INVARIANTS OF (-1)-SKEW POLYNOMIAL RINGS UNDER PERMUTATION REPRESENTATIONS

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0. Introduction

Let k be a base field of characteristic zero (unless otherwise stated) and let $k_q[\underline{x}]$ denote the q-skew polynomial ring $k_q[x_1,\dots,x_n]$ that is generated by $\{x_i\}_{i=1}^n$ and subject to the relations $x_jx_i = qx_ix_j$ for all $i<j$, where q is a nonzero element in k. In previous work [KKZ1]-[KKZ4] we have studied the invariant theory of noncommutative Artin-Schelter regular (or AS regular, for short) algebras such as $k_q[\underline{x}]$ under linear actions by finite groups G. We have shown that often the classical invariant theory of the commutative AS regular algebra $k[\underline{x}] := k[x_1,\dots,x_n]$ extends to noncommutative AS regular algebras in some analogous way. In this paper we consider the case where G is a group of permutations of $\{x_i\}_{i=1}^n$ acting on the (-1)-skew polynomial ring $k_{-1}[\underline{x}]$, which is generated by $\{x_i\}_{i=1}^n$ and subject to the relations

$$x_ix_j = -x_jx_i \tag{E0.0.1}$$

for all $i\neq j$. We have chosen to consider $k_{-1}[\underline{x}]$ because any permutation of $\{x_i\}_{i=1}^n$ preserves the relations (E0.0.1), and hence extends to an algebra automorphism of $k_{-1}[\underline{x}]$; the only q-skew polynomial algebras $k_q[\underline{x}]$ with this property are the cases when $q=\pm 1$. Hence any subgroup of the symmetric group $\mathfrak{S}_n$ acts on both $k[\underline{x}]$ and $k_{-1}[\underline{x}]$ as permutations, and our main focus is on the ring of invariants $k_{-1}[\underline{x}]^G$ when G is a subgroup of $\mathfrak{S}_n$.

The study of the fixed subring $k[\underline{x}]^G$ under permutation groups G of the commutative indeterminates $\{x_i\}_{i=1}^n$ has a long and distinguished history. Gauss showed that when G is the full symmetric group $\mathfrak{S}_n$, invariant polynomials could be expressed uniquely in terms of the n symmetric polynomials [Ne, Theorem 4.13]; the symmetric polynomials are algebraically independent so that $k[\underline{x}]^G$ is itself a polynomial ring. This result was generalized to other groups (so-called "reflection groups") by Shephard-Todd [ST] and Chevalley [Ch] in the 1950s. It follows from [KKZ2, Theorem 1.1] that $k_{-1}[\underline{x}]^G$ will not be an AS regular algebra, even for a classical reflection group like the symmetric group. However, we will show that A^G is always an AS Gorenstein domain [Theorem 1.5], while $k[\underline{x}]^G$ is not always Gorenstein [Example 1.6]. In [CA] algebra generating sets for $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$, the invariants under the full symmetric group [Theorem 3.10], and for $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$, the invariants under the alternating group $\mathfrak{A}_n$ [Theorem 4.10] have been produced. We will show that for both the full symmetric group [Theorem 3.12] and the alternating group

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[Theorem 4.15] the fixed subring is isomorphic to an AS regular algebra R modulo a central regular sequence of R (what we call a "classical complete intersection" in [KKZ4]). Moreover, we generalize some results for upper bounds on the degrees of algebra generators for $k_{-1}[\underline{x}]^G$ [Theorems 2.5 and 2.6] from results in the commutative case.

One motivation for this study was to consider the theorem of Kac-Watanabe [KW], and independently of Gordeev [G1], that provides a necessary condition for any finite group, not necessarily a permutation group, to have the property that $k[\underline{x}]^G$ is a complete intersection (the condition is that G be a group generated by so-called "bireflections"). This theorem of Kac-Watanabe-Gordeev was a first step toward the (independent) classification of finite groups G, acting linearly as automorphisms of $k[\underline{x}]$, such that $k[\underline{x}]^G$ is a complete intersections that was proven by Gordeev [G2] and Nakajima [N1, N2, NW]. We verify that an analogous result holds for $k_{-1}[\underline{x}]$ and subgroups of the symmetric group $\mathfrak{S}_n$ for $n \leq 4$ [Example 5.6], and conjecture that this result is true in general. We prove that the converse of the Kac-Watanabe-Gordeev Theorem holds for $k_{-1}[\underline{x}]$: if G is a group of permutations of the $\{x_i\}_{i=1}^n$ that is generated by quasi-bireflections then $k_{-1}[\underline{x}]^G$ is a classical complete intersection [Theorem 5.4] (this result is not true for the commutative polynomial ring $k[\underline{x}]$ [Example 5.5]).

These fixed rings of $k_{-1}[\underline{x}]$ under permutation subgroups produce a tractable class of AS Gorenstein domains that possess a variety of properties; in many cases their generators have combinatorial descriptions and their Hilbert series can be described explicitly. The following table summarizes results presented in this paper and gives a comparison between the results of $k_{-1}[\underline{x}]^G$ with that of $k[\underline{x}]^G$ for any subgroup $\{1\} \neq G \subset \mathfrak{S}_n$:

Statements about A^G	when $A = k[\underline{x}]$	when $A = k_{-1}[\underline{x}]$
Being AS Gorenstein	Not always	Always
Being AS regular	Sometimes	Never
$cci^+(A^{\mathfrak{S}_n})$	0	$\lfloor \frac{n}{2} \rfloor$
$\deg H_{A^G}(t)$	$\leq -n$	$-n$
Bound for degrees of generators	$\max\{n, \binom{n}{2}\}$	$\binom{n}{2} + \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)$
KWG theorem holds	Yes	Conjecture
Converse of KWG holds	No	Yes

where KWG stands for Kac-Watanabe-Gordeev.

1. Definitions and basic properties

An algebra A is called *connected graded* if

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots$$

and $A_i A_j \subset A_{i+j}$ for all $i, j \in \mathbb{N}$. The Hilbert series of A is defined to be

$$H_A(t) = \sum_{i \in \mathbb{N}} (\dim A_i) t^i.$$

Definition 1.1. Let A be a connected graded algebra.

(1) We call A *Artin-Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:
   (a) A has injective dimension $d < \infty$ on the left and on the right,
   (b) $\mathrm{Ext}^i_A({}_A k, {}_A A) = \mathrm{Ext}^i_A(k_A, A_A) = 0$ for all $i \neq d$, and
   (c) $\mathrm{Ext}^d_A({}_A k, {}_A A) \cong \mathrm{Ext}^d_A(k_A, A_A) \cong k(\mathfrak{l})$ for some integer $\mathfrak{l}$. Here $\mathfrak{l}$ is called the *AS index* of A.

   If in addition,
   (d) A has finite global dimension, and
   (e) A has finite Gelfand-Kirillov dimension,

   then A is called *Artin-Schelter regular* (or *AS regular*, for short) of dimension d.

(2) If A is a noetherian, AS regular graded domain of global dimension n and $H_A(t) = (1-t)^{-n}$, then we call A *a quantum polynomial ring* of dimension n.

Skew polynomial rings $k_q[\underline{x}]$, where $q \in k^\times := k \setminus \{0\}$, with $\deg x_i = 1$ are quantum polynomial rings and also Koszul algebras. Next we recall from [KKZ1] the definition of a noncommutative version of a reflection. If A is a connected graded algebra, let $\mathrm{Aut}(A)$ denote the group of all graded algebra automorphisms of A. If $g \in \mathrm{Aut}(A)$, then the trace function of g is defined to be

$$\mathrm{Tr}_A(g, t) = \sum_{i=0}^{\infty} \mathrm{tr}(g|_{A_i}) t^i \in k[[t]],$$

where $\mathrm{tr}(g|_{A_i})$ is the trace of the linear map $g|_{A_i}$. Note that $\mathrm{Tr}_A(g, 0) = 1$ and that the trace of the identity map is the Hilbert series of the algebra A. The trace of a graded algebra automorphism of a Koszul algebra can be computed from the Koszul dual using the following result.

Lemma 1.2. [JiZ, Corollary 4.4] *Let A be a Koszul algebra with Koszul dual algebra $A^!$. Let $g \in \mathrm{Aut}(A)$ and g^τ be the induced dual automorphism of $A^!$. Then*

$$\mathrm{Tr}_A(g, t) = (\mathrm{Tr}_{A^!}(g^\tau, -t))^{-1}.$$

Definition 1.3. Let A be an AS regular algebra such that

$$H_A(t) = \frac{1}{(1-t)^n f(t)}$$

where $f(1) \neq 0$. Let $g \in \mathrm{Aut}(A)$.

(1) [KKZ1, Definition 2.2] Then g is called a *quasi-reflection* of A if

$$\mathrm{Tr}_A(g, t) = \frac{1}{(1-t)^{n-1} q(t)}$$

for $q(1) \neq 0$. If A is a quantum polynomial ring, then $H_A(t) = (1-t)^{-n}$. In this case g is a *quasi-reflection* if and only if

$$\mathrm{Tr}_A(g,t) = \frac{1}{(1-t)^{n-1}(1-\lambda t)} \tag{E1.3.1}$$

for some scalar $\lambda \neq 1$. Note that we have chosen not to call the identity map a quasi-reflection.

(2) [KKZ4, Definition 3.6(b)] Then g is called a *quasi-bireflection* of A if

$$\mathrm{Tr}_A(g,t) = \frac{1}{(1-t)^{n-2}q(t)}$$

for $q(1) \neq 0$.

When A is noetherian and AS Gorenstein and g is in $\mathrm{Aut}(A)$, the homological determinant of g, denoted by $\mathrm{hdet}\, g$, is defined in [JoZ, Definition 2.3]. When $A = k[\underline{x}]$, the homological determinant of g is the inverse of determinant of the linear map, induced by g on the degree one piece $A_1 = \bigoplus_{i=1}^n kx_i$ of A, and, more generally, it is defined using a scalar map induced on the local cohomology of A; see [JoZ] for details. The homological determinant is a group homomorphism

$$\mathrm{hdet}: \quad \mathrm{Aut}(A) \to k^{\times}.$$

When A is AS regular, the conditions of the following theorem are satisfied by [JiZ, Proposition 3.3] and [JoZ, Proposition 5.5], and $\mathrm{hdet}\, g$ can be computed from the trace of g, as given in the following result.

Lemma 1.4. [JoZ, Lemma 2.6] *Let A be noetherian and AS Gorenstein and let $g \in \mathrm{Aut}(A)$. If g is k-rational in the sense of* [JoZ, Definition 1.3]*, then the rational function $\mathrm{Tr}_A(g,t)$ has the form*

$$\mathrm{Tr}_A(g,t) = (-1)^n(\mathrm{hdet}\, g)^{-1}t^{-\ell} + \text{lower terms}$$

when it is written as a Laurent series in t^{-1}.

The following result is not hard to prove.

Theorem 1.5. *Let G be any subgroup of the symmetric group $\mathfrak{S}_n$ acting on $k_{-1}[\underline{x}]$ as permutations.*

(1) *The fixed subring $k_{-1}[\underline{x}]^G$ is an AS Gorenstein domain.*
(2) *If $G \neq \{1\}$, then $k_{-1}[\underline{x}]^G$ is not AS regular.*

Proof. (1) The trace of any transposition $g = (i,j)$ in $\mathfrak{S}_n$ can be computed using the Koszul dual $(k_{-1}[\underline{x}])^!$ by Lemma 1.2, which is isomorphic to

$$k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2),$$

and found to be

$$\mathrm{Tr}_A(g,t) = \frac{1}{(1+t^2)(1-t)^{n-2}} = (-1)^{n-2}t^{-n} + \text{lower terms}.$$

It follows from Lemma 1.4 that the homological determinant of g is 1. Since $\mathfrak{S}_n$ is generated by transpositions, $\mathrm{hdet}\, g = 1$ for all $g \in \mathfrak{S}_n$.

By the last paragraph, $\mathrm{hdet}\, g = 1$ for all $g \in G$. The assertion follows from [JoZ, Theorem 3.3].

(2) Since $k_{-1}[\underline{x}]$ is a quantum polynomial ring, any quasi-reflection g has the homological determinant $\lambda \neq 1$ where λ is given in (E1.3.1). Since $\mathrm{hdet}\, g = 1$

for all $g \in G$, G contains no quasi-reflection (also see Lemma 1.7(4) below). The assertion follows from [KKZ2, Theorem 1.1]. $\square$

The analogous theorem is not true in the commutative case. As we mentioned in the introduction, $k[\underline{x}]^{\mathfrak{S}_n}$ is isomorphic to the commutative polynomial ring $k[\underline{x}]$, which is AS regular. Hence Theorem 1.5(2) fails for $k[\underline{x}]$. The next example shows that Theorem 1.5(1) fails for $k[\underline{x}]$.

Example 1.6. Set $n = 4$. Let $G = \langle(1, 2, 3, 4)\rangle$ be the cyclic subgroup of $\mathfrak{S}_4$ generated by the 4-cycle $(1, 2, 3, 4)$. Then G contains no reflections, and has elements of determinant -1, so $B_+ := k[x_1, x_2, x_3, x_4]^G$ cannot be Gorenstein by [Wa]. Or, one can also use Molien's Theorem to check that the Hilbert series of the fixed subring B_+ is

$$\frac{t^3 + t^2 - t + 1}{(1 - t)^4(1 + t)^2(1 + t^2)},$$

which does not have the symmetry property required in Stanley's criteria [S2, Theorem 4.4] for B_+ to be Gorenstein.

Note that for the same subgroup G, but acting on the noncommutative ring $k_{-1}[x_1, \ldots, x_4]$, the Hilbert series of the fixed ring $B_- := k_{-1}[x_1, \ldots, x_4]^G$ is

$$\frac{(t^2 - t + 1)(t^6 - 2t^5 + 3t^4 - 2t^3 + 3t^2 - 2t + 1)}{(1 - t)^4(1 + t^2)^2(1 + t^4)},$$

which has the symmetry property, and hence is AS Gorenstein by a noncommutative version of Stanley's criteria [JoZ, Theorem 6.2], as well as by Theorem 1.5(1).

The trace of any permutation is computed as follows.

Lemma 1.7. *Let $\mathfrak{S}_n$ act on $A = k_{-1}[\underline{x}]$ as permutations and $g \in \mathfrak{S}_n$.*

(1) *If g is an m-cycle, then*

$$\mathrm{Tr}_A(g, t) = \frac{1}{(1 + (-t)^m)(1 - t)^{n-m}}.$$

(2) *If $g = \nu_{i_1} \cdots \nu_{i_k} \mu_{j_1} \cdots \mu_{j_\ell}$ a product of disjoint cycles of length i_p and j_p, with ν_{i_p} odd permutations and μ_{j_p} even permutations, then*

$$\mathrm{Tr}_A(g, t) = \frac{1}{(1 + t^{i_1}) \cdots (1 + t^{i_k})(1 - t^{j_1}) \cdots (1 - t^{j_\ell})(1 - t)^{n-(i_1+\cdots+i_k+j_1+\cdots+j_\ell)}}.$$

(3) *The only quasi-bireflections of $k_{-1}[\underline{x}]$ in $\mathfrak{S}_n$ are the two-cycles and three-cycles.*

(4) *Permutation groups (namely, subgroups of $\mathfrak{S}_n$) contain no quasi-reflections.*

Proof. (1) This follows from Lemma 1.2 and direct computations.

(2) This follows from part (1) and a graded vector space decomposition of $k_{-1}[\underline{x}]$.

(3,4) These are consequences of part (2). $\square$

In [KKZ4] we introduced several possible generalizations of a commutative complete intersection. We review these notions here.

Definition 1.8. Let A be a connected graded noetherian algebra.

(1) We say A is a *classical complete intersection* (or a *cci*) if there is a connected graded noetherian AS regular algebra R and a sequence of regular normal homogeneous elements $\{\Omega_1, \dots, \Omega_n\}$ of positive degree such that A is isomorphic to $R/(\Omega_1, \dots, \Omega_n)$. The minimum such n is called the *cci-number* of A and denoted by $cci(A)$.
(2) We say A is a *hypersurface* if $cci(A) \leq 1$.
(3) We say A is a *complete intersection of noetherian type* (or an *nci*) if the Ext-algebra $\operatorname{Ext}^*_A(k,k) := \bigoplus_{i\geq 0} \operatorname{Ext}^i_A({}_Ak, {}_Ak)$ is noetherian.
(4) We say A is a *complete intersection of growth type* (or a *gci*) if the Ext-algebra $\operatorname{Ext}^*_A(k,k)$ has finite Gelfand-Kirillov dimension.
(5) We say A is a *weak complete intersection* (or a *wci*) if the Ext-algebra $\operatorname{Ext}^*_A(k,k)$ has subexponential growth.

In [KKZ4] we showed that a property of all of these kinds of complete intersections is the cyclotomic Gorenstein property defined below.

Definition 1.9. Let A be a connected graded noetherian algebra.

