More on domination polynomial and domination root

Saeid Alikhani[a,b,1] and Emeric Deutsch[c]

[a]Department of Mathematics, Yazd University
89195-741, Yazd, Iran

[b]School of Mathematics, Institute for Research in Fundamental Sciences (IPM)
P.O. Box: 19395-5746, Tehran, Iran.

[c] Polytechnic Institute of New York University, United States

---

ABSTRACT

Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,\lambda)=\sum_{i=0}^{n} d(G,i)\lambda^i$, where $d(G,i)$ is the number of dominating sets of G of size i. Every root of $D(G,\lambda)$ is called the domination root of G. It is clear that $(0,\infty)$ is zero free interval for domination polynomial of a graph. It is interesting to investigate graphs which have complex domination roots with positive real parts. In this paper, we first investigate complexity of the domination polynomial at specific points. Then we present and investigate some families of graphs whose complex domination roots have positive real part.

Mathematics Subject Classification: 05C60.

Keywords: Domination polynomial; Domination root; Dutch Windmill graph; Value, Complexity.

---

[1]E-mail: alikhani@yazduni.ac.ir

1 Introduction

Let G be a simple graph. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V \backslash S$ is adjacent to at least one vertex in S. An *i-subset* of $V(G)$ is a subset of $V(G)$ of cardinality i. Let $\mathcal{D}(G, i)$ be the family of dominating sets of G which are i-subsets and let $d(G, i) = |\mathcal{D}(G, i)|$. The polynomial $D(G, x) = \sum_{i=0}^{|V(G)|} d(G, i)x^i$ is defined as *domination polynomial* of G ([2, 5]). A root of $D(G, x)$ is called a *domination root* of G. We denote the set of all roots of $D(G, x)$ by $Z(D(G, x))$. For more information and motivation of domination polynomial and domination roots refer to [1, 2, 5, 10].

The value of a graph polynomial at a specific point can give sometimes a surprising information about the structure of the graph. Balister, et al. in [9] proved that for any graph G, $|q(G, -1)|$ is always a power of 2, where $q(G, x)$ is interlace polynomial of a graph G. Stanley in [14] proved that $(-1)^n P(G, -1)$ is the number of acyclic orientations of G, where $P(G, x)$ is the chromatic polynomial of G and $n = |V(G)|$. Alikhani in [6] studied the domination polynomial at -1 and gave a construction showing that for each odd number n there is a connected graph G with $D(G, -1) = n$. Obviously $(0, \infty)$ is zero free interval for domination polynomial of a graph. It is interesting to find and study graphs with domination roots in the right-half plane. In this paper we show that the computation of values of $D(G, \lambda)$

for $\lambda \in \mathbb{Q} \setminus \{-2, -1, 0\}$ is #P-hard. Also present and study graphs which have complex domination roots with positive real parts.

In Section 2 we investigate complexity of the domination polynomial at specific points. In Section 3 we consider specific families of graphs and compute their domination polynomials. We show that no nonzero real number is domination root of these kind of graphs. Also we see that these kind of graphs have domination roots in the right half-plane. In Section 4 we study the domination polynomial of other classes of graphs.

2 Complexity of domination polynomial at specific points

In this section we investigate Turing complexity of the domination polynomial. First we recall a formula for computing the domination polynomial of the following graph composition. Let G and H be graphs, with $V(G) = \{v_1, \ldots, v_n\}$. The graph $G \diamond H$ formed by substituting a copy of H for every vertex of G, is formally defined by taking a disjoint copy of H, H_v, for every vertex v of G, and joining every vertex in H_u to every vertex in H_v if and only if u is adjacent to v in G.

Theorem 1. ([4, 10]) *For any graph G, $D(G \diamond K_t, x) = D(G, (1+x)^t - 1)$.*

We shall show an application of Theorem 1 to the Turing complexity of the domination polynomial. If reader like to learn more about the basics of counting complexity theory, can refer to [8]. Since the number of dominating sets of graph G, i.e. $D(G, 1)$ is #P-complete, computing domination polynomial with respect to Turing reductions is #P-hard, even for restricted graph classes, see e.g. [12].

