Multidimensional Catalan and related numbers as Hausdorff moments

Katarzyna Górska and Karol A. Penson

ABSTRACT. We study integral representation of so-called d-dimensional Catalan numbers $C_d(n)$, defined by $\left[\prod_{p=0}^{d-1} \frac{p!}{(n+p)!}\right](dn)!$, $d = 2, 3, \ldots$, $n = 0, 1, \ldots$. We prove that the $C_d(n)$'s are the nth Hausdorff power moments of positive functions $W_d(x)$ defined on $x \in [0, d^d]$. We construct exact and explicit forms of $W_d(x)$ and demonstrate that they can be expressed as combinations of $d - 1$ hypergeometric functions of type $_{d-1}F_{d-2}$ of argument x/d^d. These solutions are unique. We analyse them analytically and graphically. A combinatorially relevant, specific extension of $C_d(n)$ for d even in the form $D_d(n) = \left[\prod_{p=0}^{d-1} \frac{p!}{(n+p)!}\right]\left[\prod_{q=0}^{d/2-1} \frac{(2n+2q)!}{(2q)!}\right]$ is analyzed along the same lines.

1. Introduction

Amongst many existing generalizations of classical Catalan numbers $C(n) = \frac{1}{n+1}\binom{2n}{n}$, those that include the parameter that in certain sense can be associated with the spacial dimension d, are particularly interesting. They permit to extend to higher dimensions $d > 2$ the notions of objects enumerated by $C(n)$ in $d = 2$. We shall be concerned in this note with one of such generalizations, called d-dimensional Catalan numbers [**1, 2, 3**], which are defined as:

$$C_d(n) = \left[\prod_{p=0}^{d-1} \frac{p!}{(n+p)!}\right](dn)!, \quad n = 0, 1, \ldots; \ d = 2, 3, \ldots, \tag{1}$$

which for $d = 2$ clearly reduce the conventional Catalan numbers $C(n)$. The form of Eq. (1) guarantees that $C_d(0) = 1$ for all d. We shall refer to Sloane's Online Encyclopedia of Integer Sequences (OEIS) [**4**] and quote initial terms, $n = 0, 1, \ldots, 7$, of several sequences $C_d(n)$, $d = 2, 3, 4$ and 5, along with the labelling of their entries in the OEIS:

- for $d = 2$: $1, 1, 2, 5, 14, 42, 132, \ldots$ (A00108), which are the Catalan numbers,
- for $d = 3$: $1, 1, 5, 42, 462, 6006, 87516, \ldots$ (A005789, A151334),
- for $d = 4$: $1, 1, 14, 462, 24024, 1662804, 140229804, \ldots$ (A005790),
- for $d = 5$: $1, 1, 42, 6006, 1662804, 701149020, 396499770810, \ldots$ (A005791),
- for general d, see A060854.

The explicit form of $C_d(n)$'s permits one to immediately write down some of their characteristics. If $\Delta(k, a) = \frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k-1}{k}$ denotes a special list of k elements, then the ordinary generating function (ogf) of $C_d(n)$'s can be written as:

$$g(d, z) = \sum_{n=0}^{\infty} C_d(n) z^n = {}_dF_{d-1}\left(\begin{array}{c} \Delta(d, 1) \\ 2, 3, \ldots, d \end{array}\middle| d^d z\right). \qquad (2)$$

Similarly, the exponential generating function (egf) of $C_d(n)$'s reads:

$$G(d, z) = \sum_{n=0}^{\infty} C_d(n) \frac{z^n}{n!} = {}_dF_d\left(\begin{array}{c} \Delta(d, 1) \\ 1, 2, \ldots, d \end{array}\middle| d^d z\right). \qquad (3)$$

The use of Stirling's formula gives the leading term of $n \to \infty$ asymptotics for $C_d(n)$:

$$C_d(n) \xrightarrow[n\to\infty]{} n^{-\frac{d^2-1}{2}} d^{dn} + \ldots, \quad d = 2, 3, \ldots . \qquad (4)$$

In Eqs. (2) and (3) we have employed the standard notation for the generalized hypergeometric function ${}_pF_q\left(\begin{array}{c} (\alpha_p) \\ (\beta_q) \end{array}\middle| x\right)$, with (α_p) and (β_q) the lists of p "upper" and q "lower" parameters, respectively, see [5]. Observe that since in ${}_dF_d$ of Eq. (3) there is a pair of lower and upper parameters differing by one, the appropriate function ${}_dF_d$ can be reduced to a combination of ${}_{d-1}F_{d-1}$'s, see the formula 7.2.13.17 on page 439 of [5].