(1) We say A is *cyclotomic* if its Hilbert series $H_A(t)$ is a rational function $p(t)/q(t)$ for some coprime polynomials $p(t), q(t) \in \mathbb{Z}(t)$ and the roots of $p(t)$ and $q(t)$ are roots of unity.
(2) We say A is *cyclotomic Gorenstein* if the following conditions hold
   (i) A is AS Gorenstein;
   (ii) A is cyclotomic.

Theorem 1.10. [KKZ4, Theorem 3.4] *Let A be R^G for some noetherian Auslander regular algebra R and a finite subgroup $G \subset \operatorname{Aut}(R)$. If A is any of the kinds of complete intersection in Definition 1.8, then it is cyclotomic Gorenstein.*

We note that in Example 1.6 although the fixed subring A^G is AS Gorenstein, it is not any of the kinds of generalized "complete intersection" of Definition 1.8 since its Hilbert series has zeros that are not roots of unity.

The following theorem of Kac-Watanabe-Gordeev is one of the motivations for this paper.

Theorem 1.11. [KW, G1] *Let G be a finite group acting linearly on $k[\underline{x}]$. If $k[\underline{x}]^G$ is a complete intersection, then G is generated by bireflections.*

A noncommutative version of Kac-Watanabe-Gordeev Theorem holds for skew polynomial rings $k_q[\underline{x}]$ when $q \neq \pm 1$ [KKZ4, Theorem 0.3], that leaves $k_{-1}[\underline{x}]$ the only unknown case. In this paper we will prove some partial results for this special skew polynomial ring. We note that in Example 1.6 the trace of a four-cycle acting on $k_{-1}[x_1, \dots, x_4]$ is $1/(1+t^4)$, which is not a quasi-bireflection, supporting a generalization of the Kac-Watanabe-Gordeev Theorem.

To conclude this section we compute the automorphism group $\operatorname{Aut}(k_{-1}[\underline{x}])$.

Lemma 1.12. (1) *$g \in \operatorname{Aut}(k_{-1}[\underline{x}])$ if and only if $g(x_i) = a_i x_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_n$ and $\{a_i\}_{i=1}^n \subset k^\times$, namely, $\operatorname{Aut}(k_{-1}[\underline{x}]) = (k^\times)^n \rtimes \mathfrak{S}_n$.*
(2) *If g is of the form in part (a), then* $\operatorname{hdet} g = \prod_{i=1}^n a_i$.

Proof. (a) Every diagonal map $g : x_i \to a_i x_i$, for $(a_1, \cdots, a_n) \in (k^\times)^n$, extends easily to a unique graded algebra automorphism of $k_{-1}[\underline{x}]$. And we have already seen that $\mathfrak{S}_n$ is a subgroup of $\operatorname{Aut}(k_{-1}[\underline{x}])$ such that $\mathfrak{S}_n \cap (k^\times)^n = \{1\}$. Thus

$(k^{\times})^n \rtimes \mathfrak{S}_n \subset \mathrm{Aut}(k_{-1}[\underline{x}])$. By [KKZ2, Lemma 3.5(e)], $\mathrm{Aut}(k_{-1}[\underline{x}]) \subset (k^{\times})^n \rtimes \mathfrak{S}_n$. The assertion follows.

(b) If $g \in (k^{\times})^n$, or $g(x_i) \to a_i x_i$ for $(a_1, \cdots, a_n) \in (k^{\times})^n$, then it is easy to see that $\operatorname{hdet} g = \prod_{i=1}^n a_i$. If $g \in \mathfrak{S}_n$, then $\operatorname{hdet} g = 1$ by the proof of Theorem 1.5(a). The assertion follows by the fact hdet is a group homomorphism. □

2. Upper bound for the algebra generators

In this section we show that Broer's and Göbel's upper bounds on the degrees of minimal generating sets of $k[\underline{x}]^G$, for arbitrary subgroup $G \subset \mathfrak{S}_n$, have analogues in this context. In this section we do not assume that $\operatorname{char} k = 0$.

The Noether upper bound on the degrees of generators does not hold for $k_{-1}[\underline{x}]$, as $k_{-1}[x_1, x_2]^{\mathfrak{S}_2}$ requires a generator of degree 3 [Example 3.1]. More generally one can ask if the degrees of generators of $k_{-1}[\underline{x}]^G$ are bounded above by $|G|$ times the dimension of the representation of G. Broer's degree bound [DK, Proposition 3.8.5] states that when f_i are primary invariants, i.e. f_i, for $1 \le i \le n$, are algebraically independent and $k[\underline{x}]^G$ is a finite module over $k[f_1, \ldots, f_n]$, then $k[\underline{x}]^G$ is generated as an algebra by homogeneous invariants of degrees at most

$$\deg(f_1) + \cdots + \deg(f_n) - n.$$

(The above statement is not true when $n = 2$ and $g : x_1 \to x_1, x_2 \to -x_2$. In this case $f_1 = x_1, f_2 = x_2^2$. Therefore we need to assume $n \ge 3$.) We show that this result generalizes for any group G (not necessarily a permutation group) when the given hypotheses are satisfied [Lemma 2.2].

Let A be any connected graded algebra. Define d_A to be the maximal degree of $A_{\ge 1}/(A_{\ge 1})^2$. Then A is generated as an algebra by homogeneous elements of degree at most d_A.

Lemma 2.1. *Let A be a noetherian connected graded AS Gorenstein algebra and B and C be graded subalgebras of A such that $C \subset B \subset A$. Assume that*

(i) *$A = B \oplus D$ as a right graded B-modules,*
(ii) *A is a finitely generated right C-module, and*
(iii) *There is a noetherian AS regular algebra R and a surjective graded algebra map $\phi : R \to C$ and* $\operatorname{gldim} R = \operatorname{injdim} A$.

Then

(1) *ϕ is an isomorphism and A_C is free.*
(2) *$d_B \le \max\{d_C, \mathfrak{l}_C - \mathfrak{l}_A\}$ where $\mathfrak{l}_A$ and $\mathfrak{l}_C$ are AS index of A and C respectively.*

Proof. (1) Let $n = \operatorname{injdim} A$. Induced by the composite map $f : R \to C \to A$ we have a convergent spectral sequence [WZ, Lemma 4.1],

$$\mathrm{Ext}^p_A(\mathrm{Tor}^R_q(A, k), A) \Longrightarrow \mathrm{Ext}^{p+q}_R(k, A).$$

Since A_R is finitely generated and R is right noetherian, $\mathrm{Tor}^R_q(A, k)$ is finite dimensional for all q. Thus $\mathrm{Ext}^p_A(\mathrm{Tor}^R_q(A, k), A) = 0$ for all $p \ne \operatorname{injdim} A = n$. The above spectral sequence collapses to the following isomorphisms

$$\mathrm{Ext}^n_A(\mathrm{Tor}^R_q(A, k), A) \cong \mathrm{Ext}^{n+q}_R(k, A).$$

For any $q > 0$, $\mathrm{Ext}^n_A(\mathrm{Tor}^R_q(A,k), A) \cong \mathrm{Ext}^{n+q}_R(k, A) = 0$. Since $\mathrm{Tor}^R_q(A,k)$ is finite dimensional, A is AS Gorenstein, we obtain that

$$\dim \mathrm{Tor}^R_q(A,k) = \dim \mathrm{Ext}^n_A(\mathrm{Tor}^R_q(A,k), A) = 0$$

for all $q > 0$. Hence A_R is projective, whence free, as R is connected graded. As a consequence, $f : R \to A$ is injective. This implies that ϕ is an isomorphism. Since ϕ is an isomorphism and A_R is free, A_C is free.

(2) Now we identify R with C. By part (1), A is a finitely generated free C-module. Since $A = B \oplus D$, both B and D are projective, whence free, graded right C-modules. Pick a C-basis for B and D, say $V_B \subset B$ and $V_D \subset D$. Then we have $B = V_B \otimes C$ and $D = V_D \otimes C$. Therefore $A = V_A \otimes C$ where $V_A = V_B \oplus V_D$. Hence

$$H_A(t) = H_{V_A}(t)H_C(t) = (H_{V_B}(t) + H_{V_D}(t))H_C(t), \quad \text{and} \quad H_B(t) = H_{V_B}(t)H_C(t).$$

Since $B = V_B \otimes C$, B is generated by V_B and C as a graded algebra. Thus we have

$$d_B \leq \max\{\deg H_{V_B}(t), d_C\} \leq \max\{\deg H_{V_A}(t), d_C\}.$$

It remains to show that $\deg H_{V_A}(t) = -\mathfrak{l}_A + \mathfrak{l}_C$. First, as $H_A(t) = H_{V_A}(t)H_C(t)$, we have $\deg H_{V_A}(t) = \deg H_A(t) - \deg H_C(t)$. Recall that C is noetherian and AS regular. Since A is a finite module over C, $H_A(t)$ is rational and the hypotheses (1°, 2°, 3°) of [JoZ, Theorem 6.1] hold. By the proof of [JoZ, Theorem 6.1] (we are not using the hypothesis that A is a domain),

$$H_A(t) = \pm t^{\mathfrak{l}_A} H_A(t^{-1})$$

where $\mathfrak{l}$ is the AS index of A. Since $H_A(t)$ is a rational function such that $H_A(0) = 1$, the above equation forces that

$$\deg H_A(t) = -\mathfrak{l}_A. \tag{E2.1.1}$$

Similarly, $\deg H_C(t) = -\mathfrak{l}_C$. The assertion follows. □

The degree of algebra generators of B is bounded by $\mathfrak{l}_C - \mathfrak{l}_A$ when $d_C \leq \mathfrak{l}_C - \mathfrak{l}_A$, which is easy to achieve in many cases. The following lemma is a generalization of Broer's upper bound [DK, Proposition 3.8.5].

Lemma 2.2 (Broer's Bound)**.** *Let A be a quantum polynomial algebra of dimension n and C an iterated Ore extension $k[f_1][f_2;\tau_2,\delta_2]\cdots[f_n;\tau_n,\delta_n]$. Assume that*

(1) *$B = A^H$ where H is a semisimple Hopf algebra acting on A,*
(2) *$C \subset B \subset A$ and A_C is finitely generated, and*
(3) *$\deg f_i > 1$ for at least two distinct i's.*

Then

$$d_{A^H} \leq \mathfrak{l}_C - \mathfrak{l}_A = \sum_{i=1}^n \deg f_i - n.$$

Proof. Since H is semisimple, $A = B \oplus D$ by [KKZ3, Lemma 2.4(a)] where $B = A^H$. Let $R = C$. Then the hypotheses Lemma 2.1(i,ii,iii) hold. By Lemma 2.1,

$$d_B \leq \max\{d_C, \mathfrak{l}_C - \mathfrak{l}_A\}.$$

It is clear that $\mathfrak{l}_A = n$. By induction on n, one sees that $H_C(t) = \frac{1}{\prod_{i=1}^n (1 - t^{\deg f_i})}$. By (E2.1.1), $\mathfrak{l}_C = -\deg H_C(t) = \sum_{i=1}^n \deg f_i$. Now it suffices to show that $d_C \leq$

$\sum_{i=1}^{n} \deg f_i - n$. For the argument sake let us assume that $\deg f_i$ is increasing as i goes up. So $d_C = \deg f_n$. Now

$$\sum_{i=1}^{n} \deg f_i - n = \sum_{i=1}^{n} (\deg f_i - 1) \geq \deg f_{n-1} - 1 + \deg f_n - 1 \geq \deg f_n.$$

The assertion follows. □

This result applies to subgroups $G \subset \mathfrak{S}_n$ acting on $k_{-1}[\underline{x}]$.

Let C be any commutative algebra over k and let n be a positive integer. Define D be the algebra generated by C and $\{y_1, \cdots, y_n\}$ subject to the relations

$$[y_i, c] = 0 \tag{E2.2.1}$$

for all $c \in C$, and

$$y_i y_j + y_j y_i = c_{ij} \tag{E2.2.2}$$

for $1 \leq i < j \leq n$, where $\{c_{ij} \mid 1 \leq i < j \leq n\}$ is a subset of the subalgebra $C[y_1^2, \cdots, y_n^2]$ (which is in the center of D).

Lemma 2.3. *Retain the above notation. Then*

(1) $\sigma : \begin{cases} y_i \mapsto -y_i & \forall\, i \\ c \ \mapsto c & \forall\, c \in C \end{cases}$ *extends uniquely to an algebra automorphism of* D*, and*

(2) *Let* $\{w_1, \cdots, w_n\}$ *be a subset of* $C[y_1^2, \cdots, y_n^2]$*. Then* $\phi : \begin{cases} y_i \mapsto w_i & \forall\, i \\ c \ \mapsto 0 & \forall\, c \in C \end{cases}$ *extends uniquely to a* σ*-derivation of* D*.*

Proof. (a) Since D is generated by C and $\{y_i\}_{i=1}^n$, the extension of σ is unique. It is clear that the extension of σ preserves relations (E2.2.1) and (E2.2.2).

(b) Since D is generated by C and $\{y_i\}_{i=1}^n$, the extension of ϕ, using the σ-derivation rule, is unique. For any $c \in C$, using the fact $\phi(c) = 0$, we have

$$\phi([y_i, c]) = \phi(y_i)c - \sigma(c)\phi(y_i) = w_i c - c w_i = 0.$$

For any i,

$$\delta(y_i^2) = \sigma(y_i)\delta(y_i) + \delta(y_i)y_i = -y_i\delta(y_i) + \delta(y_i)y_i = 0.$$

As a consequence, $\delta(c_{ij}) = 0$. Now

$$\begin{aligned} \phi(y_i y_j + y_j y_i - c_{ij}) &= \phi(y_i)y_j + \sigma(y_i)\phi(y_j) + \phi(y_j)y_i + \sigma(y_j)\phi(y_i) \\ &= w_i y_j - y_i w_j + w_j y_i - y_j w_i = 0. \end{aligned}$$

So the extension of ϕ is a σ-derivation. □

We need a lemma on symmetric functions of $k_{-1}[\underline{x}]$. For every positive integer u, let P_u denote the uth power sum $\sum_{i=1}^n x_i^u \in k_{-1}[\underline{x}]$. Let C_1 be the subalgebra of $k_{-1}[\underline{x}]$ generated by $P_2, P_4, \cdots, P_{2n-2}, P_{2n}$, C_3 be the subalgebra of $k_{-1}[\underline{x}]$ generated by $P_1, P_2, P_3, \cdots, P_{2n-1}, P_{2n}$. Define $P_i' = P_i$ is i is odd and $P_i' = P_{2i}$ if i is even. Let C_2 be the subalgebra of $k_{-1}[\underline{x}]$ generated by $P_1', P_2', \cdots, P_{n-1}', P_n'$. Note that C_1 contains P_{2i} for all i.

Lemma 2.4. *Retain the above notation.*

(1) $k_{-1}[\underline{x}]$ *is a finitely generated free module over the central subalgebra* C_1*.*
(2) *If* u *is even, then* $P_u P_v = P_v P_u$ *for any* v*.*
(3) *If* u *and* v *are odd, then* $P_u P_v + P_v P_u = 2P_{u+v}$*.*

(4) *If u is odd, then* $P_u^2 = P_{2u}$.
(5) $C_1 \subset C_2 \subset C_3 \subset k_{-1}[\underline{x}]^{\mathfrak{S}_n} \subset k_{-1}[\underline{x}]^G$.
(6) C_2 *is isomorphic to an iterated Ore extension*

$$R := k[P_4, P_8, \cdots, P_{4\lfloor \frac{n}{2} \rfloor}][P_1][P_3; \tau_3, \delta_3] \cdots [P_{n'}; \tau_{n'}, \delta_{n'}]$$

where $n' = 2\lfloor \frac{n-1}{2} \rfloor + 1$.

Proof. (1) The algebra $k_{-1}[\underline{x}]$ is a finitely generated module over $k[x_1^2, \cdots, x_n^2]$ and $k[x_1^2, \cdots, x_n^2]$ is finitely generated over $C_1 = k[P_2, P_4, \cdots, P_{2n}]$ where each P_{2i} is the ith power sum of the variables $\{x_1^2, \cdots, x_n^2\}$. Therefore $k_{-1}[\underline{x}]$ is finitely generated over C_1. By the proof of Lemma 2.1(1), $k_{-1}[\underline{x}]$ is free over C_1.

(2,3,4) By direct computations.

(5) If i is odd, $(P_i')^2 = (P_i)^2 = P_{2i}$, and if i is even, $P_i' = P_{2i}$. So $C_1 \subset C_2$. The rest is clear.