The value of a graph polynomial at a specific point can give sometimes a surprising information about the structure of the graph, see e.g. [6]. Hardness of computation of graph polynomial at specific points is another step towards understanding the complexity of a particular graph polynomial. For example in [11], has shown that the Tutte polynomial is #P-hard to compute for any rational evaluation, except those in a semi-algebraic set of low dimension which are polynomial-time computable.

Let to denote by $D(-, \lambda)$ the problem of computing for an input graph G the evaluation $D(G, \lambda)$ of the domination polynomial.

Theorem 2. *The computation of parameter $D(G, \lambda)$ is #P-hard, for every $\lambda \in \mathbb{Q} \setminus \{-2, -1, 0\}$.*

Proof. Suppose that $\lambda \in \mathbb{Q} \setminus \{-2, -1, 0\}$. We present an algorithm such that for an input graph G of order n, computes $D(G, x)$ in polynomial time in n using an oracle to $D(-, \lambda)$. Since $D(G, x)$ is #P-hard, $D(-, \lambda)$ is #P-hard. The algorithm is:

(i) For every $t \in \{1, \ldots, n+1\}$, compute $D(G \diamond K_t, \lambda) = D(G, (1+\lambda)^t - 1)$. $D(G \diamond K_t, \lambda)$ is computed using the oracle to $D(-, \lambda)$. Therefore by Theorem 1, $D(G, (1+\lambda)^t - 1)$ is computed.

(ii) Interpolate $D(G, x)$ from the values $(x_0, D(G, x_0)) = ((1+\lambda)^i - 1, D(G, (1+\lambda)^i - 1))$, for $i = 1, \ldots, n+1$. Since the values $(1+\lambda)^r - 1$

are pairwise distinct (note that $\lambda \notin \{-2, -1, 0\}$) and $D(G, x)$ has degree n, $D(G, x)$ can be interpolated from the computed values. $\square$

3 Graphs with domination roots in the right half-plane

The roots of domination polynomial was studied recently by several authors, see [1, 2, 10].

It is clear that $(0, \infty)$ is zero free interval for domination polynomial of a graph. It is interesting that to investigate graphs which have complex domination roots with positive real parts.

We consider the graphs obtained by selecting one vertex in each of n triangles and identifying them. Some call them Dutch Windmill Graphs [16]. See Figure 1. We denote these graphs by G_3^n. Note that these graphs also called friendship graphs.

We obtain the domination polynomial of theses graphs and show that there are some of these graphs whose have complex domination roots with positive real parts.

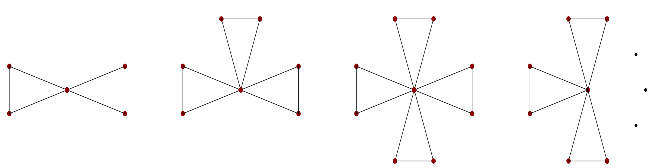

Figure 1: Dutch-Windmill graphs G_3^2, G_3^3, G_3^4 and G_3^n, respectively.

We need some preliminaries.

Theorem 3.([5]) *For every $n \in \mathbb{N}$*

$$D(K_{1,n}, x) = x^n + x(1+x)^n.$$

Theorem 4.([10]) *The domination polynomial of the star graph, $D(K_{1,n}, x)$, where $n \in \mathbb{N}$, has a real root in the interval $(-2n, -ln(n))$, for n sufficiently large.*

The domination roots of $K_{1,n}$ for $1 \leq n \leq 60$ has shown in Figure 2.