Inspired by the very fruitful interpretation of Catalan numbers $C(n)$ as moments of a positive function on $x \in [0, 4]$, which is intimately related to the famous Wigner's semicircle law [6], we set out to consider the sequences $C_d(n)$, $d > 2$ as Hausdorff power moments and have defined an objective of obtaining for $d > 2$ the equivalents of the solution for $d = 2$, quoted in Eq. (12) below.

The paper is organised as follows: in Sec. 2 we describe the method of obtaining exact and explicit solutions for $d \geq 2$. Subsequently we write down the general solution for d arbitrary and quote the specific cases of $d = 2, 3, 4$ and 5. In Section 3 we discuss some possible generalizations of $C_d(n)$'s. In Sec. 4 we close the note with short conclusions and comments about possible applications of the probability distributions found here.

2. Solutions of the Hausdorff moment problem

We are seeking the solutions of the following Hausdorff moment problem:

$$\int_0^{R(d)} x^n W_d(x)\, dx = C_d(n), \quad n = 0, 1, \ldots, \quad d = 2, 3, \ldots, \qquad (5)$$

where $R(d)$ - the upper edge of the support of $W_d(x)$ - will be determined below. The conventional estimate $R(d) = \lim_{n\to\infty} [C_d(n)]^{1/n} = d^d$ will be confirmed later by the Mellin transform analysis. As a preliminary step we shall demonstrate that the sought for $W_d(x)$ defined in Eq. (5) is positive. By using the Gauss-Legendre multiplication formula for gamma function to Eq. (1) and introducing complex s such that $n = s - 1$ we obtain

$$C_d(s-1) = (2\pi)^{\frac{1-d}{2}} d^{\frac{1}{2}-d}\left(\prod_{k=0}^{d-1} k!\right)(d^d)^s \prod_{j=0}^{d-1} \frac{\Gamma\left(s - 1 + \frac{j+1}{d}\right)}{\Gamma(s + j)}, \qquad (6)$$

which should be interpreted as the Mellin transform of $W_d(x)$ i. e. $\int_0^{\infty} x^{s-1} W_d(x) dx$, denoted by $\mathcal{M}[W_d(x); s]$, see Ref. [8]. Since $j > \frac{j+1-d}{d}$ for all $0 \leq j \leq d-1$, the individual

term labelled by j in the second product of Eq. (6) has the inverse Mellin transform [8],
see the formula 8.4.2.3 on page 631 from [5]

$$
\mathcal{M}^{-1}\left[\frac{\Gamma\left(s-1+\frac{j+1}{d}\right)}{\Gamma(s+j)}; x\right] = \frac{x^{(j+1)/d-1}(1-x)^{j-(j+1)/d}}{\Gamma\left(1+j-\frac{j+1}{d}\right)},
\tag{7}
$$

$j = 0, 1, \ldots d-1$, e. g. it is proportional to the standard probabilistic beta distribution [9]
in the variable x, which is a positive and absolutely continuous function for $0 \le x \le 1$. We
perceive now $C_d(s-1)$ as a product of d such individual terms. Then the weight $W_d(x)$
is a positive and absolutely continuous function on $[0, R(d)]$, since it is a d-fold Mellin
convolution of positive and absolutely continuous functions on $[0, 1]$. In the final result we
accommodate the prefactor $(d^d)^s$ which indicates, via elementary property of the Mellin
transform [8], that the solution of Eq. (5) will depend on x/d^d.

