CARDINALITY OF ℓ_1-SEGMENTS AND GENOCCHI NUMBERS

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Abstract. We prove that the Genocchi numbers of first and second kind give the cardinality of certain segments in permutation spaces $\mathfrak{S}_n$ with respect to the ℓ_1-distance. Experimental data suggests that those segments have maximal cardinality among all segments in the corresponding spaces.

Contents

1. Introduction

For a positive integer n let $\mathfrak{S}_n$ be the group of permutations of $[n] = \{1, 2, \ldots, n\}$. We define a distance D on $\mathfrak{S}_n$ by

$$D(u, v) = \sum_{i=1}^{n} |u(i) - v(i)| \ .$$

This distance is right invariant: $D(uw, vw) = D(u, v)$ for every $u, v, w \in \mathfrak{S}_n$. For $u \in \mathfrak{S}_n$, the segment $[\mathrm{id}, u]$ is the set

$$[\mathrm{id}, u] = \{v \in \mathfrak{S}_n \mid D(\mathrm{id}, v) + D(v, u) = D(\mathrm{id}, u)\} \ .$$

Then $v \in [\mathrm{id}, u]$ if and only if

$$\min(i, u(i)) \leqslant v(i) \leqslant \max(i, u(i)) \tag{1.1}$$

for all $i = 1, 2, \ldots, n$. Experimental data suggests that the maximal cardinality of a segment $[\mathrm{id}, u]$ in $\mathfrak{S}_n$ is attained for

$$u = w_n = \begin{cases} (m{+}1\, m{+}2\ \ldots 2m\ 1\ 2\ \ldots m)\ , & \text{if } n = 2m \\ (m{+}1\, m{+}2\ \ldots 2m{+}1\ 1\ 2\ \ldots m)\ , & \text{if } n = 2m{+}1 \end{cases} .$$

For example, $w_1 = (1)$, $w_2 = (21)$, $w_3 = (231)$, $w_4 = (3412)$, $w_5 = (34512)$, $\ldots$.

The sequence $\#[\mathrm{id}, w_n]$ starts with 1, 2, 3, 8, 17, 56, 155, 608, 2073, 9440, ... and a search on the Online Encyclopedia of Integer Sequences (at oeis.org) shows a match with the sequence "A099960: An interleaving of the Genocchi numbers of the first and second kind," from the fourth term on. The goal of this article is to give a bijective proof of this characterization of $\#[\mathrm{id}, w_n]$, the cardinality of the segment $[id, w_n]$.

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2. Genocchi Numbers

The Genocchi numbers have combinatorial descriptions in terms of Dumont permutations. The Genocchi number of the first kind G_{2n+2} is the cardinality of the set B_{2n} of permutations $\pi \in \mathfrak{S}_{2n}$ such that

$$\pi(2i) \leqslant 2i \leqslant \pi(2i-1) \tag{2.1}$$

for all $i = 1, \ldots, n$. ([BD81], [Dum74]). Note that $\pi(2n-1) = 2n$ for all $\pi \in B_{2n}$. For example, $G_6 = 3$ is the cardinality of

$$B_4 = \{(2143), (3142), (3241)\} .$$

The Genocchi median (or number of the second kind) H_{2n+1} is the cardinality of the set C_{2n} of permutations $\pi \in \mathfrak{S}_{2n}$ such that

$$\pi(2i) < 2i \leqslant \pi(2i-1) \tag{2.2}$$

for $i = 1, \ldots, n$. ([DR94]). Note that $\pi(2) = 1$ and $\pi(2n-1) = 2n$ for all $\pi \in C_{2n}$. For example, $H_7 = 8$ is the cardinality of

$$\begin{aligned} C_6 = \{&(415263), (315263), (314265), (514263), \\ &(215364), (214365), (415362), (514362)\} \end{aligned}$$

We prove that $\#[\mathrm{id}, w_{2m}] = H_{2m+3}$ and $\#[\mathrm{id}, w_{2m+1}] = G_{2m+4}$ by establishing explicit bijections between $[\mathrm{id}, w_{2m}]$ and C_{2m+2} and between $[\mathrm{id}, w_{2m+1}]$ and B_{2m+2}.