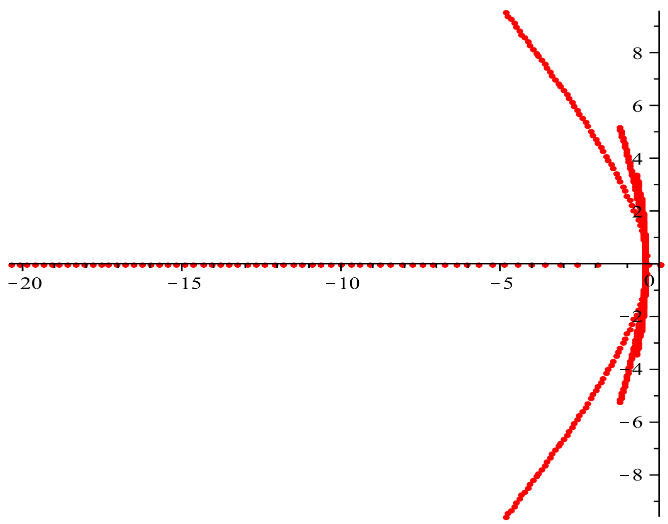

Figure 2: The domination roots of $K_{1,n}$ for $1 \leq n \leq 60$.

The join $G = G_1 + G_2$ of two graph G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

Theorem 5.([2]) *Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively.*

Then

$$D(G_1+G_2,x)=\Big((1+x)^{n_1}-1\Big)\Big((1+x)^{n_2}-1\Big)+D(G_1,x)+D(G_2,x).$$

Theorem 6. *For every $n\in\mathbb{N}$,*

$$D(G_3^n,x)=(2x+x^2)^n+x(1+x)^{2n}.$$

Proof. It is easy to see that G_3^n is join of K_1 and nK_2. Now by Theorem 5 we have the result. $\square$

In [1] the following problem has stated:

Problem. Characterize all graphs with no real domination root except zero.

One of the family with no nonzero real domination roots is $K_{n,n}$ for even n:

Theorem 7. *For every even n, no nonzero real numbers is domination root of $K_{n,n}$.*

Proof. It is easy to see that

$$D(K_{n,n},x)=\Big((1+x)^n-1\Big)^2+2x^n.$$

If $D(K_{n,n},x)=0$, then $\Big((1+x)^n-1\Big)^2=-2x^n$. Obviously this equation does not have real nonzero solution for even n. $\square$

Domination roots of complete bipartite graphs have been studied extensively in [10]. We need the following definition to state one of the main result on domination roots of $K_{n,n}$.

Definition 1. *If $\{f_n(x)\}$ is a family of (complex) polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $\{f_n(x)\}$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $R(f_n(x))$, where $R(f_n(x))$ is the union of the roots of the $f_n(x)$.*

The domination roots of $K_{n,n}$ for $1 \leq n \leq 40$ has shown in Figure 3. See also [10]. As we can see the domination roots of $K_{n,n}$ are bounded. The following theorem characterize limit of roots of the domination polynomials of $K_{n,n}$ for every $n \in \mathbb{N}$.

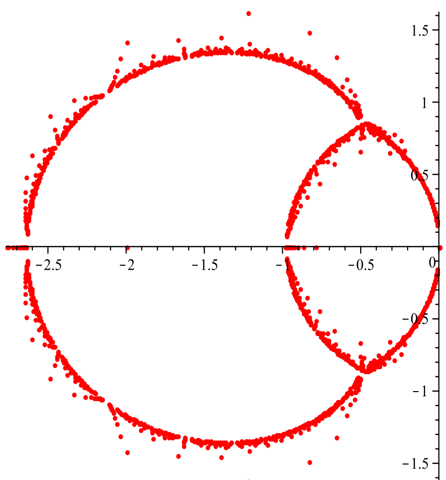

Figure 3: Domination roots of $K_{n,n}$ for $1 \leq n \leq 40$.

Theorem 8.([10]) *The complex numbers z that satisfy any of the following conditions:*

(i) $|z - (-1)| = 1$, $\Re(z) > \frac{-1}{2}$,

(ii) $z = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$,

(iii) $|1+z|^2 = |z|$, $\Re(z) < \frac{-1}{2}$,

are limits of roots of the domination polynomial of the graphs $K_{n,n}$*,* $n \in \mathbb{N}$*.*

Here we prove that, for every odd natural number n, Dutch windmill graph G_3^n have no real roots except zero.