(6) For odd integers $i < j$, part (3) says that

$$P_j P_i + P_i P_j = 2P_{i+j}.$$

We can easily determine the automorphisms τ_j and derivations δ_j by using Lemma 2.3. As a consequence, there is a surjective map $\phi : R \to C_2$. Also $\operatorname{gldim} R = n = \operatorname{gldim} k_{-1}[\underline{x}]$. By the proof of Lemma 2.1(1), $C \cong R$. □

Theorem 2.5 (Broer's Bound for $k_{-1}[\underline{x}]^G$)**.** *Let G be a subgroup of $\mathfrak{S}_n$ acting on $k_{-1}[\underline{x}]$ naturally. Suppose $|G|$ does not divides char k. Then*

$$d_{(k_{-1}[\underline{x}]^G)} \leq \frac{1}{2}n(n-1) + \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1) \sim \frac{3}{4}n^2.$$

Proof. The assertion can be checked directly for $n = 1, 2$. Assume now that $n \geq 3$. Let $A := k_{-1}[\underline{x}]$ and C be C_2 as in Lemma 2.4(6). Then C is a subalgebra of A^G for any $G \subset \mathfrak{S}_n$. Since $|G|$ does not divides char k, $H := kG$ is semisimple. Note that $\deg P_i = i$. Hence all hypotheses in Lemma 2.2 are satisfied. By Lemma 2.2,

$$d_{A^G} \leq \sum_{i=1} \deg f_i - n = \frac{1}{2}n(n+1) + \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1) - n = \frac{1}{2}n(n-1) + \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1).$$

□

This bound is sharp when $n = 2$ [Example 3.1]. For larger n, we have no examples to show this bound is sharp – and it probably is not sharp.

Next we consider a generalization of the Göbel bound [Go]. If G is a group of permutation of $\{x_i\}_{i=1}^n$ acting as automorphisms on $k[\underline{x}]$ then Göbel's Theorem states that $k[\underline{x}]^G$ is generated by the n symmetric polynomials (or the power sums) and "special polynomials". Let $\mathcal{O}_G(X^I)$ represent the orbit sum of X^I under G. "Special polynomials" are all G-invariants of the form $\mathcal{O}_G(X^I)$, where $\lambda(I) = (\lambda_i)$, the partition associated to I (i.e. arranging the elements of I in weakly decreasing order), has the properties that the last part of the partition $\lambda_n = 0$, and $\lambda_i - \lambda_{i+1} \leq 1$ for all i. It follows that an upper bound on the degree of a minimal set of generators of $k[\underline{x}]^G$ for any n-dimensional permutation representation of G is $\max\{n, \binom{n}{2}\}$. In this context the Göbel bound can be a sharp bound, as it is when the alternating group $\mathfrak{A}_n$ acts on $k[\underline{x}]$. A similar idea works for $k_{-1}[\underline{x}]$, see [CA, Corollary 3.2.4]. But we consider a modification of $\mathfrak{S}_n$.

Let $\widehat{\mathfrak{S}_n}$ be the group $\mathfrak{S}_n \rtimes \{\pm 1\}^n$, where $\{\pm 1\}^n$ is the subgroup of diagonal actions $x_i \to a_i x_i$ for all i, where $a_i = \pm 1$.

Theorem 2.6 (Göbel's Bound for $k_{-1}[\underline{x}]^G$)**.** *Let G be a subgroup of $\widehat{\mathfrak{S}_n}$. Then*

$$d_{k_{-1}[\underline{x}]^G} \leq n^2, \quad \textit{and} \quad d_{k[\underline{x}]^G} \leq n^2.$$

Proof. Let A be $k_{-1}[\underline{x}]$ or $k[\underline{x}]$. Let $C = k[P_2, P_4, \cdots, P_{2n}]$. Then A is a finitely generated free module over C such that $C \subset A^G$. By Lemma 2.2,

$$d_{A^G} \leq \sum_i \deg f_i - n = \sum_i 2i - n = n(n+1) - n = n^2.$$

□

[CA, Corollary 3.2.4] is a consequence of the above theorems.

3. Invariants under the full symmetric group $\mathfrak{S}_n$

Some results in this and the next section have been proved in [CA]. We repeat some of the arguments for completeness.

We consider the ring of invariants $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ under the full symmetric group $\mathfrak{S}_n$. Gauss proved that $k[\underline{x}]^{\mathfrak{S}_n}$ is generated by the n elementary symmetric functions σ_k for $1 \leq k \leq n$, each of which is an orbit sum (sum of all the elements in the $\mathfrak{S}_n$-orbit) of the given monomials. Recall that, for each $1 \leq k \leq n$,

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

These σ_k are algebraically independent, and hence form a commutative polynomial ring $k[\sigma_1, \ldots, \sigma_n]$. As a consequence, $cci(k[\underline{x}]^{\mathfrak{S}_n}) = 0$. Another basis of algebraically independent generators of $k[\underline{x}]^{\mathfrak{S}_n}$ is the set of the n power sums

$$P_k = \sum_{i=1}^{n} x_i^k$$

for $1 \leq k \leq n$. Hence n is the maximal degree of a set of minimal generators for the fixed subring $k[\underline{x}]^{\mathfrak{S}_n}$.

The noncommutative case is different. As we have used in the last section, P_k can be defined in the algebra $k_{-1}[\underline{x}]$ in the same way, which is also an $\mathfrak{S}_n$-invariant. However, considered as an element in $k_{-1}[\underline{x}]$, σ_k is not an $\mathfrak{S}_n$-invariant.

Example 3.1. Let $A = k_{-1}[x_1, x_2]$ and let $G = \langle g \rangle = \mathfrak{S}_2$ for $g = (1, 2)$. Now the element $\sigma_2 = x_1 x_2$ is not invariant, and, moreover, P_2 is not a generator because $P_2 = P_1^2$; it is easy to check that there are no other invariants of degree 2. We will show that the invariants are generated by $P_1 = x_1 + x_2$ and $P_3 = x_1^3 + x_2^3$, or by $S_1 = P_1 = x_1 + x_2$ and $S_2 = x_1^2 x_2 + x_1 x_2^2$. In this example the maximal degree of a minimal set of generators is 3 [Theorem 2.5], which is larger than the order of the group $|G|$ (the "Noether bound" [No] guarantees the maximal degree of a minimal set of generators of $k[\underline{x}]^G$ is $\leq |G|$). In the case of either set of generators, the generators are not algebraically independent, and the ring of invariants is not AS regular, but AS Gorenstein [Theorem 1.5]; and we will show that it is a cci in a couple ways. First, it is a hypersurface in the AS regular algebra B generated by x, y with relations $xy^2 = y^2 x$ and $x^2 y = yx^2$:

$$A^{\mathfrak{S}_2} \cong \frac{B}{(2x^6 - 3x^3 y - 3yx^3 + 4y^2)}$$

(where $P_1 \mapsto x$ and $P_3 \mapsto y$). Second, it is a factor of the iterated Ore extension $C = k[a,b][x][y;\tau,\delta]$, where τ is the automorphism of $k[a,b,x]$ defined by $\tau(a) = a, \tau(b) = b, \tau(x) = -x$, and δ is a τ-derivation of $k[a,b,x]$ defined by $\delta(a) = \delta(b) = 0$ and $\delta(x) = 2b$:

$$A^{\mathfrak{S}_2} \cong \frac{C}{(x^2 - a, y^2 - c)}.$$

Here $\{x^2 - a, y^2 - c\}$ for $c = (3ab - a^3)/2$ is a regular sequence of central elements of C. In this isomorphism $P_2 \mapsto a, P_4 \mapsto b, P_1 \mapsto x, P_3 \mapsto y$, since we have the relations [Lemma 2.4]

$$P_3P_1 + P_1P_3 = 2P_4$$
$$P_1^2 = P_2$$
$$P_3^2 = P_6 = P_2P_4 - P_2(P_2^2 - P_4)/2.$$

The aim of this section is to prove the analogous result for arbitrary n. We first repeat the analysis from [CA] and show that there are two sets of algebra generators of $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$: the n odd power sums $P_1, P_3, \cdots, P_{2n-1}$ and the n elements for $1 \le k \le n$:

$$S_k = \sum x_{i_1}^2 x_{i_2}^2 \cdots x_{i_{k-1}}^2 x_{i_k} =: \mathcal{O}_{\mathfrak{S}_n}(x_1^2 x_2^2 \cdots x_{k-1}^2 x_k)$$

where the sum is taken over all distinct $i_1, \ldots, i_k$ with $i_1 < i_2 < \cdots < i_{k-1}$, and $\mathcal{O}_{\mathfrak{S}_n}$ represents the sum of the orbit under the full symmetric group; we call these elements S_k the "super-symmetric polynomials" since they play the role that the symmetric functions play in the commutative case. Hence the maximal degree of a set of minimal generators for the full ring of invariants $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ is $2n - 1$.

Any monomial in $k_{-1}[\underline{x}]$ can be written as the form $\pm x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, where the sign is due to the fact that these x_is are (-1)-commutative. Let I denote the index $(i_k) := (i_1, \cdots, i_n)$ and let X^I denote the monomial $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. Throughout let G be a subgroup of $\mathfrak{S}_n$ unless otherwise stated. Define

$$stab_G(X^I) = \{g \in G \mid g(X^I) = X^I \text{ in } k_{-1}[\underline{x}]\}.$$

For any permutation $\sigma \in G$, $stab_G(X^I)$ and $stab_G(x_{\sigma(1)}^{i_1} x_{\sigma(2)}^{i_2} \cdots x_{\sigma(n)}^{i_n})$ are conjugate to each other. As a consequence, $|stab_G(X^I)| = |stab_G(x_{\sigma(1)}^{i_1} x_{\sigma(2)}^{i_2} \cdots x_{\sigma(n)}^{i_n})|$.

Definition 3.2. Let $\lambda(m) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be a partition m, where λ_i are weakly decreasing and $\lambda_i \ge 0$. Let X^λ be the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$. The G-orbit sum of the monomial X^λ of (total) degree m is defined by

$$\mathcal{O}_G(X^\lambda) = \mathcal{O}_G(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}) = \frac{1}{|stab_G(X^\lambda)|} \sum_{g \in G} x_{g(1)}^{\lambda_1} x_{g(2)}^{\lambda_2} \cdots x_{g(n)}^{\lambda_n}.$$

In this section we take $G = \mathfrak{S}_n$ and in the next $G = \mathfrak{A}_n$.

Remark 3.3. We divide by the order of the stabilizer of X^λ so that each element of the orbit is counted only once. Throughout we will compare monomials using the length-lexicographical order: for $I = (i_k)$ and $J = (j_k)$ we say $X^I < X^J$ if $\sum i_k < \sum j_k$, or if $\sum i_k = \sum j_k$, and if k is the smallest index for which $i_k \ne j_k$ then $i_k < j_k$; when considering elements of the same degree this order is the lexicographical order on the exponents with $x_1 > x_2 > \ldots > x_n$. Hence we will denote the $\mathfrak{S}_n$-orbit sum by $\mathcal{O}_{\mathfrak{S}_n}(X^I)$, where X^I is the leading term of the orbit sum under the (length)-lexicographic order and so I is a partition, and we call $\mathcal{O}_{\mathfrak{S}_n}(X^I)$ the $\mathfrak{S}_n$-orbit sum corresponding to the partition I. We refer to the entries in I as the "parts" of the partition (so a part may be 0).

The following lemma is easily verified.

Lemma 3.4. [CA, Theorem 2.1.3] *Let G be a finite subgroup of $\mathfrak{S}_n$. Then any G-invariant is a sum of homogeneous G-invariants and homogeneous invariants are linear combinations of G-orbit sums.*

Lemma 3.5. [CA, Lemma 2.2.2] *A $\mathfrak{S}_n$-orbit sum corresponding to a partition $\lambda(m) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is zero if and only if it has repeated odd parts. Hence a non-zero $\mathfrak{S}_n$-orbit sum corresponds to a partition with no repeated odd parts.*

Proof. An orbit sum $\mathcal{O}_{\mathfrak{S}_n}(X^I)$ is zero if and only if the $\mathfrak{S}_n$-orbit of X^I consists of monomials and their negatives, i.e. $\sigma X^I = -X^I$ for some $\sigma \in \mathfrak{S}_n$. In order for $\sigma X^I = -X^I$ there must be a repeated exponent. Consider a monomial of the form $x_1^{e_1} \cdots x_j^{e_j} \cdots x_k^{e_k} \cdots x_n^{e_n}$ where $e_j = e_k$ and both are odd. We claim that when the transposition (j,k) is applied to this monomial we get the same monomial but with a negative sign. We induct on $k-j$. If $k-j=1$ then the result is clear. Hence assume that result is true for $k-j<\ell$ and we prove it for $k-j=\ell$. We write the monomial as $x_1^{e_1} \cdots x_j^{e_j} \cdots x_{k-1}^{e_{k-1}} x_k^{e_k} \cdots x_n^{e_n}$ and consider the case when e_{k-1} is odd and the case when e_{k-1} is even. When e_{k-1} is odd then (j,k) applied to the monomial yields

$$\begin{aligned} &x_1^{e_1} \cdots x_k^{e_k} \cdots x_{k-1}^{e_{k-1}} x_j^{e_j} \cdots x_n^{e_n} \\ &\quad = -x_1^{e_1} \cdots x_k^{e_k} \cdots x_j^{e_j} x_{k-1}^{e_{k-1}} \cdots x_n^{e_n} \end{aligned}$$

which by induction is

$$\begin{aligned} &= x_1^{e_1} \cdots x_j^{e_j} \cdots x_k^{e_k} x_{k-1}^{e_{k-1}} \cdots x_n^{e_n} \\ &= -x_1^{e_1} \cdots x_j^{e_j} \cdots x_{k-1}^{e_{k-1}} x_k^{e_k} \cdots x_n^{e_n}. \end{aligned}$$

When e_{k-1} is even then (j,k) applied to the monomial yields

$$\begin{aligned} &x_1^{e_1} \cdots x_k^{e_k} \cdots x_{k-1}^{e_{k-1}} x_j^{e_j} \cdots x_n^{e_n} \\ &\quad = x_1^{e_1} \cdots x_k^{e_k} \cdots x_j^{e_j} x_{k-1}^{e_{k-1}} \cdots x_n^{e_n} \end{aligned}$$

which by induction is

$$\begin{aligned} &= -x_1^{e_1} \cdots x_j^{e_j} \cdots x_k^{e_k} x_{k-1}^{e_{k-1}} \cdots x_n^{e_n} \\ &= -x_1^{e_1} \cdots x_j^{e_j} \cdots x_{k-1}^{e_{k-1}} x_k^{e_k} \cdots x_n^{e_n}. \end{aligned}$$

Hence $\sigma X^I = -X^I$, and so for any τX^I in the $\mathfrak{S}_n$-orbit of X^I we have $-\tau X^I = \tau\sigma X^I$ is in the orbit of X^I, and hence the $\mathfrak{S}_n$-orbit sum of X^I is zero.

Clearly when indices with even exponents of the same value are permuted no sign change occurs, and so the orbit sum will not be zero unless there is at least one repeated odd exponent. □

By Lemma 3.5 the set of elements in $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ of degree k has a vector space basis corresponding to the partitions of k into at most n parts with no repeated odd entries. We next will show that both the sets S_k and P_{2k-1} for $k = 1, \ldots, n$ (corresponding to the partitions $(2, \ldots, 2, 1, 0, \ldots, 0)$ and $(2k-1, 0, \ldots, 0)$ of $2k-1$, respectively) are algebra generators of $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$.

Lemma 3.6. *Let $I = (\lambda_k)$ be a partition where no λ_i are both equal and odd. The leading term of $\mathcal{O}_{\mathfrak{S}_n}(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}) S_k$ is $x_1^{\lambda_1+2} \cdots x_{k-1}^{\lambda_{k-1}+2} x_k^{\lambda_k+1} x_{k+1}^{\lambda_{k+1}} \cdots x_n^{\lambda_n}$.*

Proof. By our assumption on I the orbit of X^I does not contain another element with the same entries as X^I. Clearly $x_1^{\lambda_1+2}\cdots x_{k-1}^{\lambda_{k-1}+2}x_k^{\lambda_k+1}x_{k+1}^{\lambda_{k+1}}\cdots x_n^{\lambda_n}$ is a summand of the product of $\mathcal{O}_{\mathfrak{S}_n}(x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n})S_k$. This product of orbits can be written as a linear combination of $\mathfrak{S}_n$-orbit sums; let $\mathcal{O}_{\mathfrak{S}_n}(X^E)$ be one of these orbit sums. The entries of the any such partition E are obtained from the partition $I=(\lambda_1,\cdots,\lambda_n)$ by adding 2 to $k-1$ entries of I, adding 1 to one entry of I, and placing these entries into numerical order. It is clear that the largest such partition E that can be obtained in this manner is $(\lambda_1+2,\ldots,\lambda_{k-1}+2,\lambda_k+1,\lambda_{k+1},\cdots,\lambda_n)$, and the leading term of this $\mathfrak{S}_n$-orbit sum occurs in the product of orbits only once. □

The following lemma follows essentially as in Gauss's proof for $k[\underline{x}]^{\mathfrak{S}_n}$; the supersymmetric polynomials $S_k\in k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ play the role of the symmetric polynomials σ_k in $k[\underline{x}]^{\mathfrak{S}_n}$.

