HIGHLY SYMMETRIC FUNDAMENTAL CELLS FOR LATTICES IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

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ABSTRACT. The fundamental cell of a lattice Γ in $\mathbb{R}^d$ is the fundamental domain $\mathbb{R}^d/\Gamma$, viewed as a compact subset of $\mathbb{R}^d$. It is shown that most lattices Γ in $\mathbb{R}^2$ and $\mathbb{R}^3$ possess fundamental cells F having more symmetries than the point group $P(\Gamma)$, i.e., the group $P(\Gamma) \subset O(d)$ fixing Γ. In particular, $P(\Gamma)$ is a subgroup of the symmetry group $S(F)$ of F of index 2 in these cases. The exceptions are rhombic lattices in the plane case and cubic lattices in the three-dimensional case.

1. Introduction

The question inspiring this work was "Given a group G acting on some space X. How much symmetries can the fundamental domain X/G have, compared to G?". Symmetry of G can be understood as symmetry of the Cayley graph of G, or symmetry of some canonical embedding of G in X. Symmetry in X can be formulated if X is a metric group, for instance. We will specify a particular instance of this question here and provide some answers.

A *lattice* in $\mathbb{R}^d$ is the $\mathbb{Z}$-span of d linearly independent vectors in $\mathbb{R}^d$. The point group $P(\Gamma)$ of a lattice Γ in $\mathbb{R}^d$ is the set of Euclidean motions fixing both Γ and the origin. In other words, $P(\Gamma) \subset O(d)$ is the set of orthogonal maps fixing Γ. It is clear that each lattice Γ has a fundamental cell having $P(\Gamma)$ as its symmetry group, see Lemma 1.4. (For more detailed definitions see below.) For instance, consider the square lattice $\mathbb{Z}^2$ in the plane $\mathbb{R}^2$. Its point group $P(\mathbb{Z}^2)$ is the dihedral group D_4 of order eight, containing rotations by 0, $\pi/2$, π, $3\pi/2$, together with four reflections. One possible fundamental cell of $\mathbb{Z}^2$ is a unit square, centred in 0. Clearly $P(\mathbb{Z}^2)$ is the symmetry group of this unit square as well.

In this paper we show that most lattices in $\mathbb{R}^2$ and $\mathbb{R}^3$ possess fundamental cells with more symmetry than the point group of the lattice. In general, these fundamental cells will be neither simply connected, nor will their interior be connected. Some of these cells are of fractal appearance. The two main results are the following.

Theorem 1.1. *Let $\Gamma \subset \mathbb{R}^2$ be a lattice with point group $P(\Gamma)$, such that Γ is not a rhombic lattice. Then there is a compact fundamental cell F of Γ with symmetry group $S(F)$ such that $P(\Gamma)$ is a subgroup of $S(F)$ of index $[S(F) : P(\Gamma)] = 2$.*

Theorem 1.2. *Let $\Gamma \subset \mathbb{R}^3$ be a lattice with point group $P(\Gamma)$, such that Γ is not a cubic lattice. Then there is a compact fundamental cell F of Γ with symmetry group $S(F)$ such that $P(\Gamma)$ is a subgroup of $S(F)$ of index $[S(F) : P(\Gamma)] = 2$.*

In the remainder of this section the necessary definitions and notations are introduced. Section 2 is dedicated to the proof of Theorem 1.1, Section 3 contains the proof of Theorem 1.2. Section 4 contains some remarks and further questions.

Notation: C_n denotes the cyclic group of order n. D_n denotes the dihedral group of order $2n$. $O(d)$ is the orthogonal group over $\mathbb{R}^d$. $O(d)$ can be identified with the group of Euclidean motions (i.e. isometries of $\mathbb{R}^d$, including reflections) fixing the origin. The closure of a set $A \subset \mathbb{R}^d$ is denoted by $\mathrm{cl}(A)$. For any set $X \subset \mathbb{R}^d$, let $S(X)$ denote the symmetry group of X; that is, the set of all Euclidean motions (including reflections and translations) $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ with $\varphi(X) = X$. The *cone* centred in x spanned by m vectors $v_1, \ldots v_m \in \mathbb{R}^d$ is defined by

$$\mathrm{cone}(x; v_1, \ldots, v_m) = \{x + \sum_{i=1}^{m} \lambda_i v_i : \lambda_i \geq 0\}.$$

A sum $A + B$, where $A, B \subset \mathbb{R}^d$, always means the *Minkowski sum*

$$A + B = \{a + b \,|\, a \in A, b \in B\}.$$

The line segment with endpoints $x, y \in \mathbb{R}^d$ is denoted by $[x, y]$. A *lattice* in $\mathbb{R}^d$ is a discrete cocompact subgroup of $\mathbb{R}^d$. Any lattice in $\mathbb{R}^d$ can be written as $\langle b_1, \ldots, b_d \rangle_{\mathbb{Z}}$, where $b_1, \ldots b_d$ span $\mathbb{R}^d$. Such a set $b_1, \ldots, b_d$ is called a *basis* of the lattice. A basis of a given lattice is not unique.

The fundamental domain of a lattice Γ is $\mathbb{R}^d/\Gamma$. By a *fundamental cell* we denote the closure of some geometric embedding of a fundamental domain into $\mathbb{R}^d$. For instance, a fundamental domain of $\mathbb{Z}^d$ is the d-torus, and a fundamental cell of the lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$ is the d-dimensional unit cube $[0, 1]^d$. A particular fundamental cell of a lattice $\Gamma = \langle b_1, \ldots, b_d \rangle_{\mathbb{Z}}$ is the *fundamental parallelepiped* $[0, b_1] + \ldots + [0, b_d]$. Note that for any fundamental cell F of a lattice Γ, $\{F + g \,|\, g \in \Gamma\}$ is a tiling of $\mathbb{R}^d$. A *tiling* of $\mathbb{R}^d$ is a packing of $\mathbb{R}^d$ which is also a covering of $\mathbb{R}^d$. In other words, a tiling is a covering of $\mathbb{R}^d$ by pairwise non-overlapping compact sets T_i. Two compact sets are *non-overlapping* if their interiors are disjoint.

Trivially, the symmetry group $S(\Gamma)$ of any lattice contains a subgroup isomorphic to Γ, namely, the group of all translations by elements of Γ. The subgroup $P(\Gamma) = S(\Gamma)/\Gamma$ is called *point group* of Γ. For lattices in $\mathbb{R}^d$, one has:

$$S(\Gamma) = P(\Gamma) \ltimes \Gamma.$$

The following fact is usually called the crystallographic restriction (see for instance [6], Section 4.5).

