Lower bounds on the Münchhausen problem

Michael Brand

michael.brand@alumni.weizmann.ac.il

Faculty of IT, Monash University

Clayton, VIC 3800

Australia

May 11, 2014

Abstract

"The Baron's omni-sequence", $B(n)$, first defined by Khovanova and Lewis (2011), is a sequence that gives for each n the minimum number of weighings on balance scales that can verify the correct labeling of n identically-looking coins with distinct integer weights between 1 gram and n grams.

A trivial lower bound on $B(n)$ is $\log_3 n$, and it has been shown that $B(n)$ is $\log_3 n + \mathrm{O}(\log \log n)$. In this paper we give a first nontrivial lower bound to the Münchhausen problem, showing that there is an infinite number of n values for which $B(n) \neq \lceil \log_3 n \rceil$.

Furthermore, we show that if $N(k)$ is the number of n values for which $k = \lceil \log_3 n \rceil$ and $B(n) \neq k$, then $N(k)$ is an unbounded function of k.

1 Introduction

Coin-weighing puzzles have been abundantly discussed in the mathematical literature over the past 60 years (see, e.g. [11, 6, 10, 5]). In coin-weighing problems one must typically identify a counterfeit coin from a set of identically-looking coins by use of balance scales, utilizing the knowledge that the counterfeit coin has distinctive weight. This can be generalized to the problem of identifying a coin, or a subset of the coins, based on distinctive weight characteristics, or, alternatively, to the problem of establishing the weight of a given coin.

This paper relates to a different kind of coin-weighing puzzle, which we call "The Münchhausen coin-weighing problem" (following, e.g., [2]). Consider the following question: given n coins with distinct integer weights between 1 gram and n grams, each labeled by a distinct integer label between 1 and n, what is the minimum number of weighings of these n coins on balance scales that can prove unequivocally that all coins are labeled by their correct weight?

This question differs from classic coin-weighing problems in that we do not need to *discover* the weights, but only to determine whether or not a given labeling of weights is the correct one. To establish the weights one would require $\Omega(n \log n)$ weighings (as can be proved by reasoning similar to that which establishes lower bounds for comparative sorting [9, 4]), whereas merely verifying an existing labeling can be performed trivially in $\mathrm{O}(n)$ weighings.

This question, inspired by a riddle that appeared in the Moscow Mathematical Olympiad [1], gives rise to an integer sequence, $B(n)$, that was studied in [8] and was dubbed there "The Baron's omni-sequence". It appears as sequence A186313 in the On-line Encyclopedia of Integer Sequences [7].

Though much progress has been made to tighten the known upper bounds on $B(n)$

[8, 3, 2], the trivial lower bound of $\log_3 n$ has proved surprisingly resilient. This lower bound stems from the straightforward observation that if the number of weighings is less than $\log_3 n$, there must be at least two coins that participate in all weighings in identical roles. (For each weighing, they are either both on the left-hand side of the scales, both on the right-hand side or both held out from the weighing.) This being the case, the weights of the two coins can be exchanged with no change to the outcome of any of the weighings, and therefore the weighings cannot provide an unequivocal verification of the weights.

In this paper we present a first nontrivial lower bound for this problem. Namely, we prove the following theorem.

Theorem 1. *For any n,*

$$3^{B(n)} \geq n + \Omega(\log \log n). \tag{1}$$

Equivalently,

$$N(k) \in \Omega(\log k), \tag{2}$$

where $N(k)$ is the number of n values for which $k = \lceil \log_3 n \rceil$ and $B(n) \neq k$.

2 Proof of the main theorem

We begin by introducing some terminology. First, following [2], we describe sequences of weighings by means of matrices. A $k \times n$ matrix, M, whose elements, M_{ij} belong to the set $\{-1, 0, 1\}$, describes a sequence of k weighings of n coins. If $M_{ij} = 1$, this indicates that coin j is to be placed on the right hand side of the scales on the i'th weighing. If it is -1, the coin is to be placed on the left hand side. A "0" indicates that on the i'th weighing the coin is to be held out.

In the case of the Münchhausen problem, it is known what weights the coins to be weighed are: the first coin weighs 1 gram, the second weighs 2 grams, and so on. We describe these weights by the vector $\vec{\boldsymbol{n}} = [1, \ldots, n]^T$. The result of the weighing sequence is therefore given by the element-wise signs of the vector $M\vec{\boldsymbol{n}}$. We describe the operation including both multiplication by $\vec{\boldsymbol{n}}$ and sign-taking by the single operator $w(M)$.

A matrix is *Münchhausen* if the sequence of weighings it describes generates a sequence of weigh results (signs) when weighing $\vec{\boldsymbol{n}}$ that is unique among all possible permutations of $\vec{\boldsymbol{n}}$. Equivalently, a matrix is Münchhausen if $w(M) = w(M\pi) \Rightarrow \pi = I$ for an $n \times n$ permutation matrix π.

The Baron's omni-sequence is the sequence that gives for each n the minimum k for which there exists a $k \times n$ Münchhausen matrix.

Theorem 1 gives a first nontrivial lower bound on k. We prove it now.

