frac{1}{(b(1 - e^{b^{-1}w}))^2} &= \frac{1}{w^2} - \frac{b^{-1}}{w} + O(1) \\ (\phi_m(bw))^2 &= 1 - 2E_1^{(m)}(q)bw + O(w^2) \\ (\tilde{\phi}_n(b^{-1}w))^2 &= 1 + 2\tilde{E}_1^{(n)}(\tilde{q})b^{-1}w + O(w^2) \\ e^{\frac{1}{4\pi i}w^2 + w(b(m+1/2) + b^{-1}(n+1/2))} &= 1 + \left(\frac{1}{2} + m\right)bw + \left(\frac{1}{2} + n\right)b^{-1}w + O(w^2) \end{aligned}$$

Combined with  $\tilde{E}_1^{(n)}(\tilde{q}) = -n + E_1^{(n)}(\tilde{q})$ , it follows that the residue  $R = \text{Res}_{w=0}(F_{1,2,m,n}(w))$  is given by

$$R = \left( b \left( \frac{1}{2} + m - 2E_1^{(m)}(q) \right) - b^{-1} \left( \frac{1}{2} + n - 2E_1^{(n)}(\tilde{q}) \right) \right)$$

The above, together with the fact that  $t_n(q) = (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2}$  satisfies  $t_n(q^{-1}) = t_n(q)$  implies Equation (14). Equation (17) follows from Equation (11) for  $(A, B) = (1, 2)$ .

3.2. **Proof of Corollary 1.8.** Assume now that  $(A, B) = (2, 3)$ . Then,

$$\begin{aligned} \frac{1}{(b(1 - e^{b^{-1}w}))^3} &= -\frac{1}{w^3} + \frac{3b^{-1}}{2w^2} - \frac{b^{-2}}{w} + O(1) \\ (\phi_m(bw))^3 &= 1 - 3E_1^{(m)}(q)bw + \frac{3}{2} \left( 3E_1^{(m)2}(q) - E_2^{(m)}(q) \right) b^2w^2 + O(w^3) \\ (\tilde{\phi}_n(b^{-1}w))^3 &= 1 + 3\tilde{E}_1^{(n)}(\tilde{q})b^{-1}w + \frac{3}{2} \left( 3\tilde{E}_1^{(n)2}(\tilde{q}) + \tilde{E}_2^{(n)}(\tilde{q}) \right) b^{-2}w^2 + O(w^3) \\ e^{\frac{2}{4\pi i}w^2 + 2w(b(m+1/2) + b^{-1}(n+1/2))} &= 1 + ((1 + 2m)b + (1 + 2n)b^{-1})w + \\ &\quad \left( 1 + \frac{b^2 + b^{-2}}{2} + \frac{1}{2\pi i} + 2b^2m^2 + 2b^{-2}n^2 + 4mn \right. \\ &\quad \left. + 2(1 + b^2)m + 2(1 + b^{-2})n \right) w^2 + O(w^3) \end{aligned}$$

If  $R = \text{Res}_{w=0}(F_{2,3,m,n}(w))$ , then

$$\begin{aligned} R_{m,n} &= -\frac{b^2}{2} \left( 1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_1^{(m)2}(q) - 3E_2^{(m)}(q) \right) \\ &\quad - \frac{1}{2\pi i} + \frac{1}{2} \left( 1 + 2m - 3E_1^{(m)}(q) \right) \left( 1 + 2n - 6E_1^{(n)}(\tilde{q}) \right) \\ &\quad + \frac{b^{-2}}{2} \left( -n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) - 9E_1^{(n)2}(\tilde{q}) + 3E_2^{(n)}(\tilde{q}) \right), \end{aligned}$$

This proves part (a) of Corollary 1.8. Part (b) follows from Equation (11) for  $(A, B) = (2, 3)$  and  $(A, B) = (1, 3)$ . Note that Theorem 1.1 states that

$$(46) \quad \mathcal{I}_{2,3}(q) = -e^{\frac{3\pi i}{4}} \langle P_{2,3}(F\tilde{F}) \rangle$$

where

$$\begin{aligned}
P_{2,3} = & -\frac{b^2}{2} \left( 1 + 4\delta + 4\delta^2 - 6\delta_1 - 12\delta\delta_1 + 9\delta_1^2 - 3\tilde{\delta}_2 \right) \\
& + \frac{1}{2} \left( 1 + 2\delta + \frac{i}{\pi} + 2\tilde{\delta} + 4\delta\tilde{\delta} - 3\delta_1 - 6\tilde{\delta}\delta_1 - 6e_2(\tilde{q}) - 6\tilde{\delta}_1 - 12\delta\tilde{\delta}_1 + 18\delta_1\tilde{\delta}_1 \right) \\
& + \frac{b^{-2}}{2} \left( -\tilde{\delta} - \tilde{\delta}^2 + 3\tilde{\delta}_1 + 6\tilde{\delta}\tilde{\delta}_1 - 9\tilde{\delta}_1^2 + 3\tilde{\delta}_2 \right).
\end{aligned}$$

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