THE PROBABILITY MEASURE CORRESPONDING TO 2-PLANE TREES

WOJCIECH MŁOTKOWSKI, KAROL A. PENSON

Abstract. We study the probability measure μ_0 for which the moment sequence is $\binom{3n}{n}\frac{1}{n+1}$. We prove that μ_0 is absolutely continuous, find the density function and prove that μ_0 is infinitely divisible with respect to the additive free convolution.

arXiv:1304.6544v2 [math.PR] 21 Jun 2013

1. Introduction

A 2-*plane tree* is a planted plane tree such that each vertex is colored black or white and for each edge at least one of its ends is white. Gu and Prodinger [3] proved, that the number of 2-plane trees on $n+1$ vertices with black (white) root is $\binom{3n+1}{n}\frac{1}{3n+1}$ (Fuss-Catalan number of order 3, sequence A001764 in OEIS [10]) and $\binom{3n+2}{n}\frac{2}{3n+2}$ (sequence A006013 in OEIS) respectively (see also [4]). We are going to study the sequence

$$\binom{3n}{n}\frac{2}{n+1}=\binom{3n+1}{n}\frac{1}{3n+1}+\binom{3n+2}{n}\frac{2}{3n+2}, \tag{1}$$

which begins with

$$2, 3, 10, 42, 198, 1001, 5304, 29070, 163438, \ldots,$$

of total numbers of such trees (A007226 in OEIS).

Both the sequences on the right hand side of (1) are positive definite (see [5, 6]), therefore so is the sequence $\binom{3n}{n}\frac{2}{n+1}$ itself. In this paper we are going to study the corresponding probability measure μ_0, i.e. such that the numbers $\binom{3n}{n}\frac{1}{n+1}$ are moments of μ_0. First we prove that μ_0 is Mellin convolution of two beta distributions, in particular μ_0 is absolutely continuous. Then we find the density function of μ_0. In the last section we prove, that μ_0 can be decomposed as additive free convolution $\mu_1 \boxplus \mu_2$ of two measures, which are both infinitely divisible with respect to $\boxplus$ and are related to the Marchenko-Pastur distribution. In particular, the measure μ_0 itself is $\boxplus$-infinitely divisible.

2. The generating function

Let us consider the generating function

$$G(z)=\sum_{n=0}^{\infty}\binom{3n}{n}\frac{2z^n}{n+1}.$$

2010 *Mathematics Subject Classification.* Primary 44A60; Secondary 46L54.
Key words and phrases. beta distribution, free convolution, Meijer function.
W. M. is supported by the Polish National Science Center grant No. 2012/05/B/ST1/00626. K. A. P. acknowledges support from PAN/CNRS under Project PICS No. 4339 and from Agence Nationale de la Recherche (Paris, France) under Program PHYSCOMB No. ANR-08-BLAN-0243-2.

According to (1), G is a sum of two generating functions. The former is usually denoted by $\mathcal{B}_3$:

$$\mathcal{B}_3(z) = \sum_{n=0}^{\infty} \binom{3n+1}{n} \frac{z^n}{3n+1}$$

and satisfies equation

$$\mathcal{B}_3(z) = 1 + z \cdot \mathcal{B}_3(z)^3. \tag{2}$$

Lambert's formula (see (5.60) in [2]) implies, that the latter is just square of $\mathcal{B}_3$:

$$\mathcal{B}_3(z)^2 = \sum_{n=0}^{\infty} \binom{3n+2}{n} \frac{2z^n}{3n+2},$$

so we have

$$G(z) = \mathcal{B}_3(z) + \mathcal{B}_3(z)^2. \tag{3}$$

Combining (2) and (3), we obtain the following equation for G:

$$2 - z - (1+2z)G(z) + 2zG(z)^2 - z^2G(z)^3 = 0, \tag{4}$$

which will be applied later on.

Now we will give formula for $G(z)$.

