# $(a, b)$-rectangle patterns in permutations and words

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**Abstract**

In this paper, we introduce the notion of a $(a, b)$-rectangle pattern on permutations that not only generalizes the notion of successive elements (bonds) in permutations, but is also related to mesh patterns introduced recently by Brändén and Claesson. We call the $(k, k)$-rectangle pattern the $k$-box pattern. To provide an enumeration result on the maximum number of occurrences of the 1-box pattern, we establish an enumerative result on pattern-avoiding signed permutations.

Further, we extend the notion of $(k, \ell)$-rectangle patterns to words and binary matrices, and provide distribution of $(1, \ell)$-rectangle patterns on words; explicit formulas are given for up to 7 letter alphabets where $\ell \in \{1, 2\}$, while obtaining distributions for larger alphabets depends on inverting a matrix we provide. We also provide similar results for the distribution of bonds over words. As a corollary to our studies we confirm a conjecture of Mathar on the number of "stable LEGO walls" of width 7 as well as prove three conjectures due to Hardin and a conjecture due to Barker. We also enumerate two sequences published by Hardin in the On-Line Encyclopedia of Integer Sequences.

**Keywords:** $(a, b)$-rectangle patterns, $k$-box patterns, bond, $k$-bond, mesh patterns, permutations, words, distribution, successions in permutations, Fibonacci numbers, LEGO

## 1 Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [2] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns (see [3] for a comprehensive introduction to the theory of patterns in permutations and words). This notion was studied further in a series of papers, e.g. in [1, 4, 5, 6, 11].

In this paper, we introduce the notion of an $(a, b)$-*rectangle* patterns in permutations, words and binary matrices. That is, let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$ be a permutation written in one-line notation, where $S_n$ denotes the set of all permutations of length $n$. Then we will consider the graph of $\sigma$, $G(\sigma)$, to be the set of points $(i, \sigma_i)$ for $i = 1, 2, \ldots, n$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1.

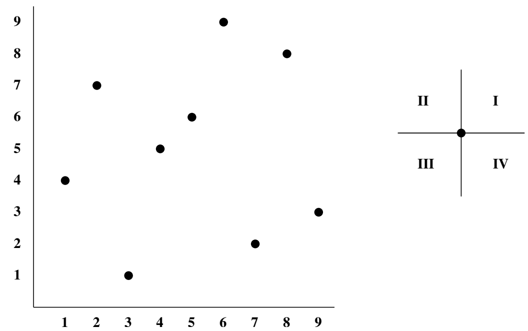


Figure 1: The graph of $\sigma = 471569283$.

Then if we draw a coordinate system centered at a point $(i, \sigma_i)$, we will be interested in the points that lie in the $(2a) \times (2b)$ rectangle centered at the origin, that is, in the set of points $(i \pm r, \sigma_i \pm s)$ such that $r \in \{0, 1, \ldots, a\}$ and $s \in \{0, 1, \ldots, b\}$. We say that $\sigma_i$ *matches the* $(a, b)$-*rectangle pattern* in $\sigma$, if there is at least one point in the $(2a) \times (2b)$ rectangle centered at the point $(i, \sigma_i)$ in $G(\sigma)$ other than $(i, \sigma_i)$. For example, when we look for matches of the (2,3)-rectangle patterns, we would look at $4 \times 6$ rectangles centered at points $(i, \sigma_i)$ as pictured in Figure 2 for a particular point.

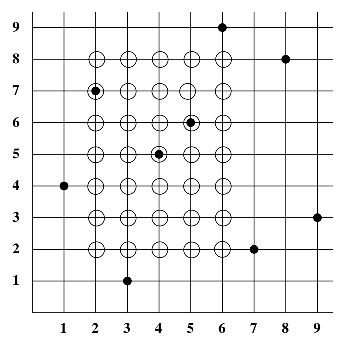


Figure 2: The $4 \times 6$ rectangle centered at the point $(4, 5)$ in the graph of $\sigma = 471569283$.

We shall refer to the $(k, k)$-rectangle pattern as the $k$-box pattern. For example, if $\sigma = 471569283$, then the 2-box centered at the point $(4, 5)$ in $G(\sigma)$ is the set of circled points pictured in Figure 3. Hence, $\sigma_i$ matches the $k$-box pattern in $\sigma$, if there is at least one point in the $k$-box centered at the point $(i, \sigma_i)$ in $G(\sigma)$ other than $(i, \sigma_i)$. For example, $\sigma_4$ matches the pattern $k$-box for all $k \geq 1$ in $\sigma = 471569283$ since the point $(5, 6)$ is present in the $k$-box centered at the point $(4,5)$ in $G(\sigma)$ for all $k \geq 1$. However, $\sigma_3$ only matches the $k$-box pattern in $\sigma = 471569283$ for $k \geq 3$ since there are no points in 1-box or 2-box centered at $(3,1)$ in $G(\sigma)$, but the point $(1,4)$ is in the 3-box centered at $(3,1)$ in $G(\sigma)$. For $k \geq 1$, we let $k$-box$(\sigma)$ denote the set of all $i$ such that $\sigma_i$ matches the $k$-box pattern in $\sigma = \sigma_1 \ldots \sigma_n$.

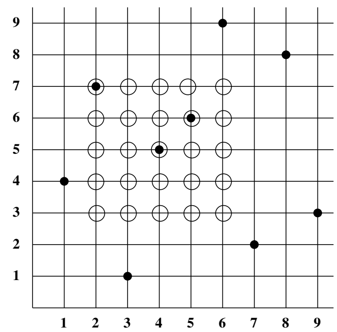

Figure 3: The 2-box centered at the point $(4,5)$ in the graph of $\sigma = 471569283$.

In this paper, we shall mainly be interested in the 1-box patterns in permutations and words. Note that $\sigma_i$ matches the 1-box pattern in a permutation $\sigma = \sigma_1 \ldots \sigma_n$ if either $|\sigma_i - \sigma_{i+1}| = 1$ or $|\sigma_{i-1} - \sigma_i| = 1$, while if $\sigma$ were a word, $\sigma_i$ matches the 1-box pattern if either $|\sigma_i - \sigma_{i+1}| \leq 1$ or $|\sigma_{i-1} - \sigma_i| \leq 1$. For any permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, let 1-box$(\sigma)$ denote the number of $i$ such that $\sigma_i$ matches the 1-box pattern in $\sigma$. More generally, $\sigma_i$ matches the $(a,b)$-rectangle pattern in $\sigma$ if there is a $\sigma_j$ such that $0 < |i-j] \leq a$ and $|\sigma_i - \sigma_j| \leq b$. We let $(a,b)$-rec$(\sigma)$ denote the number of $i$ such that $\sigma_i$ matches the $(a,b)$-rectangle pattern in $\sigma$.

Avoidance of the 1-box pattern is given by permutations without rising or falling successions which are also called bonds. That is, a *bond* in a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ is is a pair $\sigma_i\sigma_{i+1}$ of the form $s(s+1)$ or $(s+1)s$ for some $s$. We let bond$(\sigma)$ denote the number of bonds of $\sigma$. We note that in general 1-box$(\sigma) \neq$ bond$(\sigma)$. For example, if $\sigma = 214365$, then 1-box$(\sigma) = 6$ while bond$(\sigma) = 3$. However, for any permutation $\sigma \in S_n$, 1-box$(\sigma) = 0$ if and only if bond$(\sigma) = 0$.

The distributions of 1-box$(\sigma)$ and bond$(\sigma)$ for $S_2$, $S_3$, and $S_4$ are given below.

| $\sigma$ | 1-box$(\sigma)$ | bond$(\sigma)$ |
|---|---|---|
| 12 | 2 | 1 |
| 21 | 2 | 1 |

| $\sigma$ | 1-box$(\sigma)$ | bond$(\sigma)$ |
|---|---|---|
| 123 | 3 | 2 |
| 132 | 2 | 1 |
| 213 | 2 | 1 |
| 231 | 2 | 1 |
| 312 | 2 | 1 |
| 321 | 3 | 2 |

| $\sigma$ | 1-box($\sigma$) | bond($\sigma$) | | $\sigma$ | 1-box($\sigma$) | bond($\sigma$) |
|---|---|---|---|---|---|---|
| 1234 | 4 | 3 | | 2134 | 4 | 2 |
| 1243 | 4 | 2 | | 2143 | 4 | 2 |
| 1324 | 2 | 1 | | 2314 | 2 | 1 |
| 1342 | 2 | 1 | | 2341 | 3 | 2 |
| 1423 | 2 | 1 | | 2413 | 0 | 0 |
| 1432 | 3 | 2 | | 2431 | 2 | 1 |
| 3124 | 2 | 1 | | 4123 | 3 | 2 |
| 3142 | 0 | 0 | | 4132 | 2 | 1 |
| 3214 | 3 | 2 | | 4213 | 2 | 1 |
| 3241 | 2 | 1 | | 4231 | 2 | 1 |
| 3412 | 4 | 2 | | 4312 | 4 | 2 |
| 3421 | 4 | 2 | | 4321 | 4 | 3 |

Finding the number of permutations $\sigma$ of length $n$ with bond$(\sigma) = 0$ (equivalently, 1-box$(\sigma) = 0$) is equivalent to solving the *problem of Hertzsprung*, which is finding the number of ways to arrange $n$ non-attacking kings on an $n \times n$ board, with one in each row and column. Riordan [9] first derived a recurrence relation for the number $a_n$ of such permutations in 1965: $a_0 = a_1 = 1$, $a_2 = a_3 = 0$, and for $n \geq 4$,

$$a_n = (n+1)a_{n-1} - (n-2)a_{n-2} - (n-5)a_{n-3} + (n-3)a_{n-4}.$$

The initial values for $a_n$ are

$$1, 1, 0, 0, 2, 14, 90, 646, 5242, 47622, 479306, 5296790, 63779034, \ldots.$$

We refer to the sequence A002464 in the *On-Line Encyclopedia of Integer Sequences* (*OEIS*) for many references and for other interpretations/properties of this sequence of numbers. In particular, the generating function for these numbers was derived by Flajolet:

$$\sum_{n\geq 0} \frac{n!x^n(1-x)^n}{(1+x)^n}.$$

Riordan [9] obtained a more general result. That is, let $S_{n,m}$ be the number of permutations in $S_n$ with exactly $m$ bonds, and let $S[n] := S[n](t) = \sum_{m\geq 0} S_{n,m}t^m$. Then $S[0] = 1$, $S[1] = 1$, $S[2] = 2t$, $S[3] = 4t + 2t^2$, and for $n \geq 4$,

$$S[n] = (n+1-t)S[n-1] - (1-t)(n-2+3t)S[n-2] - (1-t)^2(n-5+t)S[n-3] + (1-t)^3(n-3)S[n-4].$$

In particular, the coefficient of $t$ in $S[n](t)$ gives the number of permutations of length $n$ with exactly one bond, which, in our terminology, is the number of permutations in $S_n$ with exactly two occurrences of the 1-box pattern. This is the sequence A086852 in the OEIS. Clearly, there are no permutations with exactly one occurrence of the 1-box pattern.