Proposition 1.3. *Rotations fixing a lattice in $\mathbb{R}^2$ or $\mathbb{R}^3$ are either 2-fold, 3-fold, 4-fold or 6-fold.*

The *Voronoi cell* of a lattice point x in $\mathbb{R}^d$ is the set of points in $\mathbb{R}^d$ whose distance to x is not greater than their distance to any other lattice point. It is easy to see that for any lattice $\Gamma \subset \mathbb{R}^d$ the closed Voronoi cell $V = V(0)$ of 0 is a fundamental cell of Γ with $S(V) = P(\Gamma)$. We formulate this as a lemma.

Lemma 1.4. *If Γ is a lattice in $\mathbb{R}^d$ then Γ has a fundamental cell F such that $S(F) = P(\Gamma)$.*

We will use orbifold notation to denote planar symmetry groups in the sequel, compare [3]. For instance, $*442$ denotes the symmetry group $S(\mathbb{Z}^2)$ of the square lattice $\mathbb{Z}^2$, and $*432$ denotes the symmetry group of the cube. For a translation of orbifold notation into your favourite notation, see [3] or [18]. In principle we can denote cyclic groups C_n and dihedral groups D_n in orbifold notation, too. Since the sign for C_n—regarded as the symmetry group of some object in the plane—is just n in orbifold notation, we will rather use the former abbreviation for the sake of clarity.

2. **Dimension 2**

From the theory of Coxeter groups [4, 5, 11] we know the list of all finite groups of Euclidean motions. Thus we know that each finite group of Euclidean motions in the plane is either C_n or D_n. By the crystallographic restriction (Proposition 1.3) there are just 10 candidates for such groups being point groups of a planar lattice, namely

$$C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6.$$

Note that C_2 and D_1 are equal as abstract groups, since there is only one group of order two up to isomorphisms. But since we are dealing with groups of Euclidean motions, we will use the convention that a cyclic group C_n contains rotations only, and a dihedral group D_n contains n rotations (including the identity) and n reflections. The fact that each planar lattice is fixed under a rotation through π about the origin implies that C_1, C_3, D_1 and D_3 cannot be point groups of any planar lattice. Some further thought yields the following result.

Proposition 2.1. *If Γ is a lattice in $\mathbb{R}^2$, then $P(\Gamma) \in \{C_2, D_2, D_4, D_6\}$, and $S(\Gamma) \in \{*632$, $*442$, $*2222$, $2*22$, $2222\}$.*

This result is well known. Nevertheless, since we are not aware of a decent reference, we will sketch the proof here.

Proof. We consider the distinct possibilities of properties of basis vectors of Γ. First, if Γ has a basis of two orthogonal vectors of equal length, this yields (up to similarity) the square lattice $\mathbb{Z}^2$, with point group D_4 and symmetry group $*442$. Second, if Γ has a basis of two vectors of equal length with angle $\pi/3$, this yields (up to similarity) the hexagonal lattice $A_2 = \langle (1,0)^T, (\frac{1}{2}, \frac{\sqrt{3}}{2})^T \rangle_{\mathbb{Z}}$, with point group D_6 and symmetry group $*632$. Third, if Γ has a basis of two vectors of equal length, but neither with angle $\pi/3$ nor $\pi/2$ nor $2\pi/3$, then Γ is called *rhombic lattice* and has point group D_2 and symmetry group $2*22$. A planar lattice which has orthogonal basis vectors of different length (but not of equal length) is called *rectangular lattice*. It has also point group D_2, but its symmetry group is $*2222$. In particular, the entire symmetry group of a rhombic lattice is not isomorphic to the entire symmetry group of a rectangular lattice, even though their point groups agree. (Compare [13], p 210.) All other lattices are called *oblique lattices* and have point group C_2, and symmetry group 2222. $\square$

Regarding the five cases above in connection with Lemma 1.4 one obtains that one possible fundamental cell of the hexagonal (square, rectangular, rhombic, oblique) lattice is a regular hexagon (square, rectangle, hexagon with D_2 symmetry, hexagon with C_2 symmetry). We will proof Theorem 1.1 by considering four out of these five cases. The first case—the square lattice—is due to V. Elser [7]. To the knowledge of the author his proof has not been published anywhere, so we give a detailed proof here.

Proposition 2.2 (Elser)**.** *The square lattice $\mathbb{Z}^2$ has a fundamental cell $F_\square$ with $S(F_\square) = D_8$.*

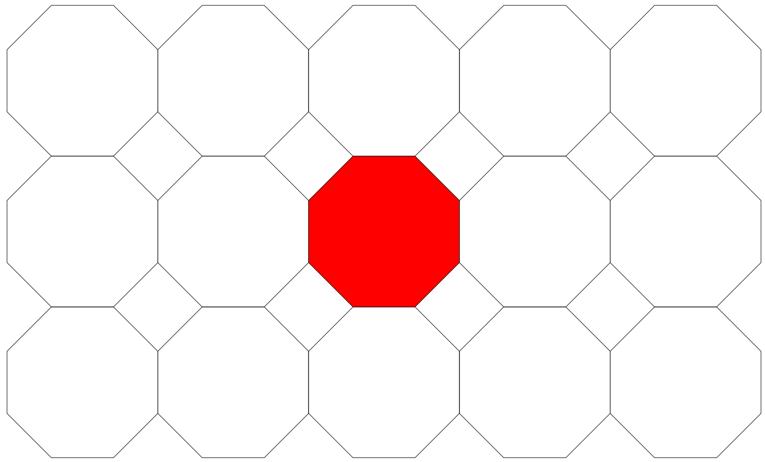

FIGURE 1. The first step of the construction of a fundamental cell for the square lattice with eight-fold symmetry. The shaded octagon is F_0.

Proof. The point group of the square lattice $\mathbb{Z}^2$ is D_4. The claim is proved by constructing a fundamental cell $F_\square$ of $\mathbb{Z}^2$ with symmetry group D_8.

Consider a packing of $\mathbb{R}^2$ by regular octagons, such that each $x \in \mathbb{Z}^2$ is the centre of some octagon. Let all octagons be of edge length $\ell := \sqrt{2} - 1$ and non-overlapping, see Figure 1. The packing looks like the Archimedean tiling 4.8^2 by octagons and squares, where the squares are the holes of the packing. The intersection of two distinct octagons is either empty, or a full edge. Denote this octagon packing by P_0.

