

Entanglement classes of permutation-symmetric qudit states: symmetric operations suffice

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We analyze entanglement classes for permutation-symmetric states for n qudits (i.e., d -level systems), with respect to local unitary operations (LU equivalence) and stochastic local operations and classical communication (SLOCC equivalence). In both cases, we show that the search can be restricted to operations where the same local operation acts on all qudits, and we provide an explicit construction for it. Stabilizers of states in the form of one-particle operations preserving permutation symmetry are shown to provide a coarse-grained classification of entanglement classes. We prove that the Jordan form of such one-particle operators is a SLOCC invariant. We find, as representatives of those classes, a discrete set of entangled states that generalize the GHZ and W states for the many-particle qudit case. In the later case, we introduce *excitation states* as a natural generalization of the W state for $d > 2$.

I. INTRODUCTION

A. Entanglement of multipartite pure states

Entanglement is perhaps the most important resource for quantum information (for a review see [1]), and its characterization is one of the most important tasks of quantum theory. Particularly difficult is the problem of characterization of entangled mixed states (for a recent review of various necessary criteria see [2]). It might seem that the problem of characterization of pure state entanglement is much simpler and more tractable, but even this statement is, generally speaking, not true, except for the case of bipartite entanglement, where the Schmidt decomposition provides a method of classification of pure entangled states of two parties [1]. In a multipartite scenario very little is known about the different classes of entanglement. Typical questions that one would like to answer concern entanglement classes of pure states which are invariant with respect to local operations. The latter are typically assumed to belong to a group (unitary, general linear, etc.). The corresponding classes of states are called then LU equivalent, SLOCC equivalent, etc., where LU denotes local unitary, and SLOCC — stochastic local operations and classical communications. Only a few rigorous results are known concerning these questions, which we list below

- For three qubits a generalization of the Schmidt decomposition has been formulated (see [3] and references therein) — this result provides a classification of invariant states with respect to local unitaries. There is a considerable amount of work regarding

this and the related problem of geometrical invariants by Sudbery and coworkers [4, 5].

- Classification of entanglement of three qubit states according to LU operations and SLOCC has been presented in Ref. [6] and [7], respectively.
- Classification of entanglement of 4 qubits according to SLOCC has been presented in Ref. [8] (see also the papers by Miyake and Wadate [9] and by Miyake [10]).
- For many qudits a multiparticle generalization of the Schmidt decomposition [4, 11] provides a general way to determine whether two states are LU equivalent.
- SLOCC equivalence can be reduced to LU equivalence of the critical points of the so-called *total variance function* [36].

There is also a considerable amount of work on many qubits states (see [12–15]), but very little is known about general many qudit states. The difficulty of classifying entanglement for multipartite pure states was evidently one of the motivations to look at restricted families of states. Such restrictions are typically introduced by considering symmetries [16], which might be physically motivated. In this spirit many authors considered totally (permutation) symmetric pure states of n qubits (see [17–22]), since such states naturally describe systems of many bosons, and appear frequently in the context of quantum optics. Similarly, quantum correlations in totally antisymmetric states (as representative states of fermions) have been intensively investigated (for a review see [23] and references therein). In the next introductory section we focus on symmetric states and their particular role in physical applications.

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B. Permutationally symmetric pure states

A many-qudit wavefunction can be permutation-symmetric for two reasons. One is when it describes a system of bosons, so that the particles are indistinguishable on a fundamental level. Second is when the particles are distinguishable but, because of a particular setting (e.g. a Hamiltonian for which the particles form an eigenstate), they happen to be in a permutation-symmetric state. The latter situation occurs for instance for the Lipkin-Meshkov-Glick model [24] of nuclear shell structure, and related models of quantum chaos [25]. It is worth stressing that the two situations are *not the same*. In the later case we are able to manipulate each particle separately in a different way, while in the first we are restricted to operations modifying each boson in the same way. The question is whether those two settings give rise to the same entanglement classes, i.e. if for symmetric states classification can be reduced to studying operations that act in the same way on all particles. Moreover, entanglement geometry of permutation-symmetric states is interesting and relevant, e.g. for quantum computation using linear optics [26]. As mentioned above, this question has been widely studied in the qubit case [18, 22, 27], but most of the results are not-applicable for qudit systems of dimension $d > 2$, a general case which we are going to address.