It is straightforward to see that the number of permutations of length $n+1$ with exactly three occurrences of the 1-box pattern is equal to the number of permutations of length $n$ with exactly two occurrences of the 1-box pattern. Indeed, to have exactly three occurrences of the pattern in a permutation $\pi$ means to have in $\pi$ a factor either of the form $a(a+1)(a+2)$ or of the form $(a+2)(a+1)a$, and no other consecutive successive elements. Removing $(a+1)$ from $\pi$ and decreasing by 1 all elements that are larger than $(a+1)$, we get a permutation containing exactly two occurrences of the 1-box pattern. This procedure is obviously reversible. Thus, the coefficient of $t$ in $S[n](t)$ also gives the number of permutations of length $n+1$ with exactly three occurrences of the 1-box pattern.

Hence, our study of 1-box/$k$-box patterns can not only be seen as an extension of the study of mesh patterns, but also as an extension of the study of consecutive successive elements (bonds) conducted in the literature. We do not define the notation of mesh patterns in this paper; however, the relevance of these patterns to our patterns is that in both cases we look for presence of points in specified regions in graphical representation of permutations.

In Theorem 2, we will enumerate permutations having the maximum number of occurrences of the 1-box pattern. To achieve this result, we obtain a result on pattern-avoiding signed permutation (see Theorem 1) thus contributing to the theory of permutation patterns (see [3]).

In Section 3 we not only provide a general solution (in matrix form) for finding the distribution of bonds and 1-box patterns over words (see Theorems 3 and 4) but also apply our studies to settle a conjecture of Mathar on the number of "stable LEGO walls" of width 7 (see Subsection 3.4), as well as to settle three conjectures of Hardin (see Subsection 3.3) and a conjecture of Barker (see Subsection 3.5). Also, in Subsection 3.5, we enumerate two sequences published by Hardin in the OEIS.

Given a word $w_1 \dots w_n \in [\ell]^n$, where $[\ell] = \{1, 2, \dots, \ell\}$, we say that the pair $w_i w_{i+1}$ is a *$k$-bond* if $|w_i - w_{i+1}| \leq k$. In Subsection 3.5, we study the distribution of 2-bonds and (1,2)-rectangle patterns in words.

# 2 Permutations with the maximum number of occurrences of the 1-box pattern

It is straightforward to see that the maximum possible number of occurrences of the 1-box pattern in a permutation of length $n$ is $n$ (e.g. the increasing permutation $12 \dots n$ achieves this maximum).

In order to enumerate permutations with the maximum number of occurrences of the 1-box pattern, we need the notion of the *hyperoctahedral group* $B_n$ whose elements can be regarded as *signed permutations* written as $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ in which each of the letters $1, 2, \dots, n$ appears, possibly barred. For example, $B_2 = \{12, \overline{1}2, 1\overline{2}, \overline{1}\overline{2}, 21, \overline{2}1, 2\overline{1}, \overline{2}\overline{1}\}$. Clearly, $|B_n| = 2^n n!$.

**Theorem 1.** *The exponential generating function for $a_n$, the number of elements in $B_n$*

*avoiding factors of the form* $i(i+1)$ *and* $(\overline{i+1})\bar{i}$ *simultaneously, is given by*

$$A(t) = \sum_{n\geq 2} \frac{a_n t^n}{n!} = 1 + 2\int_0^t \frac{e^{-z}}{(1-2z)^2}\, dz. \tag{1}$$

*The initial values for* $a_n$ *are*

$$1, 2, 6, 34, 262, 2562, 30278, 419234, 6651846, 118950658, 2366492038, \ldots.$$

*Proof.* Clearly, $a_0 = 1$ and $a_1 = 2$ since the empty signed permutation, as well as 1 and $\bar{1}$, avoid the prohibited factors. Our goal is to show that for $n \geq 2$,

$$a_n = (2n-1)a_{n-1} + 2(n-2)a_{n-2}. \tag{2}$$

In what follows, by *doubling* an element $i$ (resp. $\bar{i}$) of $\pi \in B_n$ we mean increasing all the elements of $\pi$, if any, that are greater than $i$ by 1, and substituting $i$ (resp. $\bar{i}$) by $i(i+1)$ (resp. $(\overline{i+1})\bar{i}$).

Let $b_n$ be the number of elements in $B_n$ with exactly one occurrence of either the factor $i(i+1)$ or $(\overline{i+1})\bar{i}$. We refer to the elements of such an occurrence as a *bad pair*. In particular, doubling an element results in appearance of exactly one new bad pair.

It is easy to see that

$$b_n = (n-1)a_{n-1}. \tag{3}$$

Indeed, the only way to create an object counted by $b_n$ is to pick an object counted by $a_{n-1}$ and to double one of its elements (exactly one bad pair is then created); this procedure is obviously reversible, since reversing the doubling procedure will never introduce new bad pairs.

Next, we will show the following relation between $a_n$s and $b_n$s:

$$a_n = 2b_{n-1} + 2na_{n-1} - a_{n-1}. \tag{4}$$

Indeed, remove the largest element (either $n$ or $\overline{n}$) in $\pi$ counted by $a_n$ to obtain $\pi'$. Since clearly at most one bad pair can be created, either $\pi'$ is counted by $b_{n-1}$ or by $a_{n-1}$. Thus, to generate all objects counted by $a_n$, we either take an object counted by

- $b_{n-1}$ and break the bad pair by inserting either $n$ or $\overline{n}$ between the bad pair elements; there are $2b_{n-1}$ ways to do this (note that there are no problems with $n-1$ or $\overline{n-1}$ be involved in the bad pair, since inserting either $n$ or $\overline{n}$ in this case will still not create a new bad pair), or an object counted by

- $a_{n-1}$. There are $n$ possible places we can insert either $n$ or $\overline{n}$ giving us $2na_{n-1}$ possibilities. However, inserting $n$ right after $(n-1)$ or inserting $\overline{n}$ right before $\overline{n-1}$ will give us a bad pair, and thus must not be counted: there are $a_{n-1}$ such objects (for each $\pi'$ counted by $a_{n-1}$ there is a unique bad position and a unique choice of the largest element to be inserted to create an object counted by $b_n$ rather than by $a_n$). This completes the proof of (4).

Using (3) and (4) we obtain (2).

Note that second derivative of $A(t)$ is given by

$$\begin{aligned} A''(t) &= \sum_{n\geq 0} a_{n+2}\frac{t^n}{n!} = \sum_{n\geq 0}((2n+3)a_{n+1}+2na_n)\frac{t^n}{n!} \\ &= 2t\sum_{n\geq 1} a_{n+1}\frac{t^{n-1}}{(n-1)!} + 3\sum_{n\geq 0} a_{n+1}\frac{t^n}{n!} + 2t\sum_{n\geq 1} a_n\frac{t^{n-1}}{(n-1)!} \\ &= 2tA''(t)+3A'(t)+2tA'(t). \end{aligned}$$

Solving for $A''(t)$, we see that

$$A''(t) = \frac{2t+3}{1-2t}A'(t)$$

or, equivalently,

$$\frac{A''(t)}{A'(t)} = -1 + \frac{4}{1-2t}. \tag{5}$$

Integrating both sides of (5) and using the fact that $A'(0) = 2$, we see that

$$\ln(A'(t)) = -t - 2\ln(1-2t) + \ln(2).$$

Thus

$$A'(t) = 2\frac{e^{-t}}{(1-2t)^2}. \tag{6}$$

Integrating both sides of (6) and using the fact that $A(0) = 1$, we see that

$$A(t) = 1 + 2\int_0^t \frac{e^{-t}}{(1-2t)^2}\mathrm{d}t.$$

□

**Remark 1.** *Theorem 1 is a result on pattern avoidance in signed permutations (see [3, Chapter 9.6] for relevant results). In fact, avoidance of factors of the form $i(i+1)$ and $\overline{(i+1)}\bar{i}$ can be expressed in terms of avoidance of* bivincular patterns *(see [3, Chapter 1.4] for definition; bars can be incorporated in the definition in an obvious way extending it from $S_n$ to $B_n$), and thus Theorem 1 seems to be the first instance of enumerative results on signed permutations avoiding bivincular patterns.*

**Theorem 2.** *The number of permutations in $S_n$ with the maximum number of occurrences of the* 1*-box pattern (which is $n$) is given by*

$$\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j-1}{j-1} a_j \tag{7}$$

*where $a_j$'s are given by the recurrence (2) or by the exponential generating function (1). The initial values for the number of such permutations starting with the case $n = 0$ are*

$$1, 1, 2, 2, 8, 14, 54, 128, 498, 1426, 5736, 18814, 78886, 287296, 1258018, \ldots.$$

*Proof.* Each permutation $\pi \in S_n$ having the maximum number of occurrences of the 1-box pattern can be uniquely decomposed into maximal factors of consecutive elements of size at least 2, since each element of $\pi$ must be staying next to a consecutive element. For example, the permutation $\pi = 543126798$ is decomposed into maximal factors 543, 12, 67 and 98. Let a permutation $\pi'$ be obtained from $\pi$ by substituting the $i$th largest block with $i$ if it is increasing, and with $\bar{i}$ if it is decreasing. We refer to $\pi'$ as the *basis permutation* for $\pi$ and, clearly, $\pi' \in B_n$ for some $n$. For $\pi$ as above, $\pi' = \bar{2}13\bar{4}$. Since the decomposition factors are of maximal possible length, basis permutations must avoid factors of the form $i(i+1)$ and $(\overline{i+1})\bar{i}$, and these permutations were counted by us in Theorem 1.