It turns out that such a d-fold Mellin convolution can be carried out explicitly. The key
step is first to identify the weight $W_d(x)$ as a special case of Meijer G function $G_{p,q}^{m,n}$ [5].
This is a direct consequence of (6) and reads

$$
W_d(x) = (2\pi)^{\frac{1-d}{2}} d^{\frac{1}{2}-d}\left(\prod_{k=0}^{d-1} k!\right) G_{d,d}^{d,0}\left(\frac{x}{d^d}\bigg|\begin{matrix}0, 1, \ldots, d-1\\ -\Delta(d, 0)\end{matrix}\right),
\tag{8}
$$

where $\Delta(n, a) = \frac{a}{n}, \frac{a+1}{n}, \ldots, \frac{a+n-1}{n}$. Next, the Meijer G function is converted to the hyper-
geometric form, using formulas 16.17.2 and 17.17.3 of [10], which is the Slater theorem.
We quote only the final result which is equal to:

$$
W_d(x) = \sum_{j=1}^{d-1} \frac{c_j(d)}{x^{j/d}}\, {}_{d-1}F_{d-2}\left(\begin{matrix}-\frac{j}{d}, -1-\frac{j}{d}, \ldots, -d+2-\frac{j}{d}\\ 1-\frac{1}{d}, 1-\frac{2}{d}, \ldots, 1-\frac{j-1}{d}\,;\, 1+\frac{1}{d}, 1+\frac{2}{d}, \ldots, 1+\frac{d-j-1}{d}\end{matrix}\bigg|\frac{x}{d^d}\right),
\tag{9}
$$

defined for $0 \le x \le d^d$, which implies $R(d) = d^d$ in Eq. (5). (For the reader's convenience
we point out that in Eq. (9), in the lower list of parameters of ${}_{d-1}F_{d-2}$, there are *two* se-
quences of numbers, which contain $j-1$ and $d-1-j$ terms, respectively). The numerical
coefficient $c_j(d)$ is equal to

$$
c_j(d) = (2\pi)^{\frac{1-d}{2}} d^{j-d+\frac{1}{2}}\left[\prod_{p=1}^{d-1}\frac{p!}{\Gamma\left(p+\frac{j}{d}\right)}\right]\left[\prod_{k=1}^{j-1}\Gamma\left(\frac{k}{d}\right)\right]\left[\prod_{k=j+1}^{d-1}\Gamma\left(\frac{j-k}{d}\right)\right],
\tag{10}
$$

where $j = 1, \ldots, d-1$ and $d = 2, 3, \ldots$.

The structure of parameter list of Meijer G function in Eq. (8) warrants that the as-
sumptions of formula 2.24.2.1 in [5] are satisfied:

$$
-\frac{1}{d}\sum_{k=0}^{d-1} k - \sum_{k=0}^{d-1} k = -\frac{d^2-1}{2} < 0, \quad d = 2, 3, \ldots .
\tag{11}
$$

Therefore the Mellin transform of $W_d(x)$ is well defined for $\Re(s) > \frac{d-1}{d}$.

We shall explicitly write down the solutions for $d = 2, 3, 4$ and 5, starting with $W_2(x)$:

$$
W_2(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, \quad 0 < x \le 4,
\tag{12}
$$

which is obtained in many references [7, 11], see Fig. 1. It is the only density that can
be expressed by an elementary function. Furthermore, for $d > 2$ no density can be ex-
pressed by *standard* special functions, and the hypergeometric form is the final one. For

j	1	2	3	4	
$c_j(3)$	$-\frac{3^3\sqrt{3}}{16\pi^3}\Gamma\!\left(\frac{2}{3}\right)^3$	$\frac{3^2}{10}\Gamma\!\left(\frac{2}{3}\right)^{-3}$	—	—	—
$c_j(4)$	$\frac{4^4 2}{75\pi^4}\Gamma\!\left(\frac{3}{4}\right)^4$	$-\frac{4^3}{15\pi^2}$	$\frac{4^6}{4851}\Gamma\!\left(\frac{3}{4}\right)^{-4}$	—	—
$c_j(5)$	$-\frac{5^9\sqrt{5}\Gamma\!\left(\frac{4}{5}\right)^5 A^4}{2^5 3^2 11^2 \pi^5\, B}$	$\frac{5^9\sqrt{5}\Gamma\!\left(\frac{3}{5}\right)^5 B^4}{2^6 7^3 17\pi^5\, A}$	$-\frac{5^9\sqrt{5}(AB)^{-1}}{2^7 3^4 13^2\Gamma\!\left(\frac{3}{5}\right)^5}$	$\frac{5^9\sqrt{5}(AB)^{-1}}{2^7 3^4 7^2 19\Gamma\!\left(\frac{4}{5}\right)^4}$	—
$c_j(6)$	$\frac{2^{13} 3^{16} E}{7^4 13^3 1805\pi^6}$	$-\frac{3^{19} D}{2^{10} 7^3 65\pi^6}$	$\frac{2^{21}}{7^2 5^2 3^4 \pi^3}$	$-\frac{3^{17}}{11^2 5^3 2^8 7 D}$	$\frac{2^{22} 3^{17}}{11^4 5^4 17^3 23^2 29 E}$

