ON γ-VECTORS AND THE DERIVATIVES OF THE TANGENT AND SECANT FUNCTIONS

SHI-MEI MA

ABSTRACT. In this paper we consider the γ-vectors of the types A and B Coxeter complexes as well as the γ-vectors of the types A and B associahedrons. We show that these γ-vectors can be obtained by using derivative polynomials of the tangent and secant functions. A grammatical description for these γ-vectors is discussed. Moreover, we also present a grammatical description for the well known Legendre polynomials and Chebyshev polynomials of both kinds.

Keywords: γ-vectors; Tangent function; Secant function; Eulerian polynomials

2010 Mathematics Subject Classification: 05A05; 05A15

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. The *hyperoctahedral group* B_n is the group of signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. A permutation $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ is *alternating* if $\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n)$. Similarly, an element π of B_n is alternating if $\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n)$. Denote by E_n and E_n^B the number of alternating elements in $\mathfrak{S}_n$ and B_n, respectively. It is well known (see [4, 24]) that

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \tan x + \sec x, \quad \sum_{n=0}^{\infty} E_n^B \frac{x^n}{n!} = \tan 2x + \sec 2x.$$

Derivative polynomials are an important part of combinatorial trigonometry (see [1, 8, 10, 11, 12, 13, 14] for instance). Define

$$y = \tan(x), \quad z = \sec(x).$$

Denote by D the differential operator d/dx. Clearly, we have $D(y) = 1 + y^2$ and $D(z) = yz$. In 1995, Hoffman [10] considered two sequences of *derivative polynomials* defined respectively by $D^n(y) = P_n(y)$ and $D^n(z) = zQ_n(y)$. From the chain rule it follows that the polynomials $P_n(u)$ satisfy $P_0(u) = u$ and $P_{n+1}(u) = (1 + u^2)P_n'(u)$, and similarly $Q_0(u) = 1$ and $Q_{n+1}(u) = (1 + u^2)Q_n'(u) + uQ_n(u)$.

As shown in [10], the exponential generating functions

$$P(u,t) = \sum_{n=0}^{\infty} P_n(u)\frac{t^n}{n!} \quad \text{and} \quad Q(u,t) = \sum_{n=0}^{\infty} Q_n(u)\frac{t^n}{n!}$$

are given by the explicit formulas

$$P(u,t) = \frac{u + \tan(t)}{1 - u\tan(t)} \quad \text{and} \quad Q(u,t) = \frac{\sec(t)}{1 - u\tan(t)}. \tag{1}$$

Recall that a *descent* of a permutation $\pi\in\mathfrak{S}_n$ is a position i such that $\pi(i)>\pi(i+1)$, where $1\leq i\leq n-1$. Denote by $\operatorname{des}(\pi)$ the number of descents of π. Then the equations

$$A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\operatorname{des}(\pi)}=\sum_{k=0}^{n-1}\left\langle {n \atop k}\right\rangle x^k,$$

define the *Eulerian polynomial* $A_n(x)$ and the *Eulerian number* $\left\langle {n \atop k}\right\rangle$ (see [23, A008292]). For each $\pi\in B_n$, we define

$$\operatorname{des}_B(\pi)=\#\{i\in\{0,1,2,\ldots,n-1\}|\pi(i)>\pi(i+1)\},$$

where $\pi(0)=0$. Let

$$B_n(x)=\sum_{\pi\in B_n}x^{\operatorname{des}_B(\pi)}=\sum_{k=0}^{n}B(n,k)x^k.$$

The polynomial $B_n(x)$ is called an *Eulerian polynomial of type B*, while $B(n,k)$ is called an *Eulerian number of type B* (see [23, A060187]).

Assume that

$$(Dy)^{n+1}(y)=(Dy)(Dy)^n(y)=D(y(Dy)^n(y)),\quad (Dy)^{n+1}(z)=(Dy)(Dy)^n(z)=D(y(Dy)^n(z)).$$

Note that $D(y)=z^2$ and $D(z)=yz$. Recently, we obtained the following result.