3. Cardinality of Segments

The main results and their proofs are similar. We start with the case $n = 2m+1$.
Define $\rho\colon \mathfrak{S}_{2m+1} \to \mathfrak{S}_{2m+2}$ by

$$\rho(u) = (u(1)\ u(2)\ \ldots\ u(2m)\ 2m+2\ u(2m+1))$$

In other words, ρ inserts the value $2m+2$ between on position $2m+1$ and shifts the last position to the right by one space. For example, if $m = 1$ and $u = (231) \in \mathfrak{S}_3$, then $\rho(u) = (2341) \in \mathfrak{S}_4$.

Theorem 3.1. Define the function $g\colon \mathfrak{S}_{2m+1} \to \mathfrak{S}_{2m+2}$ by $g(u) = \rho(\alpha u^{-1} \beta^{-1})$, where $\alpha = (1\,3\,\ldots\,2m+1\ 2\,4\,\ldots\,2m)$ and $\beta = (2\,4\,\ldots\,2m\,2m+1\,1\,3\,\ldots\,2m-1)$. Then

$$u \in [\mathrm{id}, w_{2m+1}] \Longleftrightarrow g(u) \in B_{2m+2} .$$

Proof. Let $u \in \mathfrak{S}_{2m+1}$. Then

$$\begin{aligned} g(u)(2m+1) &= 2m+2 \\ g(u)(2m+2) &= \alpha u^{-1} \beta^{-1}(2m+1) \leqslant 2m+1 , \end{aligned}$$

hence conditions (2.1) are satisfied by all $g(u)$ for $i = m+1$. The *value* $m+1$ can occur on any position of a permutation $u \in [\mathrm{id}, w_{2m+1}]$.

Let $1 \leqslant i \leqslant m$. Then

$$g(u)(2i) = \rho(\alpha u^{-1}\beta^{-1})(2i) = \alpha(u^{-1}(\beta^{-1}(2i))) = \alpha(u^{-1}(i)) .$$

Then the condition $g(u)(2i) \leqslant 2i$ is equivalent to

$$u^{-1}(i) \in [1, i] \cup [m+2, m+i+1] ,$$

and that is the condition for the *value* i to occur in u on a position that satisfies condition (1.1) for id and w_{2m+1}. Similarly,

$$g(u)(2i-1) = \rho(\alpha u^{-1}\beta^{-1})(2i-1) = \alpha(u^{-1}(\beta^{-1}(2i-1))) = \alpha(u^{-1}(i+m+1)) .$$

Then the condition $g(u)(2i-1) \geqslant 2i$ is equivalent to

$$u^{-1}(i+m+1) \in [i, m+1] \cup [m+i+1, 2m+1] ,$$

and that is the condition for the *value* $i+m+1$ to occur in u on a position that satisfies condition (1.1) for id and w_{2m+1}.

To summarize: $g(u)$ satisfies conditions (2.1) for positions $1, 2, \dots, 2m$ if and only if u satisfies (1.1) for values $1, 2, \dots, m, m+2, \dots, 2m$. Since $g(u)$ always satisfies (2.1) for positions $2m+1, 2m+2$ and u always satisfies (1.1) for the value $m+1$, the equivalence is established and the proof is complete. □

Corollary 3.2. The cardinality of the ℓ_1-segment $[\mathrm{id}, w_{2m+1}]$ is G_{2m+4}.

Proof. The function g induces a bijection from $[\mathrm{id}, w_{2m+1}]$ to B_{2m+2}. □

Example 3.3. For $m = 1$ we have $w_3 = (231)$ and

$$[(123), (231)] = \{(123), (132), (231)\}$$

and

$$B_4 = \{(2143), (3142), (3241)\} .$$

The function $u \to \rho((132)u^{-1}(312))$ sends (123) to (2143), (132) to (3142), and (231) to (3241), establishing a bijection between $[(123), (231)]$ and B_4.

The case $n = 2m$ is very similar. Define $\eta\colon \mathfrak{S}_{2m} \to \mathfrak{S}_{2m+2}$ by

$$\eta(u) = (u(1)+1\ 1\ u(2)+1\ \dots\ u(2m-1)+1\ 2m+2\ u(2m)+1)$$

Explicitly, η inserts the value 1 on the second position, $2m+2$ on position $2m+1$, increases the other values by 1 and shifts them to fill the remaining positions. For example, $\eta((2413)) = (315264)$.