TABLE 1.   The coefficients $c_j(d)$, see Eqs. (9) and (10) for $d = 3,\ldots,6$ and $j = 1,\ldots,d-1$; To simplify the notation in $c_j(5)$ and $c_j(6)$ we set $A = \sin\!\left(\frac{\pi}{5}\right)$, $B = \sin\!\left(\frac{2\pi}{5}\right)$ and $E = \Gamma\!\left(\frac{5}{6}\right)^6$, $D = \Gamma\!\left(\frac{2}{3}\right)^6$.

$d = 3, \ldots, 5$ the solutions read:

$$W_3(x) = \frac{c_1(3)}{x^{1/3}}\,{}_2F_1\!\left(\begin{matrix} -\frac{4}{3}, -\frac{1}{3} \\ \frac{4}{3} \end{matrix}\,\middle|\,\frac{x}{3^3}\right) + \frac{c_2(3)}{x^{2/3}}\,{}_2F_1\!\left(\begin{matrix} -\frac{5}{3}, -\frac{2}{3} \\ \frac{2}{3} \end{matrix}\,\middle|\,\frac{x}{3^3}\right), \quad 0 < x \leqslant 3^3, \qquad (13)$$

$$W_4(x) = \frac{c_1(4)}{x^{1/4}}\,{}_3F_2\!\left(\begin{matrix} -\frac{9}{4}, -\frac{5}{4}, -\frac{1}{4} \\ \frac{5}{4}, \frac{3}{2} \end{matrix}\,\middle|\,\frac{x}{4^4}\right) + \frac{c_2(4)}{x^{1/2}}\,{}_3F_2\!\left(\begin{matrix} -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2} \\ \frac{3}{4}, \frac{5}{4} \end{matrix}\,\middle|\,\frac{x}{4^4}\right) \qquad (14)$$

$$+ \frac{c_3(4)}{x^{3/4}}\,{}_3F_2\!\left(\begin{matrix} -\frac{11}{4}, -\frac{7}{4}, -\frac{3}{4} \\ \frac{1}{2}, \frac{3}{4} \end{matrix}\,\middle|\,\frac{x}{4^4}\right), \quad 0 < x \leqslant 4^4,$$

$$W_5(x) = \frac{c_1(5)}{x^{1/5}}\,{}_4F_3\!\left(\begin{matrix} -\frac{16}{5}, -\frac{11}{5}, -\frac{6}{5}, -\frac{1}{5} \\ \frac{6}{5}, \frac{7}{5}, \frac{8}{5} \end{matrix}\,\middle|\,\frac{x}{5^5}\right) + \frac{c_2(5)}{x^{2/5}}\,{}_4F_3\!\left(\begin{matrix} -\frac{17}{5}, -\frac{12}{5}, -\frac{7}{5}, -\frac{2}{5} \\ \frac{4}{5}, \frac{6}{5}, \frac{7}{5} \end{matrix}\,\middle|\,\frac{x}{5^5}\right) \qquad (15)$$

$$+ \frac{c_3(5)}{x^{3/5}}\,{}_4F_3\!\left(\begin{matrix} -\frac{18}{5}, -\frac{13}{5}, -\frac{8}{5}, -\frac{3}{5} \\ \frac{3}{5}, \frac{4}{5}, \frac{6}{5} \end{matrix}\,\middle|\,\frac{x}{5^5}\right) + \frac{c_4(5)}{x^{4/5}}\,{}_4F_3\!\left(\begin{matrix} -\frac{19}{5}, -\frac{14}{5}, -\frac{9}{5}, -\frac{4}{5} \\ \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \end{matrix}\,\middle|\,\frac{x}{5^5}\right),$$

$$0 < x \leqslant 5^5.$$

The coefficients $c_j(d)$, $j = 1,\ldots,d-1$, for $d = 3,\ldots,6$, are collected in Tab. 1. With $c_j(6)$'s given there and using Eqs. (9) and (10), the reader can easily reconstruct $W_6(x)$, which will not be reproduced here. The solution $W_3(x)$ is represented in Fig. 2.

