EQUIVALENCE CLASSES IN S_n FOR THREE FAMILIES OF PATTERN-REPLACEMENT RELATIONS

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ABSTRACT. We study a family of equivalence relations on S_n, the group of permutations on n letters, created in a manner similar to that of the Knuth relation and the forgotten relation. For our purposes, two permutations are in the same equivalence class if one can be reached from the other through a series of pattern-replacements using patterns whose order permutations are in the same part of a predetermined partition of S_c. In particular, we are interested in the number of classes created in S_n by each relation and in characterizing these classes.

Imposing the condition that the partition of S_c has one nontrivial part containing the cyclic shifts of a single permutation, we find enumerations for the number of nontrivial classes. When the permutation is the identity, we are able to compare the sizes of these classes and connect parts of the problem to Young tableaux and Catalan lattice paths.

Imposing the condition that the partition has one nontrivial part containing all of the permutations in S_c beginning with 1, we both enumerate and characterize the classes in S_n. We do the same for the partition that has two nontrivial parts, one containing all of the permutations in S_c beginning with 1, and one containing all of the permutations in S_c ending with 1.

1. INTRODUCTION

In 1970, the Robinson-Schensted-Knuth (RSK) correspondence brought the so-called Knuth relation to the forefront of mathematics. This is an equivalence relation on permutations (or, more generally, words) which connects two permutations if one can be transformed into the other by a given set of rules that allow the switching of neighboring letters (as in bubblesort) under particular conditions [Kn]. An analogue of this equivalence, the so-called forgotten relation [NS], later emerged, sharing some algebraic applications with the Knuth relation. Both the Knuth and the forgotten relation have a common structure: They are transitive, reflexive, symmetric closures of relations given by allowing the rearrangement of blocks of adjacent letters in a permutation from a certain order into another given order. This inspired various authors [LPRW], [PRW], [Ku] to systematically analyze relations of the same type, leading to numerous results on the number of equivalence classes and their sizes. This paper extends the study to consider pattern-replacement relations using patterns of arbitrary length. To our knowledge, we are the first to analyze an infinite family of pattern-replacement relations.

Before we present our results, we establish some conventions. Most of these formalize conventions made in [LPRW].

Definition 1.1. *A* word *is a finite, possibly empty string of positive integers. The elements of a word are called* letters. *We denote the ith letter of word w as w_i with w_1 as the first letter.*

Definition 1.2. *We regard any permutation of $\{1,2,\ldots,n\}$ as a word (by writing it in one-line notation). Conversely, given a word w with no two letters equal, we define the* order permutation *of w as the unique permutation $\pi \in S_n$ (where n is the length of w) such that for any i and j, we have $\pi_i < \pi_j$ if and only if $w_i < w_j$. This permutation π is also known as the* standardization *of w.*

For example, the order permutation of 425 is 213.

Definition 1.3. *If a and b are two words, then the* concatenation *of a with b is defined as the word obtained by attaching b to the end of a. It is denoted by ab or $a.b$. (*Note: *If π is a permutation of size n, then πn denotes the concatenation $\pi.n$, not the number $\pi(n)$.)*

Definition 1.4. *Let $\pi \in S_n$. Let w be a word with no two letters equal. If we can write w in the form aub for some three (possibly empty) words a, u and b such that u has order permutation π, then we say that u is a π in w. We say that w* avoids *π if there is no π in w.*

For example, 574 is a 231 in 2657431, because the order permutation of 574 is 231. However, 3124 avoids 231.

We now will define the equivalence relations that we are going to study.

Definition 1.5. *A* replacement partition *of S_k is a set partition of the symmetric group S_k for some $k \in \mathbb{N}$.*

Definition 1.6. *Given a replacement partition K of S_k and a positive integer n, the K*-equivalence *on S_n is defined as the equivalence relation on S_n generated by the following requirement: If $\phi \in S_n$ and $\psi \in S_n$ are such that $\phi = aub$ and $\psi = avb$ for some words a, b, u and v, where u and v have length k, and the order permutation of u lies in the same part of K as the order permutation of v, then we say that ϕ is equivalent to ψ. Moreover, in this case, we say that ψ results from ϕ by a K*-transformation, *or more precisely, ψ results from ϕ by a* transformation *$p \to q$, where p is the order permutation of u and q is the order permutation of v.*

When K is clear from the context, we abbreviate "K-equivalence" as "equivalence", and "K-transformation" as "transformation".

We also write $\phi \equiv \psi$ for "ϕ is equivalent to ψ".

What we call "K-equivalence" is denoted as "$K^{|\,|}$-equivalence" in [LPRW].

Example: Let $n = 5$, $k = 3$, $K = \{\{123,321\},\{132,231\},\{213\},\{312\}\}$. We will later abbreviate this by $K = \{123,321\}\{132,231\}$, leaving out the outer brackets and the one-element parts of the partition. Then, the permutation $15324 \in S_5$ is K-equivalent to $12354 \in S_5$ (because $15324 = aub$ and $12354 = avb$ with $a = 1$, $u = 532$, $v = 235$ and $b = 4$, and the order permutation of 532 lies in the same part of K as the order permutation of 235). More precisely, 12354 results from 15324 by a transformation $321 \to 123$. Similarly, 12354 is equivalent to 12453 (here, $a = 12$, and b is the empty word), and 12453 results from 12354 by a transformation $132 \to 231$. Combining these, we see that 15324 is equivalent to 12453, although 12453 does not directly result from 15324 by any transformation.