Theorem 9. *For every odd natural number* n*, no nonzero real numbers is domination root of* G_n^n*.*

Proof. By Theorem 6, for every $n \in \mathbb{N}$, $D(G_3^n, x) = (2x+x^2)^n + x(1+x)^{2n}$. If $D(G_3^n, x) = 0$, then we have

$$x = -\big(1 - \frac{1}{(1+x)^2}\big)^n.$$

First suppose that $x \geq 0$. Obviously the above equality is true just for real number 0, since for nonzero real number the left side of equality is positive but the right side is negative. Now suppose that $x < -2$. In this case the left side is less than -2 and the right side $-\big(1 - \frac{1}{(1+x)^2}\big)^n$ is greater than -1, a contradiction. Finally we shall consider $-2 < x < 0$. In this case obviously the above equality is not true for any real number. Because for odd n and for every real number $-2 < x < 0$, the left side of equality is negative but the right side is positive. $\square$

Remark. Using Maple we observed that the domination polynomial of G_3^n for $n \geq 6$ have complex roots with positive real parts. For example $D(G_3^6, x)$ has complex root with real part 0.0003550296365. See Figure 4.

Using Theorems 1 and 6 we have the following theorem:

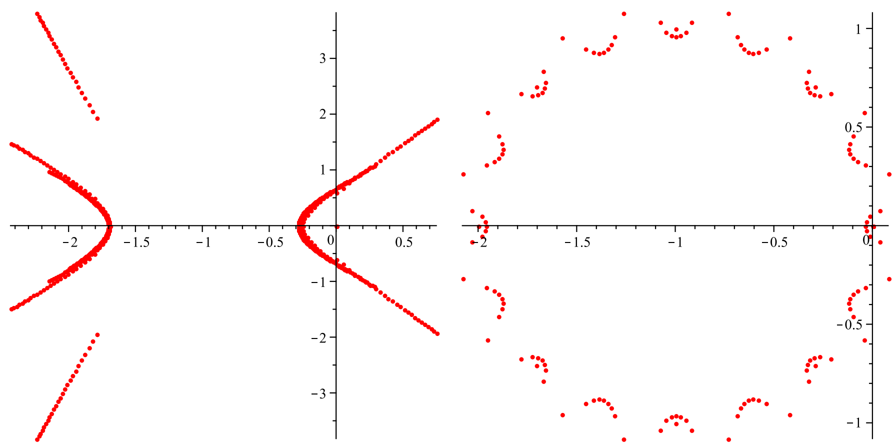

Figure 4: Domination roots of graphs G_3^n for $1 \leq n \leq 30$, and $G_3^8 \diamond K_8$, respectively.

Theorem 10. $D(G_3^n \diamond K_t, x) = ((1+x)^{2t}-1)^n + ((1+x)^t - 1)(1+x)^{2nt}$.

It is interesting that the families of graphs $G_3^n \diamond K_t$ have domination roots with positive real parts (see Figure 4).

4 The domination polynomial of other classes of graphs

In this section we study the domination polynomial of other classes of graphs.

The vertex contraction G/v of a graph G by a vertex v is the operation under which all vertices in $N(v)$ are joined to each other and then v is deleted (see [15]).

Theorem 11. ([3, 7, 13]) *For any vertex v in a graph G we have*

$$D(G,x) = xD(G/v,x) + D(G-v,x) + xD(G-N[v],x) - (x+1)p_v(G,x)$$

where $p_v(G,x)$ is the polynomial counting those dominating sets for $G - N[v]$ which additionally dominate the vertices of $N(v)$ in G.

Theorem 11 can be used to give a recurrence relation which removes triangles. We define a new operation on edges incident to a vertex u: we denote by $G \odot u$ the graph obtained from G by the removal of all edges between any pair of neighbors of u. Note u is not removed from the graph. The following recurrence relation is useful on graphs which have many triangles. This following result also appear in [13] but were proved independently.