Let the octagon centred in 0 be distinguished from the others, for instance, by colouring this octagon red and all other octagons white. Let F_0 denote the red octagon. Now consider a further packing by smaller octagons of edge length $\ell(\sqrt{2}-1)$, such that each vertex of a large octagon is the centre of some small octagon (see Figure 2). The packing of $\mathbb{R}^2$ by these small octagons is denoted by P_1.

Proceed with each small octagon O in P_1 according to the following rules: If the centre of O is the vertex of two white octagons in P_0, then O is coloured white. If the centre of O is the vertex of one white and one red octagon in P_0, then we divide O as follows: Let

$$\boldsymbol{e}_0 := \binom{1}{0}, \boldsymbol{e}_1 := 2^{-1/2}\binom{1}{1}, \boldsymbol{e}_2 := \binom{0}{1}, \boldsymbol{e}_3 := 2^{-1/2}\binom{-1}{1}, \boldsymbol{e}_{i+4} := -\boldsymbol{e}_i \quad (0 \le i \le 4).$$

For convenience, let $\boldsymbol{e}_8 := \boldsymbol{e}_0$. Denote the centre of O by x. O is divided into eight pieces $O_1, \ldots, O_8$ by the cones $\mathrm{cone}(\boldsymbol{x}; \boldsymbol{e}_i, \boldsymbol{e}_{i+1})$:

$$O_i = O \cap \mathrm{cone}(x; \boldsymbol{e}_i, \boldsymbol{e}_{i+1}) \quad (0 \le i \le 7).$$

Now, since O is centred on the vertex of two large octagons, exactly two of the eight points $x + \ell(\boldsymbol{e}_i + \boldsymbol{e}_{i+1})$ are centres of large octagons in P_0. (see Figure 2). Note, that if $x + \ell(\boldsymbol{e}_i + \boldsymbol{e}_{i+1}), x + \ell(\boldsymbol{e}_j + \boldsymbol{e}_{j+1})$ are those points, then $|i - j| = 3$. Thus the following rule is well-defined. If $x + \ell(\boldsymbol{e}_0 + \boldsymbol{e}_1)$ (or $x + \ell(\boldsymbol{e}_2 + \boldsymbol{e}_3)$, or $x + \ell(\boldsymbol{e}_4 + \boldsymbol{e}_5)$, or $x + \ell(\boldsymbol{e}_6 + \boldsymbol{e}_7)$) is the centre of a red octagon, then O_0, O_2, O_4, O_6 are coloured red, O_1, O_3, O_5, O_7 are coloured white. Analogously, if $x + \ell(\boldsymbol{e}_0 + \boldsymbol{e}_1)$ (or $x + \ell(\boldsymbol{e}_2 + \boldsymbol{e}_3)$, or $x + \ell(\boldsymbol{e}_4 + \boldsymbol{e}_5)$, or $x + \ell(\boldsymbol{e}_6 + \boldsymbol{e}_7)$) is the centre of a white octagon, then O_0, O_2, O_4, O_6 are coloured white, O_1, O_3, O_5, O_7 are coloured red. Denote the union of all red pieces of small octagons by R_1. Now let $F_1 := \mathrm{cl}((F_0 \setminus P_1) \cup R_1)$.

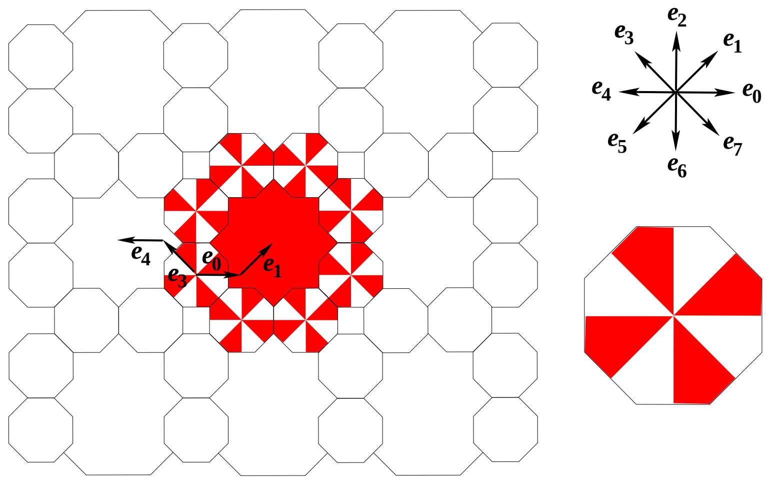

FIGURE 2. The second step of the construction of an eight-fold symmetric fundamental cell for the square lattice. The shaded part is F_1.

Note that the colouring scheme in the last step can in principle produce octagons of three different kinds: purely red octagons (all its vertices are red), purely white octagons (all its vertices are white) and octagons of mixed colour (then its vertices are alternating red-white-red-white...)

Let us proceed in an analogous manner. Let $O^{(n)}$ be the octagon of edge length $(\sqrt{2}-1)^n$. Place translates of $O^{(n)}$ on all vertices of the octagons of the previous packing P_{n-1}. Denote this octagon packing by P_n. (Figure 3 indicates the third step of this construction.) Note that the centre of each of these octagons $O^{(n)}+x$ are vertices of either one or two larger octagons in P_{n-1}. If the centre of $O^{(n)}+x$ is vertex of only one larger octagon, $O^{(n)}+x$ inherits its colour from this vertex. Similarly, if the centre of $O^{(n)}+x$ is vertex of two larger octagons, and these two vertices possess the same colour, $O^{(n)}+x$ inherits its colour from these two vertices. If the centre of $O^{(n)}+x$ is both a white vertex (of one larger octagon in P_{n-1}) and at the same a red vertex (of another larger octagon in P_{n-1}), we cut $O^{(n)}+x$ into eight pieces $O_0^{(n)}, \ldots, O_7^{(n)}$ precisely as above. Then exactly two of the eight points $x+(\sqrt{2}-1)^n(\boldsymbol{e}_i+\boldsymbol{e}_{i+1})$ are centres of some octagons in P_{n-1}. As above, if $x+(\sqrt{2}-1)^n(\boldsymbol{e}_0+\boldsymbol{e}_1)$, or $x+(\sqrt{2}-1)^n(\boldsymbol{e}_2+\boldsymbol{e}_3)$, or $x+(\sqrt{2}-1)^n(\boldsymbol{e}_4+\boldsymbol{e}_5)$, or $x+(\sqrt{2}-1)^n(\boldsymbol{e}_6+\boldsymbol{e}_7))$ is the centre of a red octagon (white octagon), then $O_0^{(n)}, O_2^{(n)}, O_4^{(n)}, O_6^{(n)}$ are coloured red (white), $O_1^{(n)}, O_3^{(n)}, O_5^{(n)}, O_7^{(n)}$ become white (red). Denote the union of all red pieces of small octagons by R_n. Now let $F_n := \mathrm{cl}((F_{n-1} \setminus P_n) \cup R_n)$.