Definition 1.7. *Given a replacement partition K and a positive integer n, the K-equivalence on S_n partitions S_n into equivalence classes. We will refer to these equivalence classes as* classes. *A class is called* trivial *if it consists of one element only.*

Definition 1.8. *Let K be a replacement partition. If w is a word with no two letters equal, then a* hit *(or more precisely, a K*-hit*) in w is a word u such that $w = aub$ for some words a and b, and such that the order permutation of u lies in a nontrivial part of K. A permutation is said to* avoid *K if it contains no hit, i.e., if it avoids every permutation in every nontrivial part of K. Otherwise it is said to be* a non-avoider *(with respect to K).*

Observe that each permutation that avoids K forms a trivial class (with respect to the K-equivalence), whereas non-avoiders lie in nontrivial classes.

This paper is devoted to cases where K is a partition of S_k for general k. The cases we consider are the following:

- K has only one nontrivial part, and this part consists of a single permutation and all of its cyclic shifts. In this case, we study the nontrivial classes in Section 2.
- K has only one nontrivial part, and this part consists of all permutations in S_k starting with 1. Here, we enumerate and characterize the classes in Section 3. (This solves a previously open case for $k = 3$.)
- K has exactly two nontrivial parts – one of them containing all permutations in S_k starting with 1, and the other containing all permutations in S_k ending with 1. Again, we are able to enumerate and characterize the classes (Section 3. (This solves a previously open case for $k = 3$.)

The centerpiece of this paper is Section 2 with the treatment of the cyclic-shifts case. The main enumeration in this case leads us to consider Young tableaux and lattice paths. Section 3 treats two additional infinite families of relations. Section 4 explains methodology and discusses computational data. Section 5 concludes with a discussion of open problems and future work.

2. Replacement Partitions Using Cyclic Shifts

In this section, we consider the case where K comprises one nontrivial part containing a permutation and its cyclic shifts. We show that there are either one or two nontrivial classes created, and we count their relative sizes for some interesting cases. We will first introduce several conventions. For this section, $c > 1$ denotes the size of the permutations in the replacement partition. Let $m = m_1 m_2 \cdots m_c$ be a permutation in S_c. We will consider the replacement partition containing m and its cyclic shifts. Specifically, let K be the replacement partition

$$\{m_1m_2m_3\cdots m_c,\ m_2m_3m_4\cdots m_1,\ m_3m_4m_5\cdots m_2,\ \ldots,\ m_cm_1m_2\cdots m_{c-1}\}.$$

We will consider the equivalence classes created by the K-equivalence in S_n where $n > c$.

Example: If $m = 13542$, then $K = \{13542, 35421, 54213, 42135, 21354\}$.

We use two specific permutations, p and q. Let p be the result of starting with the identity permutation in S_n, and reordering the 2nd through $c+1$th letters in such a way that their order permutation becomes m. In other words, $p = 1(m_1+1)(m_2+1)\cdots(m_c+1)(c+2)(c+3)\cdots n$. Let q be the permutation obtained from p by swapping the letter 1 with the letter 2. In other words, where j is the index satisfying $m_j = 1$,
$q = 2(m_1+1)(m_2+1)\cdots(m_{j-1}+1)1(m_{j+1}+1)\cdots(m_c+1)(c+2)(c+3)\cdots n$.

Definition 2.1. *A permutation w is* reachable *if it is equivalent to either p or q.*

Definition 2.2. *Let w be any word, and i be a letter. We denote by $i \rightharpoonup w$ the word obtained by increasing by 1 each letter of w which is greater than or equal to i, and then appending i to the left end of w. Similarly, $w \leftharpoondown i$ is the word obtained by increasing by 1 every letter of w which is greater than or equal to i, and then appending i to the right of w.*

Note that for the case of $n = c+1$, $p = 1 \rightharpoonup m$ and $q = 2 \rightharpoonup m$. Also note that if $w \equiv v$ for two permutations w and v, then $i \rightharpoonup w \equiv i \rightharpoonup v$ and $w \leftharpoondown i \equiv v \leftharpoondown i$ for any letter i.

Definition 2.3. *For every $a \in \{1, 2, \ldots, c\}$, let m^{+a} be the permutation $m_a m_{a+1} \cdots m_{a-1} \in S_c$.*

Observe that $m^{+1} = m$.

Lemma 2.4. *For any $j \in \{1, 2, \ldots, n-1\}$, we have $j \rightharpoonup m \equiv m \leftharpoondown (j+1)$ and $m \leftharpoondown j \equiv (j+1) \rightharpoonup m$.*

Proof. Let $a \in \{1,2,\dots,c\}$ be such that $m_{a-1} = j$ (with indices cyclic modulo c). Then, it is easy to see that $j \rightharpoonup m^{+a} = m^{+(a-1)} \leftharpoondown (j+1)$. As a consequence, $j \rightharpoonup m \equiv j \rightharpoonup m^{+a} = m^{+b} \leftharpoondown (j+1) \equiv m \leftharpoondown (j+1)$. This proves the first part of the lemma and the second is proven similarly. $\square$