Theorem 1 ([14]). *For $n\geq 1$, we have*

$$(Dy)^n(y)=2^n\sum_{k=0}^{n-1}\left\langle {n \atop k}\right\rangle y^{2n-2k-1}z^{2k+2},\quad (Dy)^n(z)=\sum_{k=0}^{n}B(n,k)y^{2n-2k}z^{2k+1}.$$

Define

$$f=\sec(2x),\quad g=2\tan(2x).$$

In this paper we will mainly consider the following differential system:

$$D(f)=fg,\quad D(g)=4f^2. \tag{2}$$

Define $h=\tan(2x)$. Note that $f^2=1+h^2$ and $g=2h$. So the following result is immediate.

Proposition 2. *For $n\geq 0$, we have $D^n(f)=2^nfQ_n(h),\quad D^n(g)=2^{n+1}P_n(h)$.*

In the next section, we collect some notation and definitions that will be needed in the rest of the paper.

2. Notation, definitions and preliminaries

The *h-polynomial* of a $(d-1)$-dimension simplicial complex Δ is the generating function $h(\Delta;x)=\sum_{i=0}^{d}h_i(\Delta)x^i$ defined by the following identity:

$$\sum_{i=0}^{d}h_i(\Delta)x^i(1+x)^{d-i}=\sum_{i=0}^{d}f_{i-1}(\Delta)x^i,$$

where $f_i(\Delta)$ is the number of faces of Δ of dimension i. There is a large literature devoted to the h-polynomials of the form

$$h(\Delta;x)=\sum_{i=0}^{\lfloor d/2\rfloor}\gamma_i x^i(1+x)^{d-2i},$$

where the coefficients γ_i are nonnegative. Following Gal [9], we call $(\gamma_0,\gamma_1,\ldots)$ the γ-vector of Δ, and the corresponding generating function $\gamma(\Delta;x)=\sum_{i\geq 0}\gamma_i x^i$ is the *γ-polynomial.* In particular, the Eulerian polynomials $A_n(x)$ and $B_n(x)$ are respectively known as the h-polynomials of Coxeter complexes of types A and B.

Let us now recall two classical result.

Theorem 3 ([7, 21])**.** *For $n\geq 1$, we have*

$$A_n(x)=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}a(n,k)x^k(1+x)^{n-1-2k}.$$

Theorem 4 ([5, 18, 19])**.** *For $n\geq 1$, we have*

$$B_n(x)=\sum_{k=0}^{\lfloor n/2\rfloor}b(n,k)x^k(1+x)^{n-2k}.$$

It is well known that the numbers $a(n,k)$ satisfy the recurrence

$$a(n,k)=(k+1)a(n-1,k)+(2n-4k)a(n-1,k-1),$$

with the initial conditions $a(1,0)=1$ and $a(1,k)=0$ for $k\geq 1$ (see [23, A101280]), and the numbers $b(n,k)$ satisfy the recurrence

$$b(n,k)=(2k+1)b(n-1,k)+4(n+1-2k)b(n-1,k-1), \tag{3}$$

with the initial conditions $b(1,0)=1$ and $b(1,k)=0$ for $k\geq 1$ (see [5, Section 4]).

The h-polynomials of the types A and B associahedrons are respectively given as follows (see [16, 17, 20, 22] for instance):

$$h(\Delta_{FZ}(A_{n-1}),x)=\frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}x^k=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}C_k\binom{n-1}{2k}x^k(1+x)^{n-1-2k}, \tag{4}$$

$$h(\Delta_{FZ}(B_n),x)=\sum_{k=0}^{n}\binom{n}{k}^2x^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2k}{k}\binom{n}{2k}x^k(1+x)^{n-2k}, \tag{5}$$

where $C_k=\frac{1}{k+1}\binom{2k}{k}$ is the kth *Catalan number* and the coefficients of x^k of $h(\Delta_{FZ}(A_{n-1}),x)$ is the *Narayana number* $N(n,k+1)$.

Define

$$F(n,k)=C_k\binom{n-1}{2k},\quad H(n,k)=\binom{2k}{k}\binom{n}{2k}.$$

There are many combinatorial interpretations of the number $F(n,k)$, such as $F(n,k)$ is number of *Motzkin paths* of length $n-1$ with k up steps (see [23, A055151]). It is easy to verify that the numbers $F(n,k)$ satisfy the recurrence relation

$$(n+1)F(n,k)=(n+2k+1)F(n-1,k)+4(n-2k)F(n-1,k-1),$$

with initial conditions $F(1,0)=1$ and $F(1,k)=0$ for $k\geq 1$, and the numbers $H(n,k)$ satisfy the recurrence relation

$$nH(n,k)=(n+2k)H(n-1,k)+4(n-2k+1)H(n-1,k-1),$$

with initial conditions $H(1,0)=1$ and $H(1,k)=0$ for $k\geq 1$ (see [23, A089627]).