By construction, each F_n has D_8-symmetry. Let $F_\square$ be the closure of the limit of the sequence F_n for $n \to \infty$. Then $F_\square$ has D_8-symmetry as well. It remains to show that $F_\square$ is a well-defined compact set, and that $F_\square$ is a fundamental cell for the square lattice. The latter is equivalent to saying that $\{F+x \,|\, x \in \mathbb{Z}^2\}$ is a tiling of $\mathbb{R}^2$.

Let us postpone the question of the compactness of $F_\square$. Clearly, in each step, $\mathbb{Z}^2+F_n$ is a packing, where the holes occupy $c \cdot (\sqrt{2}-1)^n$ of the plane, where $c=(\sqrt{2}-1)^2=3-2\sqrt{2}=0.1715728\ldots$ is the portion of the plane occupied by the holes in the packing P_0. Thus the portion of the plane occupied by holes tends to zero in the packing P_n with growing n.

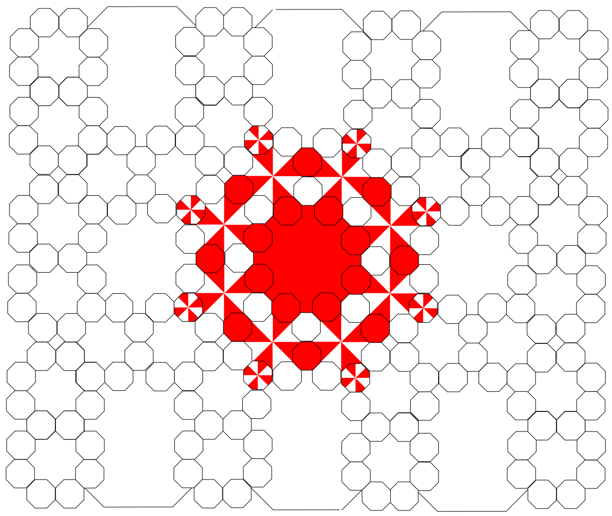

FIGURE 3. The third step of the construction of an eight-fold symmetric fundamental cell for the square lattice

Moreover, in each step, F_n and $x + F_n$ are non-overlapping ($x \in \mathbb{Z}^2$). This can be seen as follows: By construction, the red parts and the white parts are distinct, thus their closures are non-overlapping. We can repeat the construction with any other octagon coloured red, instead of the octagon at 0. Again, this new fundamental cell ($F + \boldsymbol{e}_1$, say) does not overlap with any white parts. In particular, it does not overlap with F. Consequently, F does not overlap with any of its neighbours $x + F$, ($x \in \Gamma$, $x \neq 0$). Thus the limit of $P_n = \mathbb{Z}^2 + F_n$ is a tiling of the plane, assumed $F_\square$ is a compact set.

We are left with showing that $F_\square$ is compact. Since we defined $F_\square$ as the closure of $\lim_{n\to\infty} F_n$, it suffices to show that $F_\square$ is bounded. This follows from the construction: The diameter of

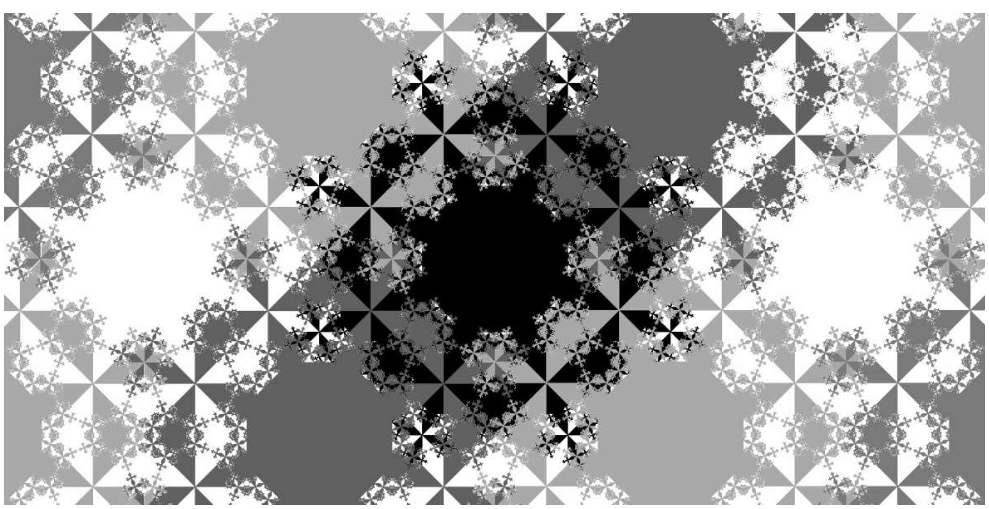

FIGURE 4. The fundamental cell $F_\square$ (black) of $\mathbb{Z}^2$ and some of its copies, illustrating how they form a tiling.

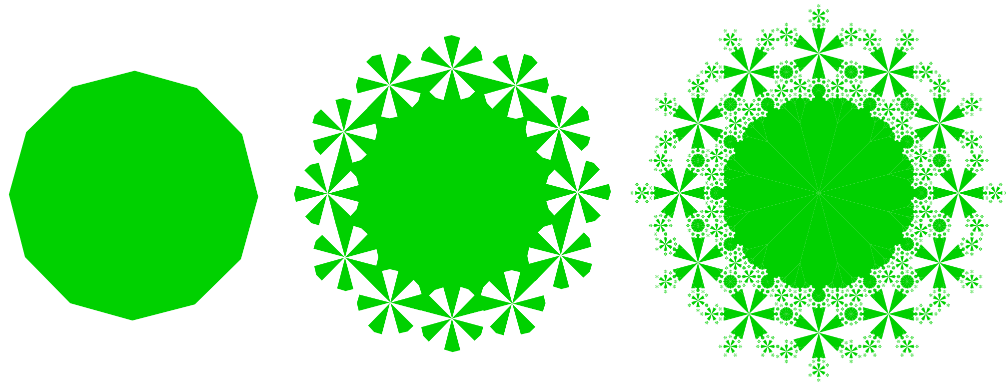

FIGURE 5. The first two iterates of the construction of the 12-fold fundamental cell $F_\triangle$ of the hexagonal lattice A_2 (left and middle), and a higher iterate (right).