Theorem 12. *Let $G = (V,E)$ be a graph and $u \in V$. Then*

$$D(G,x) = D(G-u,x) + D(G \odot u,x) - D(G \odot u - u,x)$$

Proof. Since the operation $\odot u$ only removes the edges between vertices in $N(u)$, we have the following relations:

$$(G \odot u)/u \cong G/u,\ p_u(G,x) = p_u(G \odot u,x),\ (G \odot u) - N[u] \cong (G - N[u]).$$

Using these relations and Theorem 11, we have

$$\begin{aligned} D(G,x) - D(G-u,x) &= xD(G/u,x) + xD(G-N[u],x) - (1+x)p_u(G,x) \\ &= xD((G \odot u)/u,x) + xD((G \odot u) - N[u],x) \\ &- (1+x)p_u(G \odot u,x) \end{aligned}$$

Now by Theorem 11 for $G \odot u$ we have

$$\begin{aligned} xD((G \odot u)/u, x) &+ xD((G \odot u) - N[u], x) - (1+x)p_u(G \odot u, x) \\ &= D(G \odot u, x) - D((G \odot u) - u, x) \end{aligned}$$

Therefore we have the result. □

Now we use Theorem 12 to study the domination polynomial and domination roots of some other classes of graphs:

(1) The Dutch Windmill graph with an extra edge vu, i.e pendant edge to central vertex. The 3 graphs $G - u$, $G \odot u$ and $G \odot u - u$ are (i) K_2, n times and K_1; (ii) $K_{1,2n+1}$; (iii) P_1, $2n+1$ times, respectively. So, by Theorem 12 we have

$$\begin{aligned} D(G_n, x) &= x(2x + x^2)^n + x^{2n+1} + x(1+x)^{2n+1} - x^{2n+1} \\ &= x\big((x^2 + 2x)^n + (x+1)^{2n+1}\big). \end{aligned}$$

The reader is able to see the sequence of coefficients of this polynomial in the site "The on-line encyclopedia of integer sequences" ([17]) as A213658.

It is interesting that the graph G_n has domination roots in the right-half plane (Figure 5). Also for these kind of family of graphs we have the following theorem:

Theorem 13. *For every natural n, there is exactly one nonzero real domination root of G_n.*

Proof. Suppose that α is a root of $D(G_n, x)$. So we have

$$(1 + \frac{1}{(\alpha+1)^2 - 1})^n = \frac{-1}{\alpha+1}.$$

By substituting $\alpha + 1 = t$, we shall have $t^{2n+1} + (t^2 - 1)^n = 0$. This equation has only one real root in $(0, 1)$ for odd n, and has only one real root in $(-1, 0)$ for even n. Therefore $D(G_n, x)$ has only one real root in $(-1, 0)$ or in $(-2, -1)$. $\square$

As you can see in the Figure 5 the graph G_n has domination roots in the right-half plane.

(2) The fan graph. A fan graph $F_{m,n}$ is defined as the graph join $\overline{K_m} + P_n$, where $\overline{K_m}$ is the empty graph on m vertices and is P_n the path graph on n vertices. Here we consider $F_{2,n}$. To obtain $F_{2,n}$ take two vertices u, v and join each of n vertices $1, 2, ..., n$ to both u and v. By Theorem 5, we have

$$\begin{aligned} D(F_{2,n}, x) &= (x^2 + 2x)((1 + x)^n - 1) + x^2 + D(P_n, x) \\ &= 2x((1 + x)^n - 1) + D(P_n, x). \end{aligned}$$

We can see the sequence of coefficients of this polynomial in the site "The on-line encyclopedia of integer sequences" ([17]) as A213657.

The domination roots of graph $F_{2,n}$ has shown in Figure 5.