3. Generalization of multidimensional Catalan numbers

In this paragraph we analyse the extension of $C_4(n)$ obtained by replacing $(4n)!$ in Eq. (1) by $(2n)!(2n+2)!$. The corresponding sequence

$$D_4(n) \equiv 6(2n)!(2n+2)!\left[\prod_{r=0}^{3}(n+r)!\right]^{-1}$$

has attracted attention in several contexts, as it appears in [**1, 12, 13, 14**].

The initial terms of $D_4(n)$ are $1, 1, 4, 30, 330, 4719, 81796, 1643356$, for $n = 0, 1, \ldots, 7$. It is listed as A006149 in OEIS where also additional information can
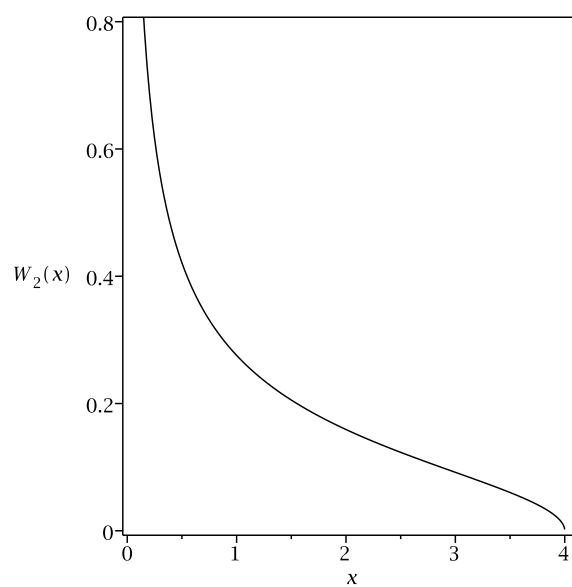
FIGURE 1. The density $W_2(x)$, s. Eq. (12).
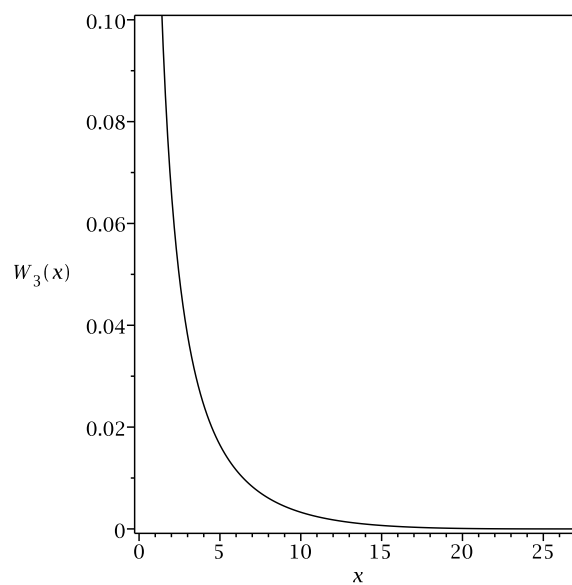
FIGURE 2. The density $W_3(x)$, s. Eq. (13).

be found. It turns out that the ogf of $D_4(n)$ can be expressed by the elliptic functions $\mathbb{E}(y)$

and $\mathbb{K}(y)$ [10]:

$$\sum_{n=0}^{\infty} D_4(n)\, z^n = \frac{1+6z}{4z^2} + \frac{(1-16z)\,(1+112z)}{240\,\pi\,z^3}\,\mathbb{K}(4\,\sqrt{z})$$
$$- \frac{1+224z+256z^2}{240\,\pi\,z^3}\,\mathbb{E}(4\,\sqrt{z}). \tag{16}$$