Lemma 3.7. *Suppose that $f\neq 0$ is a $\mathfrak{S}_n$-invariant with leading term $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$ of degree m where at least one λ_k odd. Then there is a positive integer k, a partition $\lambda^*(m-2k+1)=(\lambda_1^*,\ldots,\lambda_n^*)$ of $m-2k+1$, and a $c\in k^\times$ such that*

$$f-c\,\mathcal{O}_{\mathfrak{S}_n}(x_1^{\lambda_1^*}\cdots x_n^{\lambda_n^*})S_k$$

has leading term of smaller degree than f. As a consequence, the fixed subring $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ is generated as an algebra by the n elements S_k, for $k=1,\ldots,n$, and invariants with all even powers, $k[x_1^2,\cdots,x_n^2]^{\mathfrak{S}_n}$.

Proof. $I=(\lambda_i)$ is a partition and hence is weakly decreasing. Let k be the largest index with λ_k odd, and let

$$I^*=(\lambda_1-2,\lambda_2-2,\ldots,\lambda_{k-1}-2,\lambda_k-1,\lambda_{k+1},\ldots,\lambda_n).$$

We claim that I^* is a weakly decreasing sequence. First note that since λ_k is odd, $\lambda_k\geq 1$, and for $\ell\geq k+1$ the λ_ℓ are even and weakly decreasing, so for $\ell\geq k+1$ we have $\lambda_k\geq\lambda_\ell+1\geq\lambda_{\ell+1}+1$, and the final $n-k+1$ entries of I^* are weakly decreasing. Next, since λ_k is odd and there are no repeated odd exponents in a nonzero $\mathfrak{S}_n$-orbit sum, we have $\lambda_{k-1}\geq\lambda_k+1$ and $\lambda_{j-2}-2\geq\lambda_{j-1}-2\geq\lambda_k-1$ for $3\leq j\leq k$, so the first k entries of I^* are weakly decreasing. Hence by Lemma 3.6 we have

$$x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}=(x_1^{\lambda_1-2}\cdots x_{k-1}^{\lambda_{k-1}-2}x_k^{\lambda_k-1}\cdots x_n^{\lambda_n})(x_1^2\cdots x_{k-1}^2x_k)$$

is the leading term in $\mathcal{O}_{\mathfrak{S}_n}(X^{I^*})S_k$, and if c is the coefficient of the leading term of f then $c\,\mathcal{O}(X^{I^*})S_k-f$ has smaller order leading term. Furthermore $\mathcal{O}(X^{I^*})$ also has smaller order. Since there are only a finite number of smaller orders, the algorithm must terminate when all exponents are even. □

Since the central subring $k[x_1^2,\ldots,x_n^2]$ of $k_{-1}[\underline{x}]$ is a commutative polynomial ring and $\mathfrak{S}_n$ acts on it as permutations, the invariants $k[x_1^2,\ldots,x_n^2]^{\mathfrak{S}_n}$ are generated by either the even power sums $P_2,\cdots,P_{2n}$ or the n symmetric polynomials in the squares; in particular, if $\rho_i:=\sigma_i(x_1^2,\cdots,x_n^2)$ for the elementary symmetric function σ_i, then $k[x_1^2,\cdots,x_n^2]^{\mathfrak{S}_n}=k[\rho_1,\rho_2,\ldots,\rho_n]$. Since $P_{2k}\in k[\rho_1,\rho_2,\ldots,\rho_n]$, each P_{2k} can be expressed as a polynomial in the elementary symmetric functions, say

$$P_{2k}=f_{2k}(\rho_1,\rho_2,\ldots,\rho_n). \tag{E3.7.1}$$

Next we show that $k[x_1^2,\dots,x_n^2]^{\mathfrak{S}_n}$ is contained in the algebra generated by the n odd power sums $P_1,\dots,P_{2n-1}$, and $k[x_1^2,\dots,x_n^2]^{\mathfrak{S}_n}$ is contained in the algebra generated by the n super-symmetric polynomials S_k.

Lemma 3.8. *The fixed subring $k[x_1^2,\dots,x_n^2]^{\mathfrak{S}_n}$ is contained in the algebra generated by either the odd power sums $P_1,\cdots,P_{2n-1}$ or by the super-symmetric polynomials $S_1,\cdots,S_n$ in $k_{-1}[\underline{x}]$.*

Proof. We obtain the even power sums from the odd ones as follows: $P_2 = P_1^2$, and more generally

(E3.8.1) $$P_{2i} = (P_1P_{2i-1} + P_{2i-1}P_1)/2$$

for all $1 \le i \le n$. Also

(E3.8.2) $$\rho_j = \mathcal{O}_{\mathfrak{S}_n}(x_1^2\cdots x_j^2) = (S_1S_j + S_jS_1)/(2j)$$

for all $1 \le j \le n$. □

The next argument follows as in the case of $k[\underline{x}]$ [S1, p. 4]. Given a monomial X^I, we define $\lambda(I)$, the partition associated with X^I, to be the elements of I listed in weakly decreasing order (i.e. the partition associated to $\mathcal{O}_{\mathfrak{S}_n}(X^I)$). We define a total order on the set of monomials as $X^I < X^J$ if the associated partitions have the property that $\lambda(I)$ is lexicographically *larger* than $\lambda(J)$, or, if the partitions are equal, when I is lexicographically *smaller* than J. As an example for $n = 3$ and degree $= 4$

$$x_3^4 < x_2^4 < x_1^4 < x_2x_3^3 < x_2^3x_3 < x_1x_3^3 < x_1x_2^3 < x_1^3x_3 < x_1^3x_2 < x_2^2x_3^2$$
$$< x_1^2x_3^2 < x_1^2x_2^2 < x_1x_2x_3^2 < x_1x_2^2x_3 < x_1^2x_2x_3.$$

In the case of $k[\underline{x}]$, where all partitions represent basis elements in the subring of invariants, in a given degree $k \le n$ the "largest" partition is $(1,\dots,1,0\dots,0)$, while the "smallest" partition is $(k,0\dots,0)$. In the case of of $k_{-1}[\underline{x}]$, for monomials that correspond to nonzero invariants there are no repeated odd parts, so for odd degrees $2k-1 \le 2n-1$, the partition $(2,\dots,2,1,0,\dots,0)$ is "largest" under this order, and while the partition $(2k-1,0\dots,0)$ is smallest, and x_n^{2k-1} is the smallest monomial of degree $2k-1$. Furthermore in a product of power sums

$$P_{i_1}P_{i_2}\cdots P_{i_k}$$

the leading monomial will be $cx_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}$ for some nonzero integer c when the i_j are weakly decreasing.

Lemma 3.9. *The fixed subring $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ is generated by the n odd power sums $P_1,\dots,P_{2n-1}$.*

Proof. By Lemma 3.8 the even power sums are generated by the odd power sums $P_1,\dots,P_{2n-1}$, so it suffices to show invariants are generated by power sums P_k for $k \le 2n-1$. By Lemmas 3.7 and 3.8 the S_k are algebra generators of $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$, so it suffices to show they can be expressed in terms of power sums. Hence it suffices to describe an algorithm that writes an invariant $f \in k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ of degree $\le 2n-1$ as a product of power sums. Write the leading term of f as $ax_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ for some $a \in k^\times$. The exponents of the leading term are weakly decreasing, and each is $\le 2n-1$. The element $f - \frac{a}{c}P_{i_1}P_{i_2}\cdots P_{i_n}$ has the same total degree as f, but its leading term is less than that of f. Since there are only a finite number of monomials of smaller order for a fixed degree, the algorithm terminates with f written in terms of power sums of degree $\leq 2n-1$. $\square$

The following theorem of Cameron Atkins follows from the lemmas above, and gives us two choices of algebra generators for $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$. It is often convenient to choose the power sums, since they have fewer summands.

Theorem 3.10. [CA, Theorems 2.2.6 and 2.2.8] *The fixed subring $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ is generated by either the set of the n odd power sums $P_1, \cdots, P_{2n-1}$ or the set of the n super-symmetric polynomials $S_1, \cdots, S_n$.*

We next show that the AS Gorenstein domain $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ is a cci. First we have to construct a suitable AS regular algebra.

Let $R = k[p_1, p_2, \ldots, p_n]$ be a commutative polynomial ring, and let $a_{2i} = f_{2i}(p_1, p_2, \ldots, p_n)$ where the f_{2i} are the polynomials of (E3.7.1). Consider the following iterated Ore extension

$$B = k[p_1, \ldots, p_n][y_1 : \tau_1, \delta_1] \cdots [y_n : \tau_n, \delta_n]$$

where coefficients are written on the left, $R = k[p_1, \ldots, p_n]$ is a commutative polynomial ring, τ_j is the automorphism of $k[p_1, \ldots, p_n][y_1 : \tau_1, \delta_1] \cdots [y_{j-1} : \tau_{j-1}, \delta_{j-1}]$ defined by $\tau_j(y_i) = -y_i$ for $i < j$ and $\tau_j(r) = r$ for $r \in k[p_1, \ldots, p_n]$, and δ_j is the τ_j-derivation $\delta_j(y_i) = 2a_{2i+2j-2}$ with $\delta_j(r) = 0$ for all $r \in k[p_1, \ldots, p_n]$.

By Lemma 2.3, δ_k are τ_k-derivation for all k where $(\tau_k \delta_k)$ appeared in the definition of B.

We grade B by setting degree$(p_i) = 2i$ and degree$(y_i) = 2i-1$. With this grading the Hilbert series of B is given by

$$H_B(t) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^{2n-1})(1-t^{2n})}.$$

The algebra B is an AS regular algebra of dimension $2n$. Let $r_i = y_i^2 - a_{4i-2}$ for each $i = 1, 2, \ldots, n$; it is easy to see that r_i is a central element of B.

Lemma 3.11. *The sequence $\{r_1, r_2, \ldots, r_n\}$ is a central regular sequence in B.*

Proof. First we note that the r_i are central since a_i and y_i^2 are central

$$y_i^2 y_j = y_i(-y_j y_i + p_{i+j}) = -y_i y_j y_i + y_i p_{i+j} = -(y_i y_j + p_{i+j}) y_i = y_j y_i^2.$$

Since B is a domain, $r_1 \neq 0$ is regular in B.

Let $B_i = k[p_1, \ldots, p_n][y_1 : \tau_1, \delta_1] \cdots [y_i : \tau_i, \delta_i]$ and let $\overline{B_i} = B_i/(r_1, r_2, \ldots, r_i)_{B_i}$. Now consider the algebra $C_i = \overline{B_i}[y_{i+1} : \overline{\tau_{i+1}}, \overline{\delta_{i+1}}] \cdots [y_n : \overline{\tau_n}, \overline{\delta_n}]$, where the $\overline{\tau_j}$ and $\overline{\delta_j}$ are the induced maps. These maps are well-defined since for $j > i$ and $k \leq i, \tau_j(r_k) = r_k$ and $\delta_j(r_k) = 0$. Note that $B = B_i[y_{i+1} : \tau_{i+1}, \delta_{i+1}] \cdots [y_n : \tau_n, \delta_n]$, and hence every element of B can be written in the form $\sum_I b_I y^I$ where $b_I \in B_i, I = (e_{i+1}, e_{i+2}, \ldots, e_n)$ is a nonnegative integral vector, and $y^I = y_{i+1}^{e_{i+1}} y_{i+2}^{e_{i+2}} \cdots y_n^{e_n}$. The algebra $B/(r_1, r_2, \ldots, r_i)_B$ is isomorphic to the algebra C_i under the map

$$\sum_I b_I y^I + \langle r_1, r_2, \ldots, r_i \rangle_B \mapsto \sum_I \bar{b_I} y^I$$

where $\bar{b_I}$ denotes reduction mod $(r_1, r_2, \ldots, r_i)_{B_i}$. Now the standard polynomial degree argument in C_i shows that the image of r_{i+1} is regular in C_i. $\square$

We now can prove that $k_{-1}[\underline{x}]^{\mathfrak{S}_n} \cong B/(r_1, r_2, \ldots, r_n)$ where, by Lemma 3.11, each r_i is central in B and regular in $B/(r_1, r_2, \ldots, r_{i-1})$.

Theorem 3.12. *The algebra* $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ *is a cci.*

Proof. By Definition 3.2 and Lemma 3.5 $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ as a graded vector space has a basis of orbit sums of monomials having no repeated odd exponents. Hence its Hilbert series is the same as the generating function for the restricted partitions having no repeated odd parts. By Proposition 5.1 of the Appendix this Hilbert series is given by

$$D_n(t) \quad = \quad \frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-2})}{(1-t)(1-t^2)(1-t^3)\cdots(1-t^{2n-1})(1-t^{2n})}.$$

Let $\rho_i = \sigma_i(x_1^2, x_2^2, \ldots, x_n^2)$ where σ_i is the ith elementary symmetric polynomial. Then the algebra $k[x_1^2, x_2^2, \ldots, x_n^2]^{\mathfrak{S}_n} = k[\rho_1, \rho_2, \ldots, \rho_n]$ is a commutative polynomial ring. By Theorem 3.10, $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ is generated as an algebra by the odd power sums, and hence $k_{-1}[\underline{x}]^{\mathfrak{S}_n} = k[\rho_1, \rho_2, \ldots, \rho_n][P_1, P_3, \ldots, P_{2n-1}]$.

Consider the iterated Ore extension B constructed above and define a map $\phi : B \longrightarrow k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ by $\phi(p_i) = \rho_i$ and $\phi(y_j) = P_{2j-1}$. Note that ϕ preserves degree. Clearly ϕ takes $R = k[p_1, p_2, \ldots, p_n]$ isomorphically onto $k[\rho_1, \rho_2, \ldots, \rho_n]$, and both subrings are central. In the iterated Ore extension B, we have for $i < j$ that

$$y_j y_i + y_i y_j = 2a_{2i+2j-2} = f_{2i+2j-2}(p_1, p_2, \ldots, p_n).$$

Calculation in $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ shows that

$$P_{2j-1}P_{2i-1} + P_{2i-1}P_{2j-1} = 2P_{2i+2j-2} = 2f_{2i+2j-2}(\rho_1, \rho_2, \ldots, \rho_n);$$

hence

$$\phi(y_j)\phi(y_i) + \phi(y_i)\phi(y_j) = 2\phi(a_{2i+2j-2}).$$

Hence the skew extension relations are preserved, and we conclude that ϕ is a graded ring homomorphism. Since the odd power sums $P_1, P_3, \ldots, P_{2n-1}$ generate $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ as an algebra by Theorem 3.10, the homomorphism ϕ is an epimorphism.

Calculation yields

$$\begin{aligned} 0 \quad &= \quad P_{2i-1}^2 - P_{4i-2} = P_{2i-1}^2 - f_{4i-2}(\rho_1, \rho_2, \ldots, \rho_n) \\ &= \quad \phi(y_i^2 - a_{4i-2}) = \phi(r_i). \end{aligned}$$

Hence the ideal $(r_1, r_2, \ldots, r_n) \subseteq \ker(\phi)$, and ϕ induces a graded ring homomorphism

$$\bar{\phi} : B/(r_1, r_2, \ldots, r_n) \longrightarrow k_{-1}[\underline{x}]^{\mathfrak{S}_n}.$$

Since for each i the degree of r_i is $4i-2$ and $\{r_1, r_2, \ldots, r_n\}$ is a regular sequence, the Hilbert series of $\bar{B} = B/(r_1, r_2, \ldots, r_n)$ is given by

$$H_{\bar{B}}(t) \quad = \quad \frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-2})}{(1-t)(1-t^2)(1-t^3)\cdots(1-t^{2n-1})(1-t^{2n})}.$$

This shows that $\bar{\phi}$ is an isomorphism. □

Definition 3.13. Let A be a connected graded noetherian algebra.

(1) We say A is a *classical complete intersection*$^+$ (or a *cci*$^+$) if there is a connected graded noetherian AS regular algebra R with $H_R(t) = \frac{1}{\prod_{i=1}^n (1-t_i^d)}$ and a sequence of regular normal homogeneous elements $\{\Omega_1, \ldots, \Omega_n\}$ of positive degree such that A is isomorphic to $R/(\Omega_1, \ldots, \Omega_n)$. The minimum such n is called the *cci*$^+$*-number* of A and denoted by $cci^+(A)$.

(2) Let A be cyclotomic (e.g., A is cci). The *cyc*-number of A, denoted by $cyc(A)$, is defined to be v if the Hilbert series of A is of the form

$$H_A(t) = \frac{\prod_{s=1}^{v}(1-t^{m_s})}{\prod_{s=1}^{w}(1-t^{n_s})}$$

where $m_s \neq n_{s'}$ for all s and s'.