In this paper we consider two types of equivalence — under local unitary operations (LU equivalence) [14, 15] and under positive-operator-valued measures with post-selection. The second is called stochastic local operations and classical communication and is equivalent to multiplication by invertible matrices (not necessarily unitary or even normal or diagonalizable) on each particle.

For example, for the simplest case of two qubits, the LU-equivalence classes are distinguished by their Schmidt coefficients, where a unique representative can be written as $\lambda|00\rangle + \sqrt{1-\lambda^2}|11\rangle$, with $0 \leq \lambda \leq 1/2$. On the other hand, for SLOCC-equivalence there are only two distinct symmetric states — a product state $|00\rangle$ and the Greenberger-Horne-Zeilinger (GHZ) state $(|00\rangle + |11\rangle)/\sqrt{2}$. In general, entanglement classes of more than two particles are much more involved, even for the symmetric qubit ($d = 2$) states with three [7] or four [8] particles.

In this paper we present two results. The first one is that, when testing whether two permutation-symmetric n qudit states are equivalent under local transformations, the search can be restricted to operators which act in the same way on every particle. This property was proven for qubits [19, 28, 29] in the SLOCC variant (although the unitary version can be deduced from that proof). For a general qudit system has remained so far an open question [20, Sec. 5.1.1.]. That is, in the course of this paper, we prove the following:

Theorem 1. *Let us consider two permutation-symmetric states of n qudits (i.e. d -level particles), $|\psi\rangle$*

and $|\varphi\rangle$, for which there exist invertible $d \times d$ matrices A_1, \dots, A_n such that

$$A_1 \otimes A_2 \otimes \dots \otimes A_n |\psi\rangle = |\varphi\rangle. \quad (1)$$

Our result implies that then there exists an invertible $d \times d$ matrix A such that

$$A \otimes A \otimes \dots \otimes A |\psi\rangle = |\varphi\rangle. \quad (2)$$

If we restrict ourselves to unitary matrices A_1, \dots, A_n , then A is unitary.

For A_i unitary, (1) is a condition of the equivalence of states under reversible local operations (or LU-equivalence), which is proven to be the same as equivalence with respect to local operations and classical communication [30, 31]. Moreover, in both cases we provide a direct construction for A as a function of A_1, \dots, A_n .

Our second result stems from the consideration of stabilizers of states [21] in the form of a matrix B acting on one particle, and its inverse B^{-1} acting on another one. Only for very specific states are there such B , that are non-trivial. We show that the Jordan form of B , disregarding the values of the eigenvalues, is an invariant for SLOCC-equivalence, and analyze it in detail, providing a coarse-grained classification of the relevant entanglement classes. If each block of the Jordan form of B has a distinct eigenvalue, then there is a unique stabilized state, up to local operations. In particular, we find as entanglement class representatives a d -level generalization of the n -particle GHZ state

$$\frac{|0\rangle^n + \dots + |d-1\rangle^n}{\sqrt{d}} \quad (3)$$

and one possible generalization of the W state for $d > 2$, i.e. a state with all single particle state indices adding up to $d-1$, that is

$$\binom{n+d-2}{d-1}^{-1/2} \sum_{i_1 + \dots + i_n = d-1} |i_1\rangle |i_2\rangle \dots |i_n\rangle, \quad (4)$$

which we call an *excitation state*.

For two particles both classes coincide, as e.g.

$$|00\rangle + |11\rangle + |22\rangle \cong |02\rangle + |11\rangle + |20\rangle \quad (5)$$

Table I summarizes the entanglement classes related to Jordan blocks for the simplest non-trivial case, i.e. $n = 3$ particles (a general construction is in (41)). We adopt a special notation for the Jordan block structure. Outer brackets separate eigenspaces with different eigenvalues, while the inner brackets separate different Jordan blocks of the same eigenvalue. Each number is the dimension of a single Jordan block. Ordering of the terms does not matter, in either the inner or outer brackets. For example: $\{\{2\}\}$ is a matrix with only one Jordan block, $\{\{1, 1\}\}$ is proportional to the identity matrix and