In fact the sequence $D_4(n)$ allows for the same kind of analysis as does the ensemble of $C_d(n)$'s. The Hausdorff moment problem for $D_4(n)$, namely

$$\int_0^h x^n\, V_4(x)\, dx = D_4(n) = \frac{6(2n)!(2n+2)!}{\prod\limits_{r=0}^{3}(n+r)!}, \quad n = 0, 1, \ldots, \tag{17}$$

can be solved by the method of Mellin convolution and the use of Meijer G function elucidated above. The weight can be proven to be positive on $x \in [0, h]$ with $h = 16$ and reads:

$$V_4(x) = \frac{1}{15\pi^2}\left[\left(\frac{64}{\sqrt{x}} + 56\sqrt{x} + \frac{x^{3/2}}{4}\right)\mathbb{E}\left(\sqrt{1-\frac{x}{16}}\right) - 2\sqrt{x}(16+x)\mathbb{K}\left(\sqrt{1-\frac{x}{16}}\right)\right]. \tag{18}$$
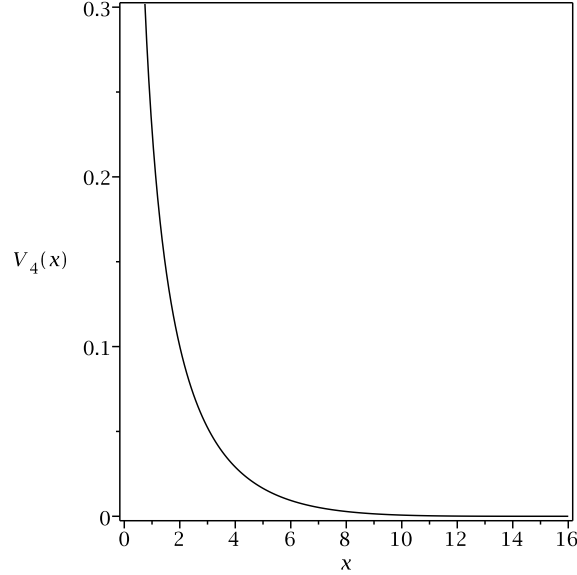
We plot the function $V_4(x)$ on Fig. 3.

FIGURE 3. The density $V_4(x)$, s. Eq. (18).

The sequence $D_4(n)$ analysed above is a special case $d = 4$ of the following generalization of $C_d(n)$ defined for *even d*:

$$D_d(n) = \left[\prod_{r=0}^{d-1}\frac{r!}{(n+r)!}\right]\left[\prod_{s=0}^{d/2-1}\frac{(2n+2s)!}{(2s)!}\right], \qquad d = 2, 4, 6, \ldots, \tag{19}$$

in which the parameter d should not be associated anymore with the spacial dimension. Several exact characteristics of the sequences $D_d(n)$ are available. The ordinary generating
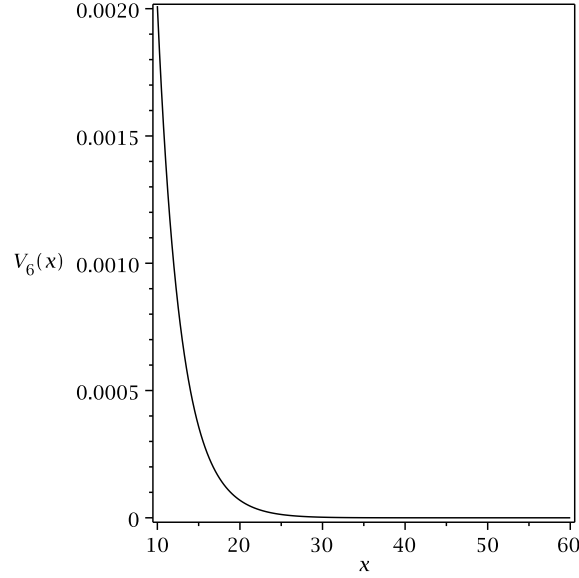
FIGURE 4.   The density $V_6(x)$, s. Eq. (23) for $d = 6$.