Finally, to create permutations of length $n$ with the maximum number of occurrences of the 1-box pattern, we choose basis permutations of length $j$, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, and decide on the lengths of the $j$ decomposition factors to be made decreasing or increasing depending on the respective elements to have or not to have bars, respectively. These lengths must be of size at least 2, and it is a standard combinatorial problem to see that the number of ways to make such a decision is $\binom{n-j-1}{j-1}$ (indeed, we reserve $2j$ elements to make sure each decomposition factor will contain at least two elements; the remaining $n-2j$ elements can be distributed among $j$ factors in the desired number of ways). Note that all permutations of interest will be generated in a bijective manner, which completes our proof of (7). □

# 3 Distribution of bonds and 1-box patterns over words

Given a word $w = w_1 \dots w_n$, let $|w| = n$ be the length of the $w$ and 1-box($w$) denote the number of occurences of the 1-box pattern in $w$. A *bond* in $w$ is a pair $w_i w_{i+1}$ of the form $s(s+1)$, $(s+1)s$, or $ss$ for some $s$. We let bond($w$) denote the number of bonds of $w$.

In Subsection 3.1 we study distribution of bonds over words, while in Subsection 3.2 we study distribution of 1-box patterns over words. Three relevant conjectures of Hardin are settled in Subsection 3.3, and a conjecture of Mathar on stable LEGO walls is settled in Subsection 3.4. In Subsection 3.5, we consider $(1,k)$-rectangle patterns for $k \geq 2$, which led us to solving a conjecture of Barker and enumerating two sequences of Hardin published in the OEIS.

## 3.1 Distribution of bonds over words.

As in the case of permutations, it is realatively straightforward to find the generating functions for the number of bonds in words over $[\ell]$ for any $\ell \geq 1$. That is, let

$$A_{\ell,1}(x,t) = \sum_{w \in [\ell]^*} x^{\mathrm{bond}(w)} t^{|w|} = \sum_{m,n \geq 0} a_\ell(m,n) x^m t^n,$$

where $[\ell]^*$ is the set of all words over the alphabet $[\ell]$. Thus $a_{\ell,1}(m,n)$ is the number of words $w$ of length $n$ over the alphabet $[\ell]$ such that bond($w$) $= m$.

The following theorem gives the distribution of bonds over words in matrix form.

**Theorem 3.** *The generating function $A_{\ell,1}(x,t)$ is equal to*

$$1+(\underbrace{1,\ldots,1}_{\ell})\mathbb{A}_{\ell,1}^{-1}(\underbrace{-t,\ldots,-t}_{\ell})^T$$

*where $\mathbb{A}_{\ell,1}$ is the following $\ell\times\ell$ matrix:*

$$\mathbb{A}_{\ell,1}=\left(\begin{array}{cccccccc} xt-1 & xt & t & t & t & \cdots & t & t \\ xt & xt-1 & xt & t & t & \cdots & t & t \\ t & xt & xt-1 & xt & t & \cdots & t & t \\ t & t & xt & xt-1 & xt & \cdots & t & t \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t & t & t & t & t & \cdots & xt & xt-1 \end{array}\right).$$

*Proof.* Let $i[\ell]^*$ denote the set of words over $[\ell]$ that begin with a letter $i$. For $1\le i\le\ell$, let

$$A_{\ell,1}^{(i)}(x,t)=\sum_{w\in i[\ell]^*}x^{\mathrm{bond}(w)}t^{|w|}=\sum_{m,n\ge0}a_{\ell,1}^{(i)}(m,n)x^mt^n.$$

Thus $a_{\ell,1}^{(i)}(m,n)$ is the number of words of length $n$ over $[\ell]$ such that $w$ begins with the letter $i$ and $\mathrm{bond}(w)=m$. Clearly,

$$A_{\ell,1}(x,t)=1+\sum_{1\le i\le\ell}A_{\ell,1}^{(i)}(x,t). \tag{8}$$

(The term 1 in (8) comes from the empty word.) Also, we have the following system of equations, where to obtain $A_{\ell,1}^{(i)}(x,t)$, we can think of taking words counted by $A_{\ell,1}^{(j)}(x,t)$, $1\le j\le\ell$, and adjoining the letter $i$ to the left of them; these functions are then to be multiplied by $xt$ if $|i-j|\le1$ (indicating that the length of such words is increased by 1 and one more bond is created), and by $t$ otherwise (to indicate change of the length keeping the number of occurrences of bonds the same); we also need to add $t$ corresponding to the one-letter word $i$.

$$\begin{array}{rcl} A_{\ell,1}^{(1)}(x,t) &=& t+xtA_{\ell,1}^{(1)}(x,t)+xtA_{\ell,1}^{(2)}(x,t)+tA_{\ell,1}^{(3)}(x,t)+tA_{\ell,1}^{(4)}(x,t)+\cdots+tA_{\ell,1}^{(\ell)}(x,t);\\ A_{\ell,1}^{(2)}(x,t) &=& t+xtA_{\ell,1}^{(1)}(x,t)+xtA_{\ell,1}^{(2)}(x,t)+xtA_{\ell,1}^{(3)}(x,t)+tA_{\ell,1}^{(4)}(x,t)+\cdots+tA_{\ell,1}^{(\ell)}(x,t);\\ A_{\ell,1}^{(3)}(x,t) &=& t+tA_{\ell,1}^{(1)}(x,t)+xtA_{\ell,1}^{(2)}(x,t)+xtA_{\ell,1}^{(3)}(x,t)+xtA_{\ell,1}^{(4)}(x,t)+\cdots+tA_{\ell,1}^{(\ell)}(x,t);\\ && \vdots \\ A_{\ell,1}^{(\ell)}(x,t) &=& t+tA_{\ell,1}^{(1)}(x,t)+tA_{\ell,1}^{(2)}(x,t)+\cdots+tA_{\ell,1}^{(\ell-2)}(x,t)+xtA_{\ell,1}^{(\ell-1)}(x,t)+xtA_{\ell,1}^{(\ell)}(x,t). \end{array}$$

Solving the system for the functions $A_{\ell,1}^{(i)}(x,t)$ and applying (8) we get the desired result. □

| $\ell$ | generating function for distribution of the number of bonds |
|---|---|
| $A_{3,1}(x,t)$ | $\frac{1-2(x-1)t-(x-1)^2t^2}{1-t-2xt-x(x-1)t^2}$ |
| $A_{4,1}(x.t)$ | $\frac{1-3(x-1)t+(x-1)^2t^2}{1-(3x+1)t+(x^2-1)t^2}$ |
| $A_{5,1}(x,t)$ | $\frac{1-3(x-1)t+2(x-1)^3t^3}{1-(3x+2)t+2(x-1)t^2+2(x+1)(x-1)^2t^3}$ |
| $A_{6,1}(x,t)$ | $\frac{1-4(x-1)t+3(x-1)^2t^2+(x-1)^3t^3}{1-2(2x+1)t+(3x^2+2x-5)t^2+(x+1)(x-1)^2t^3}$ |
| $A_{7,1}(x,t)$ | $\frac{1-4(x-1)t+2(x-1)^2t^2+4(x-1)^3t^3-(x-1)^4t^4}{1-(4x+3)t-(7-5x-2x^2)t^2+(4x+5)(x-1)^2t^3-(x+2)(x-1)^3t^4}$ |

Table 1: Distribution of the number of bonds on $\ell$-ary words, $\ell = 3, ..., 7$.

As corollaries to Theorem 3, we can obtain, e.g. using Mathematica, explicit generating functions for $\ell$ letter alphabets, where $3 \leq \ell \leq 7$. These are presented in Table 1. Note that $A_{1,1}(x,t)$ and $A_{2,1}(x,t)$ are trivial since any word $w$ of length $n$ over the alphabet $\{1\}$ or the alphabet $\{1,2\}$ has $n-1$ bonds. We also give expansions of the functions $A_\ell(x,t)$ for $\ell = 3, ..., 7$:

$$\begin{aligned}
A_{3,1}(x,t) &= 1+3t+(2+7x)t^2+\left(2+8x+17x^2\right)t^3+\left(2+10x+28x^2+41x^3\right)t^4 \\
&+ \left(2+12x+42x^2+88x^3+99x^4\right)t^5+\left(2+14x+58x^2+154x^3+262x^4+239x^5\right)t^6 \\
&+ \left(2+16x+76x^2+240x^3+524x^4+752x^5+577x^6\right)t^7 \\
&+ \left(2+18x+96x^2+348x^3+908x^4+1692x^5+2104x^6+1393x^7\right)t^8+\cdots;
\end{aligned}$$

$$\begin{aligned}
A_{4,1}(x,t) &= 1+4t+2(3+5x)t^2+2\left(5+14x+13x^2\right)t^3+4\left(4+17x+26x^2+17x^3\right)t^4 \\
&+ 2\left(13+72x+162x^2+176x^3+89x^4\right)t^5 \\
&+ 2\left(21+145x+422x^2+662x^3+565x^4+233x^5\right)t^6 \\
&+ 4\left(17+140x+503x^2+1016x^3+1239x^4+876x^5+305x^6\right)t^7 \\
&+ 2\left(55+527x+2247x^2+5567x^3+8717x^4+8757x^5+5301x^6+1597x^7\right)t^8+\cdots;
\end{aligned}$$