Clearly we have $cci^+(A) \geq cci(A)$. It is a conjecture that every noetherian AS regular algebra has Hilbert series of the form $\frac{1}{\prod_{i=1}^{n}(1-t_i^d)}$. If this conjecture holds, then being cci$^+$ is equivalent to being cci and $cci^+(A) = cci(A)$. One can easily show that the expression of $H_A(t)$ in Definition 3.13(2) is unique (as we assume that $m_s \neq n_{s'}$ for all s, s'). It follows from the definition that $cci^+(A) \geq cyc(A)$. Finally we would like to calculate $cci^+(k_{-1}[\underline{x}]^{\mathfrak{S}_n})$.

Theorem 3.14. $cci^+(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) = cyc(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) = \lfloor \frac{n}{2} \rfloor$.

Proof. First we prove the claim that $cci^+(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) \leq \lfloor \frac{n}{2} \rfloor$.

Let C_2 be the subalgebra of $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ defined before Lemma 2.4. By Lemma 2.4(6), it is isomorphic to the iterated Ore extension

$$k[P_4, P_8, \cdots, P_{4\lfloor \frac{n}{2} \rfloor}][P_1][P_3; \tau_3, \delta_3] \cdots [P_{n'}; \tau_{n'}, \delta_{n'}]$$

where $n' = 2\lfloor \frac{n-1}{2} \rfloor + 1$. By Lemma 2.4(5), C_2 contains P_{2i} for all $i \geq 1$. Let $F_{n'} := C_2$, and for any odd integer $n' < j \leq 2n-1$, we inductively construct a sequence of iterated Ore extensions $F_j = F_{j-2}[P_j, \tau_j, \delta_j]$ where τ_j is defined by $\tau_j(P_s) = (-1)^s P_s$ for all even s and for all odd $s \leq j-2$, and where the τ_j-derivation δ_j is defined by $\delta_j(P_s) = \begin{cases} 0 & \text{if } s \text{ is even} \\ 2P_{s+j} & \text{if } s \text{ is odd.} \end{cases}$ It follows from the induction and Lemma 2.3 that τ_j is an automorphism of F_{j-2} and δ_j is a τ_j-derivation of F_{j-2}. Therefore F_j (and whence F_{2n-1}) is an iterated Ore extension (which is a noetherian AS regular algebra with Hilbert sires of the form $(\prod_{i=1}^{n}(1-t^{d_i}))^{-1}$). Let $u_s = P_{2s-1}^2 - P_{4s-2}$ for all integers from $s = \lfloor \frac{n-1}{2} \rfloor + 2$ to $s = n$. The proof of Lemma 3.11 shows that $\{u_{\lfloor \frac{n-1}{2} \rfloor + 2}, \cdots, u_n\}$ is a central regular sequence of F_{2n-1}. It is easy to see that $F_{2n-1}/(u_{\lfloor \frac{n-1}{2} \rfloor + 2}, \cdots, u_n) \cong k_{-1}[\underline{x}]^{\mathfrak{S}_n}$. Therefore $cci^+(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) \leq n - (\lfloor \frac{n-1}{2} \rfloor + 1) = \lfloor \frac{n}{2} \rfloor$ and we proved the claim.

By Theorem 3.12

$$\begin{aligned} H_{k_{-1}[\underline{x}]^{\mathfrak{S}_n}}(t) = H_{\bar{B}}(t) &= \frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-2})}{(1-t)(1-t^2)(1-t^3)\cdots(1-t^{2n-1})(1-t^{2n})} \\ &= \frac{\prod_{s=\lfloor \frac{n-1}{2} \rfloor + 2}^{n}(1-t^{4s-2})}{\prod_{j=1}^{\lfloor \frac{n}{2} \rfloor}(1-t^{4j}) \prod_{i=1}^{n}(1-t^{2i-1})} \end{aligned}$$

which is an expression satisfying the condition in Definition 3.13(2). Hence

$$cyc(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) = \lfloor \frac{n}{2} \rfloor.$$

The assertion follows from the claim and the fact $cci^+(A) \geq cyc(A)$. $\square$

4. Invariants under $\mathfrak{A}_n$

First let us review the classical case. Let $\mathfrak{A}_n$ be the alternating group. Any element of $k[\underline{x}]^{\mathfrak{A}_n}$ can be written uniquely as $h_1 + Dh_2$, where h_1 and h_2 are symmetric polynomials and D is the "Vandermonde determinant"

$$D = D(x_1, \cdots, x_n) = \prod_{i<j}(x_i - x_j)$$

[S1, p. 5]. Hence the maximal degree of a minimal set of generators of $k[\underline{x}]^{\mathfrak{A}_n}$ is $\binom{n}{2}$. A polynomial f is called "antisymmetric" if $\tau f = -f$ for every odd permutation $\tau \in \mathfrak{S}_n$ [S1, p. 5]; D is the smallest degree antisymmetric element of $k[\underline{x}]^{\mathfrak{A}_n}$. Moreover, D^2 is a symmetric polynomial, the Hilbert series of $k[\underline{x}]^{\mathfrak{A}_n}$ is

$$\frac{1+t^r}{\prod_{i=1}^n (1-t^i)} = \frac{1-t^{2r}}{(1-t^r)\prod_{i=1}^n (1-t^i)}$$

for $r = \binom{n}{2}$ [Be, pp. 104-5], and hence $k[\underline{x}]^{\mathfrak{A}_n}$ is isomorphic to the complete intersection

$$\frac{k[\sigma_1, \ldots, \sigma_n][y]}{(y^2 - D^2)}$$

under the map that associates y to D (and the symmetric polynomial in the x_i to σ_i). Following Definition 3.13, one easily gets

$$cci(k[\underline{x}]^{\mathfrak{A}_n}) = cci^+(k[\underline{x}]^{\mathfrak{A}_n}) = cyc(k[\underline{x}]^{\mathfrak{A}_n}) = 1.$$

The group $\mathfrak{A}_n$ is generated by 3-cycles, which have trace

$$\mathrm{Tr}_{k[\underline{x}]}(g, t) = \frac{1}{(1-t^3)(1-t)^{n-3}},$$

and hence are bireflections of $k[\underline{x}]$; the 3-cycles are a generating set of bireflections that the Kac-Watanabe-Gordeev Theorem states must exist since $k[\underline{x}]^{\mathfrak{A}_n}$ is a complete intersection.

In this section we consider the analogous situation for $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ for $n \geq 3$. As a general setup, we are working with the noncommutative algebra $k_{-1}[\underline{x}]$ unless otherwise stated. Again there is an overlap between [CA] and this section.

The trace of a 3-cycle g acting on $k_{-1}[\underline{x}]$ is also

$$\mathrm{Tr}_{k_{-1}[\underline{x}]}(g, t) = \frac{1}{(1-t^3)(1-t)^{n-3}},$$

hence $\mathfrak{A}_n$ is generated by quasi-bireflections of $k_{-1}[\underline{x}]$. The aim of this section is to show that $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ is a cci, which is consistent with the conjectured generalization of the Kac-Watanabe-Gordeev Theorem. Here the smallest degree antisymmetric polynomial is $\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})$, and the subring of invariants $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ is generated by $\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})$, and either the $n-1$ super-symmetric polynomials $S_1, \ldots, S_{n-1}$ or the power sums $P_1, \ldots, P_{2n-3}$, and so an upper bound on the degrees of generators of $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ is $2n-3$. We will show that the Hilbert series of $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ is given by

$$\text{(E4.0.1)} \qquad H_{k_{-1}[\underline{x}]^{\mathfrak{A}_n}}(t) = \frac{(1+t)(1+t^3)\cdots(1+t^{2n-3})(1+t^n)(1+t^{n-1})}{(1-t^2)(1-t^4)\cdots(1-t^{2n})}.$$

We construct invariants under $\mathfrak{A}_n$ as $\mathcal{O}_{\mathfrak{A}_n}(X^I)$, the sum of the orbit of a monomial X^I under $\mathfrak{A}_n$ [Definition 3.2]; we note that the number of terms in this sum is the index of the $\mathfrak{A}_n$-stabilizer of X^I in $\mathfrak{A}_n$.

Lemma 4.1. [CA, Lemma 4.1.1] *If there is an odd permutation that stabilizes X^I then $f = \mathcal{O}_{\mathfrak{A}_n}(X^I)$ is also invariant under the full symmetric group $\mathfrak{S}_n$.*

Proof. Since the index of the subgroup $stab_{\mathfrak{A}_n}(X^I)$ in $stab_{\mathfrak{S}_n}(X^I)$ is less than or equal to $[\mathfrak{S}_n : \mathfrak{A}_n] = 2$, if there is an odd permutation that stabilizes X^I then the index $[stab_{\mathfrak{S}_n}(X^I) : stab_{\mathfrak{A}_n}(X^I)] = 2$, and the order of the orbit X^I under $\mathfrak{S}_n = [\mathfrak{S}_n : stab_{\mathfrak{S}_n}(X^I)]$ is the same as the order of the orbit of X^I under $\mathfrak{A}_n = [\mathfrak{A}_n : stab_{\mathfrak{A}_n}(X^I)]$, so the orbit sum of X^I under $\mathfrak{S}_n$ is the same as that under $\mathfrak{A}_n$; hence $\mathcal{O}_{\mathfrak{A}_n}(X^I)$, the orbit sum of X^I under $\mathfrak{A}_n$, is $\mathfrak{S}_n$-invariant. □

Here is an immediate consequence.

Corollary 4.2. *If $I = (i_j)$ with at least 2 indices $i_j = i_k$, an even number, then $\mathcal{O}_{\mathfrak{A}_n}(X^I)$ is an $\mathfrak{S}_n$-invariant. In particular if there are at least 2 indices $i_j = i_k = 0$ then $\mathcal{O}_{\mathfrak{A}_n}(X^I)$ is an $\mathfrak{S}_n$-invariant.*

Lemma 4.3. [CA, Lemma 4.1.2] *An $\mathfrak{A}_n$-orbit sum $\mathcal{O}_{\mathfrak{A}_n}(X^I) = 0$ if and only if I has at least two indices $i_j = i_k$ an even number, and two indices $i_r = i_s$ an odd number.*

Proof. If I has repeated even indices then $\mathcal{O}_{\mathfrak{A}_n}(X^I) = \mathcal{O}_{\mathfrak{S}_n}(X^I)$ by Corollary 4.2. Since I has repeated odd indices then $\mathcal{O}_{\mathfrak{S}_n}(X^I) = 0$ by Lemma 3.5.

Conversely, suppose that $\mathcal{O}_{\mathfrak{A}_n}(X^I) = 0$, then X^I and $-X^I$ are in the $\mathfrak{A}_n$-orbit of X^I, hence in the $\mathfrak{S}_n$-orbit of X^I. Hence for every τX^I in the $\mathfrak{S}_n$-orbit of X^I we also have $-\tau X^I$ in the $\mathfrak{S}_n$-orbit, and so the $\mathfrak{S}_n$ orbit sum is 0, which forces at least two indices to have the same odd value by Lemma 3.5. We have $\tau X^I = -X^I$ for an even permutation τ. Write τ as a product of disjoint cycles

$$\tau = \nu_1 \cdots \nu_{2m} \mu_1 \cdots \mu_k$$

where the ν_i are odd permutations and the μ_j are even permutations. Note that since $\tau X^I = -X^I$ exponents in I must be constant over the support of each cycle. Suppose there are no repeated even indices in I, so that all repeats are of odd indices. Hence for each $\mu_j = (a_1, \cdots, a_{2s_j+1})$, an even cycle, μ_j can be written as an even number of transpositions, interchanging variables with the same odd exponent. By the proof of Lemma 3.5 each of these transpositions maps X^I to $-X^I$, and hence $\mu_j X^I = X^I$, For similar reasons each $\nu_i X^I = -X^I$. It follows that $\tau X^I = \nu_1 \cdots \nu_{2m} \mu_1 \cdots \mu_k X^I = X^I$, a contradiction. Hence I must also contain at two indices with the same even number. □

Note that $\mathfrak{A}_n$-orbit sums do not necessarily correspond to partitions, e.g. when $n = 4$ the orbit sums $\mathcal{O}_{\mathfrak{A}_n}(x_1^4 x_2^3 x_3^2 x_4)$ and $\mathcal{O}_{\mathfrak{A}_n}(x_1^4 x_2^3 x_3 x_4^2)$ are different (and $\mathcal{O}_{\mathfrak{A}_n}(x_1^4 x_2^3 x_3^2 x_4) + \mathcal{O}_{\mathfrak{A}_n}(x_1^4 x_2^3 x_3 x_4^2) = \mathcal{O}_{\mathfrak{S}_n}(x_1^4 x_2^3 x_3^2 x_4)$).

Adapting the classical definition, an element $g \in k_{-1}[\underline{x}]$ is called *symmetric* (respectively, *antisymmetric*) if $\tau(g) = g$ (respectively, $\tau(g) = -g$) for every odd permutation $\tau \in \mathfrak{S}_n$. Note that g is symmetric if and only if g is $\mathfrak{S}_n$-invariant. If g is antisymmetric, then g is $\mathfrak{A}_n$-invariant. The following lemma follows easily.

Lemma 4.4. *Let f, g, h be elements in $k_{-1}[\underline{x}]$.*

(1) *Linear combinations of antisymmetric invariants are antisymmetric. Hence if $f+g$ and g are antisymmetric invariants, then f is an antisymmetric invariant.*
(2) *If $f = gh$ with g an antisymmetric invariant and h a symmetric invariant, then f is an antisymmetric invariant.*
(3) *If $f = gh$ with f and g antisymmetric invariants then h a symmetric invariant.*

The following lemma follows as in the case of $k[\underline{x}]$ and the proof is omitted.

Lemma 4.5. [CA, Theorem 4.1.4] *If f is an $\mathfrak{A}_n$-invariant and σ is the transposition $(1,2)$ then $\sigma f = \tau f$ for any odd permutation τ. Furthermore $f + \sigma f$ is symmetric and $f - \sigma f$ is antisymmetric. As a consequence, each invariant $f \in k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ can be be written uniquely as the sum of a symmetric invariant and an antisymmetric invariant.*

Example 4.6. For $n \geq 3$ the following are examples of antisymmetric invariants: some $\mathfrak{A}_n$-orbits are antisymmetric (e.g. $\mathcal{O}_{\mathfrak{A}_n}(x_1^4x_2x_3)$ and $\mathcal{O}_{\mathfrak{A}_n}(x_1^3x_2^3x_3)$) and antisymmetric elements can be constructed from the lemma above (e.g. $f - \sigma f$ for $f = \mathcal{O}_{\mathfrak{A}_n}(x_1^4x_2^3x_3^2\ ))$.

For the rest of this section, we assume that $n \geq 3$ as $\mathfrak{A}_2$ is trivial. In the case $k_{-1}[\underline{x}]$ we have the two antisymmetric orbit sums given in the lemma below; the orbit sums of these monomials are symmetric polynomials when $\mathfrak{A}_n$ acts on $k[\underline{x}]$.

Lemma 4.7. *The $\mathfrak{A}_n$ orbit sums*

$$\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_n) = x_1x_2\cdots x_n \quad \text{and} \quad \mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})$$

are both antisymmetric $\mathfrak{A}_n$-invariants. And $\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})$ is the smallest degree antisymmetric invariant.

Proof. It is easy to show that $x_1x_2\cdots x_n$ is an antisymmetric $\mathfrak{A}_n$-invariant, whence $\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_n) = x_1x_2\cdots x_n$. So we focus on $\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})$.

We note that

$$\mathcal{O}_{\mathfrak{A}_3}(x_1x_2) = x_1x_2 - x_1x_3 + x_2x_3.$$

For $n \geq 4$ applying the even permutation $(1,2)(n-1,n)$ to $x_1x_2\cdots x_{n-1}$ we obtain

$$(1,2)(n-1,n)(x_1x_2\cdots x_{n-1}) = x_2x_1\cdots x_{n-2}x_n = -x_1x_2\cdots x_{n-2}x_n$$

and similarly

$$(1,2)(n-2,n)(x_1x_2\cdots x_{n-1}) = x_2x_1\cdots x_{n-3}x_nx_{n-1} = x_1x_2\cdots x_{n-3}x_{n-1}x_n$$

and

$$\begin{aligned}(1,2)(j,n)(x_1x_2\cdots x_{n-1}) &= x_2x_1\cdots x_{j-1}x_nx_{j+1}\cdots x_n \\ &= (-1)^{n-j}x_1x_2\cdots x_{j-1}x_{j+1}\cdots x_n\end{aligned}$$

so that the n monomials with jth missing variable occur in the $\mathfrak{A}_n$-orbit with the sign $(-1)^{n-j}$. Since $x_1\cdots x_{n-1}$ has repeated odd exponents we have seen that the monomials in the $\mathfrak{S}_n$-orbit of $x_1\cdots x_{n-1}$ occur with both plus and minus signs, and the $\mathfrak{S}_n$-orbit sum of $x_1x_2\cdots x_{n-1}$ is 0. Hence the $\mathfrak{S}_n$-orbit of $x_1\cdots x_{n-1}$ has $2n$ elements, and so the $\mathfrak{S}_n$-stabilizer of $x_1\cdots x_{n-1}$ has $(n-1)!/2$ elements, and clearly the $(n-1)!/2$ even permutations of $\{1,\ldots,n-1\}$ stabilize $x_1\cdots x_{n-1}$ so must constitute its stabilizer. Hence the stabilizer in $\mathfrak{A}_n$ must also have $(n-1)!/2$

elements, and hence the $\mathfrak{A}_n$-orbit of $x_1 \cdots x_{n-1}$ must be the n elements we have computed, and hence

$$\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1}) = (x_1 \cdots x_{n-1}) - (x_1 \cdots x_{n-2}x_n) + (x_1 \cdots x_{n-3}x_{n-1}x_n) + \cdots + ((-1)^{n-1}x_2x_3 \cdots x_n).$$

Then to see the effect of any transposition (i, j) on this orbit sum, consider a summand of the orbit sum that contains both i and j and note, as in the argument above, that the transposition (i, j) changes the sign of this term; since any element in an orbit represents the orbit, any transposition reverses the sign on the $\mathfrak{A}_n$-orbit sum of $x_1 \cdots x_{n-1}$, and hence $\mathcal{O}_{\mathfrak{A}_n}(x_1x_2 \cdots x_{n-1})$ is an antisymmetric $\mathfrak{A}_n$-invariant.