3. γ-VECTORS

Define the generating functions

$$a_n(x)=\sum_{k\geq 0}a(n,k)x^k,\quad b_n(x)=\sum_{k\geq 0}b(n,k)x^k.$$

The first few $a_n(x)$ and $b_n(x)$ are respectively given as follows:

$$a_1(x)=1,a_2(x)=1,a_3(x)=1+2x,a_4(x)=1+8x;$$

$$b_1(x)=1,b_2(x)=1+4x,b_3(x)=1+20x,b_4(x)=1+72x+80x^2.$$

Combining (1) and [5, Prop. 3.5, Prop. 4.10], we immediately get the following result.

Theorem 5. *For $n\geq 1$, we have*

$$a_n(x)=\frac{1}{x}\left(\frac{\sqrt{4x-1}}{2}\right)^{n+1}P_n\left(\frac{1}{\sqrt{4x-1}}\right),\quad b_n(x)=(4x-1)^{\frac{n}{2}}Q_n\left(\frac{1}{\sqrt{4x-1}}\right).$$

Assume that

$$(fD)^{n+1}(f)=(fD)(fD)^n(f)=fD((fD)^n(f)),$$
$$(fD)^{n+1}(g)=(fD)(fD)^n(g)=fD((fD)^n(g)).$$

We can now present the main result of this paper.

Theorem 6. *For $n\geq 1$, we have*

$$D^n(f)=\sum_{k=0}^{\lfloor n/2\rfloor}b(n,k)f^{2k+1}g^{n-2k},$$
$$D^n(g)=2^{n+1}\sum_{k=0}^{\lfloor n-1/2\rfloor}a(n,k)f^{2k+2}g^{n-1-2k},$$
$$(fD)^n(f)=n!\sum_{k=0}^{\lfloor n/2\rfloor}H(n,k)f^{n+1+2k}g^{n-2k},$$
$$(fD)^n(g)=2(n+1)!\sum_{k=0}^{\lfloor (n-1)/2\rfloor}F(n,k)f^{n+2+2k}g^{n-1-2k}.$$

Proof. We only prove the assertion for $D^n(f)$ and the others can be proved in a similar way. It follows from (2) that $D(f)=fg$ and $D^2(f)=fg^2+4f^3$. For $n\geq 0$, we define $\widetilde{b}(n,k)$ by

$$D^n(f)=\sum_{k=0}^{\lfloor n/2\rfloor}\widetilde{b}(n,k)f^{2k+1}g^{n-2k}, \tag{6}$$

Then $\widetilde{b}(1,0)=1$ and $\widetilde{b}(1,k)=0$ for $k\geq 1$. It follows from (6) that

$$D(D^n(f))=\sum_{k=0}^{\lfloor n/2\rfloor}(2k+1)\widetilde{b}(n,k)f^{2k+1}g^{n-2k+1}+4\sum_{k=0}^{\lfloor n/2\rfloor}(n-2k)\widetilde{b}(n,k)f^{2k+3}g^{n-2k-1}.$$

We therefore conclude that $\widetilde{b}(n+1,k)=(2k+1)\widetilde{b}(n,k)+4(n+2-2k)\widetilde{b}(n,k-1)$ and complete the proof by comparing it with (3). □

Define

$$N_n(x)=\frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}(x+1)^k(x-1)^{n-1-k},$$
$$L_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2(x+1)^k(x-1)^{n-k}.$$

Taking $f^2=1+h^2$ and $g=2h$ in Theorem 6 leads to the following result and we omit the proof of it, since it is a straightforward application of (4) and (5).

Corollary 7. *For $n\geq 1$, we have*

$$(fD)^n(f)=n!f^{n+1}(-\imath)^nL_n(\imath h),$$
$$(fD)^n(g)=2(n+1)!f^{n+2}(-\imath)^{n-1}N_n(\imath h),$$

where $\imath=\sqrt{-1}$.

It should be noted that the polynomial $\frac{1}{2^n}L_n(x)$ is the famous *Legendre polynomial* [23, A100258]. Therefore, from Corollary 7, we see that the Legendre polynomial can be generated by $(fD)^n(f)$.