$$\begin{aligned}
A_{5,1}(x,t) &= 1+5t+(12+13x)t^2+5\left(6+12x+7x^2\right)t^3+\left(74+222x+234x^2+95x^3\right)t^4 \\
&+ \left(184+724x+1134x^2+824x^3+259x^4\right)t^5 \\
&+ \left(456+2236x+4574x^2+4902x^3+2750x^4+707x^5\right)t^6 \\
&+ \left(1132+6624x+16800x^2+23480x^3+19290x^4+8868x^5+1931x^6\right)t^7 \\
&+ \left(2808+19124x+57696x^2+99716x^3+106666x^4+71418x^5+27922x^6+5275x^7\right)t^8+\cdots;
\end{aligned}$$

$$\begin{aligned}
A_{6,1}(x,t) &= 1+6t+4(5+4x)t^2+4\left(17+26x+11x^2\right)t^3+2\left(115+263x+209x^2+61x^3\right)t^4 \\
&+ 4\left(195+590x+696x^2+378x^3+85x^4\right)t^5 \\
&+ 2\left(1321+4987x+7742x^2+6218x^3+2585x^4+475x^5\right)t^6 \\
&+ 2\left(4477+20230x+39031x^2+41156x^3+25211x^4+8534x^5+1329x^6\right)t^7 \\
&+ 2\left(15169+79871x+183933x^2+240507x^3+193107x^4+95997x^5+27503x^6+3721x^7\right)t^8+\cdots;
\end{aligned}$$

| $\ell$ | generating function for permutations which avoid the 1-box pattern |
|---|---|
| $A_{3,1}(0,t)$ | $\frac{1+2t-t^2}{1-t}$ |
| $A_{4,1}(0.t)$ | $\frac{1+3t+t^2}{1-t-t^2}$ |
| $A_{5,1}(0,t)$ | $\frac{1+3t-2t^3}{1-2t-2t^2+2t^3}$ |
| $A_{6,1}(0,t)$ | $\frac{1+4t+3t^2-t^3}{1-2t-5t^2+t^3}$ |
| $A_{7,1}(x,t)$ | $\frac{1+4t+2t^2-4t^3-t^4}{1-3t-7t^2+5t^3+2t^4}$ |

Table 2: Distribution of $\ell$-ary words which avoid the 1-box pattern for $\ell = 3, ..., 7$.

| $\ell$ | number of $\ell$-ary words avoiding the 1-box pattern | sequence in [10] |
|---|---|---|
| 3 | 1, 3, 2, 2, 2, 2, 2, 2, 2, 2, ... | |
| 4 | 1, 4, 6, 10, 16, 26, 42, 68, 110, 178, ... | A006355, $n \geq 1$ |
| 5 | 1, 5, 12, 30, 74, 184, 456, 1132, 2808, 6968, ... | A118649, $n \geq 1$ |
| 6 | 1, 6, 20, 68, 230, 780, 2642, 8954, 30338, 102804, ... | |
| 7 | 1, 7, 30, 130, 562, 2432, 10520, 45514, 196898, 851828, ... | |

Table 3: Avoidance of the 1-box patterns in $\ell$-ary words for lengths $n$ up to 9.

$$\begin{aligned}
A_{7,1}(x,t) &= 1 + 7t + (30 + 19x)t^2 + \left(130 + 160x + 53x^2\right) t^3 + \left(562 + 1034x + 656x^2 + 149x^3\right) t^4 \\
&+ \left(2432 + 5940x + 5598x^2 + 2416x^3 + 421x^4\right) t^5 \\
&+ \left(10520 + 32068x + 39942x^2 + 25526x^3 + 8400x^4 + 1193x^5\right) t^6 \\
&+ \left(45514 + 166236x + 257634x^2 + 217088x^3 + 105512x^4 + 28172x^5 + 3387x^6\right) t^7 \\
&+ \left(196898 + 838274x + 1553178x^2 + 1625554x^3 + 1039904x^4 + 409176x^5 + 92190x^6 + 9627x^7\right) t^8 + \cdots .
\end{aligned}$$

As noted in the introduction, the number of permutations $\sigma \in S_n$ such that 1-box$(\sigma) = 0$ equals the number of permutations $\sigma \in S_n$ such that bond$(\sigma) = 0$. The same applies to words. Thus, plugging in $x = 0$ in the functions in Table 1, one gets generating functions for avoidance of the 1-box pattern (alternatively, we can plug in $x = 0$ in the matrix $\mathbb{A}_{\ell,1}$ in Theorem 3 to get the most general case and to work out particular small values of $\ell$); in Table 3, we list initial values of the respective sequences indicating connections to the OEIS [10]. In particular, the connection to the sequence A118649 led us to solving a conjecture of Mathar (published in [10, A118649]) to be discussed in Subsection 3.4.

In [8], Knopfmacher, Mansour, Munagi, and Prodinger studied generating functions for smooth $\ell$ words where a word $w = w_1 \dots w_n \in [\ell]^n$ is smooth if $|w_i - w_{i+1}| \leq 1$ for $1 \leq i < n$. Thus in our notation, $w \in [\ell]^n$ is smooth if bond$(w) = n - 1$. Let $M_{n,1,\ell}$ denote the number of $w \in [\ell]^n$ such that bond$(w) = n - 1$ and $sm_\ell(t) = 1 + \sum_{n \geq 1} M_{n,1,\ell} t^n$. Then

Knopfmacher, Mansour, Munagi, and Prodinger [8, Theorem 2.2] proved that

$$sm_\ell(t) = 1 + \frac{t(\ell-(3\ell+2)t)}{(1-3t)^2} + \frac{2t^2}{(1-3t)^2}\frac{1+U_{\ell-1}\left(\frac{1-t}{2t}\right)}{U_\ell\left(\frac{1-t}{2t}\right)} \tag{9}$$

where $U_r(t)$ is the Chebyshev polynomial of the second kind defined by

$$U_r(\cos(\theta)) = \frac{\sin((r+1)\theta)}{\sin(\theta)}.$$

Alternatively, one can define the polynomials by recursion by setting $U_0(t) = 1$, $U_1(t) = 2t$, $U_2(t) = 4t^2 - 1$, and

$$U_r(t) = 2tU_{r-1}(t) - U_{r-2}(t) \text{ for } r \geq 3.$$

We can obtain the same generating functions from our generating function $B_{\ell,1}(x,t)$. That is, clearly

$$B_{\ell,1}(1/x, xt) = 1 + \sum_{n\geq 1}\sum_{w\in[\ell]^n} x^{n-\mathrm{bond}(w)}t^n$$

so that

$$C_{\ell,1}(x,t) := \frac{1}{x}(B_{\ell,1}(1/x,xt)-1) = \sum_{n\geq 1}\sum_{w\in[\ell]^n} x^{n-1-\mathrm{bond}(w)}t^n.$$

Hence

$$sm_\ell(t) = 1 + C_{\ell,1}(0,t).$$

## 3.2 Distribution of 1-box patterns over words.

One can use similar methods to find the distribution of 1-box$(w)$ for $w \in [\ell]^*$. In this case we have to keep track of more information. This is due to the fact that extra contribution to 1-box$(w)$ caused by adding an extra letter at the front of a word $w$ depends on the first two letters of $w$. For example, 1-box$(12) = x^2t^2$ and 1-box$(112) = x^3t^3$ so that adding 1 to the front of $w = 12$ increased $x^{\text{1-box}(w)}t^{|w|}$ by a factor of $xt$. However, 1-box$(13) = t^2$ and 1-box$(113) = x^2t^3$ so that adding 1 to the front of $w = 13$ increased $x^{\text{1-box}(w)}t^{|w|}$ by a factor of $x^2t$.

For $1 \leq i, j \leq \ell$, let

$$B_{\ell,1}^{(ij)} = \sum_{w\in ij[\ell]^*} WT(w)$$

where $WT(w) = x^{\text{1-box}(w)}t^{|w|}$ and $ij[\ell]^*$ denotes the set of words over $[\ell]$ that begin with letters $ij$. For any statement $S$, let $\chi(S) = 1$ if $S$ is true and $\chi(S) = 0$ if $S$ is false. Then we claim that for all $1 \leq i, j \leq \ell$,

$$\begin{aligned} B_{\ell,1}^{(ij)}(x,t) = {} & x^{2\chi(|i-j|\leq 1)}t^2 + \\ & \sum_{k=1}^{\ell}(t\chi(|i-j|>1) + xt\chi(|i-j|\leq 1)\chi(|j-k|\leq 1) + \\ & \quad x^2t\chi(|i-j|\leq 1)\chi(|j-k|>1))B_{\ell,1}^{(jk)}(x,t). \end{aligned} \tag{10}$$

That is, the words in $ij[\ell]^*$ are of the form $ij$ plus words $ijkv$ where $k \in [\ell]$ and $v \in [\ell]^*$. Now

$$WT[ij] = \begin{cases} t^2 & \text{if } |i-j| > 1 \text{ and} \\ x^2t^2 & \text{if } |i-j| \leq 1. \end{cases}$$

Similarly,

$$WT[ijkv] = \begin{cases} tWT[jkv] & \text{if } |i-j| > 1, \\ xtWT[jkv] & \text{if } |i-j| \leq 1 \text{ and } |j-k| \leq 1, \text{ and} \\ x^2tWT[jkv] & \text{if } |i-j| \leq 1 \text{ and } |j-k| > 1. \end{cases}$$

The set of equations of the form (10) can be written out in matrix form. That is, let $\vec{B}_{\ell,1}$ be the row vector of length $\ell^2$ of the $B_{\ell,1}^{(ij)}(t,x)$ where the elements are listed in the lexicographic order of the pairs $(ij)$. For example, $\vec{B}_3$ equals

$$(B_{3,1}^{(11)}(x,t), B_{3,1}^{(12)}(x,t), B_{3,1}^{(13)}(x,t), B_{3,1}^{(21)}(x,t), B_{3,1}^{(22)}(x,t), B_{3,1}^{(23)}(x,t), B_{3,1}^{(31)}(x,t), B_{3,1}^{(32)}(x,t), B_{3,1}^{(33)}(x,t)).$$

Similarly, let $\vec{I}_{\ell,1}$ be the row vector of length $\ell^2$ of the terms $t^2x^{2\chi(|i-j|\leq 1)}$ again listed in the lexicographic order on the pairs $ij$. For example,

$$\vec{I}_{3,1} = (x^2t^2, x^2t^2, t^2, x^2t^2, x^2t^2, x^2t^2, t^2, x^2t^2, x^2t^2).$$