Lemma 2.5. *All non-avoiders are reachable for the case of $n = c+1$.*

Proof. Let $n = c+1$. In S_n, every non-avoider has one of the forms $i \rightharpoonup m^{+a}$ and $m^{+a} \leftharpoondown i$ for an $i \in \{1,2,\dots,n\}$ and an $a \in \{1,2,\dots,c\}$, and thus is equivalent to either $i \rightharpoonup m$ or $m \leftharpoondown i$. Hence, we only need to prove that $i \rightharpoonup m$ and $m \leftharpoondown i$ are reachable for every i. Repeatedly using Lemma 2.4, we obtain the chains of equivalences

$$1 \rightharpoonup m \equiv m \leftharpoondown 2 \equiv 3 \rightharpoonup m \equiv m \leftharpoondown 4 \equiv \cdots, \tag{1}$$

$$m \leftharpoondown 1 \equiv 2 \rightharpoonup m \equiv m \leftharpoondown 3 \equiv 4 \rightharpoonup m \equiv \cdots. \tag{2}$$

For every i, each of the permutations $i \rightharpoonup m$ and $m \leftharpoondown i$ appears in one of these chains, and thus is equivalent to either $1 \rightharpoonup m = p$ or $2 \rightharpoonup m = q$. $\square$

Corollary 2.6. *If c is even and $m = \mathrm{id}$, there is only one nontrivial class in S_n for $n = c+1$.*

Proof. We follow the proof of Lemma 2.5, but notice that now the equivalence chains (1) and (2) are merged because the identity in S_n can be written as either $1 \rightharpoonup m$ (an element of the chain (1)) or $m \leftharpoondown n$ (an element of the chain (2)). $\square$

Theorem 2.7. *When $n > c$, all non-avoiders are reachable.*

Proof. We will prove this by inducting on n, with Lemma 2.5 serving as a base case (the case $n = c+1$). Assume that the theorem holds for S_{n-1} and that $n \geq c+2$. Let $x \in S_n$ be a non-avoider. If x only has a hit in the final c letters, then applying Lemma 2.4 to the final $c+1$ letters, we slide the hit to the left, placing it in the first $n-1$ letters. So, without loss of generality, x has a hit in the first $n-1$ letters. Applying the inductive hypothesis to the left-most $n-1$ letters of x, one can get a permutation x' of the form $a\cdots u$ where $a \in \{1,2,3\}$, u is the right-most letter of x, and the letters not shown are in increasing order, except with the first c rearranged to form a hit. Noting that the sequence $x'_2 x'_3 \cdots x'_{c+1}$ forms a hit, and applying the inductive hypothesis to the right-most $n-1$ letters of x', one can get a permutation x'' of the form $ab\cdots$ where $a, b \in \{1,2,3\}$ and the letters not shown start off with a hit containing the letters up to $c+2$ and proceed with the remaining $n-c-2$ letters in increasing order. In showing that all such permutations are reachable, it can be assumed without loss of generality that $n = c+2$ (as otherwise, we can simply restrict ourselves to considering the left-most $c+2$ letters). Then, $x'' = ab\cdots = j \rightharpoonup (k \rightharpoonup m)$ for some $j \in \{1,2,3\}$ and $k \in \{1,2\}$.

Now, we only need to show that the 6 permutations in S_{c+2} of the form $j \rightharpoonup (k \rightharpoonup m)$ for $j \in \{1,2,3\}$ and $k \in \{1,2\}$ are reachable. Let $j \in \{1,2,3\}$ and $k \in \{1,2\}$, and let $w = j \rightharpoonup (k \rightharpoonup m)$. If $k \not\equiv n-1 \mod 2$, then applying chains (1) and (2) we go from w to $j \rightharpoonup (m \leftharpoondown (n-1)) = (j \rightharpoonup m) \leftharpoondown n$, and to $(j' \rightharpoonup m) \leftharpoondown n$ for some $j' \in \{1,2\}$, thus reaching p or q. If $k \equiv n-1 \mod 2$ and $n \geq 8$, we apply the chains to go from w to $j \rightharpoonup (m \leftharpoondown k') = (j \rightharpoonup m) \leftharpoondown (k'+1)$ for some $k' \in \{3,4\}$, then to $(j' \rightharpoonup m) \leftharpoondown (k'+1) = (j'+1) \rightharpoonup (m \leftharpoondown (k'+1))$ for some $j' \in \{5,6\}$, then to $(j'+1) \rightharpoonup (m \leftharpoondown (n-1)) = ((j'+1) \rightharpoonup m) \leftharpoondown n$, and to $(j'' \rightharpoonup m) \leftharpoondown n$ for some $j'' \in \{1,2\}$, thus reaching p or q. The case of $c < 6$ can be easily shown by computer (or for $c \neq 3$, in a way similar to but slightly more complicated than the one above). $\square$

Remark 2.8. As a result of Theorem 2.7, we can start to characterize the classes created under these transformations. For even c, there is either one or two classes containing all non-avoiders, and one class for every avoiding permutation. Note that if $m = \mathrm{id}$, then by corollary 2.6, $p \equiv q$, so there is only one class containing non-avoiders. There are also known cases of m which result in two nontrivial classes in S_n, for example when $m = 145236$. For odd c, because oddness and evenness are maintained by each transformation, there is one class containing all even non-avoiders, one class containing all odd non-avoiders, and one class for each avoiding permutation.

We will now start counting the relative sizes of the nontrivial classes in the case when $m = \mathrm{id}$.