4. Context-free grammars

Many combinatorial objects permit grammatical interpretations (see [2, 3, 15] for instance). The grammatical method was systematically introduced by Chen [2] in the study of exponential structures in combinatorics. Let A be an alphabet whose letters are regarded as independent commutative indeterminates. A *context-free grammar* G over A is defined as a set of substitution rules that replace a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. For example, if $G=\{u\rightarrow uv, v\rightarrow v\}$, then

$$D(u)=uv, D(v)=v, D^2(u)=u(v+v^2), D^3(u)=u(v+3v^2+v^3).$$

It follows from Theorem 6 that the γ-vectors of Coxeter complexes (of types A and B) and associahedrons (of types A and B) can be respectively generated by the grammars

$$G_1=\{u\rightarrow uv, v\rightarrow 4u^2\}$$

and

$$G_2=\{u\rightarrow u^2v, v\rightarrow 4u^3\}. \tag{7}$$

There are many sequences can be generated by the grammar (7). A special interesting result is the following.

Proposition 8. *Let G be the same as in* (7). *Then*

$$D^n(uv) = n! \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} 4^k \binom{n+1}{2k} u^{n+1+2k} v^{n+1-2k}.$$

Let $T_n(x)$ and $U_n(x)$ be the *Chebyshev polynomials of the first and second kind* of order n, respectively. We can now conclude the following result, which is based on Proposition 8. The proof runs along the same lines as that of Theorem 6.

Theorem 9. *If $G = \{u \to u^2 v, v \to u^3\}$, then*

$$D^n(uv) = n! \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1}{2k} u^{n+1+2k} v^{n+1-2k},$$

$$D^n(u^2) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} u^{n+2+2k} v^{n-2k}.$$

In particular,

$$D^n(uv) \mid_{u^2=x^2-1, v=x} = n!(x^2-1)^{\frac{n+1}{2}} T_{n+1}(x),$$

$$D^n(u^2) \mid_{u^2=x^2-1, v=x} = n!(x^2-1)^{\frac{n+2}{2}} U_n(x).$$

Taking $u = \sec^2(x)$ and $v = 2\tan(x)$, it is clear that $D(u) = uv$ and $D(v) = 2u$. One can easily to verify another grammatical description of the γ-vectors of the type A Coxeter complex.

Theorem 10. *If $G = \{u \to uv, v \to 2u\}$, then*

$$D^n(u) = \sum_{k=0}^{\lfloor n/2 \rfloor} a(n+1,k) u^{k+1} v^{n-2k}.$$

Define

$$T(n,k) = \binom{n}{k} \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}. \tag{8}$$

It is well known that $T(n,k)$ is the number of paths of length n with steps $U = (1,1), D = (1,-1)$ and $H = (1,0)$, starting at $(0,0)$, staying weakly above the x-axis (i.e. left factors of *Motzkin paths*) and having k H steps (see [23, A107230]). It follows from (8) that

$$(n+1)T(n,k) = (2n+1-k)T(n-1,k-1) + 2T(n-1,k) + 4(k+1)T(n-1,k+1). \tag{9}$$

We end our paper by giving the following result.

Theorem 11. *If $G = \{t \to tu^2, u \to u^2 v, v \to 4u^3\}$, then*

$$D^n(t^2u^2) = (n+1)!t^2 \sum_{k=0}^{n} T(n,k) u^{2n+2-k} v^k,$$

$$D^n(t^2u) = n!t^2 \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} u^{2n+1-k} v^k,$$

Proof. We only prove the assertion for $D^n(t^2u^2)$ and the corresponding assertion for $D^n(t^2u)$ can be proved in a similar way. It is easy to verify that $D(t^2u^2) = 2t^2(u^4 + u^3v)$ and $D^2(t^2u^2) = 3!t^2(2u^6 + 2u^5v + u^4v^2)$. For $n \geq 0$, we define

$$D^n(t^2u^2) = (n+1)!t^2 \sum_{k=0}^{n} \widetilde{T}(n,k)u^{2n+2-k}v^k.$$

Note that

$$\frac{D^{n+1}(t^2u^2)}{(n+1)!t^2} = \sum_k (2n+2-k)\widetilde{T}(n,k)u^{2n+3-k}v^{k+1}+$$
$$2\sum_k \widetilde{T}(n,k)u^{2n+4-k}v^k + 4\sum_k k\widetilde{T}(n,k)u^{2n+5-k}v^{k-1}.$$