F_0 (a regular octagon with edge length one) is $h := 2\sqrt{1-\frac{\sqrt{2}}{2}}$. In each step, the diameter of F_n grows by $h(\sqrt{2}-1)^n$. Thus the diameter of $F_\Box$ is

$$\operatorname{diam}(F) = \sum_{n\geq 0} h(\sqrt{2}-1)^n = \frac{h}{1-(\sqrt{2}-1)} = \sqrt{2+\sqrt{2}} = 1.847759\ldots \tag{1}$$

In particular, $F_\Box$ is bounded. $\Box$

Proposition 2.3. *The hexagonal lattice A_2 has a compact fundamental cell $F_\triangle$ with $S(F_\triangle) = D_{12}$.*

Proof (sketch). The proof is very similar to the proof of Proposition 2.2. One starts with a packing Q_0 by regular dodecagons, each one centred on a point of the hexagonal lattice A_2, leaving triangular holes (corresponding to the Archimedean tiling 3.12^2 by triangles and dodecagons). Then one places smaller dodecagons on the vertices of the larger ones, either white, red or mixed, compare Figure 5. Iteration yields a fundamental cell $F_\triangle$ of A_2 with $S(F_\triangle) = D_{12}$, see Figure 5, right. Proving that $F_\triangle$ is a fundamental cell is completely analogous to the proof of Proposition 2.2. The diameter of $F_\triangle$ is

$$\operatorname{diam}(F_\triangle) = \frac{2}{\sqrt{3}} \sum_{n\geq 0}(2-\sqrt{3})^n = \frac{1}{3}(3+\sqrt{3}) = 1.57735\ldots \tag{2}$$

$\Box$

Interestingly, the set $F_\triangle$ appears in an entirely different context in [1].

Proof (of Theorem 1.1). We consider the four cases when Γ is a square lattice, a hexagonal lattice, a rectangular lattice and an oblique lattice.

Case 1: $\Gamma = \mathbb{Z}^2$: This is Proposition 2.2. We have $S(F_\Box) = D_8$ and $P(\Gamma) = D_4$.

Case 2: $\Gamma = A_2$: This is Proposition 2.3. We have $S(F_\triangle) = D_{12}$ and $P(\Gamma) = D_6$.

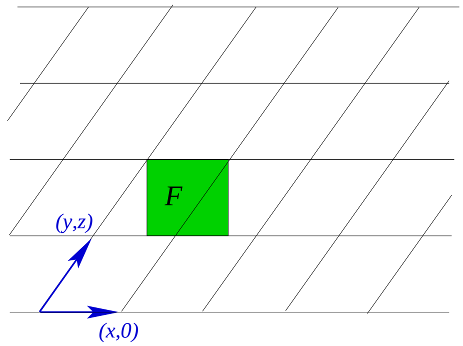

FIGURE 6. A fundamental cell for an oblique lattice, with D_2-symmetry.

Case 3: Let Γ be an oblique lattice. Without loss of generality one basis of Γ is $b_1 = \binom{x}{0}$, $b_2 = \binom{y}{z}$, $z \neq 0$. (A canonical fundamental cell is the parallelogram $F' = [0, b_1] + [0, b_2]$.) Then, let F be the rectangle with vertices $\binom{0}{0}$, $\binom{x}{0}$, $\binom{x}{z}$, $\binom{0}{z}$, see Figure 6.

It is easy to see that $\Gamma + F = \mathbb{R}^2$, and the copies of F do not overlap. Thus F is a fundamental cell of Γ. We have $S(F) = D_2$ and $P(\Gamma) = C_2$.

Case 4: Let Γ be a rectangular lattice, with basis $\binom{a}{0}$, $\binom{0}{b}$. In a similar way as in the proof of Proposition 2.2, we will construct a fundamental domain with D_4-symmetry, whereas $P(\Gamma) = D_2$.

Without loss of generality, let $a > b$. First, consider a square packing P_0 with squares of edge length b, where each $x \in \Gamma$ is the centre of some square in P_0 (see Figure 7). For later purposes, let $r_0 = a$, $r_1 = b$, and $v_1 = \lfloor \frac{a}{b} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less or equal to x. Distinguish the square with centre 0 from the others, say, by colouring it black and the other squares grey. Let F_0 denote the black square.

In the following it is convenient to distinguish between a 'step' of the procedure and an 'iteration' of the procedure. A step consists of one or more iterations. An iteration means placing squares of some edge length ℓ on all vertices of the squares in the current packing $P_{m,j}$, yielding a packing $P_{m,j+1}$. The next iteration in the same step places squares on all vertices of squares in $P_{m,j+1}$, and so on. These new squares have the same edge length ℓ as in the last iteration (see Figure 8). If such an iteration is no longer possible without producing overlaps, one goes over to the next step. In the next step we place squares with an an edge length strictly smaller than ℓ (see Figure 7). Several iterations—using squares of the same edge length—are considered as one single step of the procedure.

In the first step, the first iteration is placing squares of edge length $r_1 = b$ on each vertex of the squares in P_0. Denote the collection of these new squares by $P_{1,1}$. There are two possibilities: A small square $S \in P_{1,1}$ is centred on two coincident vertices of grey squares of P_0. Then S is coloured grey. Or the small square S is centred on the vertex of a grey square W which coincides with a vertex of a black square G. Then S gets two colours: One quarter of S, namely, $S \cap W$, is coloured grey. The opposite quarter of S is also coloured grey. The two remaining quarters, $S \cap G$ and its opposite, are coloured black (see Figure 7). Denote the union of the black pieces (black squares or black parts of squares) of $P_{1,1}$ by $B_{1,1}$. Let $F_{1,1} := \mathrm{cl}((F_0 \setminus P_{1,1}) \cup B)$. If $v_1 > 1$, do further iterations (placing squares of edge length

$r_1 = b$ on the vertices in $P_{1,1}$ and so on), altogether v_1 times. In each iteration we get a packing $P_{1,m}$ and a set $F_{1,m} := \mathrm{cl}((F_{1,m-1} \setminus P_{1,m}) \cup B_{1,m})$, where $B_{1,m}$ always denotes the union of all black pieces (black squares or black parts of squares) in $P_{1,m}$. After v_1 iterations we cannot proceed by placing squares of edge length b on vertices in P_{1,v_1} without producing overlaps. Thus we proceed with the second step.