d	Block structure	A class representative
2	$\{\{2\}\}$	$ 001\rangle + 010\rangle + 001\rangle$
	$\{\{1, 1\}\}$	(not unique) any state
	$\{\{1\}, \{1\}\}$	$ 000\rangle + 111\rangle$
3	$\{\{3\}\}$	$ 002\rangle + 020\rangle + 200\rangle$ $+ 011\rangle + 101\rangle + 110\rangle$
	$\{\{2, 1\}\}$	(not unique)
	$\{\{1, 1, 1\}\}$	(not unique) any state
	$\{\{2\}, \{1\}\}$	$ 001\rangle + 010\rangle + 100\rangle + 222\rangle$
	$\{\{1, 1\}, \{1\}\}$	(not unique)
	$\{\{\{1\}, \{1\}, \{1\}\}\}$	$ 000\rangle + 111\rangle + 222\rangle$

TABLE I: A summary of entanglement classes related to Jordan blocks, for the case of three qubits ($d = 2$) and qutrits ($d = 3$). The notation is explained in the main text (6). The general construction for the unique states is given in (41).

$\{\{1\}, \{1\}\}$ is a matrix with two different eigenvalues, that is ($\lambda_1 \neq \lambda_2$):

$$\{\{2\}\} \equiv \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \quad (6)$$

$$\{\{1, 1\}\} \equiv \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} \quad (7)$$

$$\{\{1\}, \{1\}\} \equiv \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (8)$$

The number of different Jordan block structures for a given d is given by double partitions [32].

This paper is organized as follows: Sec. II proves that it is sufficient to study invariance under symmetric transformations. Section III discusses the entanglement classes which can be obtained by studying stabilizer operators related to one-particle transformations. Section IV is devoted to conclusions and further work.

II. SYMMETRIC OPERATIONS SUFFICE

We start with an approach similar to the one from [19]. Let us consider two permutation-symmetric states, $|\psi\rangle$ and $|\varphi\rangle \in \mathcal{S}$, with \mathcal{S} denoting the symmetric subspace of the full Hilbert space. If (1) holds, then any different permutation of A_1, \dots, A_n will also work. In order to show this property explicitly, we may use $|\psi\rangle = P_\sigma|\psi\rangle$ and $|\varphi\rangle = P_{\sigma^{-1}}|\varphi\rangle$, where P_σ is a permutation matrix for the permutation of particles σ , i.e. $P_\sigma|i_1 i_2 \dots i_n\rangle = |i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(n)}\rangle$.

Since all A_i are invertible, it means in particular that

$$(A_2^{-1} \otimes A_1^{-1} \otimes \dots \otimes A_n^{-1}) (A_1 \otimes A_2 \otimes \dots \otimes A_n) |\psi\rangle = |\psi\rangle \quad (9)$$

or, equivalently,

$$(B \otimes B^{-1} \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I}) |\psi\rangle = |\psi\rangle, \quad (10)$$

where $B = A_2^{-1} A_1$.

From now on, we will use a subscript in parentheses to indicate the position of an operator in the tensor product, e.g.,

$$B_{(2)} \equiv \mathbb{I} \otimes B \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I}, \quad (11)$$

where the total number of factors is n .

First, let us show that if an operation on one particle can be reversed by applying the inverse operation on a *different* particle, then that single-particle operation must preserve permutation symmetry of the state.

Lemma 1. *For a symmetric $|\psi\rangle \in \mathcal{S}$ the equality (10)*

$$B_{(1)} B_{(2)}^{-1} |\psi\rangle = |\psi\rangle \quad (12)$$

holds if and only if

$$B_{(1)} |\psi\rangle \in \mathcal{S}. \quad (13)$$

Proof.

$$B_{(1)} |\psi\rangle \in \mathcal{S} \Leftrightarrow B_{(1)} |\psi\rangle = B_{(2)} |\psi\rangle \quad (14)$$

$$\Leftrightarrow B_{(1)} B_{(2)}^{-1} |\psi\rangle = |\psi\rangle \quad (15)$$

□

Now we will show that the action of the aforementioned single-particle operation $B_{(1)}$ can be expressed as an operation acting in the same way on every particle $S^{\otimes n}$. Intuitively, we must search for an n -th root of B . But not all such n -th roots will work, as the following example shows:

Consider $S = \sigma_x$, which is a square root of $B = I$, acting on $|00\rangle \in \mathcal{S}$. While $B_{(1)}|00\rangle \in \mathcal{S}$, $S_{(1)}|00\rangle = |10\rangle \notin \mathcal{S}$, and $S \otimes S|00\rangle = |11\rangle$, which, despite being symmetric, is not the desired state. Thus, the relevant question is: *which one is the appropriate n -th root?*

Before we can proceed, we need a few lemmas.