Convention 2.9. *From here until the end of this section, we let* $m = \mathrm{id}$ *and* c *be odd.* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *denotes the* n*th Catalan number. All Young tableaux mentioned are Standard Young tableaux in the English notation.*

First, two auxiliary notions:

Definition 2.10. *A* hit-opener *is a permutation with a hit starting at the first letter and with no other hits.*

Definition 2.11. *A* hit-hugger *is a permutation with only two hits, one starting at the first letter and one starting at the* $n-c+1$*th letter.*

Definition 2.12. *The* rot *of a permutation* $x \in S_n$ *is* $(234\cdots n1) \circ x$.

Because $m = \mathrm{id}$, the operation rot maintains the position and the existence of hits in a permutation even though the letters of the hits are changed.

Definition 2.13. *We will denote the sets of hit-huggers, non-avoiders, and hit-openers in* S_j *as* H_j, N_j, *and* P_j, *respectively. A superscript of "odd" or "even" on the right restricts the set to elements of that parity. A subscript (respectively superscript) on the left restricts the set to elements that begin (respectively end) with the letter in the subscript (respectively superscript). For example,* ${}_1N_{n-1}^{even}$ *refers to the set of even non-avoiders that begin with* 1 *in* S_{n-1}.

Theorem 2.14. *For even* n*, there are the same number of odd and even non-avoiders in* S_n.

Proof. A bijection from the even to the odd non-avoiders in S_n can be constructed by mapping each even non-avoider ω to $\mathrm{rot}(\omega)$ since rot changes parity. ☐

Lemma 2.15. *If* n *is odd and* $c > \frac{n}{2}$, *then* $|{}^nH_n^{even}| - |{}^nH_n^{odd}| = |{}^nN_n^{odd}| - |{}^nN_n^{even}|$.

Proof. Every non-avoider ψ in S_n starting with 1 is either a hit-opener or has the form $1 \rightharpoonup \pi$ for some non-avoider $\pi \in S_{n-1}$. Since $1 \rightharpoonup \sigma$ has the same parity as σ for every $\sigma \in S_{n-1}$, this yields $|{}_1N_n^{odd}| = |N_{n-1}^{odd}| + |{}_1P_n^{odd}|$, and the same equality holds with "odd" replaced by "even". Taking the difference between these equalities, and noting that Theorem 2.14 yields that there are the same number of even and odd non-avoiders in S_{n-1}, we obtain
$|{}_1N_n^{odd}| - |{}_1N_n^{even}| = |N_{n-1}^{odd}| + |{}_1P_n^{odd}| - |N_{n-1}^{even}| - |{}_1P_n^{even}| = |{}_1P_n^{odd}| - |{}_1P_n^{even}|$.

If σ is a hit-opener in S_{n-1}, then $\sigma \leftharpoondown n$ is either a hit-opener or a hit-hugger in S_n ending with n. Each hit-opener and hit-hugger in S_n ending in n can be written as $\sigma \leftharpoondown n$ for exactly one such σ. Since $\sigma \leftharpoondown n$ has the same parity as σ, this results in $|P_{n-1}^{odd}| = |{}^nP_n^{odd}| + |{}^nH_n^{odd}|$. Again, the same equality holds with "odd" replaced by "even". Taking the difference between these two equalities, we obtain $|P_{n-1}^{even}| - |P_{n-1}^{odd}| = |{}^nP_n^{even}| + |{}^nH_n^{even}| - |{}^nP_n^{odd}| - |{}^nH_n^{odd}|$. Because of rot, there are the same number of even and odd hit-openers in S_{n-1}. Therefore, the left hand side of this equality is 0, so we obtain $|{}^nP_n^{even}| - |{}^nP_n^{odd}| = |{}^nH_n^{odd}| - |{}^nH_n^{even}|$.

Note that rot maintains parity since n is odd. So, $|^{n}N_n^{odd}| - |^{n}N_n^{even}| = \frac{1}{n} \cdot |N_n^{odd}| - \frac{1}{n} \cdot |N_n^{even}|$ $= |_1N_n^{odd}| - |_1N_n^{even}|$. Similarly, $|_1P_n^{odd}| - |_1P_n^{even}| = |^{n}P_n^{odd}| - |^{n}P_n^{even}|$. Thus, by transitivity of equality, $|^{n}H_n^{even}| - |^{n}H_n^{odd}| = |^{n}N_n^{odd}| - |^{n}N_n^{even}|$. □

Lemma 2.16. *In the case of $c > \frac{n}{2}$ where n is odd, there is a bijection between hit-huggers in S_n ending in n and $2 \times (n-c-1)$ Young tableaux. The parity of the Latin reading word (the word obtained by reading the entries of a Young tableau from left to right like a paragraph) of a $2 \times (n-c-1)$ Young tableau is the same as that of its corresponding hit-hugger in S_n.*

Proof. In the hit-hugger $a_1a_2a_3 \cdots a_{n-c}a_{n-c+1} \cdots a_ca_{c+1}a_{c+2} \cdots a_{n-1}a_n$, the last c letters strictly increase from left to right since n is the largest letter, so $a_{n-c+1} < \cdots < a_c < a_{c+1} < a_{c+2} < \cdots < a_{n-1} < a_n$. In order for letters a_{n-c} through a_{n-1} to be avoiding, we must have $a_{n-c} > a_{n-c+1}$. We know that, $a_1a_2a_3 \cdots a_c$ has to be a hit, so $a_{n-c+1} < a_{n-c+2} < \ldots < a_c < a_1 < a_2 < a_3 < \ldots < a_{n-c}$. Since a_{n-c+1} through a_c are less than all of the other letters, they are the smallest $2c-n$ letters in value.