Theorem 3.4. Define the function $h\colon \mathfrak{S}_{2m} \to \mathfrak{S}_{2m+2}$ by $h(u) = \eta(\alpha u^{-1}\beta^{-1})$ where $\alpha = (1\,3\,\dots\,2m{-}1\,2\,4\,\dots\,2m)$, $\beta = (3\,5\,\dots\,2m{-}1\,1\,2m\,2\,4\,\dots\,2m{-}2)$. Then

$$u \in [\mathrm{id}, w_{2m}] \iff h(u) \in C_{2m+2} .$$

Proof. Let $u \in \mathfrak{S}_{2m}$. Then

$$\begin{aligned} &h(u)(1) = \alpha u^{-1}\beta^{-1}(1) + 1 = \alpha(u^{-1}(m)) + 1 \geqslant 2 \\ &h(u)(2) = 1 < 2 \\ &h(u)(2m+1) = 2m+2 \geqslant 2(m+1) \\ &h(u)(2m+2) = \alpha u^{-1}\beta^{-1}(2m) + 1 = \alpha(u^{-1}(m+1))+1 < 2m+2 , \end{aligned}$$

hence conditions (2.2) are satisfied by all $h(u)$ for $i = 1$ and $i = m+1$. The *values* m and $m+1$ can occur on any position of a permutation $u \in [\mathrm{id}, w_{2m}]$.

Let $2 \leqslant i \leqslant m$. Then

$$h(u)(2i) = \eta(\alpha u^{-1}\beta^{-1})(2i) = \alpha(u^{-1}(\beta^{-1}(2i-1))) + 1 = \alpha(u^{-1}(i-1)) + 1 .$$

Then the condition $h(u)(2i) < 2i$ is equivalent to

$$u^{-1}(i-1) \in [1, i-1] \cup [m+1, m+i-1] ,$$

and that is the condition for the *value* $i-1$ to occur in u on a position that satisfies condition (1.1) for id and w_{2m}. Similarly,

$$h(u)(2i-1) = \eta(\alpha u^{-1}\beta^{-1})(2i-1) = \alpha(u^{-1}(\beta^{-1}(2i-2))) + 1 = \alpha(u^{-1}(i+m)) + 1 \; .$$

Then the condition $h(u)(2i) \geqslant 2i$ is equivalent to

$$u^{-1}(i+m) \in [i, m] \cup [m+i, 2m] \; ,$$

and that is the condition for the *value* $i+m$ to occur in u on a position that satisfies condition (1.1) for id and w_{2m}.

To summarize: $h(u)$ satisfies conditions (2.2) for positions $3, 4, \ldots, 2m$ if and only if u satisfies (1.1) for values $1, 2, \ldots, m-1, m+2, \ldots, 2m$. Since $h(u)$ always satisfies (2.2) for positions $1, 2, 2m+1, 2m+2$ and u always satisfies (1.1) for values $m, m+1$, the equivalence is established and the proof is complete. □

Corollary 3.5. The cardinality of the ℓ_1-segment $[\mathrm{id}, w_{2m}]$ is H_{2m+3}.

Proof. The function h induces a bijection from $[\mathrm{id}, w_{2m}]$ to C_{2m+2}. □

Example 3.6. For $m = 2$ we have $w_4 = (3412)$ and

$$[(1234), (3412)] = \{(1234), (1324), (1423), (1432), (2314), (2413), (3214), (3412)\} \; .$$

The function $u \to \eta((1324)u^{-1}(2413))$ sends the permutations in $[(1234), (3412)]$ to the corresponding permutations in

$$C_6 = \{(415263), (315263), (314265), (514263), (215364), (214365), (415362), (514362)\}$$

References

[BD81] Daniel Barsky and Dominique Dumont. Congruences pour les nombres de Genocchi de 2e espèce. In *Study Group on Ultrametric Analysis. 7th–8th years: 1979–1981 (Paris, 1979/1981) (French)*, pages Exp. No. 34, 13. Secrétariat Math., Paris, 1981.

[DR94] Dominique Dumont and Arthur Randrianarivony. Dérangements et nombres de Genocchi. *Discrete Math.*, 132(1-3):37–49, 1994.

[Dum74] Dominique Dumont. Interprétations combinatoires des nombres de Genocchi. *Duke Math. J.*, 41:305–318, 1974.

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