Proposition 2.1. *For the generating function of the sequence (1) we have*

$$G(z) = \frac{12\cos^2\alpha + 6}{\left(4\cos^2\alpha - 1\right)^2}, \tag{5}$$

where $\alpha = \frac{1}{3}\arcsin\left(\sqrt{27z/4}\right)$.

Proof. Denoting $(a)_n := a(a+1)\ldots(a+n-1)$ we have

$$\frac{2(3n)!}{(n+1)!(2n)!} = \frac{-2\left(\frac{-2}{3}\right)_{n+1}\left(\frac{-1}{3}\right)_{n+1} 27^{n+1}}{3(n+1)!\left(\frac{-1}{2}\right)_{n+1} 4^{n+1}}.$$

Therefore

$$G(z) = \frac{2 - 2\cdot {}_2F_1\left(\frac{-2}{3}, \frac{-1}{3}; \frac{1}{2}\middle|\, \frac{27z}{4}\right)}{3z}.$$

Now we apply formula

$${}_2F_1\left(\frac{-2}{3}, \frac{-1}{3}; \frac{-1}{2}\middle|\, u\right) = \frac{1}{3}\sqrt{u}\sin\left(\frac{1}{3}\arcsin\left(\sqrt{u}\right)\right) + \sqrt{1-u}\cos\left(\frac{1}{3}\arcsin\left(\sqrt{u}\right)\right),$$

which can be proved by hypergeometric equation (note that both the functions $w \mapsto w\sin\left(\frac{1}{3}\arcsin(w)\right)$, $w \mapsto \cos\left(\frac{1}{3}\arcsin(w)\right)$ are even, so the right hand side is well defined for $|u| < 1$). Putting $\alpha = \frac{1}{3}\arcsin(\sqrt{u})$, $u = 27z/4$, we have $\sqrt{u} = \sin 3\alpha$, $\sqrt{1-u} = \cos 3\alpha$, which after elementary calculations gives (5). $\square$

3. The measure

In this part we are going to study the (unique) measure μ_0 for which $\left\{\binom{3n}{n}\frac{1}{n+1}\right\}_{n=0}^{\infty}$ is the moment sequence. We will show that μ_0 can be expressed as the Mellin convolution of two beta distributions. Then we will provide explicit formula for the density function $V(x)$ of μ_0.

Recall (see [1]), that for $\alpha, \beta > 0$, the *beta distribution* $\mathrm{Beta}(\alpha, \beta)$ is the absolutely continuous probability measure defined by the density function

$$f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1},$$

for $x \in (0,1)$. The moments of $\mathrm{Beta}(\alpha, \beta)$ are

$$\int_0^1 x^n f_{\alpha,\beta}(x)\,dx = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\alpha+\beta+n)} = \prod_{i=0}^{n-1} \frac{\alpha+i}{\alpha+\beta+i}.$$

For probability measures ν_1, ν_2 on the positive half-line $[0,\infty)$ the *Mellin convolution* is defined by

$$(\nu_1 \circ \nu_2)(A) := \int_0^\infty \int_0^\infty \chi_A(xy)d\nu_1(x)d\nu_2(y) \tag{6}$$

for every Borel set $A \subseteq [0,\infty)$ (χ_A denotes the indicator function of the set A). This is the distribution of the product $X_1 \cdot X_2$ of two independent nonnegative random variables with $X_i \sim \nu_i$. In particular, if $c > 0$ then $\nu \circ \delta_c$ is the *dilation* of ν:

$$(\nu \circ \delta_c)(A) = \mathbf{D}_c\nu(A) := \nu\left(\frac{1}{c}A\right),$$

where δ_c denotes the Dirac delta measure at c.

If both the measures ν_1, ν_2 have all *moments*

$$s_n(\nu_i) := \int_0^\infty x^n\,d\nu_i(x)$$

finite then so has $\nu_1 \circ \nu_2$ and

$$s_n(\nu_1 \circ \nu_2) = s_n(\nu_1) \cdot s_n(\nu_2)$$

for all n. The method of Mellin convolution has been recently applied to a number of related problems, see for example [6, 8].