In order for $a_2a_3 \cdots a_{n-c}a_{n-c+1} \cdots a_ca_{c+1}$ to be avoiding, we must have $a_2 < a_{c+1}$. For similar reasons, $a_3 < a_{c+2}$, $a_4 < a_{c+2}, \ldots, a_{n-c} < a_{n-1}$. Letter a_1 is the $2c-n+1$th letter in value since $a_1 < a_2 < a_{c+1}$, and $a_n = n$ is the largest valued letter. The rest of the letters have relations that allow them to be placed in a $2 \times (n-c-1)$ Young tableau, with letters a_2 through a_{n-c} in the top row, and letters a_{c+1} through a_{n-1} in the second row. Note that if and only if a permutation is a hit-hugger, it will satisfy these conditions.

It remains to show the claim about parity. Since $(n-c)(2c-n)$ is even, sliding the subsequence $a_{n-c+1} \cdots a_c$ to be at the start of the permutation will not change the parity of the permutation. In this new permutation, $a_{n-c+1} \cdots a_ca_1a_2a_3 \cdots a_{n-c}a_{c+1}a_{c+2} \cdots a_{n-1}a_n$, all inversions are within the letters a_2 through a_{n-1}, since $a_{n-c+1} \cdots a_c$ are the smallest letters in increasing order and a_n is the largest letter. Furthermore, the subword a_2 through a_{n-1} is exactly the Latin reading word of the Young tableau corresponding to the hit-hugger. Thus, the parity of the Latin reading word of a $2 \times (n-c-1)$ Young tableau is the same as that of its corresponding hit-hugger in S_n. □

Remark 2.17. The lattice paths we consider are only those that stay above or on the main diagonal.

Lemma 2.18. *For odd n, there is a bijection between $2 \times (n-c-1)$ Young tableaux and lattice paths inside a $(n-c-1) \times (n-c-1)$ grid. The parity of the area above the lattice path equals the parity of the Latin reading word of the corresponding Young tableau.*

Proof. Given any Young tableau, start in the top left corner of the grid. Traverse the Young tableau in order of increasing entries. For every number in the top row, draw a step horizontally to the right in the grid, and for every number in the bottom row, draw a step vertically down. We have created a way to bijectively map each Young tableaux to a lattice path. We will now look at parity. For every pair of numbers $i < j$ where i is in the bottom row and j is in the top row, there is an inversion in the Latin reading word of the Young tableau, and there is a corresponding unit square above the lattice path in the same row and column as the pair of steps that correspond to i and j. This means the parity of the area above the lattice path is the same as that of the corresponding Latin reading word. □

Definition 2.19. *A* leg *of a lattice path is a maximal sequence of consecutive steps in the same direction.*

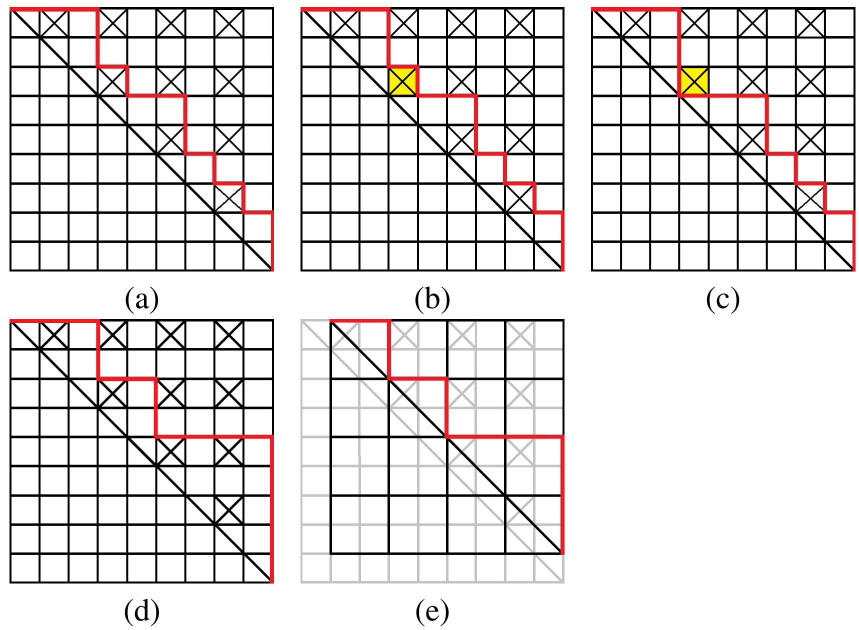

FIGURE 1. Lattice paths in a grid.

Lemma 2.20. *For odd n, there are $C_{\frac{n-c-2}{2}}$ more lattice paths inside a $(n-c-1) \times (n-c-1)$ grid with even than with odd area above the path.*

Proof. Given a lattice path l, draw X's inside the squares that are in even columns and odd rows. An example is shown in Figure 1(a).

Take the first time the path touches two edges of the same square with an X in it (highlighted in Figure 1(b)) and draw the path l' that is the same as l except that it touches the other two edges of that square (shown in Figure 1(c)). Note that l' also gets mapped to l, and that the area above l has the opposite parity as the area above l'.

Thus, there are the same number of even and odd lattice paths that touch at least one square with an X on two edges. Consider all of the paths that do not touch any squares with an X on two edges (for example, Figure 1(d)). Notice that these paths must not have any odd-length legs except for the first and last legs and that there is an even area above each such path.