Then one can write a set of equations of the form (10) in the form

$$(\vec{I}_{\ell,1})^T = \mathbb{B}_{\ell,1}(\vec{B}_{\ell,1})^T$$

where $\mathbb{B}_{\ell,1}$ is an $\ell^2 \times \ell^2$ matrix. For example, $\mathbb{B}_{3,1}$ is the matrix

$$\begin{pmatrix} xt-1 & xt & x^2t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & xt & xt & xt & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & t & t & t \\ xt & xt & x^2t & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & xt & xt-1 & xt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & x^2t & xt & xt \\ t & t & t & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & xt & xt & xt & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2t & xt & xt-1 \end{pmatrix}.$$

Note that since setting $x = t = 0$ in $\mathbb{B}_{\ell,1}$ will gives an $\ell \times \ell$ diagonal matrix with $-1$s on the diagonal, $\mathbb{B}_{\ell,1}$ is invertible. Thus

$$(\vec{B}_{\ell,1})^T = \mathbb{B}_{\ell,1}^{-1}(\vec{I}_{\ell,1})^T.$$

Let $\vec{1}_{\ell,1}$ denote the vector of length $\ell^2$ consisting of all 1s. Then

$$\sum_{1\leq i,j\leq \ell} B_{\ell,1}^{(ij)}(x,t) = \vec{1}_{\ell,1}\mathbb{B}_{\ell,1}^{-1}(\vec{I}_{\ell,1})^T.$$

Taking into account the empty word and all the words of length 1 will yeild the following theorem.

**Theorem 4.** *For all $\ell \geq 2$,*

$$B_{\ell,1}(x,t) = 1 + \ell t + \vec{1}_{\ell,1}\mathbb{B}_{\ell,1}^{-1}(\vec{I}_{\ell,1})^T.$$

We have used Theorem 4 to compute $B_{\ell,1}(x,t)$ for $\ell = 3, 4,$ and 5.

$$B_{3,1}(x,t) = \frac{1 + 2(1-x)t - (1+4x-5x^2)t^2 + 2x(1-x)^2t^3 + x^2(1-x)^2t^4}{1-(1+2x)t + 2x(1-x)t^2 + x^2(1-x)t^3};$$

$$B_{4,1}(x,t) = \frac{1 + 3(1-x)t + (1-9x+8x^2)\,t^2 - 3x(1-x)^2t^3 + x^2(1-x)^2t^4}{1-(1+3x)t - (1-3x+2x^2)\,t^2 - x\,(3-4x+x^2)\,t^3 - x^2(1-x)^2t^4};$$

$$B_{5,1}(x,t) = \frac{f_{5,1}(x,t)}{g_{5,1}(x,t)}$$

where

$$\begin{aligned} f_{5,1}(x,t) \;&=\; 1 + 3(1-x)t + 9x(1-x)t^2 - 2(1-x)^2(1+2x)t^3 + \\ &\phantom{=\;} 6x(1-x)^2(1+x)t^4 - 4(1-x)^3x^3t^6 \end{aligned}$$

and

$$\begin{aligned} g_{5,1}(x,t) \;&=\; 1 - (2+3x)t - \left(2-6x+4x^2\right)t^2 - \left(-2-6x+8x^2\right)t^3 - \\ &\phantom{=\;} 6(1-x)^2x(1+x)t^4 - 4(1-x)^2x^3t^5 + 4(1-x)^3x^3t^6. \end{aligned}$$

Using the generating functions above, we have computed some of the initial terms in their Taylor series expansions.

$$\begin{aligned}
&B_{3,1}(x,t) = 1 + 3t + (2+7x^2)t^2 + (2+8x^2+17x^3)t^3 + \\
&(2+10x^2+20x^3+49x^4)t^4 + (2+12x^2+26x^3+64x^4+139x^5)t^5 + \\
&(2+14x^2+32x^3+88x^4+200x^5+393x^6)t^6 + \\
&(2+16x^2+38x^3+114x^4+290x^5+614x^6+1113x^7)t^7 + \\
&(2+18x^2+44x^3+142x^4+392x^5+932x^6+1880x^7+3151x^8)t^8 + \cdots .
\end{aligned}$$

$$\begin{aligned}
&B_{4,1}(x,t) = 1 + 4t + (6+10x^2)t^2 + (10+28x^2+26x^3)t^3 + \\
&(16+68x^2+72x^3+100x^4)t^4 + (26+144x^2+174x^3+338x^4+342x^5)t^5 + \\
&(42+290x^2+368x^3+930x^4+1256x^5+1210x^6)t^6 + \\
&(68+560x^2+740x^3+2232x^4+3612x^5+4932x^6+4240x^7)t^7 + \\
&(110+1054x^2+1428x^3+4996x^4+8984x^5+15246x^6+18820x^7+14898x^8)t^8 + \cdots ,
\end{aligned}$$

$$\begin{aligned}
&B_{5,1}(x,t) = 1 + 5t + \left(12+13x^2\right)t^2 + 5\left(6+12x^2+7x^3\right)t^3 + \\
&\left(74+222x^2+160x^3+169x^4\right)t^4 + \left(184+724x^2+592x^3+974x^4+651x^5\right)t^5 + \\
&\left(456+2236x^2+1932x^3+4238x^4+4048x^5+2715x^6\right)t^6 + \\
&\left(1132+6624x^2+5968x^3+16036x^4+18372x^5+18982x^6+11011x^7\right)t^7 + \\
&\left(2808+19124x^2+17688x^3+56072x^4+71724x^5+94282x^6+83828x^7+45099x^8\right)t^8 + \cdots .
\end{aligned}$$

### 3.3 Solving three conjectures of Hardin.

Note that

$$\overline{B}_{k,1}(x,t) := B_{k,1}[1/x, xt] = \sum_{w\in[k]^*} x^{n-(1\text{-box}(w))} t^n$$

so that $\overline{B}_{k,1}(0,t)$ is the generating function of all words in $w = w_1 \dots w_n \in [k]^*$ such that 1-box$(w) = n$, i.e. each letter of $w$ differs from at least one neighbor by 1 or less. We have computed $\overline{B}_{k,1}(0,t)$ for $k = 3, 4, 5$.

$$\overline{B}_{3,1}(0,t) = \frac{1-2t+5t^2+2t^3+t^4}{1-2t-2t^2-t^3}.$$

The initial terms of this series are $1, 0, 7, 17, 49, 139, 393, 1113, 3151, 8921, \dots$. This is the sequence A221591 which was apparently computed directly from its combinatorial definition by R. H. Hardin. If $\overline{B}_{3,1}(0,t) = \sum_{n\geq 0} b_{3,1,n}t^n$, then Hardin observed empirically that $b_{3,1,n} = 2b_{3,1,n-1} + 2b_{3,1,n-2} + b_{3,1,n-3}$ for $n > 4$. This recursion follows immediately from the generating function for $\overline{B}_{3,1}(0,t)$ so that we have proved Hardin's conjecture.

$$\overline{B}_{4,1}(0,t) = \frac{1-3t+8t^2-3t^3+t^4}{1-3t-2t^2+t^3-t^4}.$$

The initial terms of this series are $1, 0, 10, 26, 100, 342, 1210, 4240, 14898, 52306, \dots$. This is the sequence A221569 which was also computed directly form its combinatorial definition by R. H. Hardin. If $\overline{B}_{4,1}(0,t) = \sum_{n\geq 0} b_{4,1,n}t^n$, then Hardin observed empirically that $b_{4,1,n} = 3b_{4,1,n-1} + 2b_{4,1,n-2} - b_{4,1,n-3} + b_{4,1,n-4}$ for $n > 5$. This recursion follows immediately from the generating function for $\overline{B}_{4,1}(0,t)$ so that we have also proved this conjecture of Hardin.

$$\overline{B}_{5,1}(0,t) = \frac{1-3t+9t^2-4t^3+6t^4+4t^6}{1-3t-4t^2-6t^4-4t^5-4t^6}.$$

The initial terms of this series are $1, 0, 13, 35, 169, 651, 2715, 11011, 45099, 184063, \dots$. This is the sequence A221592 which was also computed directly form its combinatorial definition by R. H. Hardin. If $\overline{B}_{5,1}(0,t) = \sum_{n\geq 0} b_{5,1,n}t^n$, then Hardin observed empirically that $b_{4,1,n} = 3b_{4,1,n-1} + 4b_{4,1,n-2} + 6b_{4,1,n-4} + 6b_{4,1,n-5} + 5b_{4,1,n-6}$ for $n > 6$. This recursion follows immediately from the the generating function for $\overline{B}_{5,1}(0,t)$ so that we have also proved this conjecture of Hardin.

### 3.4 Solving an enumerative conjecture on LEGO.

A "stable LEGO wall" is a wall in which seams do not match up from one level to the next. Stable LEGO walls of width 7 and heights 1 and 2 when using bricks of length 2, 3, and 4 can be found in Figure 4 (the numbers should be ignored there for the moment).

**Lemma 5.** *There is a bijection between words over the alphabet $A = \{1, 2, 3, 4, 5\}$ of length $n$ that avoid the* 1*-box pattern and stable LEGO walls of width* 7 *and height $n$ when using bricks of length* 2*,* 3*, and* 4*.*

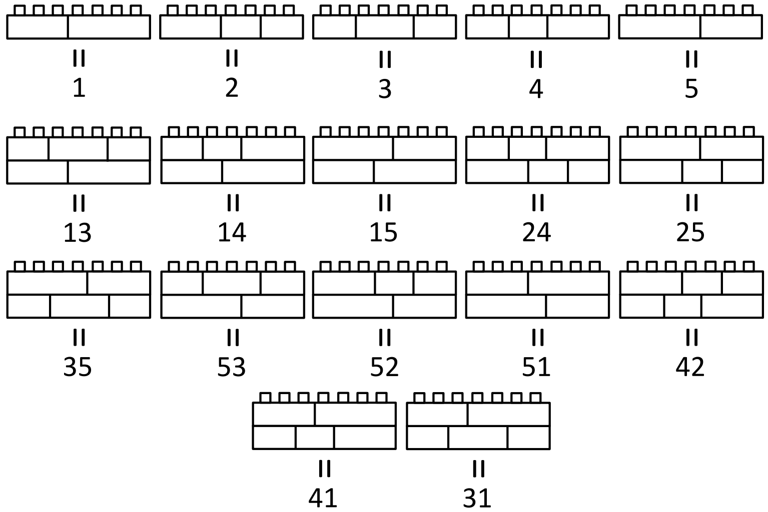


Figure 4: Stable LEGO walls of width 7 and heights 1 and 2.