Now we can describe the probability measure corresponding to the sequence $\binom{3n}{n}\frac{1}{n+1}$.

Proposition 3.1. *Define μ_0 as the Mellin convolution*

$$\mu_0 = \mathrm{Beta}(1/3, 1/6) \circ \mathrm{Beta}(2/3, 4/3) \circ \delta_{27/4}. \tag{7}$$

Then the numbers $\binom{3n}{n}\frac{1}{n+1}$ are moments of μ_0:

$$\int_0^{27/4} x^n\,d\mu_0(x) = \binom{3n}{n}\frac{1}{n+1}.$$

Proof. It is sufficient to check that

$$\frac{(3n)!}{(n+1)!(2n)!} = \prod_{i=0}^{n-1} \frac{1/3+i}{1/2+i} \cdot \prod_{i=0}^{n-1} \frac{2/3+i}{2+i} \cdot \left(\frac{27}{4}\right)^n.$$

□

In view of formula (7), the measure μ_0 is absolutely continuous and its support is the interval $[0, 27/4]$. Now we are going to find the density function $V(x)$ of μ_0.

Theorem 3.2. *Let*

$$V(x) = \frac{\sqrt{3}}{2^{10/3}\pi x^{2/3}}\left(3\sqrt{1-4x/27}-1\right)\left(1+\sqrt{1-4x/27}\right)^{1/3} + \frac{1}{2^{8/3}\pi x^{1/3}\sqrt{3}}\left(3\sqrt{1-4x/27}+1\right)\left(1+\sqrt{1-4x/27}\right)^{-1/3},$$

$x \in (0, 27/4)$. *Then V is the density function of μ_0, i.e.*

$$\int_0^{27/4} x^n\, V(x)\, dx = \binom{3n}{n}\frac{1}{n+1}$$

for $n = 0, 1, 2, \ldots$.

The density $V(x)$ of μ_0 is represented in Fig. 1.B.

Proof. Putting $n = s-1$ and applying the Gauss-Legendre multiplication formula

$$\Gamma(mz) = (2\pi)^{(1-m)/2} m^{mz-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)\ldots\Gamma\left(z+\frac{m-1}{m}\right)$$

we obtain

$$\binom{3n}{n}\frac{1}{n+1} = \frac{\Gamma(3n+1)}{\Gamma(n+2)\Gamma(2n+1)} = \frac{\Gamma(3s-2)}{\Gamma(s+1)\Gamma(2s-1)} = \frac{2}{27}\sqrt{\frac{3}{\pi}}\left(\frac{27}{4}\right)^s \frac{\Gamma(s-2/3)\Gamma(s-1/3)}{\Gamma(s-1/2)\Gamma(s+1)} := \psi(s).$$

Then ψ can be extended to an analytic function on the complex plane, except the points $1/3-n$, $2/3-n$, $n = 0, 1, 2, \ldots$.

Now we are going to apply a particular type of the Meijer G-function, see [9] for details. Let $\widetilde{V}$ denote the inverse Mellin transform of ψ. Then we have

$$\begin{aligned}\widetilde{V}(x) &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s}\psi(s)\, ds \\ &= \frac{2}{27}\sqrt{\frac{3}{\pi}}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\Gamma(s-2/3)\Gamma(s-1/3)}{\Gamma(s-1/2)\Gamma(s+1)}\left(\frac{4x}{27}\right)^{-s} ds \\ &= \frac{2}{27}\sqrt{\frac{3}{\pi}}\, G^{2,0}_{2,2}\left(\frac{4x}{27}\,\middle|\,\begin{matrix}-1/2, & 1\\ -2/3, & -1/3\end{matrix}\right),\end{aligned}$$

where $x \in (0, 27/4)$ (consult [11] for the role of c in the integrals). On the other hand, for the parameters of the G-function we have