In the $n+1$-th step ($n \geq 1$), let $v_n = \lfloor \frac{r_{n-1}}{r_n} \rfloor$. Furthermore, let $r_{n+1} = r_{n-1} - v_n r_n$. Place squares of edge length r_{n+1} on the vertices of all squares in $P_{n,1}$. Do v_n iterations, to obtain the packing P_{n,v_n}, following the same rules as above (with 1 replaced by n appropriately).

Note that after step $n-1$ we are left with a packing $\Gamma + F_{n,v_n}$, which has rectangular holes of size $r_n \times r_{n+1}$. Note also that the sequence $(r_j)_j$ is—by definition of the r_j—the output of the Euclidean algorithm applied to a, b.

If $\frac{a}{b} \in \mathbb{Q}$ this yields a fundamental cell $F := F_{n,v_n}$ after finitely many steps. By construction, F is a fundamental cell of Γ, and it has the desired symmetry group, namely, D_4. Clearly it is compact, since it is the finite union of compact sets.

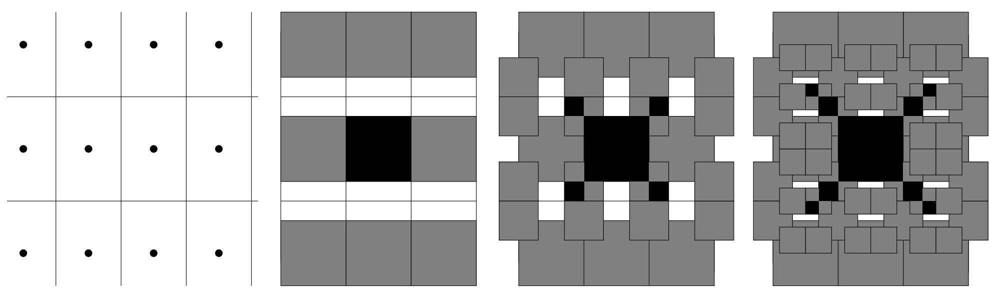

FIGURE 7. A rectangular lattice and its Voronoi cells (left), the first three steps of the construction of a fundamental cell with D_4-symmetry (right).

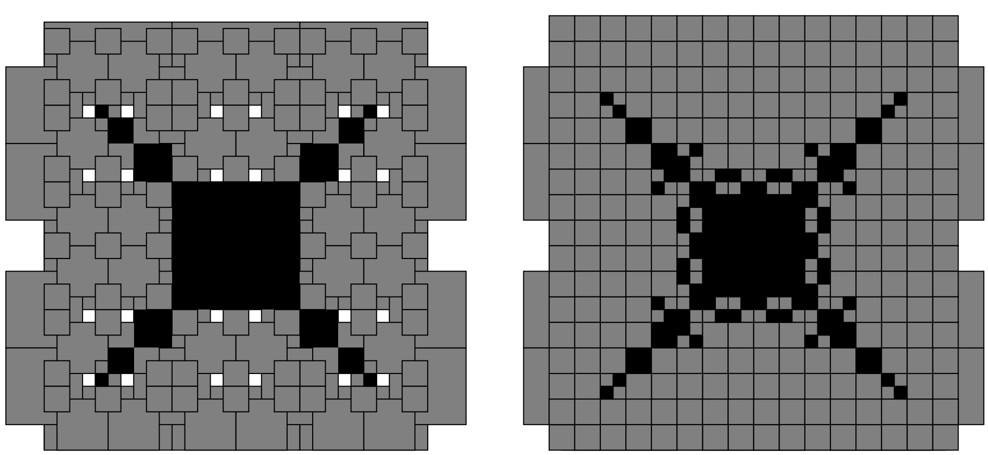

FIGURE 8. The last two iterations of the construction of the D_4-symmetric fundamental cell of a lattice with basis vectors of length 5 and 8. These two iterations correspond to one 'step' in the proof of Theorem 1.1.

If $\frac{a}{b} \notin \mathbb{Q}$ then the construction needs infinitely many steps, and we get a fundamental cell F in the limit. As above, F is closed and D_4-symmetric by construction. F is compact, since F is bounded. F is bounded since the sequence $(r_j)_j$ is the output of the Euclidean algorithm. The length of F in direction $\binom{a}{0}$ is

$$a + b + r_2 + \cdots = \sum_{i=0}^{\infty} r_i.$$

It is known that the worst case—i.e., the slowest convergence of the partial fractions in the continued fraction expansion —is attained if $a/b = \frac{1}{2}(\sqrt{5}+1) =: \tau$. Here "slow convergence" holds in a pretty strong sense, i.e., $\frac{1}{a}\sum\limits_{i=0}^{\infty} r_i$ attains its maximal value for $a/b = \frac{1}{2}(\sqrt{5}+1)$, see Hurwitz' theorem [10, Theorem 193].

Hence the length of F in direction $\binom{a}{0}$ is maximal for $a/b = \tau$. Then we obtain $r_i = a\tau^{-i}$, and the length of F in direction $\binom{a}{0}$ is

$$a\sum_{i=0}^{\infty} r_i = a\sum_{i=0}^{\infty} \tau^{-i} = a\frac{1}{1-\tau^{-1}} = a\tau^2.$$

Since F has the symmetry of a square, the diameter of F is at most $\sqrt{2}a\tau^2$. $\square$

It is not obvious how to devise some similar general construction of a fundamental cell for any rhombic lattice.

3. **Dimension 3**

Similar to the proof of Theorem 1.1, the proof of Theorem 1.2 consists of considering all possible cases. Fortunately, we can utilise Theorem 1.1 to cover most cases: we can use thickened, three-dimensional versions of the plane fundamental cells. In $\mathbb{R}^3$ there are 32 finite groups of Euclidean motions obeying the crystallographic restriction in Proposition 1.3, see [6], Section 15.6. Only seven of them occur as point groups of lattices. Table 1 summarises the situation: The second column contains the name of the lattice, more precisely: the name of the family of lattices with a common symmetry group (the names as being used in crystallography). The third column contains the point group of the lattice in orbifold notation, the fourth column contains the order of the point group. The last column indicates the two-dimensional fundamental cell of Theorem 1.1 which yields a three-dimensional fundamental cell F for the current three-dimensional lattice, and the order $|S(F)|$ in parentheses.