There can be no smaller degree antisymmetric $\mathfrak{A}_n$-invariant since any smaller degree monomial X^I must have at least two zero entries in I, hence $\mathcal{O}(X^I)$ must be $\mathfrak{S}_n$-symmetric, and so no linear combination of such orbits can be antisymmetric. □

The antisymmetric orbit sum $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_n)$ can be generated from the super-symmetric polynomials and $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})$.

Lemma 4.8. *The antisymmetric orbit sum $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_n) = x_1 \cdots x_n$ is generated by the super-symmetric polynomial $P_1 = S_1 = \mathcal{O}_{\mathfrak{S}_n}(x_1) = \mathcal{O}_{\mathfrak{A}_n}(x_1)$ and the antisymmetric orbit sum $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})$ as follows*

$$\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_n) = \frac{1}{2n}(\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})S_1 + (-1)^{n-1}S_1\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})).$$

Proof. Computing

$$\begin{aligned}&\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})\mathcal{O}_{\mathfrak{A}_n}(x_1)\\ &\quad = (x_1x_2 \cdots x_{n-1} - x_1x_2 \cdots x_{n-2}x_n + \cdots + (-1)^{n-1}x_2 \cdots x_n)(x_1 + \cdots + x_n)\end{aligned}$$

we see that the monomial $x_1 \cdots x_n$ occurs n times (each with positive sign) as a summand in this product when expanded, and

$$x_1 \cdots x_{n-1}x_1 = (-1)^{n-2}x_1^2 \cdots x_{n-1}$$

so the respective orbits sums occur in the expanded product. Since there are n^2 monomials in the product $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})\mathcal{O}_{\mathfrak{A}_n}(x_1)$, and $n(n-1)$ summands in $\mathcal{O}_{\mathfrak{A}_n}(x_1^2x_2 \cdots x_{n-1})$ these orbit sums account for all the terms, and so

$$\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})\mathcal{O}_{\mathfrak{A}_n}(x_1) = n\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_n) + (-1)^{n-2}\mathcal{O}_{\mathfrak{A}_n}(x_1^2x_2 \cdots x_{n-1}).$$

Similarly

$$\mathcal{O}_{\mathfrak{A}_n}(x_1)\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1}) = \mathcal{O}_{\mathfrak{A}_n}(x_1^2x_2 \cdots x_{n-1}) + (-1)^{n-1}n\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_n),$$

and the result follows. □

Next we note that the super-symmetric polynomial $S_n = \mathcal{O}_{\mathfrak{S}_n}(x_1^2 \cdots x_{n-1}^2x_n)$ can be generated by antisymmetric invariants $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_n)$ and $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})$.

Lemma 4.9. *The super-symmetric polynomial $S_n = \mathcal{O}_{\mathfrak{S}_n}(x_1^2 \cdots x_{n-1}^2x_n)$ can be generated by antisymmetric invariants $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_n)$ and $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})$ as follows*

$$\mathcal{O}_{\mathfrak{S}_n}(x_1^2 \cdots x_{n-1}^2x_n) = (-1)^{(n-2)(n-1)/2}(\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1}))(\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_n)).$$

Proof. The monomial $x_1^2 \cdots x_{n-1}^2 x_n$ is stabilized by $(1,2)$ so

$$S_n = \mathcal{O}_{\mathfrak{S}_n}(x_1^2 \cdots x_{n-1}^2 x_n) = \mathcal{O}_{\mathfrak{A}_n}(x_1^2 \cdots x_{n-1}^2 x_n),$$

and

$$S_n = \sum_{i=1}^{n} x_1^2 \cdots x_{i-1}^2 x_i x_{i+1}^2 \cdots x_n^2.$$

This expression is a sum of n terms, each with $x_1 \cdots x_n$ as a factor. Consider the product $(\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1}))(x_1 \cdots x_n)$, and observe when this product is expanded one term is

$$\begin{aligned}(x_1 \cdots x_{n-1})(x_1 \cdots x_n) &= (-1)^{n-2}(x_1^2 x_2 \cdots x_{n-1})(x_2 \cdots x_n)\\ &= (-1)^{n-2}(-1)^{n-3}(x_1^2 x_2^2 \cdots x_{n-1})(x_3 \cdots x_n)\\ &= (-1)^{(n-2)(n-1)/2}(x_1^2 \cdots x_{n-1}^2 x_n),\end{aligned}$$

the last equality holding by induction. Since $(\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1}))(x_1 \cdots x_n)$ is an invariant, the entire orbit sum of this monomial must occur as terms in this expanded product, accounting for the n terms in S_n yielding the result. □

Here we are ready to prove a result of Cameron Atkins [CA].

Theorem 4.10. [CA, Theorem 4.2.7] *The fixed subring $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ is generated by the super-symmetric polynomials $S_1, \cdots, S_{n-1}$ and the antisymmetric $\mathfrak{A}_n$-invariant $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})$ (or the odd power sums $P_1, P_3, \ldots, P_{2n-3}$ and $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})$).*

Proof. By Lemma 4.8 $S_1 = P_1$ and $\mathcal{O}(x_1 \cdots x_{n-1})$ generate $x_1 x_2 \cdots x_n$, which in turn by Lemma 4.9 generate S_n. By Theorem 3.10 $S_1, S_2, \ldots, S_n$ generate all the symmetric invariants. Hence it suffices to show that any antisymmetric $\mathfrak{A}_n$-invariant f can be obtained.

We will induct on the degree of f, noting that the result is true in degrees $\leq n-1$ since $\mathcal{O}(x_1 \cdots x_{n-1})$ is the only antisymmetric $\mathfrak{A}_n$-invariant of degree $\leq n-1$.

Let X^I be the leading term of f under the length-lexicographic order. If σ is an transposition $\sigma f = -f$ also has leading term X^I. Hence by applying transpositions, we may assume that f has leading term X^I where I is weakly decreasing (and hence corresponds to a partition). We can write f as a linear combination of distinct orbit sums $f = \sum c_I \mathcal{O}_{\mathfrak{A}_n}(X^I)$ where X^I is the highest degree monomial in the orbit, and where $c_I \in k$ [Lemma 3.4]. Since f is antisymmetric $\sigma f = \sum c_I \sigma \mathcal{O}_{\mathfrak{A}_n}(X^I) = -f$ so that $2f = f - (-f) = \sum c_I(\mathcal{O}_{\mathfrak{A}_n}(X^I) - \sigma \mathcal{O}_{\mathfrak{A}_n}(X^I))$. Hence without loss of generality we may assume that $f = \mathcal{O}_{\mathfrak{A}_n}(X^I) - \sigma(\mathcal{O}_{\mathfrak{A}_n}(X^I))$, with X^I the leading term of f, and with $I = (\lambda_i)$ weakly decreasing; (since X^I is the leading term of f, and f is antisymmetric, $\sigma(\mathcal{O}_{\mathfrak{A}_n}(X^I)) \neq \mathcal{O}_{\mathfrak{A}_n}(X^I)$). Since $x_1^2 \cdots x_n^2$ is central and symmetric, we can factor it out of f, obtaining an antisymmetric invariant of smaller degree. Hence we may assume without loss of generality that $\lambda_n = 0$ or 1. If $\lambda_n = 1$, then each x_i occurs in all terms of f, so we can factor out $(x_1 \cdots x_n)$ from f and write $f = h(x_1 \cdots x_n)$ for some $\mathfrak{A}_n$-invariant h. It follows that h is symmetric, and we are done. Hence, assume that $\lambda_n = 0$ and $I = (\lambda_1, \ldots, \lambda_{n-1}, 0)$.

Now we induct on the order of I. The lowest order possible for I is when $I = (\lambda_1, 0, \cdots, 0)$. Since $n \geq 3$, we have $\lambda_{n-1} = 0$. If $\lambda_{n-1} = 0$, then the transposition $\tau = (n-1, n)$ stabilizes X^I and hence $\mathcal{O}(X^I)$ is $\mathfrak{S}_n$-invariant [Lemma 4.1]. Consequently, $f = 0$ and we are done. Therefore we can assume that $\lambda_i \neq 0$ for all $i = 1, \cdots, n-1$. Let $I^* = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{n-1} - 1, 0)$, which is a weakly decreasing sequence, and let $h = \mathcal{O}_{\mathfrak{A}_n}(X^{I^*}) + \sigma\mathcal{O}_{\mathfrak{A}_n}(X^{I^*})$, which is $\mathfrak{S}_n$-invariant (it is possible that $\mathcal{O}(X^I)$ itself is $\mathfrak{S}_n$-invariant – e.g. if $\lambda_{n-1} = 1$ or I^* has two even entries that are equal – in this case $h = 2\mathcal{O}_{\mathfrak{A}_n}(X^{I^*})$). Let $g = h\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})$, which is an antisymmetric $\mathfrak{A}_n$-invariant that is a product of a $\mathfrak{S}_n$-invariant and $\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})$. We claim that $\pm f$ is a summand of g and that all other terms have lower order; by induction these claims will complete the proof. Notice that the terms g_1 and g_2 occur in g where

$$g_1 = (x_1^{\lambda_1 - 1} x_2^{\lambda_2 - 1} \cdots x_{n-1}^{\lambda_{n-1}-1})(x_1 \cdots x_{n-1})$$

$$g_2 = (x_2^{\lambda_1 - 1} x_1^{\lambda_2 - 1} \cdots x_{n-1}^{\lambda_{n-1}-1})(x_1 \cdots x_{n-1}),$$

and hence their $\mathfrak{A}_n$-orbit sums occur in g. Note that $g_1 = \pm X^I$ and $g_2 = \pm\sigma X^I$ and $\sigma g_1 = -g_2$, and hence $\pm f$ is a summand of g. Finally notice that X^I is clearly the leading term of g and so all the other terms of g are of lower order. Hence $f \pm g$ is antisymmetric of lower order, hence of the desired form by induction.

The argument of Lemma 3.9 shows that $S_1, S_2, \ldots, S_{n-1}$ can be obtained from $P_1, P_3, \ldots, P_{2n-3}$. □

In the above proof we have shown that antisymmetric invariants correspond to partitions

$$I \mapsto \mathcal{O}_{\mathfrak{A}_n}(X^I) - \sigma\mathcal{O}_{\mathfrak{A}_n}(X^I)$$

for any odd permutation σ. This antisymmetric invariant will be non-zero if and only if $0 \neq \mathcal{O}_{\mathfrak{A}_n}(X^I)$ is not $\mathfrak{S}_n$-invariant, i.e. $\mathcal{O}_{\mathfrak{A}_n}(X^I)$ has no odd permutations stabilizing it. By the lemma below this is equivalent to I having no repeated even indices (by Lemma 4.3 this condition also assures $\mathcal{O}_{\mathfrak{A}_n}(X^I) \neq 0$.)

Lemma 4.11. *Let X^I be the highest degree lexicographic ordered term in the $\mathfrak{A}_n$-orbit of X^I. Then $\sigma\mathcal{O}_{\mathfrak{A}_n}(X^I) = \mathcal{O}_{\mathfrak{A}_n}(X^I)$ for an odd permutation σ if and only if I has at least two entries $\lambda_j = \lambda_k$ that are an even number (including 0).*

Proof. If $\lambda_j = \lambda_k$ is even then $(j,k)X^I = X^I$ so $\mathfrak{S}_n = \mathfrak{A}_n \cup \mathfrak{A}_n(j,k)$ and the $\mathfrak{A}_n$-orbit of X^I is the same as the $\mathfrak{S}_n$-orbit of X^I so $(j,k)\mathcal{O}_{\mathfrak{A}_n}(X^I) = \mathcal{O}_{\mathfrak{A}_n}(X^I)$, and, in fact, any permutation stabilizes the orbit sum.

Conversely, suppose that there is an odd permutation σ with $\sigma\mathcal{O}_{\mathfrak{A}_n}(X^I) = \mathcal{O}_{\mathfrak{A}_n}(X^I)$. Since σX^I is in the $\mathfrak{A}_n$-orbit of X^I we must have $\sigma X^I = \tau X^I$ for τ an even permutation. Hence $\tau^{-1}\sigma X^I = X^I$ so X^I is stabilized by an odd permutation. Suppose that I has no repeated even entries, and write $\sigma = \nu_1 \cdots \nu_{2m+1}\mu_1 \cdots \mu_k$ as a product of disjoint cycles, where ν_i are odd permutations and μ_i are even. Noting that entries of I in the support of each cycle must be constant and all repeated entries are assumed to be odd, we see that each $\mu_i X^I = X^I$ because μ_i is the product of an even number of transpositions of variables with the same odd exponents and so each transposition changes the sign; since there are an even number of sign changes $\mu_i X^I = X^I$. However $\nu_i X^I = -X^I$ since ν_i is the product of an odd number of interchanges of variables to the same odd power, and hence results in an odd number of sign changes. Hence

$$\sigma X^I = \nu_1 \cdots \nu_{2m+1}\mu_1 \cdots \mu_k X^I = \nu_1 \cdots \nu_{2m+1} X^I = (-1)^{2m+1} X^I = -X^I,$$

contradicting $\sigma X^I = X^I$. Hence I must have at least one repeated even entry. □

We note that in the commutative case the antisymmetric nonzero invariants $\mathcal{O}_{\mathfrak{A}_n}(X^I)-\sigma\mathcal{O}_{\mathfrak{A}_n}(X^I)$ that corresponding to a partition I are those with all entries of I distinct.

We next compute the Hilbert series for $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ and use it to show that $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ is a cci. For specific values of n the coefficients of these series do not seem to be in the *Online Encyclopedia of Integer Sequences.*

Lemma 4.12. *The Hilbert series of $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ is given by*

$$H_{k_{-1}[\underline{x}]^{\mathfrak{A}_n}}(t) \quad = \quad \frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-2})(1+t^n)(1+t^{n-1})}{(1-t)(1-t^2)(1-t^3)\cdots(1-t^{2n-1})(1-t^{2n})(1+t^{2n-1})}.$$

Proof. By remarks above in each dimension the invariants are vector space direct sums of the symmetric invariants and the antisymmetric invariants, so the Hilbert series $H_{k_{-1}[\underline{x}]^{\mathfrak{A}_n}}(t)$ for the invariants under $\mathfrak{A}_n$ is the sum of $H_{k_{-1}[\underline{x}]^{\mathfrak{S}_n}}(t)$ and the generating function $S_n(t)$ for $s_n(k)$, the number of partitions of k with at most n parts having no repeated even parts (not even 0). By Proposition 6.3 of the Appendix we have

$$S_n(t) = D_n(t)\frac{t^{n-1}(1+t)}{(1+t^{2n-1})}.$$

Hence

$$\begin{aligned}
H_{k_{-1}[\underline{x}]^{\mathfrak{A}_n}}(t) &= D_n(t)+S_n(t) = D_n(t)+D_n(t)\frac{t^{n-1}(1+t)}{(1+t^{2n-1})} \\
&= D_n(t)\left(1+\frac{t^{n-1}(1+t)}{(1+t^{2n-1})}\right) \\
&= D_n(t)\frac{(1+t^n)(1+t^{n-1})}{(1+t^{2n-1})} \\
&= \frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-2})(1+t^n)(1+t^{n-1})}{(1-t)(1-t^2)(1-t^3)\cdots(1-t^{2n-1})(1-t^{2n})(1+t^{2n-1})}.
\end{aligned}$$

Canceling yields the expression in equation (E4.0.1). □

Consider the algebras given by

$$\begin{aligned}
B_{n-1} &= k[p_1,\cdots,p_n][y_1:\tau_1,\delta_1]\cdots[y_{n-1}:\tau_{n-1},\delta_{n-1}] \\
B_{n+1} &= B_{n-1}[y_{n+1};\tau_{n+1}] \\
B_{n+2} &= B_{n+1}[y_{n+2};\tau_{n+2},\delta_{n+2}].
\end{aligned}$$

For $i\le n-1$ define τ_i and δ_i as for the algebra B considered in the previous section (note that B is not a subalgebra of C since y_n is not adjoined). Define the τ_{n+1} by letting it be the identity on $R=k[p_1,\cdots,p_n]$ and $\tau_{n+1}(y_i)=(-1)^{n-1}y_i$ for $i\le n-1$. Then τ_{n+1} extends uniquely to an algebra automorphism of B_{n-1}. Define the algebra automorphism τ_{n+2} of D_n by letting it be the identity on R and letting

$$\tau_{n+2}(y_i)=\begin{cases}(-1)^n y_i & \text{if } i\le n-1,\\ (-1)^{n+1}y_{n+1} & \text{if } i=n+1.\end{cases}$$

The derivation δ_{n+2} is given by letting $\delta_{n+2}(a)=0$ for all $a\in R$, $\delta_{n+2}(y_i)=(-1)^{n-1}2na_{2i-2}y_{n+1}$ for $i\le n-1$, and $\delta_{n+2}(y_{n+1})=0$. Recall that $a_{2i-2}=f_{2i-2}(p_1,p_2,\dots p_n)$ where f_{2i-2} is given by (E3.7.1).