$$(-2/3-1/3)-(-1/2+1) = -3/2 < 0$$

and hence the assumptions of formula 2.24.2.1 in [9] are satisfied. Therefore we can apply the Mellin transform on $\widetilde{V}(x)$:

$$\begin{aligned}\int_0^{27/4} x^{s-1}\widetilde{V}(x)\, dx &= \frac{2}{27}\sqrt{\frac{3}{\pi}}\int_0^{27/4} x^{s-1} G^{2,0}_{2,2}\left(\frac{4x}{27}\,\middle|\,\begin{matrix}-1/2, & 1\\ -2/3, & -1/3\end{matrix}\right) dx \\ &= \frac{2}{27}\sqrt{\frac{3}{\pi}}\left(\frac{27}{4}\right)^s\int_0^1 u^{s-1} G^{2,0}_{2,2}\left(u\,\middle|\,\begin{matrix}-1/2, & 1\\ -2/3, & -1/3\end{matrix}\right) du = \psi(s)\end{aligned}$$

whenever $\Re s > 2/3$. Consequently, $\widetilde{V} = V$.

Now we use Slater's formula (see [9], formula 8.2.2.3) and express V in terms of the hypergeometric functions:

$$\begin{aligned} V(x) &= \frac{2}{27}\sqrt{\frac{3}{\pi}}\frac{\Gamma(1/3)}{\Gamma(1/6)\Gamma(5/3)}\left(\frac{4x}{27}\right)^{-2/3} {}_2F_1\left(\frac{-2}{3},\frac{5}{6};\frac{2}{3}\middle|\frac{4x}{27}\right) \\ &+\frac{2}{27}\sqrt{\frac{3}{\pi}}\frac{\Gamma(-1/3)}{\Gamma(-1/6)\Gamma(4/3)}\left(\frac{4x}{27}\right)^{-1/3} {}_2F_1\left(\frac{-1}{3},\frac{7}{6};\frac{4}{3}\middle|\frac{4x}{27}\right) \\ &= \frac{\sqrt{3}}{4\pi x^{2/3}}\,{}_2F_1\left(\frac{-2}{3},\frac{5}{6};\frac{2}{3}\middle|\frac{4x}{27}\right) + \frac{1}{2\pi\sqrt{3}x^{1/3}}\,{}_2F_1\left(\frac{-1}{3},\frac{7}{6};\frac{4}{3}\middle|\frac{4x}{27}\right). \end{aligned}$$

Applying the formula

$${}_2F_1\left(\frac{t-2}{2},\frac{t+1}{2};t\middle|z\right) = \frac{2^t}{2t}\left(t-1+\sqrt{1-z}\right)\left(1+\sqrt{1-z}\right)^{1-t}$$

(see [6]) for $t = 2/3$ and $t = 4/3$ we conclude the proof. □

4. Relations with free probability

In this part we are going to describe relations of μ_0 with free probability. In particular we will show that μ_0 is infinitely divisible with respect to the additive free convolution.

Let us briefly describe the additive and multiplicative free convolutions. For details we refer to [12, 7].

Denote by $\mathcal{M}^c$ the class of probability measures on $\mathbb{R}$ with compact support. For $\mu \in \mathcal{M}^c$, with moments

$$s_m(\mu) := \int_{\mathbb{R}} t^m\, d\mu(t),$$

and with the *moment generating function*:

$$M_\mu(z) := \sum_{m=0}^{\infty} s_m(\mu) z^m = \int_{\mathbb{R}} \frac{d\mu(t)}{1-tz},$$

we define its *R-transform* $R_\mu(z)$ by the equation

$$R_\mu\big(zM_\mu(z)\big) + 1 = M_\mu(z). \tag{8}$$

Then the *additive free convolution* of $\mu', \mu'' \in \mathcal{M}^c$ is defined as the unique $\mu' \boxplus \mu'' \in \mathcal{M}^c$ which satisfies