Since the list of finite groups of Euclidean motions in $\mathbb{R}^3$ is known, we know that there is no such group containing the group $*432$ as a subgroup of index 2. (The only candidates—the ones of order 96—are the (non-primitive) groups C_{96}, D_{48} and $C_2 \times D_{24}$, regarded as symmetry groups of solids in $\mathbb{R}^3$.) The corresponding lattices are the so-called *cubic lattices*: the *primitive cubic lattice* $\mathbb{Z}^3$, the *body centred cubic lattice* $\mathbb{Z}^3 \cup \big(\mathbb{Z}^3 + (\frac{1}{2},\frac{1}{2},\frac{1}{2})^T\big)$ (bcc) and the *face centred cubic lattice* (fcc). So we cannot expect to find fundamental cells for these three cubic lattices with more symmetry than their point group $*432$.

Proof (of Theorem 1.2). We consider 6 cases (numbers 4-14 in Table 1 and Figure 9, identified if they have equal point groups). This will yield the entries of the last column of Table 1,

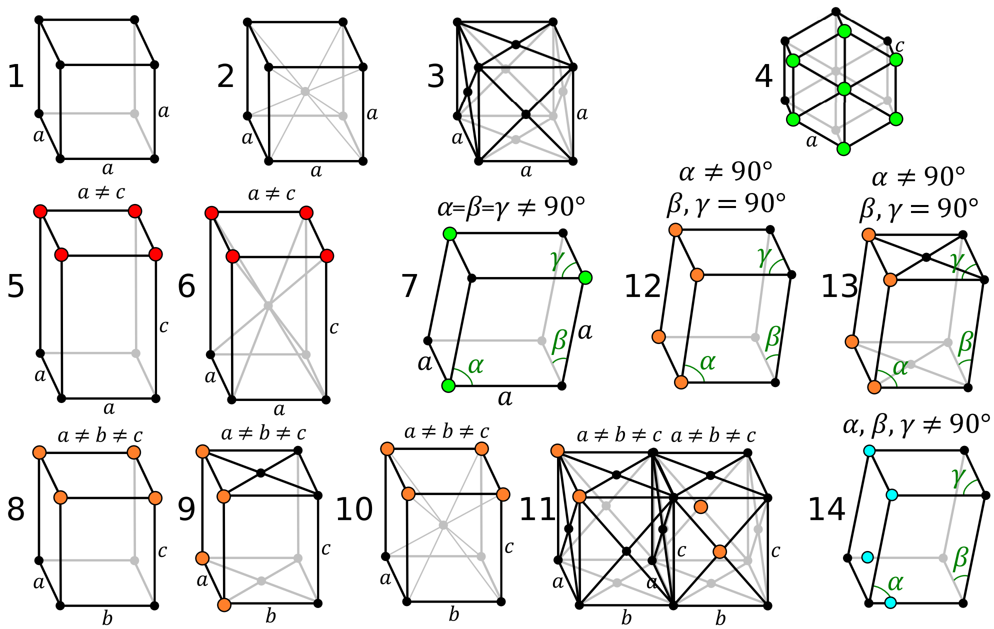

FIGURE 9. Illustrations of the 14 types of lattices in $\mathbb{R}^3$. The shaded nodes indicate how the lattices consist of layers of two-dimensional lattices. Angles omitted in the figure are assumed to be $\pi/2$ (or $\pi/3$ in 4). Edges labelled with equal letters are of equal length. The image is taken from [17] and only slightly modified.

which shows the name of the two-dimensional fundamental cell used, and the order of the symmetry group of the corresponding three-dimensional fundamental cell (in parentheses).

Nr	Name	Point group	Order	2-dim fundamental cell (number of symmetries $\lvert S(F)\rvert$)
1	$\mathbb{Z}^3$	$*432$	48	—
2	body centred cubic	$*432$	48	—
3	face centred cubic	$*432$	48	—
4	Hexagonal	$*622$	24	12fold (48)
5	Tetragonal primitive	$*422$	16	8fold (32)
6	Tetragonal body-centred	$*422$	16	8fold (32)
7	Rhombohedral	$2*3$	12	6fold (24) / 12fold(48)
8	Orthorhombic primitive	$*222$	8	4fold (16)
9	Orthorhombic base-centred	$*222$	8	4fold (16)
10	Orthorhombic body-centred	$*222$	8	4fold (16)
11	Orthorhombic face-centred	$*222$	8	4fold (16)
12	Monoclinic primitive	$2*$	4	2fold (8)/4fold(16)
13	Monoclinic base-centred	$2*$	4	2fold (8)/4fold(16)
14	Triclinic primitive	2	2	mon.(4) / 2fold (8) / 4fold (16)

TABLE 1. The 14 types of lattices with respect to their point groups.

Case 1: Hexagonal (4). The lattice consists of (equidistant) layers of hexagonal lattices. Attaching a thickened version of the fundamental cell $F_\triangle$—say, $F := F_\triangle \times [0, \ell]$, where ℓ is the distance of two adjacent layers—to each lattice point yields a tiling of $\mathbb{R}^3$. The symmetry group $S(F)$ of F is $*12\,2$, the new symmetries coming from rotating F along an axis which is parallel to the layers of hexagonal lattices about π ("turning F upside down").

Case 2: Tetragonal (5-6). These lattices consist of equidistant layers of square lattices. Thus we can use the thickened version of the fundamental cell $F_\Box$ of the square lattice, its symmetry being $*82$, having order 32.

Case 3: Rhombohedral (7). This lattice consists of equidistant layers of the hexagonal lattice A_2. So we can either use a thickened fundamental cell of A_2 with D_6-symmetry, yielding a three-dimensional fundamental cell F with $S(F) = *62$, $|S(F)| = 24$ and index $[S(F) : P(\Gamma)] = 2$. Or we can use a thickened version of $F_\triangle$ as in case 1, yielding a fundamental cell with $S(F) = *12\,2$, $|S(F) = 48$ and index $[S(F) : P(\Gamma)] = 4$.

Case 4: Orthorhombic (8-11). These lattices consist of equidistant layers of rectangular lattices. Thus we can use the thickened version of the fundamental cell of the rectangular lattice, its symmetry group being $*42$ of order 16.

The rectangular lattices are indicated in Figure 9 by shaded points. In number 11 the rectangular lattice is rather hard to spot. One may ask whether lattice number 11 does indeed consist of *equidistant* layers of rectangular lattices, but this is clear from the lattice property.

Case 5: Monoclinic (12-13). These two lattices also consist of equidistant layers of rectangular lattices. Thus we can reason as in the preceding case.

Alternatively, we can use rectangular cuboids as fundamental cells, having symmetry group $*222$ of order 8.