Lemma 2. *If $|\psi\rangle \in \mathcal{S}$, $X_{(1)}|\psi\rangle \in \mathcal{S}$ and $Y_{(2)}|\psi\rangle \in \mathcal{S}$, then $X_{(1)}Y_{(2)}|\psi\rangle \in \mathcal{S} \Leftrightarrow$ the commutator acting on the state vanishes $[X_{(1)}, Y_{(1)}]|\psi\rangle = 0$.*

Proof. If the final state is symmetric, we may permute the first two particles without altering the result:

$$0 = X_{(1)} (Y_{(2)}|\psi\rangle) - Y_{(1)} (X_{(2)}|\psi\rangle) \quad (16)$$

$$= X_{(1)} (Y_{(1)}|\psi\rangle) - Y_{(1)} (X_{(1)}|\psi\rangle) \quad (17)$$

$$= [X_{(1)}, Y_{(1)}]|\psi\rangle. \quad (18)$$

□

To see how commutativity is important, take as an example

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (19)$$

acting on $|\psi\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ (i.e. $|\psi\rangle$, $X_{(1)}|\psi\rangle$ and $Y_{(2)}|\psi\rangle$ are symmetric, but $X \otimes Y|\psi\rangle = (|00\rangle + |01\rangle + |11\rangle)/\sqrt{2}$ is not).

Lemma 3. *Moreover, for $n \geq 3$ the commutator acting on the state always vanishes, i.e. $[X_{(1)}, Y_{(1)}]|\psi\rangle = 0$.*

Proof.

$$X_{(1)}Y_{(1)}|\psi\rangle = X_{(1)}Y_{(3)}|\psi\rangle = X_{(2)}Y_{(3)}|\psi\rangle \quad (20)$$

$$= Y_{(3)}X_{(2)}|\psi\rangle = Y_{(1)}X_{(1)}|\psi\rangle \quad (21)$$

□

Lemma 4. *If $X_{(1)}|\psi\rangle$ is symmetric, then $X_{(1)}^p|\psi\rangle$ is symmetric for all natural p (integer if X is invertible).*

Proof. We use mathematical induction with respect to p , starting at $p = 0$. Since X commutes with X^p (even without the restriction to a specific state), then using Lemma 2, $X_{(1)}^p|\psi\rangle \in \mathcal{S}$ implies that $X_{(1)}^{p+1}|\psi\rangle \in \mathcal{S}$. If X is invertible, we may use the same argument for X and X^{-p} , respectively. □

Corollary 1. *Moreover, we get*

$$X_{(1)}^p|\psi\rangle = X^{p_1} \otimes X^{p_2} \otimes \cdots \otimes X^{p_n}|\psi\rangle, \quad (22)$$

for any integers p_i (can be negative if X is invertible) that add up to p .

Corollary 2. *In particular, $f(X)_{(1)}|\psi\rangle \in \mathcal{S}$ for any analytic function $f(z)$.*

Theorem 2. *For any X and $|\psi\rangle \in \mathcal{S}$ it holds that if*

$$X_{(1)}|\psi\rangle = |\phi\rangle \in \mathcal{S} \quad (23)$$

then there exists a single-particle operator S such that $S^n = X$, $S_{(1)}|\psi\rangle \in \mathcal{S}$ and

$$(S \otimes S \otimes \cdots \otimes S)|\psi\rangle = |\phi\rangle. \quad (24)$$

Proof. The proof outline is the following: we prove that, among the n -th roots of operator X , there is (at least) one, S , which can be expressed as a polynomial of X ; following Corollary 2, we get $S_{(1)}|\psi\rangle \in \mathcal{S}$ and the rest of the theorem follows.

The n -th root function is multivalued, so we can not use it to prove the theorem as it stands. Let us, then, prove that there exists a polynomial function f , such that $[f(X)]^n = X$.