Using Theorem 5, we have the following corollary:

Corollary 1. *For every natural* $m, n \in \mathbb{N}$,

$$D(F_{m,n}, x) = ((1 + x)^m - 1)((1 + x)^n - 1) + x^m + D(P_n, x).$$

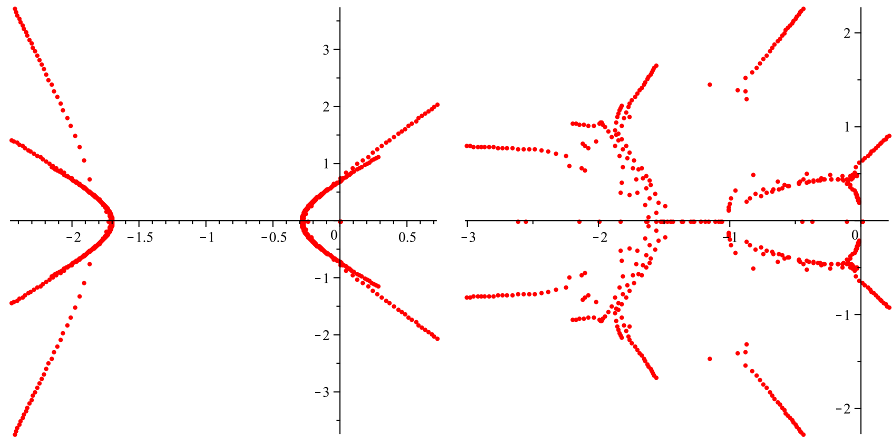

Figure 5: Domination roots of graphs G_n and $F_{2,n}$ for $1 \leq n \leq 30$, respectively.

(3) The Gem graph G. Consider the path P_{n+1} and an additional vertex u; join u to each vertex of the path. The 3 graphs in Theorem 12 are: (i) P_{n+1}; (ii) the star $K_{1,n+1}$; (iii) P_1, $(n+1)$ times. So,

$$\begin{aligned} D(G,x) &= D(P_{n+1},x) + x^{n+1} + x(1+x)^{n+1} - x^{n+1} \\ &= D(P_{n+1},x) + x(1+x)^{n+1}. \end{aligned}$$

The reader is able to see the sequence of coefficients of this polynomial in the site "The on-line encyclopedia of integer sequences" ([17]) as A213662.

(4) The Gem graph with an extra edge vu which is denoted by G'. The 3 graphs in Theorem 12 are (i) P_{n+1} and P_1; (ii) $K_{1,n+2}$; (iii) P_1, $(n+2)$ times. So,

$$\begin{aligned} D(G',x) &= xD(P_{n+1},x) + x^{n+2} + x(1+x)^{n+2} - x^{n+2} \\ &= x\big(D(P_{n+1},x) + (1+x)^{n+2}\big). \end{aligned}$$

(5) Join a vertex u with two consecutive vertices of the cycle C_n (i.e. a triangle placed on an edge of C_n). Let to denote this graph by G. The 3 graphs in Theorem 12 are: (i) C_n; (ii) C_{n+1}; (iii) P_n. So,

$$D(G,x) = D(C_n,x) + D(C_{n+1},x) - D(P_n,x).$$

We can see the sequence of coefficients of this polynomial in the site "The on-line encyclopedia of integer sequences" ([17]) as A213664.

(6) The wheel graph W_n. The 3 graphs in Theorem 12 are (i) $C(n-1)$; (ii) $K_{1,n-1}$; (iii) P_1, $n-1$ times. So,

$$\begin{aligned} D(W_n,x) &= D(C_{n-1},x) + x^{n-1} + x(1+x)^{n-1} - x^{n-1} \\ &= D(C_{n-1},x) + x(1+x)^{n-1}. \end{aligned}$$

As we can see in the Figure 6, there are graphs in the families of G'_n and W_n which their domination roots are in the right half-plane.