$$R_{\mu'\boxplus\mu''}(z) = R_{\mu'}(z) + R_{\mu''}(z).$$

If the support of $\mu \in \mathcal{M}^c$ is contained in the positive halfline $[0, +\infty)$ then we define its *S-transform* $S_\mu(z)$ by

$$M_\mu\left(\frac{z}{1+z}S_\mu(z)\right) = 1 + z \qquad \text{or} \qquad R_\mu\big(zS_\mu(z)\big) = z. \tag{9}$$

on a neighborhood of 0. If μ', μ'' are such measures then their *multiplicative free convolution* $\mu' \boxtimes \mu''$ is defined by

$$S_{\mu'\boxtimes\mu''}(z) = S_{\mu'}(z) \cdot S_{\mu''}(z).$$

Recall, that for dilated measure we have: $M_{\mathbf{D}_c\mu}(z) = M_\mu(cz)$, $R_{\mathbf{D}_c\mu}(z) = R_\mu(cz)$ and $S_{\mathbf{D}_c\mu}(z) = S_\mu(z)/c$. The operations $\boxplus$ and $\boxtimes$ can be regarded as free analogs of the classical and Mellin convolution.

For $t>0$ let ϖ_t denote the *Marchenko-Pastur distribution* with parameter t:

$$\varpi_t = \max\{1-t,0\}\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx, \tag{10}$$

with the absolutely continuous part supported on $\left[(1-\sqrt{t})^2,(1+\sqrt{t})^2\right]$. Then

$$M_{\varpi_t}(z) = \frac{2}{1+z-tz+\sqrt{\left(1-z-tz\right)^2-4tz^2}} \tag{11}$$

$$= 1+\sum_{n=1}^{\infty} z^n \sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}\frac{t^k}{n}, \tag{12}$$

$$R_{\varpi_t}(z) = \frac{tz}{1-z}, \qquad S_{\varpi_t}(z) = \frac{1}{t+z}. \tag{13}$$

In free probability the measures ϖ_t play the role of the Poisson distributions. Note that from (13) the family $\{\varpi_t\}_{t>0}$ constitutes a semigroup with respect to $\boxplus$, i.e. we have $\varpi_s \boxplus \varpi_t = \varpi_{s+t}$ for $s,t>0$.

Theorem 4.1. *The measure μ_0 is equal to the additive free convolution $\mu_0=\mu_1\boxplus\mu_2$, where $\mu_1=\mathbf{D}_2\varpi_{1/2}$, so that*

$$\mu_1 = \frac{1}{2}\delta_0 + \frac{\sqrt{8-(x-3)^2}}{4\pi x}\chi_{(3-\sqrt{8},3+\sqrt{8})}(x)\,dx, \tag{14}$$

and $\mu_2=\frac{1}{2}\delta_0+\frac{1}{2}\varpi_1$, i.e.

$$\mu_2 = \frac{1}{2}\delta_0 + \frac{\sqrt{4x-x^2}}{4\pi x}\chi_{(0,4)}(x)\,dx. \tag{15}$$

The measures μ_1,μ_2 are infinitely divisible with respect to the additive free convolution $\boxplus$, and consequently, so is μ_0.

The absolutely continuous parts of the measures μ_1,μ_2 are represented in Fig. 1.A.

Proof. The moment generating function of μ_0 is $M_{\mu_0}(z)=G(z)/2$. Then we have $M_{\mu_0}(0)=1$ and by (4)

$$2-z-2(1+2z)M_{\mu_0}(z)+8zM_{\mu_0}(z)^2-8z^2M_{\mu_0}(z)^3=0.$$

Let $T(z)$ be the inverse function for $M_{\mu_0}(z)-1$, so that $T(0)=0$ and $M_{\mu_0}\big(T(z)\big)=1+z$. Then

$$2-T(z)+(-1-2T(z))2(1+z)+8T(z)(1+z)^2-8T(z)^2(1+z)^3=0,$$

which gives

$$8(1+z)^3T(z)^2-(8z^2+12z+3)T(z)+2z=0$$

and finally

$$T(z)=\frac{8z^2+12z+3-\sqrt{9+8z}}{16(1+z)^3}=\frac{4z}{8z^2+12z+3+\sqrt{9+8z}}.$$