Let $\{\lambda_i\}$ be the eigenvalues of X , with algebraic multiplicities $\{m_i\}$ (i.e. the size of the largest Jordan block related to such eigenvalue). Matrix function theory [33, Chapter 1] states that the action of any analytical function f on a matrix X is completely determined by the set of values $\{f(\lambda_i)\}$, along with the derivatives $\{f^{(k)}(\lambda_i)\}$, up to degree m_i . Let us choose, for each i separately, $f(\lambda_i)$ and $f^{(k)}(\lambda_i)$ from the same branch of the complex n -th root function. It is always possible to find a polynomial f that takes exactly those values and derivatives at the eigenvalues of X , e.g., via Hermite interpolation. Thus, we can define $S \equiv f(X)$, and we have $S^n = X$, as required.

Combining this result with the corollaries, we get that

$$S^{\otimes n}|\psi\rangle = S_{(1)}^n|\psi\rangle = X_{(1)}|\psi\rangle. \quad (25)$$

□

The converse of theorem 2 is false. Take, e.g., $S = \sigma_x$ and $|\psi\rangle = |00\rangle$. It is true that $S \otimes S|00\rangle = |11\rangle \in \mathcal{S}$, yet there is no B such that $B_{(1)}|00\rangle = |11\rangle$.

A. Explicit formula for symmetrization

In this section we provide the explicit form of A , given all A_i . Let $B_{ij} \equiv A_i^{-1}A_j$. Thus, operator B in the previous section would correspond to B_{12} with the new notation. Transforming (1) we get

$$A_1 \otimes A_2 \otimes \cdots \otimes A_n|\psi\rangle \quad (26)$$

$$= A_1 \otimes A_1 B_{12} \otimes \cdots \otimes A_1 B_{1n}|\psi\rangle \quad (27)$$

$$= A_1^{\otimes n} B_{12} B_{13} \cdots B_{1n}|\psi\rangle. \quad (28)$$

All $B_{1j}|\psi\rangle$ are symmetric states, similarly to $B_{(1)}|\psi\rangle$. Consequently, the last part can be reshuffled as

$$A_1^{\otimes n} (B_{11} B_{12} \cdots B_{1n})_{(1)}|\psi\rangle. \quad (29)$$

Note that no requirements are imposed about their commutativity. Using Lemma 2 we get $A = A_1 S$, where S is an appropriate n -th root of $B_{11} B_{12} \cdots B_{1n}$.

Moreover, when all A_i are unitary, then S is unitary, since roots of unitary matrices can be chosen to be unitary given that $f(UXU^\dagger) = Uf(X)U^\dagger$ for all unitary U . This finalizes the proof of Theorem 1.

III. SYMMETRY CLASSES FROM SINGLE-PARTICLE STABILIZERS

A well-known strategy in the search for entanglement classes is to study the dimension of the stabilizers of a state [21], i.e.: operators X such that $X|\psi\rangle = |\psi\rangle$. In our case it is natural to consider stabilizers in the form of $X = B_{(1)}B_{(2)}^{-1}$, and state that B stabilizes $|\psi\rangle \in \mathcal{S}$ as a convenient shorthand notation. Following Lemma 1, B stabilizes $|\psi\rangle \in \mathcal{S}$ if and only if $B_{(1)}|\psi\rangle \in \mathcal{S}$. Bear in mind that a set of B stabilizing a particular state is guaranteed to form a group only for $n \geq 3$, as follows from Lemma 3.

Let us consider the Jordan normal form J of B . We have shown that all local operations for symmetric states are equivalent to the action of the same single-particle operation on all qudits: $A^{\otimes n}$. Consequently, if a state is stabilized by B , a SLOCC transformed state is stabilized by some ABA^{-1} , i.e.: the Jordan form of the stabilizer is preserved.

Below, we prove the following facts relating the Jordan form of B to the stabilized states. First, we show that

the precise eigenvalues are not important — only their degeneracies matter (see the notation from Table I). We show, second, that stabilized states never mix eigenspaces of different eigenvalues. In particular, this means that the problem can be split into a direct sum over distinct eigenvalues. Third, we show the explicit form of a state stabilized by a single Jordan block. Fourth, we show that when eigenvalues are non-degenerate, there is a unique state related to it (up to SLOCC). Fifth, we proceed to write down states for multiple Jordan blocks with the same eigenvalue. This will complete the characterization of states stabilized by any B .