*Proof.* Encode the eligible LEGO configurations of height 1 by the elements of $A$ as shown in Figure 4, which gives a bijection between the objects in the case of $n = 1$.

More generally, given a word $w = w_1 w_2 \dots w_n$ avoiding the 1-box pattern, we let the $i$-th level from below of the wall corresponding to $w$ be given by the configuration corresponding to the letter $w_i$ defined in Figure 4. For example, the correspondence for the case $n = 2$ is shown in Figure 4.

It is straightforward to check that the prohibited factors of words, namely 12, 23, 34, 45, 54, 43, 32, and 21, correspond to the prohibited configurations in LEGO, and vice versa. □

Using Lemma 5, the function corresponding to $\ell = 5$ and $x = 0$ in Table 1, and taking care of the offset (removing the number 2 in the sequence [10, A118649] and shifting down the indices of the larger numbers), we can confirm a conjecture of R. J. Mathar that stable LEGO walls satisfying the assumptions of Lemma 5 are counted by the following generating function:

$$\frac{1 + 3t - 2t^3}{1 - 2t - 2t^2 + 2t^3}.$$

### 3.5 $(1, k)$-rectangle patterns for $k \geq 2$; solving a conjecture of Barker and enumerating two sequences of Hardin.

Given a word $w = w_1 \dots w_n \in [\ell]^n$, let $k$-bond$(w) = |\{i : |w_i - w_{i+1}| \leq k\}|$. It is straightforward to generalize Theorems 3 and 4 to find the distribution of $k$-bond$(w)$ and $(1, k)$-rec$(w)$, the number of $(1, k)$-rectangle patterns in $w$, over words $w$ in $[\ell]^*$. That is, we claim that the same method of proof can also be used to find the generating function

$$A_{\ell,k} = \sum_{w \in [\ell]^*} x^{k\text{-bond}(w)} t^{|w|} = \sum_{m,n \geq 0} a_{\ell,k}(m,n) x^m t^n$$

for $k \geq 2$. Thus $a_{\ell,k}(m,n)$ is the number of words $w \in [\ell]^n$ such that $k$-bond$(w) = m$.

Let $\mathbb{A}_{\ell,k}$ be the $\ell \times \ell$ matrix whose entries on the main diagonal consists of all $xt-1$'s, whose entries on the first $k$ superdiagonals and the first $k$ subdiagonals are $xt$, and whose remaining entries are $t$. Then we have the following theorem.

**Theorem 6.** *For all $\ell, k \geq 1$,*

$$A_{\ell,k}(x,t) = 1 + (\underbrace{1,\ldots,1}_{\ell})\mathbb{A}_{\ell,k}^{-1}(\underbrace{-t,\ldots,-t}_{\ell})^T.$$

*Proof.* For $i = 1, 2, \ldots, \ell$, let

$$A_{\ell,k}^{(i)}(x,t) = \sum_{w \in i[\ell]^*} x^{k\text{-bond}(w)} t^{|w|} = \sum_{m,n \geq 0} a_{\ell,k}^{(i)}(m,n) x^m t^n.$$

When $k \geq 2$, we can follow the proof of Theorem 3 and find simple recurrences for the functions $A_{\ell,k}^{(i)}(x,t)$. Indeed, in this case we may have more possibilities to create an occurrence of the $k$-box pattern while adjoining letter $i$ from the left side, so that in the terminology of the proof of Theorem 3,

$$\begin{aligned} A_{\ell,k}^{(i)}(x,t) &= t + tA_{\ell,k}^{(1)}(x,t) + \cdots + tA_{\ell,k}^{(i-k-1)}(x,t) + \\ & xtA_{\ell}^{(i-k)}(x,t) + \cdots + xtA_{\ell,k}^{(i+k)}(x,t) + \\ & tA_{\ell,k}^{(i+k+1)}(x,t) + \cdots + tA_{\ell}^{(\ell)}(x,t). \end{aligned}$$

Thus, for an arbitrary $k$, the first row in the matrix $A$ in Theorem 3 is the vector

$$(xt-1, \underbrace{xt,\ldots,xt}_{k}, t, \ldots, t),$$

the second row is the vector

$$(xt, xt-1, \underbrace{xt,\ldots,xt}_{k}, t, \ldots, t),$$

and, more generally, any row in $A$ in this case is of the form

$$(t, \ldots, t, \underbrace{xt,\ldots,xt}_{k}, xt-1, \underbrace{xt,\ldots,xt}_{k}, t, \ldots, t).$$

□

| $\ell$ | generating function $A_{\ell,2}(x,t)$ for $\ell=4,5,6,7$. |
|---|---|
| 4 | $\frac{1-3t(-1+x)-2t^2(-1+x)^2}{1-t-3tx-2t^2(-1+x)x}$ |
| 5 | $\frac{1-4t(-1+x)+t^3(-1+x)^3}{1+t^2(-1+x)+t^3(-1+x)^2x-t(1+4x)}$ |
| 6 | $\frac{1-4t(-1+x)-t^2(-1+x)^2+t^3(-1+x)^3}{1-t^2(-1+x)^2+t^3(-1+x)^2(1+x)-2t(1+2x)}$ |
| 7 | $\frac{1-5t(-1+x)+2t^2(-1+x)^2+4t^3(-1+x)^3-2t^4(-1+x)^4}{1-2t^4(-1+x)^3(1+x)+2t^3(-1+x)^2(1+2x)-t(2+5x)+2t^2(-2+x+x^2)}$ |

Table 4: Distribution of the 2-bond on $\ell$-ary words, $\ell=4,5,6,7$.

For example, the generating function $A_{\ell,2}(x,t)$ is equal to

$$1+(\underbrace{1,\ldots,1}_{\ell})A_{\ell,2}^{-1}(\underbrace{-t,\ldots,-t}_{\ell})^T$$

where $A_{\ell,2}$ is the following $\ell\times\ell$ matrix:

$$\mathbb{A}_{\ell,2}=\begin{pmatrix} xt-1 & xt & xt & t & t & t & t & \cdots & t & t & t \\ xt & xt-1 & xt & xt & t & t & t & \cdots & t & t & t \\ xt & xt & xt-1 & xt & xt & t & t & \cdots & t & t & t \\ t & xt & xt & xt-1 & xt & xt & t & \cdots & t & t & t \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ t & t & t & t & t & t & t & \cdots & xt & xt & xt-1 \end{pmatrix}.$$

We have used Theorem 6 to compute the generating functions $A_{\ell,2}(x,t)$ for $\ell=4,5,6,7$.

$$\begin{aligned} A_{4,2}(x,t) \quad = \quad & 1+4t+2(1+7x)t^2+2\left(1+6x+25x^2\right)t^3+2\left(1+7x+31x^2+89x^3\right)t^4+ \\ & 2\left(1+8x+42x^2+144x^3+317x^4\right)t^5+2\left(1+9x+54x^2+222x^3+633x^4+1129x^5\right)t^6+ \\ & 2\left(1+10x+67x^2+316x^3+1095x^4+2682x^5+4021x^6\right)t^7+ \\ & 2\left(1+11x+81x^2+427x^3+1707x^4+5145x^5+11075x^6+14321x^7\right)t^8+\cdots; \end{aligned}$$

$$\begin{aligned} A_{5,2}(x,t) \quad = \quad & 1+5t+(6+19x)t^2+5\left(2+8x+15x^2\right)t^3+\left(16+88x+226x^2+295x^3\right)t^4+ \\ & \left(26+176x+606x^2+1156x^3+1161x^4\right)t^5+\left(42+342x+1428x^2+3644x^3+5600x^4+4569x^5\right)t^6+ \\ & \left(68+644x+3170x^2+9840x^3+20250x^4+26172x^5+17981x^6\right)t^7+ \\ & \left(110+1190x+6708x^2+24456x^3+61446x^4+106686x^5+119266x^6+70763x^7\right)t^8+\cdots; \end{aligned}$$

$$\begin{aligned} A_{6,2}(x,t) \quad = \quad & 1+6t+12(1+2x)t^2+4\left(7+22x+25x^2\right)t^3+\left(62+294x+522x^2+418x^3\right)t^4+ \\ & 4\left(35+214x+552x^2+706x^3+437x^4\right)t^5+2\left(157+1191x+3926x^2+7154x^3+7245x^4+3655x^5\right)t^6+ \\ & \left(706+6364x+25702x^2+59624x^3+85166x^4+71804x^5+30570x^6\right)t^7+ \\ & 2\left(793+8295x+39525x^2+111571x^3+202491x^4+239637x^5+173575x^6+63921x^7\right)t^8+\cdots; \end{aligned}$$

| $\ell$ | generating function for permutations which avoid the $(1,2)$-rectangle pattern |
|---|---|
| $A_{4,2}(0,t)$ | $\frac{1+3t-2t^2}{1-t}$ |
| $A_{5,2}(0,t)$ | $\frac{1+4t-t^3}{1-t-t^2}$ |
| $A_{6,2}(0,t)$ | $\frac{1+4t-t^2-t^3}{1-2t-t^2+t^3}$ |
| $A_{7,2}(0,t)$ | $\frac{1+5t+2t^2-4t^3-2t^4}{1-2t-4t^2+2t^3+2t^4}$ |

Table 5: Enumeration of $\ell$-ary words which avoid the $(1,2)$-rectangle pattern for $\ell = 4,5,6,7$.