This means that the number of these paths is equal to the difference between the number of odd and even paths. By deleting the first column and last row, and shrinking the dimensions of the grid each by a factor of two (shown in Figure 1(e)), we see that we only have to count the number of lattice paths inside a $\frac{n-c-2}{2} \times \frac{n-c-2}{2}$ grid, which is just $C_{\frac{n-c-2}{2}}$. $\square$

Remark 2.21. Note that a result equivalent to Lemma 2.20 was proven by [SS]. However, we choose to provide our proof above both because our proof is very different than the previous one and because [SS] does not appear to be easily accessible for the public.

Theorem 2.22. *When $c > \frac{n}{2}$ and n is odd, the odd nontrivial equivalence class will have $nC_{\frac{n-c-2}{2}}$ more permutations than the even nontrivial class.*

Proof. By Lemma 2.15, $|{}^nH_n^{even}| - |{}^nH_n^{odd}| = |{}^nN_n^{odd}| - |{}^nN_n^{even}|$. Therefore, due to rot preserving the parity of a permutation, it suffices to show that there are $C_{\frac{n-c-2}{2}}$ more even than odd hit-huggers ending with the letter n. By Lemma 2.16, the parity of the Latin reading word of a $2\times(n-c-1)$ Young tableau is the same as that of a corresponding hit-hugger ending with n in S_n. By Lemma 2.18, this is the same as the parity of the area above a corresponding lattice path inside a $(n-c-1)\times(n-c-1)$ grid. Since by Lemma 2.20, there are $C_{\frac{n-c-2}{2}}$ more even than odd area-ed lattice paths inside a $(n-c-1)\times(n-c-1)$ grid, we are done. □

3. Two More Infinite Families of Relations

In this section, we consider two infinite families of relations. For each family, we consider the number of equivalence classes created in S_n and the size of the equivalence classes for each relation in the family. In each subsection, we consider the family of relations indicated in the title. It is interesting to note that both the $\{123,132\}$-equivalence and the $\{123,132\}\{231,321\}$-equivalence are members of the families that we study, the first of which [LPRW] considered but was unable to enumerate.

3.1. $\{\mathbf{x}|\mathbf{x}\in \mathbf{S_{c+1}},\ \mathbf{x_1}=\mathbf{1}\}$-Equivalence.

Note that we define c such that hits are of size $c+1$.

Definition 3.1. *We call a permutation k*-squished *if for each $j\le k$, j is in the first $c(j-1)+1$ letters.*

Let $f(n)$ be the number of classes in S_n under the $\{x|x\in S_{c+1},\ x_1=1\}$-equivalence. Let $g(k,n)$ be the number of classes in S_n containing permutations that are k-squished under the $\{x|x\in S_{c+1},\ x_1=1\}$-equivalence.

Lemma 3.2. *Let w be a k-squished permutation where $c+1\le n\le ck+1$. Then, w is equivalent the identity under the $\{x|x\in S_{c+1},\ x_1=1\}$-equivalence.*

Proof. Let w be such a permutation. Let r be $\lceil (n-1)/c\rceil$. Assume that the lemma holds in S_{n-1} (with a trivial inductive base case of S_{c+1}). Applying the inductive hypothesis to the first $n-1$ letters of w, we reach a permutation w' which has each letter $j\le r$ in position $c(j-1)+1$; and which with exception of the final letter, has the other letters in increasing order. In w', r is in position $(\lceil (n-1)/c\rceil-1)c+1$ which is in the final $c+1$ positions but not in the final position; $r-1$ is c positions before it. Using the hit starting with $r-1$, we rearrange w' to have r in position $n-c$, while not changing the relative order of any other letters. Then, since r is followed by c letters, all greater than it, and is among the final two letters is n, we may rearrange the final two letters so that n is in the final position. Finally, we may apply the inductive hypothesis to the first $n-1$ letters and reach the identity. □

Lemma 3.3.

$$g(k,n)=\begin{cases}(n-1)!, & \text{if } n<c+1\\ 1, & \text{if } c+1\le n\le ck+1\\ g(k+1,n)+\sum\limits_{j=ck+2}^{n} g(k,j-1)\cdot g(1,n-j+1)\cdot\binom{n-k-1}{n-j}, & \text{if } n>ck+1\end{cases}$$

Proof. The case of $n<c+1$ is trivial because no transformations can be applied to a permutation of size less than $c+1$. Of the remaining cases, the case of $ck+1\ge n$ yields one class because all permutations being counted are equivalent to the identity by Lemma 3.2 (as shown by [LPRW] for the case of $c=2$). Now, we consider the case of $ck+1<n$.

Consider the permutations which are $k+1$-squished. Such permutations trivially fall into $g(k+1,n)$ classes.