function reads

$$\tilde{g}(d, z) = {}_{d/2}F_{d/2-1}\left(\begin{array}{c} \frac{1}{2}, \frac{3}{2}, \ldots, \frac{d-1}{2}, 1 \\ \frac{d}{2} + 1, \frac{d}{2} + 2, \ldots, d \end{array}\middle|2^d z\right), \qquad (20)$$

whereas the corresponding exponential generating function is equal to

$$\tilde{G}(d, z) = {}_dF_d\left(\begin{array}{c} \frac{1}{2}, \frac{3}{2}, \ldots, \frac{d-1}{2}, 1, 2, \ldots, \frac{d}{2} \\ 1, 2, 3, \ldots, d \end{array}\middle|2^d z\right). \qquad (21)$$

The leading term of the $n \to \infty$ asymptotics for $D_d(n)$ can be obtained by using the Stirling's formula and it has the following form

$$D_d(n) \xrightarrow[n\to\infty]{} n^{-d(d-1)/4}\, 2^{dn}, \quad d = 4, 6, \ldots. \qquad (22)$$

It is remarkable that the Hausdorff moment problem for $D_d(n)$, i. e.

$$\int_0^{\kappa(d)} x^n\, V_d(x)\, dx \;=\; D_d(n), \quad n = 0, 1, \ldots; \quad d = 4, 6, \ldots$$

can be exactly solved as well in terms of positive functions $V_d(x)$ defined on $x \in [0, 2^d]$, i. e. $\kappa(d) = 2^d$, which read:

$$V_d(x) = \frac{2^{-d} \prod\limits_{r=0}^{d-1} r!}{\prod\limits_{k=0}^{d/2-1} \Gamma\left(k + \frac{1}{2}\right) k!}\, G_{d/2,d/2}^{d/2,0}\left(\frac{x}{2^d}\middle|\begin{array}{c} \frac{d}{2}, \frac{d}{2} + 1, \ldots, d - 1 \\ -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots, \frac{d-3}{2} \end{array}\right). \qquad (23)$$

Here the condition 2.24.2.1 in [5] implies $-\frac{d}{4}(d + 1) < 0$, $d = 4, 6, \ldots$, which is always satisfied. In addition, the Mellin transform of $V_d(x)$ is well defined for $\Re(s) > \frac{1}{2}$. The proof of positivity of $V_d(x)$ can be carried out along the lines exposed in the previous Section.

Since in Eqs. (23) in both parameter lists in the Meijer G function there are index pairs that differ by an integer, this Meijer G function cannot be represented by a sum of

generalized hypergeometric functions. However the expression (23) can be easily manipulated algebraically and represented graphically [**15**]. In Fig. 4 we display $V_6(x)$ in the range $x \in [10, 60]$. Observe the rapid decrease of this function for $x \gtrsim 25$, followed by a large region where it is practically flat and equals to zero. Similar behavior is observed for higher values of d.

4. Discussion and Conclusions

We have treated in this work essentially two generalizations of conventional Catalan numbers, which are related to such notions as Young tableaux, hook lengths, generalized Dyck paths, etc. [**16**]. They all turn out to be moments of positive functions on supports included in the positive half line. The relevant weight functions have been obtained explicitly and analyzed graphically. All these positive functions are unique solutions of Hausdorff moment problems. The key tools in this approach had been the inverse Mellin transform and the encoding with Meijer G functions. The positivity of solutions has been rigorously proven using the method of Mellin convolution, applied to related problems previously [**17, 18, 19**].

It needs to be specified that the function $W_2(x)$ of Eq. (12) is the known Marchenko-Pastur distribution [**7, 18**] which describes the level statistic of random Wishart matrices $W = GG^\dagger$, where G is a square, $N \times N$ random Ginibre matrix. As far as applications for random matrices are concerned two problems appear to be relevant for the distributions found in the present work.

First it would be intriguing to know if the distributions $W_d(x)$ for $d \geq 3$, and $V_d(x)$ for $d = 4, 6, \ldots$ would correspond to limit spectral densities of certain (if any) ensembles of random matrices. A second possibility is to extend the analysis of products of square random matrices to products of rectangular $N \times M$ random matrices with $r = N/M$. A case in point is a detailed analysis of products of rectangular Gaussian random matrices carried out in [**20**]. Therefore, once the relevant matrix ensemble has been properly identified, it is quite feasible to undertake the analysis of appropriate products of rectangular matrices. This would lead, in the spirit of [**20**] to, for instance, $W_3^{(r)}(x)$ parametrized by r, with $W_3^{(1)}(x) \equiv W_3(x)$. Both of these problems are under active consideration.