| $\ell$ | number of $\ell$-ary words avoiding the (1,2)-rectangle pattern | sequence in [10] |
|---|---|---|
| 4 | 1, 4, 2, 2, 2, 2, 2, 2, 2, 2, ... | |
| 5 | 1, 5, 6, 10, 16, 26, 42, 68, 110, 178, ... | A006355, $n \geq 2$ |
| 6 | 1, 6, 12, 28, 62, 140, 314, 706, 1586, 3564, ... | A052994, $n \geq 2$ |
| 7 | 1, 7, 20, 62, 186, 566, 1712, 5192, 15728, 47688, ... | |

Table 6: Avoidance of the (1,2)-rectangle patterns in $\ell$-ary words for lengths $n$ up to 9.

$$\begin{aligned} A_{7,2}(x,t) \quad = \quad & 1+7t+(20+29x)t^2+\left(62+156x+125x^2\right)t^3+\left(186+710x+962x^2+543x^3\right)t^4+ \\ & \left(566+2820x+5658x^2+5400x^3+2363x^4\right)t^5+\left(1712+10648x+27710x^2+38526x^3+28766x^4+10287x^5\right)t^6+ \\ & \left(5192+38520x+124086x^2+222928x^3+239930x^4+148100x^5+44787x^6\right)t^7+ \\ & \left(15728+135852x+519888x^2+1149548x^3+1594738x^4+1409754x^5+744298x^6+194995x^7\right)t^8+\cdots. \end{aligned}$$

Clearly the number of words $w \in [\ell]^n$ such that $k$-bond$(w) = 0$ equals the number of words $w \in [\ell]^n$ such that $(1,k)$-rec$(w) = 0$. Plugging in $x = 0$ in the functions in Table 4 one gets generating functions for avoidance of the $(1,2)$-rectangle pattern. In Table 6, we list initial values of the respective sequences indicating connections to the OEIS [10].

We note that the sequence A052994 has no combinatorial interpretation in the OEIS so now we have given a combinatorial interpretation to this sequence. Also, comparing Tables 3 and 6, and using an interpretation of [10, A006355], one has the truth of the following proposition that we explain combinatorially.

**Proposition 1.** *For $n \geq 2$, the following objects are equinumerous:*

*(i) words of length $n$ over the alphabet $[5]$ that avoid the $(1,2)$-rectangle pattern;*

*(ii) words of length $n$ over the alphabet $[4]$ that avoid the 1-box pattern;*

*(iii) binary words of length $n+3$ that contain no singletons, that is, any 0 has a 0 staying next to it, and any 1 has a 1 staying next to it.*

| | | | | |
|---|---|---|---|---|
| 13 | 00011 | | 131 | 000111 |
| 14 | 00000 | | 141 | 000000 |
| 24 | 00111 | | 142 | 000011 |
| 31 | 11000 | | 241 | 001111 |
| 41 | 11111 | | 242 | 001100 |
| 42 | 11100 | | 313 | 110011 |
| | | | 314 | 110000 |
| | | | 413 | 111100 |
| | | | 414 | 111111 |
| | | | 424 | 111000 |

Table 7: Mapping 1-box avoiding permutations over [4] to binary strings without singletons.

*Thus, according to [10, A006355], any of these objects is counted by $F_{n-1}+F_{n+2}$ where $F_n$ is the nth Fibonacci number defined as $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.*

*Proof.* Equinumeration of (i) and (ii) follows directly from the observation that the letter 3 never appears in words described by (i), so that we can take any such word, make the substitution of letters $4 \to 3$ and $5 \to 4$ to get a proper word described by (ii); this operation is clearly reversible.

Equinumeration of (ii) and (iii) is established by the following bijective map from (ii) to (iii). Let a word $w = w_1w_2 \dots w_n$ described by (ii) is given and we want to obtain its binary image $u = u_1u_2 \dots u_{n+3}$. If $w_1 \in \{1, 2\}$ then $u_1u_2 = 00$; if $w_1 \in \{3, 4\}$ then $u_1u_2 = 11$. Also, no matter what $u_{n+2}$ is, we set $u_{n+3} = u_{n+2}$. To recover the letters $u_3, u_4, \dots, u_{n+2}$, we read $w$ from left to right letter by letter: if $w_i \in \{1, 4\}$, then $u_{i+2} = u_{i+1}$; if $w_i \in \{2, 3\}$, then $u_{i+2} \neq u_{i+1}$. For example, the word 3413142 avoiding the 1-box pattern is mapped to 1100011100. In Table 7, we provide our map for all words of length $n = 2, 3$.

We do not provide a proper proof of the fact that the map described by us from (ii) to (iii) is a bijection just giving a couple of remarks why this is the case. Indeed, if $w_i \in \{2, 3\}$ and $i < n$ then $w_{i+1} \in \{1, 4\}$ and thus $u_{i+2} = u_{i+3}$. This, together with the fact that $u_{n+3} = u_{n+2}$ makes sure that $u$ has no singletons. $\square$

We say that a word $w = w_1 \dots w_n \in [\ell]^n$ is *$k$-smooth* if $|w_i - w_{i+1}| \leq k$ for $1 \leq i < n$. Thus in our notation, $w \in [\ell]^n$ is $k$-smooth if $k$-bond$(w) = n - 1$. Let $M_{n,k,\ell}$ denote the number of $w \in [\ell]^n$ such that $k$-bond$(w) = n-1$ and $sm_{\ell,k}(t) = 1+\sum_{n\geq 1} M_{n,k,\ell}t^n$. Clearly,

$$B_{\ell,k}(1/x, xt) = 1 + \sum_{n\geq 1} \sum_{w\in[\ell]^n} x^{n-k\text{-bond}(w)}t^n$$

so that

$$C_{\ell,k}(x,t) = \frac{1}{x}(B_{\ell,k}(1/x, xt) - 1) \sum_{n\geq 1} \sum_{w\in[\ell]^n} x^{n-1-k\text{-bond}(w)}t^n.$$

Hence

$$sm_{\ell,k}(t) = 1 + C_{\ell,k}(0, t).$$

| $\ell$ | generating function for words $w \in [\ell]^n$ such that 2-bond$(w) = n-1$. |
|---|---|
| $sm_{4,2}(t)$ | $\frac{1+t}{1-3t-2t^2}$ |
| $sm_{5,2}(t)$ | $\frac{1+t-t^2}{1-4t+t^3}$ |
| $sm_{6,2}(t)$ | $\frac{1+2t-t^2-t^3}{1-4t-t^2+t^3}$ |
| $sm_{7,2}(t)$ | $\frac{1+2t-4t^2-2t^3+2t^4}{1-5t+2t^2+4t^3-2t^4}$ |

Table 8: Distribution of words $w \in [\ell]^n$ such that 2-bond$(w) = n-1$, $\ell = 4, 5, 6, 7$.

| $\ell$ | number of words $w \in [\ell]^n$ such that 2-bond$(w) = n-1$ | sequence in [10] |
|---|---|---|
| 4 | 1, 4, 14, 50, 178, 634, 2258, 8042, 28642,102010, ... | A055099, $n \geq 0$ |
| 5 | 1, 5, 19, 75, 295, 1161, 4569, 17981, 70763, 278483, ... | A126392, $n \geq 0$ |
| 6 | 1, 6, 24, 100, 418, 1748, 7310, 30570, 127842, 534628, ... | A126393, $n \geq 0$ |
| 7 | 1, 7, 29, 125, 543, 2363, 10287, 44787, 194995, 848979, ... | A126394, $n \geq 0$ |

Table 9: Number of words $w \in [\ell]^n$ such that $k$-bond$(w) = n-1$ for $n$ up to 9.

We have used our generating functions for $B_{\ell,2}(x,t)$ to compute $sm_{\ell,2}(t)$ for $\ell = 4, 5, 6, 7$, which we record in Table 9. In the case $\ell = 4$, our objects match a combinatorial interpretation for the sequence A055099. For the sequence A126392, the generating function $sm_{5,2}(t) = \frac{1+t-t^2}{1-4t+t^3}$ was conjectured by Colin Barker, so we have proved his conjecture. The sequences A126393 and A126394 were apparently computed from their combinatorial definitions by R. H. Hardin, so that we now have found explicit formulas for their generating functions.

One can also modify the proof of Theorem 4 to find the generating function for the distribution of $(1,k)$-rec$(w)$ for $w \in [\ell]^*$. That is, suppose $k \geq 2$, and for $1 \leq i, j \leq \ell$,

$$B_{\ell,k}^{(ij)} = \sum_{w \in ij[\ell]^*} WT_k(w)$$

where $WT_k(w) = x^{(1,k)\text{-rec}(w)} t^{|w|}$. Then we claim that for all $1 \leq i, j \leq \ell$,

$$\begin{aligned} B_{\ell,k}^{(ij)}(x,t) &= x^{2\chi(|i-j| \leq k)} t^2 + \\ &\sum_{k=1}^{\ell} (t\chi(|i-j| > k) + xt\chi(|i-j| \leq k)\chi(|j-k| \leq k) + \\ &\qquad x^2 t\chi(|i-j| \leq k)\chi(|j-k| > k)) B_{\ell,k}^{(jk)}(x,t). \end{aligned} \tag{11}$$

That is, the words in $ij[\ell]^*$ are of the form $ij$ plus words $ijmv$ where $m \in [\ell]$ and $v \in [\ell]^*$.