Theorem 3. *The set of states stabilized by B does not depend on the particular values of its eigenvalues, as long as (non-)degeneracy is preserved.*

Proof. We will show that mapping eigenvalues to different ones does not break the stabilizer's condition. Let us choose a complex function $f(z)$ such that (i) for all eigenvalues $f(\lambda_i) = \tilde{\lambda}_i$, and (ii) $f^{(k)}(\lambda_i) = \delta_{0k}$ for all k up to the algebraic multiplicity of each λ_i . Now, $f(B)_{(1)}$ is also a stabilizer of $|\psi\rangle$, with the same Jordan blocks, but arbitrarily set eigenvalues. \square

In particular, for $d = 2$ the only two non-trivial Jordan forms of B are related to the GHZ state (two different eigenvalues) and the W state (single eigenvalue). We proceed to show that stabilized states never mix subspaces with different eigenvalues.

Given a subspace V , let us denote by $\text{Sym}^n(V)$ the permutation-symmetric subspace of $V^{\otimes n}$. Then we have the following:

Theorem 4. *For a given Jordan form J with generalized eigenspaces V_1, \dots, V_p for distinct eigenvalues, stabilized states are of the form*

$$|\psi\rangle \in \bigoplus_{i=1}^p \text{Sym}^n(V_i). \quad (30)$$

That is, they contain no vectors mixing components from Jordan blocks of different eigenvalues.

Proof. Let $|\mu\rangle$ and $|\nu\rangle$ be one-particle states ($\mu, \nu \in \{0, \dots, d-1\}$) that belong to blocks of J with different eigenvalues. Let us take $f(B)$ mapping all subspaces to zero, except the one to which $|\mu\rangle$ belongs, which we map to 1. Suppose that $|\psi\rangle$ has a component containing a product of $|\mu\rangle$ and $|\nu\rangle$ (at different sites). Then, in particular, it has $|\mu\rangle|\nu\rangle|\xi\rangle$ and $|\nu\rangle|\mu\rangle|\xi\rangle$, for some symmetric ξ (perhaps containing $|\mu\rangle$ or $|\nu\rangle$ as well).

But

$$f(B)_{(1)}(|\mu\rangle|\nu\rangle|\xi\rangle + |\nu\rangle|\mu\rangle|\xi\rangle) \quad (31)$$

$$= |\mu\rangle|\nu\rangle|\xi\rangle. \quad (32)$$

The right hand side cannot be paired with any other terms in order to make a symmetric state. So, $f(B)_{(1)}|\psi\rangle$ is not symmetric, which contradicts the assumption.

Thus, a stabilized state can not contain a term with a product of elements from two Jordan subspaces with different eigenvalues. \square

Thus, when J has d distinct eigenvalues, the stabilized state is a generalized GHZ state:

$$|\psi\rangle = \alpha_0|0\rangle^n + \dots + \alpha_{d-1}|d-1\rangle^n. \quad (33)$$

When we consider local unitary equivalence, then the set of $\{|\alpha_i|^2\}_{i \in \{0, \dots, d-1\}}$ distinguishes classes, whereas for SLOCC, the state is equivalent to any other with the same number of non-zero α_i .

Now, it suffices to focus on a *Jordan subspace* related to a single eigenvalue. Still, for a single eigenvalue there may be more than one Jordan blocks, i.e. invariant subspaces. We start with the analysis of a *single Jordan block*, and then generalize our result to more blocks with the same eigenvalue.

Theorem 5. *Let K be a $k \times k$ Jordan block with eigenvalue zero, i.e. $\sum_{i=1}^{k-1} |i-1\rangle\langle i|$. Its stabilized states are*

$$|\psi\rangle = \sum_{j=0}^{k-1} \alpha_j |E_j\rangle, \quad (34)$$

where $|E_j\rangle$ is a symmetric state with j excitations, i.e. a symmetrized sum of all basis states for which the sum of the particle indices is j :

$$|E_j\rangle = \sum_{i_1 + \dots + i_n = j} |i_1\rangle|i_2\rangle \dots |i_n\rangle. \quad (35)$$

Proof. First, let us show that all states $K_{(1)}|E_j\rangle$ are symmetric, as long as $j < k$.

$$K_{(1)}|E_j\rangle = \sum_{i_1 + \dots + i_n = j} |i_1 - 1\rangle|i_2\rangle \dots |i_n\rangle \quad (36)$$

$$= \sum_{i'_1 + \dots + i_n = j-1} |i'_1\rangle|i_2\rangle \dots |i_n\rangle = |E_{j-1}\rangle, \quad (37)$$

where we use $|-1\rangle \equiv 0$ as a convenient shorthand notation. Now we can apply a substitution $i'_1 = i_1 - 1$ and change the summation limit (thus requiring $j < k$).