Consider the position of the $k+1$ in a permutation w which is k-squished but not $k+1$-squished. In w, clearly none of the k smallest letters can ever be moved to the right of position $c(k-1)+1$. So, in w, because $k+1$ is in position at least $ck+2$, $k+1$ can never move under the transformations considered and the movement of letters to its left and right are completely independent. If the position of $k+1$ is j, then permutations that meet the restrictions of w fall into $g(k,j-1)\cdot g(1,n-j+1)\cdot\binom{n-k-1}{n-j}$ classes because we get to choose which letters are on each side of j ($n-j$ letters out of a possible $n-k-1$ letters are to the right of j) and then partition the classes from there using the function g. Thus, if we consider each possible j, such permutations break into

$$\sum_{j=ck+2}^{n} g(k,j-1)\cdot g(1,n-j+1)\cdot\binom{n-k-1}{n-j} \text{ classes.}$$

□

Theorem 3.4.

$$f(n)=\begin{cases} n!, & \textit{if}\, n<c+1\\ \sum\limits_{j=1}^{n} f(j-1)\cdot g(1,n-j+1)\cdot\binom{n-1}{j-1}, & \textit{if}\, n\geq c+1\end{cases}$$

Proof. As before, the case of $n<c+1$ is trivial. Consider the case of $n\geq c+1$. Under the transformations considered, the position of 1 does not change. Furthermore, the letters to the left and right of 1 move independently of each other. So, permutations with 1 in position j fall in to $f(j-1)\cdot g(1,n-j+1)$ classes for a given choice of which letters are to the left and right of 1. There are $\binom{n-1}{j-1}$ such choices, resulting in $f(j-1)\cdot(1,n-j+1)\cdot\binom{n-1}{j-1}$ classes. So, if we sum for each possible j, we get

$$\sum_{j=1}^{n} f(j-1)\cdot g(1,n-j+1)\cdot\binom{n-1}{j-1}\text{classes.}$$

□

Remark 3.5. Given a permutation in S_n, one can figure out the size of the class that it is in. If the permutation is n-squished, then the enumeration is easily calculated (simply the number of n-squished permutations). Otherwise, we take the lowest k such that the permutation is not k-squished, and the size of the class is the product of the size of the class containing the contiguous subword starting with k and going to the end of the permutation and the size of the class containing the contiguous subword going from the start of the permutation to the letter before k (because k can never be involved in a hit with letters to its left).

3.2. $\{\mathbf{x}|\mathbf{x}\in\mathbf{S_{c+1}},\ \mathbf{x_1}=\mathbf{1}\}$ $\{\mathbf{x}|\mathbf{x}\in\mathbf{S_{c+1}},\ \mathbf{x_{c+1}}=\mathbf{1}\}$**-Equivalence.** Note that we define c such that the replacement partition is of S_{c+1}.

Definition 3.6. *Pick j letters between 1 and $j+k$ inclusive, $l_1,l_2,\dots,l_j$, such that $l_{i<j}<l_{i+1}$. Let $r_1,r_2,\dots,r_k$ be the remaining k of the letters between 1 and $j+k$ such that $r_{i<k}<r_{i+1}$. If l_i is among the first $c(i-1)+1$ positions for $i\leq j$ and r_i is among he final $c(i-1)+1$ positions for $i\leq k$, then a permutation is j,k-squished. The letters $r_{a\leq j}$ and $l_{b\leq k}$ are referred to as r_a and l_b respectively.*

Note that if $ck+cj+1\geq n\geq c+1$, there may be more than one choice of r_i or l_i for a given i that allows for a permutation to be j,k-squished.

Let $f(n)$ be the number of classes in S_n under the $\{x|x \in S_{c+1},\ x_1 = 1\}$ $\{x|x \in S_{c+1},\ x_{c+1} = 1\}$-equivalence. Let $g(j,k,n)$ be the number of classes in S_n containing permutations that are j,k-squished for a given choice of $l_1, l_2, \ldots, l_j$ and $r_1, r_2, \ldots, r_k$.

Lemma 3.7. *If $ck + cj + 1 \geq n \geq c + 1$, then $g(j,k,n) = 1$.*

Proof. Let $ck + cj + 1 \geq n \geq c + 1$. By Lemma 3.2, we can rearrange the first $jc + 1$ letters to form any j-squished permutation. Similarly, we can rearrange the final $kc + 1$ letters to form any backwards k-squished permutation. In this way, we can move any letter greater than $j + k$ to any position by first moving it to position $jc + 1$ (or n if $jc + 1 > n$) and then to a position of our choosing. Then, by Lemma 3.2, the letters less than or equal to $j + k$ can then be moved anywhere such that the permutation remains j,k-squished with the same choices of $r_1, \ldots, r_k$ and $l_1, \ldots, l_j$. Hence, $g(j,k,n) = 1$. ☐

Lemma 3.8.

$$g(j,k,n) = \begin{cases} (n-1)!, & \text{if } n < c+1 \\ 1, & \text{if } ck+cj+1 \geq n \geq c+1 \\ g(j+1,k,n) + g(j,k+1,n) + & \\ \sum_{x=cj+2}^{n-ck-1} g(j,1,x) \cdot g(1,k,n-x+1) \cdot \binom{n-j-k-1}{x-j-1}, & \text{otherwise} \end{cases}$$

Proof. The case of $n < c + 1$ is trivial because no transformations can be applied to permutations of size $< c + 1$. The case of $ck + jk + 1 \geq n \geq c + 1$ yields one class by Lemma 3.7. Now, we consider the remaining case of $ck + cj + 1 > n$.

Note that $j + k + 1$ cannot move between being in the first $cj + 1$ positions, in the final $ck + 1$ positions, or in-between under the transformations considered. If $j + k + 1$ is in the first $cj + 1$ letters, then it can only be moved to position at most $cj + 1$ using letters less than it that are to its left to create hits. Using a letter less than it but to its right however, $j + k + 1$ cannot then be involved in a hit because $n > ck + cj + 1$. Similarly, $j + k + 1$ is restricted to being only in the final $ck + 1$ letters if starts in the final $ck + 1$ letters. Since $n > ck + cj + 1$, the final $ck + 1$ and first $cj + 1$ letters can not overlap.