Lemma 4.13. *Retain the above notation.*

(1) τ_{n+2} *is an algebra automorphism of* B_{n+1}.
(2) δ_{n+2} *is a* τ_{n+2}*-derivation of* B_{n+1}.

Proof. (1) It is straightforward to check that τ_{n+2} is an algebra automorphism of B_{n+1}.

(2) The relations of B_{n+1} are of the form

$$\begin{aligned} y_i a - a y_i = 0,\ \forall\, i = 1, \cdots, n-1, n+1,\ a \in R \\ y_i y_j + y_j y_i = 2\, a_{2i+2j-2},\ \forall\, 1 \le i, j \le n-1, \\ y_{n+1} y_i + (-1)^n y_i y_{n+1} = 0,\ \forall i = 1, \cdots, n-1. \end{aligned}$$

The proof of δ_{n+2} preserving the relations $y_i a - a y_i = 0$ is similar to the proof of Lemma 2.3(2). Now we show that δ_{n+2} preserves other relations. For $i, j \le n-1$,

$$\begin{aligned} &\delta_{n+2}(y_i y_j + y_j y_i - 2a_{2i+2j-2}) \\ &\quad = \delta_{n+2}(y_i) y_j + \tau_{n+2}(y_i)\delta_{n+2}(y_j) + \delta_{n+2}(y_j) y_i + \tau_{n+2}(y_j)\delta_{n+2}(y_i) \\ &\quad = (-1)^{n-1} 2n a_{2i-2} y_{n+1} y_j + (-1)^n y_i (-1)^{n-1} 2n a_{2j-2} y_{n+1} \\ &\quad\quad + (-1)^{n-1} 2n a_{2j-2} y_{n+1} y_i + (-1)^n y_j (-1)^{n-1} 2n a_{2i-2} y_{n+1} \\ &\quad = 0 \end{aligned}$$

For $i \le n-1$, we have

$$\begin{aligned} &\delta_{n+2}(y_{n+1} y_i + (-1)^n y_i y_{n+1}) \\ &\quad = \tau_{n+2}(y_{n+1})\delta_{n+2}(y_i) + (-1)^n \delta_{n+2}(y_i) y_{n+1} \\ &\quad = (-1)^{n+1} y_{n+1} (-1)^{n-1} 2n a_{2i-2} y_{n+1} + (-1)^n (-1)^{n-1} 2n a_{2i-2} y_{n+1} y_{n+1} \\ &\quad = 0. \end{aligned}$$

□

The above lemma verifies that δ_{n+2} is a τ_{n+2}-derivation. Let $C = B_{n+2}$. The algebra C is AS regular of dimension $2n+1$. Grade C by letting degree$(y_i) = 2i-1$ for $i \le n-1$, degree$(y_{n+1}) = n$, and degree$(y_{n+2}) = n-1$. Then the Hilbert series of C is given by

$$H_C(t) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^{2n-3})(1-t^{2n-2})(1-t^{2n})(1-t^n)(1-t^{n-1})}.$$

Since $\mathfrak{A}_n \le \mathfrak{S}_n$, the algebra $k[x_1^2, x_2^2, \ldots, x_n^2]^{\mathfrak{S}_n}$ is a subalgebra of $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$. Then $k[x_1^2, x_2^2, \ldots, x_n^2]^{\mathfrak{S}_n} = k[\rho_1, \rho_2, \ldots, \rho_n]$, a commutative polynomial ring where $\rho_i = \sigma_i(x_1^2, x_2^2, \ldots, x_n^2)$ and σ_i is the ith elementary polynomial. Observe that

(E3.12.1) $$\mathcal{O}_{\mathfrak{A}_n}(x_1 \cdots x_{n-1})^2 = \pm \mathcal{O}_{\mathfrak{A}_n}((x_1 \cdots x_{n-1})^2) = \pm \mathcal{O}_{\mathfrak{A}_n}(x_1^2 \cdots x_{n-1}^2)$$

because

$$\begin{aligned} &(x_1 \cdots x_k \cdots x_{n-1})(x_1 \cdots x_{k-1} x_{k+1} \cdots x_n) \\ &= (-1)^{n-2}(x_1 \cdots x_k \cdots x_{n-1} x_n)(x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n-1}) \\ &= (-1)^n (-1)^{(n-k)+(k-1)}(x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n-1} x_n)(x_1 \cdots x_{k-1} x_k x_{k+1} \cdots x_{n-1}) \\ &= -(x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n-1} x_n)(x_1 \cdots x_{k-1} x_k x_{k+1} \cdots x_{n-1}), \end{aligned}$$

so the orbits of the cross-terms cancel out, leaving only an orbit in the x_i^2 that is symmetric. Hence we can write $\mathcal{O}_{\mathfrak{A}_n}(x_1^2 x_2^2 \cdots x_{n-1}^2) = g(\rho_1, \rho_2, \ldots, \rho_n)$ for a polynomial g. Similarly, $(x_1 x_2 \cdots x_n)^2 = \pm x_1^2 x_2^2 \cdots x_n^2 = h(\rho_1, \rho_2, \ldots, \rho_n)$ for a polynomial h.

As in the previous section let for $i \leq n-1$ let $r_i = y_i^2 - a_{4i-2}$. Let

$$b_1 = g(p_1, p_2, \ldots, p_n) \tag{E4.13.1}$$

and

$$b_2 = h(p_1, p_2, \ldots, p_n) \tag{E4.13.2}$$

and consider two additional relations $r_{n+1} = y_{n+1}^2 - b_2$ and $r_{n+2} = y_{n+2}^2 - b_1$.

The proof of the following lemma is the same as that of Lemma 3.11.

Lemma 4.14. *The sequence $\{r_1, r_2, \ldots, r_{n-1}, r_{n+1}, r_{n+2}\}$ is a central regular sequence in C.*

We are now ready to show that $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ is a cci.

Theorem 4.15. *The algebra $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ is a cci.*

Proof. Note that $\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})$ and $x_1x_2\cdots x_n$ are elements of $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$. Consider the algebra C constructed above and define a map $\phi : C \longrightarrow k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ as follows: for $i \leq n$ let $\phi(p_i) = \rho_i$; for $i \leq n-1$ let $\phi(y_i) = P_{2i-1}$; let $\phi(y_{n+1}) = x_1x_2\cdots x_n$; and let $\phi(y_{n+2}) = \mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})$. Note that ϕ takes $k[p_1, p_2, \ldots, p_n]$ isomorphically onto $k[\rho_1, \rho_2, \ldots, \rho_n]$. In the proof of Theorem 3.12 it was shown that ϕ preserves the skew polynomial relations associated to y_i for $i \leq n-1$. Calculating shows that $(x_1x_2\cdots x_n)P_{2i-1} = (-1)^{n-1}(x_1x_2\cdots x_n)P_{2i-1}$, and hence ϕ preserves the relation associated to y_{n+1}. Further calculation shows that

$$\begin{aligned}\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})P_{2i-1} = (-1)^n P_{2i-1}&\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})\\ &+ (-1)^{n-1}2nP_{2i-2}\ \cdot (x_1x_2\cdots x_n).\end{aligned}$$

Since $\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})(x_1x_2\cdots x_n) = (-1)^{n-1}(x_1x_2\cdots x_n)\mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})$ and $P_{2i-2} = f_{2i-2}(\rho_1, \rho_2, \ldots, \rho_n)$, the relation associated to y_{n+2} is preserved by ϕ. Hence ϕ is a graded ring homomorphism. The homomorphism ϕ is onto by Theorem 4.10. By (E3.12.1)

$$\begin{aligned} 0 &= \mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1})^2 - \mathcal{O}_{\mathfrak{A}_n}(x_1^2x_2^2\cdots x_{n-1}^2)\\ &= \mathcal{O}_{\mathfrak{A}_n}(x_1x_2\cdots x_{n-1}) - g(\rho_1, \rho_2. \ldots, \rho_n) = \phi(y_{n+2}^2 - b_1) = \phi(r_{n+2}).\end{aligned}$$

Similarly, $\phi(r_{n+1}) = \phi(y_{n+1}^2 - b_2) = 0$. As in the proof of Theorem 3.12 $\phi(r_i) = 0$ for $i \leq n-1$. Hence $(r_1, r_2, \ldots, r_{n-1}, r_{n+1}, r_{n+2}) \subseteq \ker(\phi)$, and ϕ induces a graded ring homomorphism $\bar{\phi} : \overline{C} \longrightarrow k_{-1}[\underline{x}]^{\mathfrak{A}_n}$ where $\overline{C} = C/(r_1, r_2, \ldots, r_{n-1}, r_{n+1}, r_{n+2})$.

We have degree$(r_i) = 4i-2$ for $i \leq n-1$, degree$(r_{n+1}) = 2n$, and degree$(r_{n+2}) = 2n-2$. Since $\{r_1, r_2, \ldots, r_{n-1}, r_{n+1}, r_{n+2}\}$ is a regular sequence, the Hilbert series of $\overline{C}$ is given by

$$\begin{aligned} H_{\overline{C}}(t) &= \frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-6})(1-t^{2n})(1-t^{2n-2})}{(1-t)(1-t^2)\cdots(1-t^{2n-2})(1-t^{2n})(1-t^n)(1-t^{n-1})}\\ &= \frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-6})(1-t^{2n})(1-t^{2n-2})}{(1-t)(1-t^2)\cdots(1-t^{2n-2})(1-t^{2n})(1-t^n)(1-t^{n-1})}\,\frac{(1-t^{4n-2})}{(1-t^{4n-2})}\\ &= \frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-2})(1+t^n)(1+t^{n-1})}{(1-t)(1-t^2)(1-t^3)\cdots(1-t^{2n-1})(1-t^{2n})(1+t^{2n-1})}.\end{aligned}$$

This is the Hilbert series of $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$, and hence the ring homomorphism $\bar{\phi}$ is an isomorphism as desired. The assertion follows. □

Theorem 4.16. $\lfloor \frac{n}{2} \rfloor = cyc(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) \leq cci^{+}(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) \leq \lfloor \frac{n}{2} \rfloor + 1.$

Proof. First we prove the claim that $cci^+(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) \leq \lfloor \frac{n}{2} \rfloor + 1$.

Following the proof of Theorem 3.14, let C_2 be the subalgebra of $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ defined before Lemma 2.4, which is (isomorphic to) the iterated Ore extension

$$k[P_4, P_8, \cdots, P_{4\lfloor \frac{n}{2} \rfloor}][P_1][P_3; \tau_3, \delta_3] \cdots [P_{n'}; \tau_{n'}, \delta_{n'}]$$

where $n' = 2\lfloor \frac{n-1}{2} \rfloor + 1$. Let F_{2n-3} be the iterated Ore extension defined in the proof of Theorem 3.14. (We are not going to use F_{2n-1}, instead we will define two new algebras H_{2n-1} and H_{2n+1}.) Recall from the proof of Theorem 4.15 that p_i is the image of P_{2i} for all $i = 1, \cdots, n$. By Lemma 2.4(5), P_{2i} are in C_2 for all i. Define $H_{2n-1} = F_{2n-3}[Q_{2n-1}; \phi_{2n-1}]$ where $\phi_{2n-1} : P_i \mapsto (-1)^{i(n-1)} P_i$ for all even i and all odd $i \leq 2n-3$ if P_i appeared in F_{2n-3}. It is easy to check that ϕ_{2n-1} is an algebra automorphism of F_{2n-3} and therefore H_{2n-1} is an iterated Ore extension. Define $H_{2n+1} = H_{2n-1}[Q_{2n+1}; \phi_{2n+1}, \lambda_{2n+1}]$ where ϕ_{2n+1} is an algebra automorphism determined by $\phi_{2n+1} : \begin{cases} P_i \mapsto (-1)^{in} P_i & \text{for even } i \text{ or odd } i \leq 2n-3 \\ Q_{2n-1} \mapsto (-1)^{n+1} Q_{2n-1} \end{cases}$ (see the proof of Lemma 4.13(1)), and ϕ_{2n+1}-derivation λ_{2n+1} is determined by

$$\lambda_{2n+1} : \begin{cases} P_i \mapsto 0 & \text{if } i \text{ is even and } i \leq 2n \\ P_i \mapsto (-1)^{n+1} 2n Q_{2n-1} f_{2i-2}(P_2, \cdots, P_{2n}) & \text{if } i \text{ is odd and } i \leq 2n-3 \\ Q_{2n-1} \mapsto 0 \end{cases},$$

where f_{2i-2} is given by (E3.7.1). Similar to the proof of Lemma 4.13(2), one can show that λ_{2n+1} is a ϕ_{2n+1}-derivation, therefore H_{2n+1} is an iterated Ore extension. Let $u_s = P_{2s-1}^2 - P_{4s-2}$ for all integers from $s = \lfloor \frac{n-1}{2} \rfloor + 2$ to $s = n-1$. Let u_{n+1} be $Q_{2n-1}^2 - b_2$ where $b_2 \in C_1 \subset C_2$ is defined in (E4.13.2). Let u_{n+2} be $Q_{2n+1}^2 - b_1$ where $b_1 \in C_1 \subset C_2$ is defined in (E4.13.1).

The proof of Lemma 3.11 (see also Lemma 4.14) shows that

$$\{u_{\lfloor \frac{n-1}{2} \rfloor + 2}, \cdots, u_{n-1}, u_{n+1}, u_{n+2}\}$$

is a central regular sequence of H_{2n+1}. It is straightforward to see that

$$H_{2n+1}/(u_{\lfloor \frac{n-1}{2} \rfloor + 2}, \cdots, u_{n-1}, u_{n+1}, u_{n+2}) \cong k_{-1}[\underline{x}]^{\mathfrak{S}_n}.$$

Therefore $cci^+(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) \leq n+1-(\lfloor \frac{n-1}{2} \rfloor + 1) = \lfloor \frac{n}{2} \rfloor + 1$ and we proved the claim.

By Theorem 4.15

$$\begin{aligned} H_{k_{-1}[\underline{x}]^{\mathfrak{A}_n}}(t) &= H_{k_{-1}[\underline{x}]^{\mathfrak{S}_n}}(t) \frac{(1+t^n)(1+t^{n-1})}{1+t^{2n-1}} \\ &= \frac{\prod_{s=\lfloor \frac{n-1}{2} \rfloor + 2}^{n} (1-t^{4s-2})}{\prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} (1-t^{4j}) \prod_{i=1}^{n} (1-t^{2i-1})} \frac{(1-t^{2n-1})(1-t^{2n})(1-t^{2(n-1)})}{(1-t^{4n-2})(1-t^n)(1-t^{n-1})} \\ &= \frac{\prod_{s=\lfloor \frac{n-1}{2} \rfloor + 2}^{n-1} (1-t^{4s-2})}{\prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} (1-t^{4j}) \prod_{i=1}^{n-1} (1-t^{2i-1})} \frac{(1-t^{2n})(1-t^{2(n-1)})}{(1-t^n)(1-t^{n-1})} \\ &= \frac{\prod_{s=\lfloor \frac{n-1}{2} \rfloor + 2}^{n-1} (1-t^{4s-2})}{\prod_{j=1}^{\lfloor \frac{n}{2} - 1 \rfloor} (1-t^{4j}) \prod_{i=1}^{n-1} (1-t^{2i-1})} \frac{(1-t^{2(2\lfloor \frac{n-1}{2} \rfloor + 1)})}{(1-t^n)(1-t^{n-1})} \end{aligned}$$

which is an expression satisfying the condition in Definition 3.13(2). Hence

$$\lfloor \frac{n}{2} \rfloor = cyc(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) \leq cci^+(k_{-1}[\underline{x}]^{\mathfrak{S}_n}) \leq \lfloor \frac{n}{2} \rfloor + 1.$$

□

Question 4.17. Let A be either $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$ and $k_{-1}[\underline{x}]^{\mathfrak{A}_n}$. Let $E(A)$ be the Ext-algebra $\operatorname{Ext}^*_A(k,k)$.