5. Acknowledgement

References

1. J. P. Allouche, *Transcendence of formal power series with rational coefficients,* Theor. Comput. Sci. 218 (1999), pp. 143–160.
2. R. A. Sulanke, *Generalizing Narayana and Schröder Numbers to Higher Dimensions,* Electron. J. Comb. 11 (2004), pp. R54.
3. V. Klebanov, *Extending the Reach and Power Deductive Program Verification*, doctoral dissertation, Universität Koblenz-Landau, Germany (2009, unpublished), http://kola.opus.hbz-nrw.de/volltexte/2009/477/.
4. N. J. A. Sloane, On-line Encyclopedia of Integer Sequences (OEIS), published electronically at http://oeis.org/ (2013).
5. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series, vol. 3: More Special Functions*, Gordon and Breach, New York 1998.
6. L. Arnold, *On Wigner's Semicircle Law for Eigenvalues of Random Matrices*, Z. Wahrscheinlichkeit **19** (1971) 191.

7. V. A. Marchenko and L. A. Pastur, *Distribution of eigenvalues for some sets of random matrices*, Mat. Sb. 72 (1967) 507.

8. I. A. Sneddon, *The Use of Integral Transforms*, Tata McGraw-Hill, New Delhi 1972.

9. Ch. Kleiber and S. Kotz, *Statistical Size Distributions in Economics and Actuarial Sciences*, John Wiley & Sons, New Jersey 2003.

10. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and Ch. W. Clark, *NIST Handbook of Mathematical Functions*, NIST and Cambridge University Press, Cambridge 2010.

11. K. A. Penson and J. M. Sixdeniers, *Integral Representations of Catalan and Related Numbers,* J. Int. Seq. 4 (2001), Article 01.2.5.

12. M. De Saint-Catherine and G. Viennot, in *Combinatoire énumérative.* Proceedings of the Colloque de combinatoire énumérative, held at Université du Québec à Montréal, May 28 June 1, 1985, (Lecture Notes in Mathematics, vol. 1234, p. 58-67), edited by Gilbert Labelle and Pierre Leroux, Springer, Berlin 1986.

13. O. Bernardi and N. Bonichon, *Catalan's intervals and realizers of triangulations*, Formal Power Series and Algebraic Combinatorics, Nankai University, Tianjin, China 2007, (http://www-igm.univ-mlv.fr/~fpsac/FPSAC07/SITE07/PDF-Proceedings/Talks/59.pdf).

14. M. Bousquet-Melou and M. Misha, *Walks with small steps in the quarter plane,* Contemp. Math. 520 (2010) pp. 1–40.

15. We have made extensive use of Maple® in this work.

16. E. Deutsch (OEIS and private communication to the authors),

17. K. Górska, K. A. Penson, A. Horzela, G. H. E. Duchamp, P. Blasiak, and A. I. Solomon, *Quasiclassical asymptotics and coherent states for bounded discrete spectra,* J. Math. Phys. 51 (2010), pp. 122102.

18. K. A. Penson and K. Życzkowski, *Product of Ginibre matrices: Fuss-Catalan and Raney distributions,* Phys. Rev. E 83 (2011), pp. 061118.

19. W. Młotkowski, K. A. Penson, and K. Życzkowski, *Densities of the Raney distributions,* arXiv: 1211.7259.

20. Z. Burda, A. Jarosz, G. Livan, M. A. Nowak, and A. Swiech, *Eigenvalues and singular values of products of rectangular Gaussian random matrices*, Phys. Rev. E 82 (2010) 061114.

H. Niewodniczański Institute of Nuclear Physics,, Polish Academy of Sciences,, ul. Eljasza-Radzikowskiego 152,, 31342 Kraków, Poland

E-mail address: katarzyna.gorska@ifj.edu.pl

Laboratoire de Physique Théorique, de la Matière Condensée (LPTMC),, Université Pierre et Marie Curie,, CNRS UMR 7600, Tour 13 - 5ième ét.,, Boîte Courrier 121, 4 place Jussieu,, F 75252 Paris Cedex 05, France

E-mail address: penson@lptl.jussieu.fr