Let w be a j,k-squished permutation for a given choice of $l_1, l_2, \ldots, l_j$, and $r_1, r_2, \ldots, r_k$. Now, consider the position of $j + k + 1$ in w. If $j + k + 1$ is in the first $cj + 1$ letters, then there are $g(j,k+1,n)$ possibilities of w. If $j + k + 1$ is in the final $ck + 1$ letters, then there are $g(j,k+1,n)$ possibilities for w. If $j + k + 1$ is neither in the first $cj + 1$ letters or the final $ck + 1$ letters, than under the relation considered, it can only ever act as the 1 in a hit. In this case, let x be the position of $j+k+1$. There are $g(j,1,x) \cdot g(1,k,n-x+1) \cdot \binom{x-j-k-1}{n-j-1}$ possibilities for w. This is because out of the highest $n - j - k - 1$ letters, we get to choose $x - j - 1$ of them to be to the left of $k + j + 1$ and the remaining to be to the right. Then, there are $g(j,1,x)$ possibilities for arranging the letters to the left of $k + j + 1$ and $g(1,k,n-x+1)$ possibilities for arranging the letters to the right of $k + j + 1$. Bearing in mind the possibilities for x, we get

$$g(j,k,n) = g(j+1,k,n) + g(j,k+1,n) + \sum_{x=cj+2}^{n-ck-1} g(j,1,x) \cdot g(1,k,n-x+1) \cdot \binom{n-j-k-1}{x-j-1}.$$

☐

Theorem 3.9.

$$f(n) = \begin{cases} n!, & \text{if } n < c+1 \\ \sum_{j=1}^{n} g(0,1,j) \cdot g(1,0,n-j+1) \cdot \binom{n-1}{j-1}, & \text{if } n \geq c+1 \end{cases}$$

Proof. The case of $n < c+1$ is trivial. So, consider the case of $n \geq c+1$. Under the transformations considered, the position of 1 does not change. Furthermore, the letters to the left and right of 1 move independently of each other. So, permutations with 1 in position j fall into $g(0,1,j) \cdot g(1,0,n-j+1)$ classes for a given choice of which letters are to the left and right of 1. There are $\binom{n-1}{j-1}$ such choices, resulting in $g(0,1,j) \cdot g(1,0,n-j+1) \cdot \binom{n-1}{j-1}$ classes. If we sum for each possible j, we get the formula provided. □

Remark 3.10. Given a permutation in S_n, one can figure out the size of the class that it is in. If the permutation is j,k-squished such that $jc+kc+1 \geq n$, then the enumeration is easily calculated (simply the number of j,k-squished permutations for each choice for each of the smallest $\lfloor (n-1)/c \rfloor + 1$ letters of whether the letter is a r_i or l_i). Otherwise, we take the lowest a such that the permutation is not j,k-squished for any $j+k \geq a$, and the size of the class is the product of the size of the class containing the order permutation of the contiguous starting with a and going to the end of the permutation and the size of the class containing the order permutation of the contiguous subword going from the start of the permutation to the letter before a (because a can never play the role of anything but 1 in a hit).

4. Experimental Data and Conjectures

Consider the case in Section 2 where $m = \text{id}$. Figure 2 shows computational results for the number of even and odd hit-huggers beginning with 1 in S_n. Here, we consider the case where $n > \frac{c}{2}$, c and n are odd, and $c > 3$. The importance of this calculation stems from Lemma 2.15.

FIGURE 2. The numbers of even and odd hit-hugging permutations in S_n beginning with 1 in the situation of Section 2 where $m = \text{id} \in S_c$.

n	c	# of even	# of odd	# of even − # of odd
19	7	2951215617	2951215365	252
13	5	129963	129949	14
17	7	9687436	9687390	46
21	9	437585672	437585530	142
11	5	1097	1103	-6
15	7	41854	41883	-29
19	9	1166613	1166743	-130
23	11	28050890	28051452	-562

Conjecture 4.1. *Let k be a fixed positive odd integer. Let c be an arbitrary odd positive integer and $n = 2c+k$. Then, the number of odd hit-huggers beginning with 1 in S_n subtracted from the number of even ones has the same sign for any pick of c.*

Conjecture 4.2. *Let n, c be odd such that $n = 2c+1$. Then, the difference between the number of even and odd hit-huggers beginning with 1 in S_n is the number of ways of choosing at most $\frac{(c-3)}{2}$ elements from a set of size c.*

5. Conclusion and Future Work

It would be interesting to continue studying equivalence classes in S_n created by a replacement partition whose only nontrivial part contains the cyclic shifts of the identity in S_c. For some cases of n and c, the relative sizes of the classes behave in enigmatic patterns. There are also the following directions for future work:

(1) [LPRW] deals with relations that allow re-ordering only adjacently valued letters or, alternatively, re-ordering any subword (rather than only a contiguous set of adjacent letters). This is an important direction of research to continue.
(2) One can also study equivalence classes of labelings of graphs. The role of k adjacent letters is then taken by a path of length k in the graph. In particular, the case of K_n seems approachable.
(3) Some equivalence classes have additional structure. Can one classify permutations in a given equivalence class based on characteristics such as inversions, length (number of inversions), the locations of hits, ascents, Major index, etc.?

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