Case 6: Triclinic (14). This lattice—or rather: these lattices—consist of layers of oblique lattices. We may use a right prism over a parallelogram as a fundamental cell, or a skew prism over a rectangle, or even a cuboid (erected on the rectangles of Figure 6). This yields symmetry groups $*2$ of the fundamental cell of order 4, or $*22$ of order 8, or $*222$ of order 16, respectively. $\Box$

4. **Conclusions and Outlook**

The above results can imply several further questions. Here we mention a few we assume to be of possible interest.

4.1. **Rhombic lattices.** The reason that we excluded cubic lattices in Theorem 1.2 is that there exist no fundamental cells F for the cubic lattices $\mathbb{Z}^3$, bcc or fcc such that $S(F)$ contains $P(\Gamma)$ ($\Gamma \in \{\mathbb{Z}^3, \mathrm{bcc}, \mathrm{fcc}\}$) as a proper subgroup of finite index. This is just because there are no such groups $S(F) \subset O(3)$. There still may be fundamental cells which have more symmetries in the sense that $|S(F)| > |P(\Gamma)|$ (but the author doubts it).

The reason that we excluded rhombic lattices in Theorem 1.1 is that we were not able to find a general construction for fundamental cells F for any rhombic lattice Γ such that $S(F)$ is larger than $P(\Gamma)$. One can construct such fundamental cells for several particular cases, but a general construction seems hard to obtain. If one tries to use the same idea as in the other non-obtuse cases—start with a packing of polygons of higher symmetry (octagon, dodecagon, square) and refine—one runs into problems because the packings are in general not vertex-to-vertex from some point on.

4.2. **Even more symmetry.** Are there fundamental cells F with $[S(F) : P(G)] > 2$? We have found a few ones: In the case of oblique plane lattices there is a rectangular fundamental cell with $[S(F) : P(G)] = 4$. In the case of the triclinic primitive lattice there is a cuboidal fundamental cell with $[S(F) : P(G)] = 8$. What is the maximal value of the index $[S(F) : P(G)]$ in $\mathbb{R}^d$ $(d \geq 2)$?

4.3. **Higher Dimensions.** The results in the present paper has been obtained by considering all different classes of lattices with respect to their symmetry group. There are 5 such classes in $\mathbb{R}^2$, 14 such classes in $\mathbb{R}^3$, 64 such classes in $\mathbb{R}^4$, 189 such classes in $\mathbb{R}^5$ and 826 such classes in $\mathbb{R}^6$ [2, 8, 12, 15, 16]. At some point it seems desirable to find more general arguments than case-by-case considerations. However, it is very likely that in higher dimensions there are several lattices Γ with fundamental cells F such that $P(\Gamma)$ is a proper subgroup of $S(F)$.

4.4. **Non-Euclidean spaces.** The constructions used in this paper work also in spherical or hyperbolic spaces, using spherical or hyperbolic regular n-gons. The fact that these n-gons are not similar to each other on different length scales does not matter. All we need is that there are edge-to-edge packings by regular n-gons on different length scales.

4.5. **Fractal Dimension.** The fundamental cells $F_\square$ of the square lattice, $F_\triangle$ of the hexagonal lattice and those of the rectangular lattices with incommensurate basis lengths are of fractal appearance. It might be possible to compute the Hausdorff dimensions of the boundaries of these cells, as well as other fractal dimensions, like the box-counting dimension or the affinity dimension [9], see also [14] and references therein. The two latter dimensions are particularly easy to compute if one finds an iterated function system (IFS) generating the fractal under consideration, see [14]. Up to the knowledge of the author, no IFS for $F_\square$ or $F_\triangle$ or the fundamental cells of rectangular lattices are known yet.

4.6. **Alternative Constructions.** The constructions used in this paper can be altered in many ways. For instance, there are other ways to partition the octagons, dodecagons and squares into two regions of different colours than the one used in the proof of Theorem 1.1. All that is required is to keep the mirror symmetry of the partition, and take care that no overlaps occur. One possibility is just to interchange the colours.

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References

[1] M. Baake, R. Klitzing, M. Schlottmann: Fractally shaped acceptance domains of quasiperiodic square-triangle tilings with dodecagonal symmetry, *Physica A* **191** (1992) 554-558.

[2] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, H. Zassenhaus: *Crystallographic Groups of Four-Dimensional Space*, Wiley, New York (1978).

[3] H. Brugiel, J.H. Conway, C. Goodman-Strauss: *Symmetry of Things*, A.K. Peters, Wellesley MA (2008).
[4] H.S.M. Coxeter: Discrete groups generated by reflections, *Ann. Math.* **35** (1934) 588-621.
[5] H.S.M. Coxeter: The complete enumeration of finite groups of the form $r_i^2 = (r_i r_j)^{k_{ij}} = 1$, *J. London Math. Soc.* **10** (1935) 21-25.
[6] H.S.M. Coxeter: *Introduction to Geometry*, Wiley, New York (1969).
[7] V. Elser: Exceptionally symmetric fundamental regions for the root lattices in 2D, *Tagungsbericht Oberwolfach* 20 (2001).
[8] P. Engel: Geometric crystallography, in: P.M. Gruber, J.M. Wills (editors): *Handbook of Convex Geometry*, Vol. B, North-Holland, Amsterdam (1993) pp. 989-1041.
[9] K.J. Falconer: *Fractal Geometry. Mathematical Foundations and Applications*, John Wiley, Chichester (1990).
[10] G.H. Hardy, E.M. Wright: *An Introduction to the Theory of Numbers*, Oxford University Press (1975).
[11] J.E. Humphreys: *Reflection Groups and Coxeter Groups*, Cambridge University Press (1990).
[12] W. Plesken, W. Hanrath: The lattices of six-dimensional space, *Math. Comp.* **43** (1984) 573-587.
[13] R.L.E. Schwarzenberger: Colour symmetry, *Bull. London Math. Soc.* **16** (1984) 209-240.
[14] B. Sing: *Pisot Substitutions and Beyond*, PhD thesis, Univ. Bielefeld 2006, `http://nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:hbz:361-11555`
[15] N.J.A. Sloane, S. Plouffe: *The Encyclopedia of Integer Sequences*, `http://oeis.org/A004030`.
[16] B. Souvignier: Enantiomorphism of crystallographic groups in higher dimensions with results in dimensions up to 6, *Acta Cryst. A* **59** (2003) 210-220.
[17] Wikipedia: Crystal system, `http://en.wikipedia.org/wiki/Crystal_system`, version of 6. May 2013.
[18] Wikipedia: Orbifold notation, `http://en.wikipedia.org/wiki//Orbifold_notation`, version of 7. May 2013.

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