Let us now show that all stabilized states $|\psi\rangle$ have the form of (34). We proceed by induction with respect to n , the number of particles. For $n = 1$ (inductive basis), all basis states are stabilized. Now, let us assume that the condition works up to a given n . As $K_{(1)}$ reduces the total number of excitations by 1, it suffices to look at subspaces of fixed j . Together with the inductive assumption (in particular, the fact that the first n particles must remain in a permutation symmetric state after application of $K_{(1)}$) we get a general form

$$|\xi\rangle = \sum_{l=0}^j \beta_l |E_{j-l}\rangle|l\rangle. \quad (38)$$

To find the actual constraints on β_l , we just note that the assumed symmetry of $K_{(1)}|\xi\rangle$ implies that

$$K_{(1)}|\xi\rangle = \sum_{l=0}^{j-1} \beta_l |E_{j-l-1}\rangle |l\rangle \quad (39)$$

$$= K_{(n+1)}|\xi\rangle = \sum_{l=1}^j \beta_l |E_{j-l}\rangle |j-1\rangle. \quad (40)$$

Again, with a simple shift of index, and using the orthogonality of the components, we get $\beta_l = \beta_{l+1}$. Thus, there is only one state (up to a factor) for a given j that remains symmetric after $K_{(1)}$. \square

When considering SLOCC-equivalence, we may take $|E_{k-1}\rangle$ as a representative of the states stabilized by $K_{(1)}$. The reason is that all other states (with $\alpha_{k-1} \neq 0$) can be built from it via an operator $\sum_{j=0}^{k-1} \alpha_{k-1-j} K_{(1)}^j$. This operator is invertible, since its determinant is α_{k-1}^k .

Throughout the derivation, we work with unnormalized states for convenience. The properly normalized excitation state is given in Eq. (4).

Theorem 6. *There is a unique (up to SLOCC operations) state stabilized by B if and only if each Jordan block of B has a distinct eigenvalue.*

An $(n > 2)$ -particle state $|\psi\rangle \in \mathcal{S}$, stabilized by B , is unique (up to SLOCC) if and only if each block of its Jordan form has a distinct eigenvalue and no other B' exists with a greater number of eigenvalues or a lesser number of Jordan blocks.

The formulation may seem complicated, but we want to exclude *degenerate* states which are also stabilized by other operators. For the excitation state we want to ensure that the amplitude of the $(d-1)$ excitations is non-zero (otherwise it is stabilized also by a matrix with two eigenvalues), or, for the GHZ states, that all amplitudes are non-zero (otherwise, two eigenvalues can be merged into one, forming a single Jordan block). For example, a three qutrit pure state $|000\rangle + |111\rangle$ is stabilized by a matrix with its Jordan block structure $\{\{1\}, \{1\}, \{1\}\}$ (as for the GHZ state). However, unlike $|000\rangle + |111\rangle + |222\rangle$ (the GHZ state), it is also stabilized by a matrix with one fewer Jordan block, $\{\{1\}, \{2\}\}$.

Proof. “ \Leftarrow ”

We have already shown that the GHZ-like state with all amplitudes different from zero is unique, as well as the excitation state with non-zero amplitude for the highest excitation. It follows as well for any state without blocks of the same eigenvalue, as the problem can be split into a problem for each eigenvalue.

If any amplitude is zero in the GHZ-like case, the state is also stabilized by a B with a Jordan block of dimension 2.

If the amplitude for the highest excitation is zero, in the excitation state, the state is also stabilized with a B with one more eigenvalue.