Thus, we get

$$(n+2)\widetilde{T}(n+1,k) = (2n+3-k)\widetilde{T}(n,k-1) + 2\widetilde{T}(n,k) + 4(k+1)\widetilde{T}(n,k+1).$$

Comparing with (9), we see that the coefficients $\widetilde{T}(n,k)$ satisfy the same recurrence relation and initial conditions as $T(n,k)$, so they agree. □

It should be noted that the numbers $\binom{n}{k}2^{n-k}$ are elements of the f-vector for the n-dimensional cubes (see [23, A038207])

References

[1] K.N. Boyadzhiev, Derivative Polynomials for tanh, tan, sech and sec in Explicit Form, Fibonacci Quart. 45 (2007) 291–303.

[2] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1993) 113–129.

[3] W.Y.C. Chen, R.X.J. Hao and H.R.L. Yang, Context-free Grammars and Multivariate Stable Polynomials over Stirling Permutations, `arXiv:1208.1420v2`.

[4] C.-O. Chow, Counting involutory, unimodal, and alternating signed permutations, Discrete Math. 36 (2006) 2222–2228.

[5] C.-O. Chow, On certain combinatorial expansions of the Eulerian polynomials, Adv. in Appl. Math. 41 (2008) 133–157.

[6] K. Dilks, T.K. Petersen, J.R. Stembridge, Affine descents and the Steinberg torus, Adv. in Appl. Math. 42 (2009) 423–444.

[7] D. Foata and M. P. Schützenberger, Théorie géometrique des polynômes eulériens, Lecture Notes in Math. vol. 138, Springer, Berlin, 1970

[8] G.R. Franssens, Functions with derivatives given by polynomials in the function itself or a related function, Anal. Math. 33 (2007) 17–36.

[9] S.R. Gal, Real root conjecture fails for five and higher-dimensional spheres, Discrete Comput. Geom. 34 (2005) 269–284.

[10] M.E. Hoffman, Derivative polynomials for tangent and secant, Amer. Math. Monthly 102 (1995) 23–30.

[11] M.E. Hoffman, Derivative polynomials, Euler polynomials, and associated integer sequences, Electron. J. Combin. 6 (1999) #R21.

[12] S.-M. Ma, Derivative polynomials and enumeration of permutations by number of interior and left peaks, Discrete Math. 312 (2012) 405–412.

[13] S.-M. Ma, An explicit formula for the number of permutations with a given number of alternating runs, J. Combin. Theory Ser. A 119 (2012), 1660–1664.

[14] S.-M. Ma, A family of two-variable derivative polynomials for tangent and secant, Electron. J. Combin. 20(1) (2013), #P11.

[15] S.-M. Ma, Some combinatorial arrays generated by context-free grammars, European J. Combin. 34 (2013) 1081–1091.

[16] T. Mansour, Y. Sun, Identities involving Narayana polynomials and Catalan numbers, Discrete Math. 309 (2009) 4079–4088.

[17] E. Marberg, Actions and identities on set partitions, Electron. J. Combin. 19 (2012), #P28.

[18] E. Nevo, T.K. Petersen, On γ-Vectors Satisfying the Kruskal-Katona Inequalities, Discrete Comput. Geom. 45 (2011) 503–521.

[19] T.K. Petersen, Enriched P-partitions and peak algebras, Adv. Math. 209 (2007) 561–610.

[20] A. Postnikov, V. Reiner, L. Williams, Faces of generalized permutohedra, Documenta Math. 13 (2008) 207–273.

[21] L.W. Shapiro, W.J. Woan, S. Getu, Runs, slides and moments, SIAM J. Algebraic Discrete Methods 4 (1983) 459–466.

[22] R. Simion, A type-B associahedron, Adv. in Appl. Math. 30 (2003) 2–25.

[23] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[24] R.P. Stanley, A Survey of Alternating Permutations, in *Combinatorics and Graphs*, R. A. Brualdi *et. al.* (eds.), Contemp. Math., Vol. 531, *Amer. Math. Soc.*, Providence, RI, 2010, pp. 165–196.

School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Hebei 066004, P. R. China

E-mail address: shimeima@yahoo.com.cn (S.-M. Ma)