Therefore we can find the S-transform of μ_0:

$$S_{\mu_0}(z)=\frac{1+z}{z}T(z)=\frac{8z^2+12z+3-\sqrt{9+8z}}{16z(1+z)^2}=\frac{4(1+z)}{8z^2+12z+3+\sqrt{9+8z}}$$

and from (9) we get the R-transform:

$$R_{\mu_0}(z) = \frac{4z - 1 + \sqrt{1-2z}}{2(1-2z)}.$$

Now we observe that $R_{\mu_0}(z)$ can be decomposed as follows:

$$R_{\mu_0}(z) = \frac{z}{1-2z} + \frac{1-\sqrt{1-2z}}{2\sqrt{1-2z}} = R_1(z) + R_2(z).$$

Comparing with (13) we observe that $R_1(z)$ is the R-transform of $\mu_1 = \mathbf{D}_2\varpi_{1/2}$, which implies that μ_1 is $\boxplus$-infinitely divisible.

Consider the Taylor expansion of $R_2(z)$:

$$R_2(z) = \sum_{n=1}^{\infty} \binom{2n}{n} 2^{-n-1} z^n = \frac{z}{2} + z^2 \sum_{n=0}^{\infty} \binom{2(n+2)}{n+2} 2^{-n-3} z^n.$$

Since the numbers $\binom{2n}{n}$ are moments of the *arcsine distribution*

$$\frac{1}{\pi\sqrt{x(4-x)}} \chi_{(0,4)}(x)\, dx,$$

the coefficients of the last sum constitute a positive definite sequence. So $R_2(z)$ is R-transform of a probability measure μ_2, which is $\boxplus$-infinitely divisible (see Theorem 13.16 in [7]). Now using (8) we obtain

$$M_{\mu_2}(z) = \frac{1 + 2z - \sqrt{1-4z}}{4z} = \frac{1}{2} + \frac{1-\sqrt{1-4z}}{4z} = \frac{1}{2} + \frac{1}{1+\sqrt{1-4z}}.$$

Comparing with (11) for $t = 1$ we see that $\mu_2 = \frac{1}{2}\delta_0 + \frac{1}{2}\varpi_1$. □

Let us now consider the measures μ_1, μ_2 separately. For $\mu_1 = \mathbf{D}_2\varpi_{1/2}$ the moment generating function is

$$M_{\mu_1}(z) = \frac{2}{1 + z + \sqrt{1 - 6z + z^2}} = 1 + \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1}\frac{2^{n-k}}{n},$$

so the moments are

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, \ldots.$$

This is the A001003 sequence in OEIS (little Schroeder numbers), $s_n(\mu_1)$ is the number of ways to insert parentheses in product of $n+1$ symbols. There is no restriction on the number of pairs of parentheses. The number of objects inside a pair of parentheses must be at least 2.