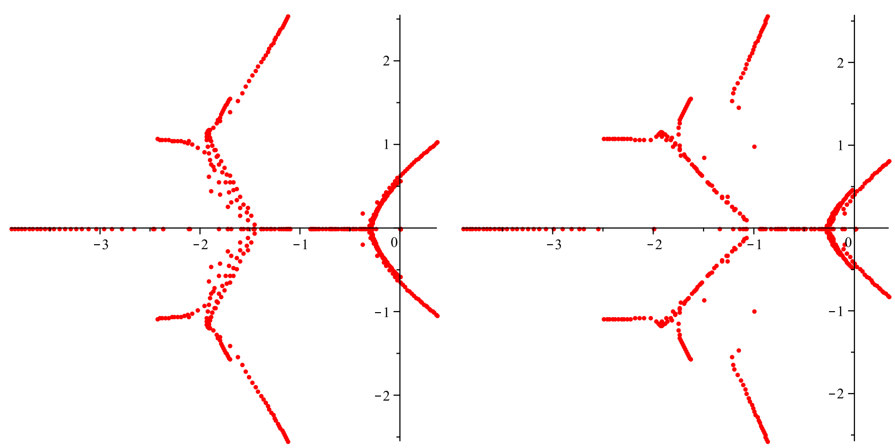

Figure 6: Domination roots of graphs G'_n and W_n for $1 \leq n \leq 30$, respectively.

Acknowledgement. The authors would like to express their gratitude to the referee for helpful comments. The research of the first author was in part supported by a grant from IPM (No. 91050015) and partially supported by Yazd University Research Council. The authors would like to thank S. Jahari for some useful Maple programme.

References

[1] S. Akbari, S. Alikhani, M. R. Oboudi and Y.H. Peng, *On the zeros of domination polynomial of a graph*, Contem. Math., American Mathematical Society, 531 (2010) 109-115.

[2] S. Akbari, S. Alikhani and Y. H. Peng, *Characterization of graphs using Domination Polynomial*, European J. Combin. Vol 31 (2010) 1714-1724.

[3] S. Alikhani, *Dominating sets and domination polynomials of graphs: Domination polynomial: A new graph polynomial*, LAMBERT Academic Publishing, ISBN: 9783847344827 (2012).

[4] S. Alikhani, *On the domination polynomial of some graph operations*, ISRN Combin., Volume 2013, Article ID 146595, 3 pages `http://dx.doi.org/10.1155/2013/146595`.

[5] S. Alikhani, Y. H. Peng, *Introduction to Domination Polynomial of a Graph*, Ars Combin., to appear. Available at `http://arxiv.org/abs/0905.2251`.

[6] S. Alikhani, *The domination polynomial of a graph at -1*, Graphs Combin., (2013) 29:1175-1181.

[7] S. Alikhani, *On the domination polynomials of non P_4-free graphs*, Iran. J. Math. Sci. Inf., Vol. 8, No. 2 (2013) 49–55.

[8] S. Arora, B. Barak, *Computational Complexity - A Modern Approach*, Cambridge University Press (2009).

[9] P.N. Balister, B. Bollobas, J. Cutler, L. Pebody, *The interlace polynomial of graphs at -1*, European J. Combin., 23, 761–767 (2002).

[10] J. Brown, J. Tufts, *On the Roots of Domination Polynomials*, Graphs Combin., DOI 10.1007/s00373-013-1306-z (2013).

[11] F. Jaeger, D.L. Vertigan, D.J.A. Welsh, *On the computational complexity of the Jones and Tutte polynomials*, Mathematical Proceedings of the Cambridge Philosophical Society 108:35-53 (1990).

[12] S. Kijima, Y. Okamoto, T. Uno, *Dominating set counting in graph classes*, In Bin Fu and Ding-Zhu Du, editors, COCOON, volume 6842 of Lecture Notes in Computer Science, pages 13–24 (2011).

[13] T. Kotek, J. Preen, F. Simon,P. Tittmann, M. Trinks, *Recurrence relations and splitting formulas for the domination polynomial*, Elec. J. Combin. 19(3) (2012), # P47.

[14] R.P. Stanley, *Acyclic orientations of graphs*, Discrete Math., 5, 171-178 (1973).

[15] M. Walsh, *The hub number of a graph*, Int. J. Math. Comput. Sci., 1, 2006, 117-124.

[16] http://mathworld.wolfram.com/DutchWindmillGraph.html

[17] The On-Line Encyclopedia of Integer Sequences. http://oeis.org, 2012.