Now

$$WT_k[ij] = \begin{cases} t^2 & \text{if } |i-j| > k \text{ and} \\ x^2t^2 & \text{if } |i-j| \leq k. \end{cases}$$

Similarly,

$$WT_k[ijkv] = \begin{cases} tWT_k[jkv] & \text{if } |i-j| > k, \\ xtWT_k[jkv] & \text{if } |i-j| \leq k \text{ and } |j-k| \leq k, \text{ and} \\ x^2tWT_k[jkv] & \text{if } |i-j| \leq k \text{ and } |j-k| > k. \end{cases}$$

The set of equations of the form (11) can be written out in matrix form. That is let $\vec{B}_{\ell,1}$ be the row vector of length $\ell^2$ of the $B^{(ij)}_{\ell,1}(t,x)$ where the elements are listed in the lexicographic order of the pairs $(ij)$. Let $\vec{I}_{\ell,k}$ be the row vector of length $\ell^2$ of the terms $t^2x^{2\chi(|i-j|\leq k)}$ again listed in the lexicographic order on the pairs $ij$. For example,

$$\vec{I}_{4,2} = (x^2t^2, x^2t^2, x^2t^2, t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, t^2, x^2t^2, x^2t^2, x^2t^2).$$

Then one can write a set of equations of the form (11) in the form

$$(\vec{I}_{\ell,x})^T = \mathbb{B}_{\ell,k}(\vec{B}_{\ell,k})^T$$

where $\mathbb{B}_{\ell,k}$ is an $\ell^2 \times \ell^2$ matrix. For example, $\mathbb{B}_{4,2}$ is the matrix

$$\begin{pmatrix}
xt-1 & xt & xt & x^2t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & xt & xt & xt & xt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & xt & xt & xt & xt & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t \\
xt & xt & xt & x^2t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & xt & xt-1 & xt & xt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & xt & xt & xt & xt & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & x^2t & xt & xt & xt \\
xt & xt & xt & x^2t & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & xt & xt & xt & xt & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & xt & xt & xt-1 & xt & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & x^2t & xt & xt & xt \\
t & t & t & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & xt & xt & xt & xt & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & xt & xt & xt & xt & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2t & xt & xt & xt-1
\end{pmatrix}$$

Note that $\mathbb{B}_{\ell,k}$ is invertible since setting $x = t = 0$ in $\mathbb{B}_{\ell,k}$ will give the $\ell \times \ell$ diagonal matrix with $-1$s on the diagonal. Thus

$$(\vec{B}_{\ell,k})^T = \mathbb{B}^{-1}_{\ell,k}(\vec{I}_{\ell,k})^T.$$

Let $\vec{1}_{\ell,1}$ denote the vector of length $\ell^2$ consisting of all 1s. Then

$$\sum_{1\leq i,j\leq \ell} B^{(ij)}_{\ell,1}(x,t) = \vec{1}_{\ell,1}\mathbb{B}^{-1}_{\ell,k}(\vec{I}_{\ell,k})^T.$$

Taking into account the empty word and all the words of length 1 will yeild the following theorem.

**Theorem 7.** *For all $\ell \geq 2$,*

$$B_{\ell,k}(x,t) = 1 + \ell t + \vec{1}_{\ell,k}\mathbb{B}^{-1}_{\ell,k}(\vec{I}_{\ell,k})^T.$$

Note that $B_{\ell,2}(x,t) = \frac{1}{1-\ell xt}$ for $\ell = 1,2,3$ since in such words every letter matches $(1,2)$-rectangle pattern. We have used Theorem 7 to compute $B_{\ell,2}(x,t)$ for $\ell = 4,5$:

$$B_{4,2}(x,t) = \frac{1-3t(-1+x)+6t^3(-1+x)^2x+4t^4(-1+x)^2x^2+t^2\left(2+9x-11x^2\right)}{1-t-3tx-3t^2(-1+x)x+2t^3(-1+x)x^2};$$

$$\begin{aligned}&B_{5,2}(x,t) = \\ &-\frac{1+t(-1+x)\left(-4+t\left(16x+t(-1+x)\left(-1+x\left(-2+t^3(-1+x)x^2+4t(1+x)\right)\right)\right)\right)}{-1+t\left(1+4x+t(-1+x)\left(-1+x\left(3+t^3(-1+x)x^2+t(4+x)\right)\right)\right)}.\end{aligned}$$

Using the generating functions above, we have computed some of the initial terms in their Taylor series expansions:

$$\begin{aligned}&B_{4,2}(x,t) = 1+4t+2\left(1+7x^2\right)t^2+2\left(1+6x^2+25x^3\right)t^3+2\left(1+7x^2+22x^3+98x^4\right)t^4+\\ &2\left(1+8x^2+27x^3+93x^4+383x^5\right)t^5+2\left(1+9x^2+32x^3+117x^4+396x^5+1493x^6\right)t^6+\\ &2\left(1+10x^2+37x^3+142x^4+519x^5+1659x^6+5824x^7\right)t^7+\\ &2\left(1+11x^2+42x^3+168x^4+652x^5+2247x^6+6930x^7+22717x^8\right)t^8+\cdots;\end{aligned}$$

$$\begin{aligned}&B_{5,2}(x,t) = 1+5t+\left(6+19x^2\right)t^2+5\left(2+8x^2+15x^3\right)t^3+\left(16+88x^2+160x^3+361x^4\right)t^4+\\ &\left(26+176x^2+358x^3+876x^4+1689x^5\right)t^5+\\ &\left(42+342x^2+724x^3+2106x^4+4496x^5+7915x^6\right)t^6+\\ &\left(68+644x^2+1416x^3+4586x^4+11328x^5+22976x^6+37107x^7\right)t^7+\\ &\left(110+1190x^2+2680x^3+9562x^4+25712x^5+60762x^6+116672x^7+173937x^8\right)t^8+\cdots.\end{aligned}$$

We also can compute the generating functions of the number of words that avoid the $(2,1)$-rectangle patterns for words $w \in [5]^*$. That is, we have that

$$\begin{aligned}B_{5,2}(0,t) &= \frac{1+4t-t^3}{1-t-t^2} \\ &= 1+5t+6t^2+10t^3+16t^4+26t^5+42t^6+68t^7+110t^8+\cdots.\end{aligned}$$

Note that
$$\overline{B}_{\ell,k}(x,t) := B_{\ell,k}(1/x,xt) = \sum_{w\in[\ell]^*} x^{n-((1,k)\text{-rec}(w))}t^n$$
so that $\overline{B}_{\ell,k}(0,t)$ is the generating function of all words in $w = w_1 \ldots w_n \in [k]^*$ such that $(1,k)$-rec$(w) = n$, i.e. each letter of $w$ differs from at least one neighbor by $k$ or less. We have computed $\overline{B}_{\ell,2}(0,t)$ for $k = 4,5$.

$$\overline{B}_{4,2}(0,t) = \frac{1-3t+11t^2+6t^3+4t^4}{1-3t-3t^2-2t^3}.$$

The initial terms of this series are $1, 0, 14, 50, 196, 766, 2986, 11648, 44343, 177218, 691252, \ldots$. This sequence does not appear in the OEIS.

$$\overline{B}_{5,1}(0,t) = \frac{1 - 3t + 9t^2 - 4t^3 + 6t^4 + 4t^6}{1 - 3t - 4t^2 - 6t^4 - 4t^5 - 4t^6}.$$

The initial terms of this series are $1, 0, 19, 75, 361, 1689, 7915, 37107, 173937, 815345, \ldots$. This sequence also does not appear in the OEIS.

Our methods obviously extend to allow us to write a matrix equation for the generating function $B_{\ell,a,b}(x,t) = \sum_{w\in[\ell]^*} x^{(a,b)\text{-rec}(w)} t^{|w|}$. However, it becomes computationally unfeasiable even in the case of 2-box$(w)$. That is, one has to keep track of the first four letters to be able to compute the necessary recursions. For example, let

$$B^{rstu}_{\ell,2\text{-box}}(x,t) = \sum_{w\in rstu[\ell]^*} x^{2\text{-box}(w)} t^{|w|},$$

where $rstu[\ell]^*$ is the set of all words over $[\ell]$ that begin with letters $rstu$. Then it is easy to see that

$$B^{rstu}_{\ell,2\text{-box}}(x,t) = x^{2\text{-box}(rstu)} t^4 + \sum_{v=1}^{\ell} \theta(rstuv) B^{stuv}_{\ell,2\text{-box}}(x,t)$$

where $\theta(rstuv)$ is computed according the following four cases.

**Case 1.** $|r-s| > 2$ and $|r-t| > 2$. In this case, $\theta(rstuv) = t$.

**Case 2.** $|r-s| > 2$ and $|r-t| \le 2$. In this case, $\theta(rstuv) = xt$ if $t$ matches the 2-box pattern in $stuv$ and $\theta(rstuv) = x^2t$ if $t$ does not match the 2-box pattern in $stuv$. That is, for any word $w \in [\ell]^*$, the presence of $r$ does not effect whether $s$ will match the 2-box pattern in $rstuw$, but it does effect the question of whether $t$ matches the 2-box pattern in $rstuvw$.

**Case 3.** $|r-s| \le 2$ and $|r-t| > 2$. In this case, $\theta(rstuv) = xt$ if $s$ matches the 2-box pattern in $stu$ and $\theta(rstuv) = x^2t$ if $s$ does not match the 2-box pattern in $stu$. That is, for any word $w \in [\ell]^*$, the presence of $r$ does not effect whether $t$ will match the 2-box pattern in $rstuw$, but it does effect the question of whether $s$ matches the 2-box pattern in $rstuvw$.

**Case 4.** $|r-s| \le 2$ and $|r-t| \le 2$. In this case $\theta(rstuv) = xt$ if both $s$ and $t$ match the 2-box pattern in $stuv$, $\theta(rstuv) = x^2t$ if exactly one of $s$ and $t$ match the 2-box pattern in $stuv$, and $\theta(rstuv) = x^3t$ if neither $s$ nor $t$ match the 2-box pattern in $stuv$.

This recursion allows us to write a simple matrix type equation for the generating function $B_{\ell,2\text{-box}}(x,t)$; however, it requires that we have to invert an $\ell^4 \times \ell^4$ matrix which is not really feasible even for small $\ell$. Indeed, the generating function $B_{\ell,2\text{-box}}(x,t)$ is trivial for

$\ell \leq 3$, so the smallest non-trivial $\ell$ is $\ell = 4$ which requires we would have to invert a $4^4 \times 4^4$-matrix.

# 4 Conclusion

The goal of this paper was to introduce $k$-box patterns and to study them, mainly in the case of $k = 1$, on permutations and words. In the upcoming paper [7], we study 1-box patterns on pattern-avoiding permutations (more precisely, on 132-avoiding permutations and on *separable permutations*).

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