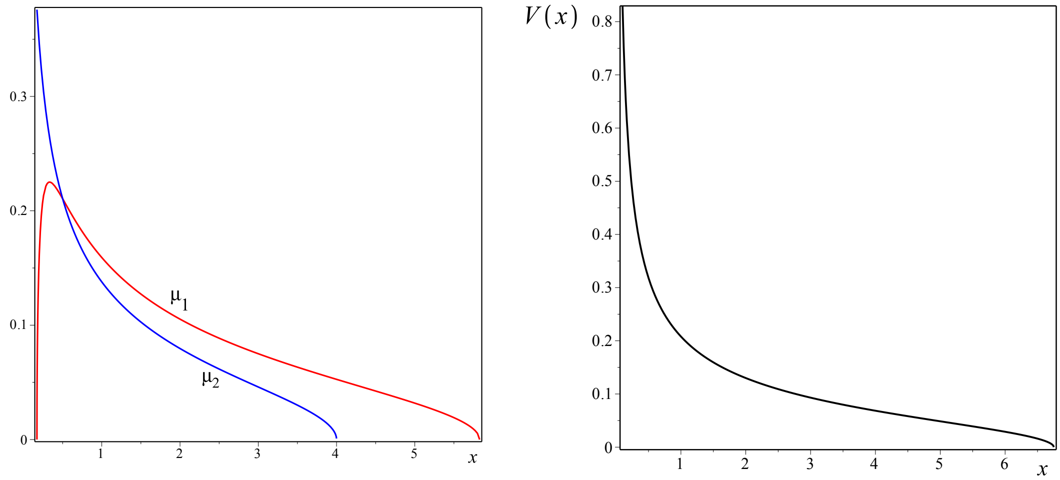
On the subject of μ_2, applying (9) we can find the S-transform:

$$S_{\mu_2}(z) = \frac{2(1+z)}{(1+2z)^2} = \frac{1+z}{1/2+z} \cdot \frac{1}{1+2z}.$$

One can check, that $\frac{1+z}{1/2+z}$ is the S-transform of $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, which yields

$$\mu_2 = \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) \boxtimes \mu_1. \tag{16}$$

We would like to thank G. Aubrun, C. Banderier, K. Górska and H. Prodinger for fruitful interactions.



(A) The densities of μ_1, μ_2 (B) The density of $\mu_0 = \mu_1 \boxplus \mu_2$

FIGURE 1. The densities of μ_1, μ_2 and $\mu_0 = \mu_1 \boxplus \mu_2$

REFERENCES

[1] N. Balakrishnan, V. B. Nevzorov, *A primer on statistical distributions,* Wiley-Interscience, Hoboken, N. J. 2003.

[2] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics. A Foundation for Computer Science,* Addison-Wesley, New York 1994.

[3] N. S. S. Gu and H. Prodinger, *Bijections for 2-plane trees and ternary trees,* European J. of Combinatorics **30** (2009), 969-985.

[4] N. S. S. Gu, H. Prodinger, S. Wagner, *Bijection for a class of labeled plane trees,* European J. of Combinatorics **31** (2010) 720–732.

[5] W. Młotkowski, *Fuss-Catalan numbers in noncommutative probability,* Documenta Math. **15** (2010) 939–955.

[6] W. Młotkowski, K. A. Penson, K. Życzkowski, *Densities of the Raney distributions,* arXiv:1211.7259.

[7] A. Nica, R. Speicher, *Lectures on the Combinatorics of Free Probability*, Cambridge University Press, 2006.

[8] K. A. Penson, K. Życzkowski, *Product of Ginibre matrices: Fuss-Catalan and Raney distributions* Phys. Rev. E **83** (2011) 061118, 9 pp.

[9] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, *Integrals and Series,* Gordon and Breach, Amsterdam (1998) Vol. 3: More special functions.

[10] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences,* (2013), published electronically at: http://oeis.org/.

[11] I. N. Sneddon, *The use of Integral Transforms,* Tata McGraw-Hill Publishing Company, 1974.

[12] D. V. Voiculescu, K. J. Dykema, A. Nica, *Free random variables*, CRM, Montréal, 1992.

INSTYTUT MATEMATYCZNY, UNIWERSYTET WROCŁAWSKI, PLAC GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND

E-mail address: mlotkow@math.uni.wroc.pl

LABORATOIRE DE PHYSIQUE THÉORIQUE DE LA MATIÈRE CONDENSÉE (LPTMC), UNIVERSITÉ PIERRE ET MARIE CURIE, CNRS UMR 7600, TOUR 13 - 5IÈME ÉT., BOÎTE COURRIER 121, 4 PLACE JUSSIEU, F 75252 PARIS CEDEX 05, FRANCE

E-mail address: penson@lptl.jussieu.fr