Proof of Theorem 1. Consider, first, the trivial lower bound for the Baron's omni-sequence. In matrix terminology, we claim that if a $k \times n$ matrix, M, is Münchhausen, then $n \leq 3^k$. The reason for this is that if $n > 3^k$, at least two of M's columns are identical. A permutation π permuting the columns of M by switching identical columns will have no effect on it: we have $M = M\pi$, and therefore necessarily also $w(M) = w(M\pi)$.

The relevant observation regarding this proof is that it demonstrates that the columns of M must be distinct. Because they all belong to the set $\{-1, 0, 1\}^k$, of size 3^k, the set, C, of choices for the set of M's columns (ignoring their order) is limited by $|C| \leq \binom{3^k}{n}$.

Consider, now, row permutations on M. For an M with a large k, there are many row permutations of M that do not change $w(M)$. For example, consider that each row of M generates a sign that has only 3 possibilities. As such, there will be at least $\lceil k/3 \rceil$ rows that share the same generated sign. Any σ_1, σ_2 of the $(\lceil k/3 \rceil)!$ possible row permutations on M that keep all rows other than these $\lceil k/3 \rceil$ as fixed points share the same $w(\sigma_1 M) = w(\sigma_2 M) = w(M)$. We define R to be the set of all row permutations that satisfy $w(\sigma M) = w(M)$, noting that $|R| \geq (\lceil k \rceil /3 \rceil)!$.

We claim that for M to be Münchhausen,

$$|C| \geq |R|. \tag{3}$$

If we define $l = 3^k - n$, then $|C| \leq \binom{3^k}{n}$ implies $|C| < 3^{kl}$. On the other hand, $\log_3 |R|$ is $\Omega(k \log k)$, so Equation (3) implies that l is $\Omega(\log k)$.

Because a lower bound for l is also a lower bound for $N(k)$ of Equation (2), Equation (3) directly implies Equation (2), which, in turn, implies Equation (1), because k is $\Omega(\log n)$. In other words, proving Equation (3) is tantamount to proving the entire theorem. We now proceed to establish this claim.

We define the relation $f : R \to C$ as follows. For $\sigma \in R$, $f(\sigma)$ is the set of columns of σM. Because changing the order of the weighings clearly has no effect on whether or not a set of weighings establishes unequivocally the weights of n coins, σM is Münchhausen if and only if M is Münchhausen, so by definition the set of columns of σM is necessarily a member of C. Instead of showing Equation (3), we make the stronger claim that f is one-to-one.

To prove this, let us assume to the contrary that f is not one-to-one. This indicates the existence of two row permutations $\sigma_1, \sigma_2 \in R$ for which $f(\sigma_1) = f(\sigma_2)$. Because the application of a permutation is invertible, we know that $\sigma_1 M \neq \sigma_2 M$. The two are therefore related by a column permutation, π, which is not the identity, as follows:

$$\sigma_1 M \pi = \sigma_2 M.$$

Let $\sigma \stackrel{\text{def}}{=} \sigma_2^{-1} \sigma_1$, then

$$\sigma M \pi = M.$$

Recall that by definition of R, we have $w(M) = w(\sigma M)$, so

$$w(\sigma M \pi) = w(M) = w(\sigma M),$$

so by definition σM cannot be a Münchhausen matrix. However, as argued earlier, σM is Münchhausen if and only if M is Münchhausen, so the above implies that M, too, is not Münchhausen, contradicting the assumption. $\square$

3 Conclusions

With the new Theorem 1, the best known bounds now place n between $3^k - \Omega(\log k)$ and $3^k / \mathrm{O}(\text{polylog } k)$, for n to satisfy $B(n) = k$. This still leaves a significant window for further refinement. At the current time, it is not even known whether $B(n)$ is a monotone sequence.

References

[1] Problems from the last round of LIV Moscow Mathematical Olympiad. *Kvant*, 9:70–71, 1991.

[2] M. Brand. Münchhausen matrices. *The Electronic Journal of Combinatorics*, 19(4), 2012. P40.

[3] M. Brand. Tightening the bounds on the Baron's omni-sequence. *Discrete Mathematics*, 312(7):1326–1335, 2012.

[4] T.H. Cormen, C.E. Leiserson, R.L. Rivest, and C. Stein. *Introduction to Algorithms*, Section 8.1: Lower bounds for sorting, pages 165–168. MIT Press and McGrow-Hill, 2nd edition, 2001.

[5] F.J. Dyson. The problem of the pennies. *The Mathematical Gazette*, 30(291):231–234, 1946. #1931.

[6] L. Halbeisen and N. Hungerbühler. The general counterfeit coin problem. *Discrete Mathematics*, 147(1–3):139–150, 1995.

[7] T. Khovanova and J.B. Lewis. A186313. `oeis.org/A186313`. [Online; accessed 10 March 2012].

[8] T. Khovanova and J.B. Lewis. Baron Münchhausen redeems himself: Bounds for a coin-weighing puzzle. *The Electronic Journal of Combinatorics*, 18(1), 2011. P37.

[9] D.E. Knuth. *The Art of Computer Programming, Volume 3: Sorting and Searching*, Section 5.3.1: Minimum-Comparison Sorting, pages 180–197. Addison-Wesley, 2nd edition, 1997.

[10] C.A.B. Smith. The counterfeit coin problem. *Mathematical Gazette*, 31:31–39, 1947.

[11] H. Steinhaus. *Mathematical Snapshots*. Dover Publications, 3rd edition, 1999.