“ \Rightarrow ”

If there are two blocks with the same eigenvalue, then we can take two one-particle eigenvectors $|\mu\rangle$ and $|\nu\rangle$ having the same eigenvalue. Let us look at the projection of $|\psi\rangle$ on the subspace spanned by $\text{Sym}^n(\text{lin}\{|\mu\rangle, |\nu\rangle\})$. Then, in particular, a linear combination with non-zero coefficients of elements with zero, one and two $|\nu\rangle$ states among all other $|\mu\rangle$ does not give rise to more blocks or eigenvalues, but gives rise to some states which cannot be interchanged with local operations. \square

Corollary 3. *The number of Jordan block structures with non-degenerate eigenvalues is the same as the number of integer partitions of d [34].*

A general construction of such a state is

$$\bigoplus_{i=1}^{\#\text{blocks}} |E_{k_i}\rangle, \quad (41)$$

where k_i is the dimension of the i -th Jordan block, in descending order. In particular, for GHZ states there are only blocks of size $k_i = 1$, whereas for the excitation state there is only one block, $k_1 = d$.

It is also relevant to ask about stabilized states for B whose Jordan decomposition contains two different blocks with the same eigenvalue.

Let us use a one-particle basis given by $|i^{(b)}\rangle$, where i denotes the excitation level (i.e. the largest i such that J^i acting on this vector is non-zero) and b the Jordan block to which it belongs. First, we notice that the sum of the excitations j in a given state is decreased by 1 after the action of $K_{(1)}$. Second, we notice that the excitations can be distributed among all Jordan subspaces which are *big enough* (i.e. all blocks of size strictly smaller than j). Moreover, the distribution among such Jordan subspaces needs to be permutation-invariant.

Theorem 7. *An unnormalized state of excitation j distributed among s blocks (with weights n_1, n_2, \dots adding up to n , related to distribution of excitations among Jordan blocks) reads*

$$|E_j^{n_1, n_2, \dots}\rangle = \sum_{\vec{b}: \#i=n_i} \sum_{i_1+\dots+i_n=j} |i_1^{(b_1)}\rangle \dots |i_n^{(b_n)}\rangle. \quad (42)$$

We will show by induction that only states of the form $|E_j^{n_0, n_1, \dots}\rangle$ are stabilized by such J .

For example, one excitation $j = 1$ among two particles, distributed among two modes ($n_1 = 1, n_2 = 1$) reads

$$|E_1^{1,1}\rangle = |0^{(1)}1^{(2)}\rangle + |1^{(1)}0^{(2)}\rangle \quad (43)$$

$$+ |0^{(2)}1^{(1)}\rangle + |1^{(2)}0^{(1)}\rangle. \quad (44)$$

Proof. The induction basis is for $n = 1$ and holds trivially (as it works for all states). So let us assume that (42) holds for n .

For $n + 1$ particles, a generic state with fixed j and n_1, n_2, \dots is

$$\sum_{l=0}^j \sum_{b=1} \beta_{l,b} |E_{j-l}^{n_1 - \delta_{b1}, n_1 - \delta_{b1}, \dots}\rangle |l^{(b)}\rangle. \quad (45)$$

Applying $J_{(1)}$ and $J_{(n+1)}$ on the state above, we get a relation $\beta_{l,b} = \beta_{l+1,b} = \beta_b$. Moreover, from the condition of permutation symmetry for blocks (i.e. components with the same (b)) we get that all β need to be the same, so the state is of the form (42). \square

This finalizes the classification of symmetric states for which (10) holds.

IV. CONCLUSION AND FURTHER WORK

This paper proves an open conjecture regarding the classification of pure symmetric states under local operations. We show that the study of homogeneous operations, i.e.: those where the same single-particle operator acts on each particle, suffices.

Furthermore, it introduces and analyzes entanglement classification by checking which one-particle operations preserve permutation symmetry. In that classification we obtain a sequence of states, unique up to SLOCC. On one

extreme we find the multiparticle GHZ state, whereas on the other there is a $(d - 1)$ excitation state, which is a natural generalization of the W state resulting from the classification scheme.

Moreover, some questions are left open:

- Whether invariance under all local operations (that is, not only invertible operations) on symmetric states can be represented as the same transformation for each particle.
- Whether the application of k -particle transformations on permutation-symmetric states which are reversible by acting on another part will give rise to a different entanglement classification.

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