(1) Is $E(A)$ noetherian?
(2) What is the GK-dimension of $E(A)$?

5. Converse of Kac-Watanabe-Gordeev Theorem

Kac-Watanabe-Gordeev showed that when $k[\underline{x}]^G$ is a complete intersection then G must be generated by classical bireflections. We next prove the converse of this result for $k_{-1}[\underline{x}]^G$ when $G \subset \mathfrak{S}_n$ and note that the converse is not true for $k[\underline{x}]^G$. By Lemma 1.7(3) a quasi-bireflection must be a 2-cycle or a 3-cycle. We conclude by showing that for subgroups G of $\mathfrak{S}_4$ acting on $k_{-1}[x_1,x_2,x_3,x_4]$, the fixed subring $k_{-1}[x_1,x_2,x_3,x_4]^G$ is a cci if and only if G is generated by quasi-bireflections, and when G is not generated by quasi-bireflections, $k_{-1}[x_1,x_2,x_3,x_4]^G$ is not cyclotomic Gorenstein, hence $k_{-1}[x_1,x_2,x_3,x_4]^G$ is not any of the kinds of complete intersections described in Definition 1.8. The following result on permutation groups may be well-known, but is included for completeness.

Proposition 5.1. *Let G be a subgroup of $\mathfrak{S}_n$.*

(1) *If G is generated by 3-cycles, then G is an internal direct product of alternating groups.*
(2) *If G is generated by 3-cycles and 2-cycles, then G is an internal direct product of alternating and symmetric groups.*

We first prove some lemmas. Let X be any subset of $\{i\}_{i=1}^n := \{1,\cdots,n\}$. We use $\mathfrak{S}_X$ for the full symmetric group of X.

Proof. Suppose that G is generated by 3-cycles and 2-cycles. We may assume that $G = \langle \tau_1,\tau_2,\dots,\tau_\ell\rangle$ where $\tau_1,\tau_2,\dots,\tau_\ell$ are all of the 3-cycles and 2-cycles in G. Let $X=\{1,2,\dots,n\}$. We will show that there are disjoint nonempty subsets $X_1,X_2,\dots,X_k$ of X such that $G = G_1\times G_2\times\cdots\times G_k$ where G_i is the alternating or symmetric group on X_i. Given a permutation σ define $M(\sigma)=\{x\in X : \sigma(x)\neq x\}$, the set of elements that are moved by σ. Let $Y = \bigcup_{\sigma\in G} M(\sigma)$ and define a relation $\sim$ on Y by $x\sim y$ if there exists 3-cycles and/or 2-cycles $\sigma_1,\sigma_2,\dots,\sigma_m$ such that $x\in M(\sigma_1), y\in M(\sigma_m)$ and $M(\sigma_i)\cap M(\sigma_{i+1})\neq\emptyset$ for $i=1,2,\dots,m-1$. In this case we say that there is a path from x to y. It is easy to see that $\sim$ is an equivalence relation on Y. Let $X_1,X_2,\dots,X_k$ be the equivalence classes. We view the X_i as the path connected components of Y. Clearly either $M(\tau_j)\subseteq X_i$ or $M(\tau_j)\cap X_i=\emptyset$ for all i,j. Let $G_i = \langle \tau_j : M(\tau_j)\subseteq X_i\rangle$.

Case 1: Suppose that G is generated by 3-cycles. It will be sufficient to show that each G_i is an alternating group. Furthermore, there is no loss of generality in assuming that there is one component Y. We will induct on ℓ. If $|Y|=3$, (the smallest possible) then $G=\langle\tau\rangle\cong A_3$. If $|Y|=4$, we may assume that $Y=\{1,2,3,4\}, \tau_1=(1,2,3)$ and $\tau_2=(2,3,4)$. In this case $|\langle\tau_1\rangle\langle\tau_2\rangle| = 9$ and G must be all of A_4. Inductively assume that whenever $G=\langle\tau_1,\tau_2,\dots,\tau_\ell\rangle$ has one component Y with $4\le |Y| = s\le n$, then $G\cong A_s$. We may let $Y=\{1,2,\dots,s\}$. Now suppose that $G' = \langle\tau_1,\tau_2,\dots,\tau_{\ell+1}\rangle$ where $\tau_{\ell+1}$ is a 3-cycle, and $Y' = \bigcup_1^{\ell+1} M(\tau_j)$

is connected. Let $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_m}$ be a maximal path in Y'. Then $\cup_{j\neq i_m} M(\tau_j)$ must be connected, for otherwise, we could extend the path. Hence there is no loss of generality in assuming that $\tau_{\ell+1}$ is such that $Y = \bigcup_{i\neq \ell+1} M(\tau_i)$ is connected with $|Y| = s$. Let $G = \langle \tau_1, \tau_2, \ldots, \tau_\ell\rangle$. There are two subcases.

Case 1.1: $|Y'| = s+1$. We may assume, renumbering if necessary, that $\tau_{\ell+1} = (s-1, s, s+1)$. We will show that G' contains all elements that are products of two disjoint 2-cycles. The set of all such generates a normal subgroup of A_{s+1}, and hence we would have $G' = A_{s+1}$. By induction we have all disjoint products $(i,j)(k,\ell)$ where $i,j,k,\ell \leq s$. If $i,j \leq s-1$, then $(i,j)(s-1,s)(s-1,s,s+1) = (i,j)(s,s+1)$. Then $(s-1,k)(i,j)\cdot(i,j)(s,s+1) = (s-1,k)(s,s+1)$. The conjugation $(k,s,\ell)(i,j)(s,s+1)(k,\ell,s) = (i,j)(\ell,s+1)$ gives the remaining products. Thus $G' = A_{s+1}$ and the result follows by induction.

Case 1.2: $|Y'| = s+2$. We may assume that $\tau_{\ell+1} = (s,s+1,s+2)$. By induction G is A_s and we have the following chain from 1 to $s-1$:

$$(1,2,3),(2,3,4),\ldots,(s-3,s-2,s-1).$$

Computing

$$(1,2)(s-1,s)(s,s+1,s+2)(1,2)(s-1,s) = (s-1,s+1,s+2),$$

and $(s-1,s+1,s+2) \in G'$. We have that $Y'' = \{1,2,\ldots,s-1\} \cup \{s+1,s+2\}$ is a connected component, and by induction $G'' = \langle A_{s-1}, (s-1,s+1,s+2)\rangle$ is a copy of A_{s+1}. Then $G = \langle G'', \tau_{\ell+1}\rangle$ is the alternating group A_{s+2} by Case 1.1.

Case 2: Once again there is no loss of generality in assuming that there is one connected component. We may also suppose that G contains at least one 2-cycle by Case 1. Again the proof is by induction on ℓ. If $|Y| = 2$, the result is clear. Since $(1,2)(2,3) = (1,2,3)$, we see that if $|Y| = 3$, then $G = S_3$. Inductively assume that whenever $G = \langle \tau_1, \tau_2, \ldots, \tau_\ell\rangle$ with $3 \leq |Y| \leq n$ then G is a symmetric group. Now suppose that $G' = \langle \tau_1, \tau_2, \ldots, \tau_{\ell+1}\rangle$ with $Y' = \bigcup_1^{\ell+1} M(\tau_j)$ connected. Again we may assume that $Y = \bigcup_{j\neq \ell+1} M(\tau_j)$ is connected with $|Y| = s \leq n$. Then by induction or by case 1 we have that $G = \langle \tau_1, \tau_2, \ldots, \tau_\ell\rangle$ is either a symmetric group or an alternating group (if all τ_i for $i \leq \ell$ are 3-cycles). We have two subcases.

Case 2.1: $\tau_{\ell+1}$ is a 3-cycle. By the argument in Case 1, G' contains the full alternating group. Since G' must also contain a 2-cycle, it is the full symmetric group.

Case 2.2: $\tau_{\ell+1}$ is a 2-cycle. Without loss of generality we may assume that $\tau_{\ell+1} = (s,s+1)$. As noted, G is either the symmetric group or the alternating group. In this case G' must contain

$$(1,2)(s-1,s)(s,s+1) = (1,2)(s-1,s,s+1).$$

Squaring yields that $(s-1,s+1,s) \in G'$. By Case 1.1, G' contains the full alternating group. Since it also contains a 2-cycle, it must be the full symmetric group.

The result follows by induction. $\square$

Let A and B be two graded algebra. Define $A \otimes_{-1} B$ be the $\mathbb{Z}^2$-graded twist of the tensor product $A \otimes B$ by the twisting system

$$\sigma := \{\sigma_{i,j} = Id^i \xi_{-1}^j \mid (i,j) \in \mathbb{Z}^2\}$$

where ξ_{-1} maps $a \otimes b \mapsto (-1)^{|a|+|b|} a \otimes b$ for all $a \otimes b \in A \otimes B$. The following lemmas are easy to check.

Lemma 5.2. *Retain the above notation.*

(1) $A \otimes_{-1} B = A \otimes B$ *as* $\mathbb{Z}^2$*-graded vector spaces.*
(2) *Identifying* A *with* $A \otimes 1 \subset A \otimes_{-1} B$ *and identifying* B *with* $1 \otimes B \subset A \otimes_{-1} B$*. Then* A *and* B *are subalgebras of* $A \otimes_{-1} B$*, and the algebra* $A \otimes_{-1} B$ *is equal to the vector space generated by the products* AB (*and* BA *respectively*).
(3) *Under the identification in part (2),* $ab = (-1)^{|a|\,|b|} ba$ *for all* $a \in A$ *and* $b \in B$.

Lemma 5.3. *Let* $m < n$.

(1) $k_{-1}[x_1, \cdots, x_m] \otimes_{-1} k_{-1}[x_{m+1}, \cdots, x_n] \cong k_{-1}[x_1, \cdots, x_n]$.
(2) *If* $G_1 \subset \mathrm{Aut}(A)$ *and* $G_2 \subset \mathrm{Aut}(B)$*, then* $(A \otimes_{-1} B)^{G_1 \times G_2} = A^{G_1} \otimes_{-1} B^{G_2}$.
(3) [KKZ3, Lemma 2.7] *If* A *and* B *are AS regular, then so is* $A \otimes_{-1} B$.
(4) *Suppose* $A = R/(\Omega_1, \cdots, \Omega_m)$ *and* $B = C/(f_1, \cdots, f_d)$ *where* R *and* C *are AS regular and* $\{\Omega_i\}_{i=1}^m$ *and* $\{f_j\}_{j=1}^d$ *are regular normal sequences of positive even degrees. If* $R \otimes C$ *is noetherian, then* $A \otimes_{-1} B$ *is a factor ring of a noetherian AS regular algebra modulo a regular normal sequences of positive even degrees. As a consequence,* $A \otimes_{-1} B$ *is a cci.*

For any subset X of $[1, \cdots, n]$, let $\mathfrak{S}_X$ denote the symmetric group of X (all permutations of X).

Theorem 5.4. *If* G *is a subgroup of* $\mathfrak{S}_n$ *generated by quasi-bireflections, then* $k_{-1}[\underline{x}]^G$ *is a cci.*

Proof. We use induction on n. Suppose the assertion holds for $G \subset \mathfrak{S}_m$ for all $m \leq n-1$. Now let G be a subgroup of $\mathfrak{S}_n$ generated by quasi-bireflections. If G is $\{1\}$, the assertion is trivial. If $G = \mathfrak{S}_n$ or $\mathfrak{A}_n$, the assertion follows from Theorems 3.12 and 4.15. Otherwise, by Proposition 5.1, there is a disjoint union $X \cup Y = [1, \cdots, n]$ such that G is a product of G_1 and G_2, where G_1 and G_2 are subgroups $\mathfrak{S}_X$ and $\mathfrak{S}_Y$ respectively, and further G_1 is either $\mathfrak{S}_X$ or $\mathfrak{A}_X$ and G_2 is generated by quasi-bireflections of $k_{-1}[x_i \mid i \in Y]$ (or equivalently, 2- or 3-cycles of $\mathfrak{S}_Y$). By induction, both A^{G_1} and B^{G_2} are cci, where $A = k_{-1}[x_i \mid i \in X]$ and $B = k_{-1}[x_i \mid i \in Y]$. It follows from Lemma 5.2 and 5.3 that $k_{-1}[\underline{x}]^G \cong A^{G_1} \otimes_{-1} B^{G_2}$ is a cci. □

The following example shows that for $k[\underline{x}]$ permutation groups generated by classical bireflections need not have a fixed ring that is a complete intersection.

Example 5.5. Let $\mathfrak{S}_5$ act on $A := k[x_1, x_2, x_3, x_4, x_5]$ by permuting the variables. Let $G = \langle (1,2)(3,4), (2,3)(4,5) \rangle$. These two generators are classical bireflections. Note that $(1,2)(3,4) \cdot (2,3)(4,5) = (1,2,4,5,3)$. Calculating shows that $\langle (1,2)(3,4), (1,2,4,5,3) \rangle$ is a copy of the dihedral group D_5 of order 10 and is in fact all of G. Using Molien's Theorem we have

$$\begin{aligned} H_{A^G}(t) &= \frac{1}{10}\left(\frac{1}{(1-t)^5}+\frac{5}{(1-t)^3(1+t)^2}+\frac{4}{1-t^5}\right) \\ &= \frac{t^6-t^5+2t^3-t+1}{(1-t)^2(1-t^2)^2(1-t^5)}. \end{aligned}$$

The numerator is an irreducible polynomial that is not cyclotomic; in fact, none of its zeros are roots of unity. Hence A^G cannot be a complete intersection.

We conclude by computing the invariants of $A = k_{-1}[x_1, x_2.x_3, x_4]$ under each of the subgroups of $\mathfrak{S}_4$. In this case the conjectured generalization of the Kac-Watanabe-Gordeev Theorem becomes both necessary and sufficient. We show that A^H is a cci if and only if H is generated by quasi-bireflections (i.e. 2-cycles or 3-cycles); when H is not generated by quasi-bireflections A^H is not cyclotomic Gorenstein – hence not any kind of complete intersection by Theorem 1.10.

Example 5.6. For the following subgroups H of $\mathfrak{S}_4$ we consider the fixed subring A^H. We show that A^H is either a cci or not cyclotomic Gorenstein (and hence none of the kinds of complete intersection we considered in Definition 1.8).

- If H is the full symmetric group or the alternating group, both generated by quasi-bireflections, we have shown that A^H is a cci. Similarly, cyclic subgroups generated by a 2-cycle (so isomorphic to $\mathfrak{S}_2$) or by a 3-cycle (so isomorphic to $\mathfrak{A}_3$) are also easily seen to give ccis when they act on $A = k_{-1}[x_1, x_2, x_3, x_4]$ (we showed they did when they acted on $A = k_{-1}[x_1, x_2]$ and $A = k_{-1}[x_1, x_2, x_3]$ and the results extend by fixing the remaining variable(s)).
- Let H be the subgroup of order 2 generated by an element that is a product of two disjoint 2-cycles, e.g. (12)(34); this subgroup is not generated by quasi-bireflections of A (it is generated by a bireflection of $k[x_1, x_2, x_3, x_4]$). Molien's Theorem shows that the Hilbert series of A^H is
$$\frac{1-2t+4t^2-2t^3+t^4}{(1-t)^4(1+t^2)^2}$$
which has zeros that are not roots of unity. Hence A^H is not cyclotomic Gorenstein.
- We have already noted (Example 1.6) that the subgroup H generated by a 4-cycle is not generated by quasi-bireflections, and that the invariants A^H are not cyclotomic Gorenstein.
- Let H be the Klein-Four subgroup generated by two disjoint 2-cycles (e.g. $H = \langle (12), (34)\rangle$). Then H is generated by quasi-bireflections of A, the generators of A^H are $x_1 + x_2, x_3 + x_4, x_1^3 + x_2^3, x_3^3 + x_4^3$, Hilbert series of A^H is
$$\frac{1-t+t^2}{(1-t)(1+t^2)^2},$$
and
$$A^H \cong \frac{k[p_1, p_2, q_1, q_2][y_1][y_2; \tau_1, \delta_1][z_1; \tau_2, \delta_2][z_2; \tau_3, \delta_3]}{\langle y_1^2 - a_1, y_2^2 - a_2, z_1^2 - b_1, z_2^2 - b_2\rangle},$$
where p_1, p_2 (resp., q_1, q_2) correspond to the first two symmetric polynomials in x_1^2, x_2^2 (resp., x_3^2, x_4^2), y_1, y_2 (resp., z_1, z_2) correspond to $x_1+x_2, x_1^3+x_2^3$ (resp., $x_3 + x_4, x_3^3 + x_4^3$).

- The Klein-Four subgroup of even permutations
$$H = \{1, (12)(34), (13)(24), (14)(23)\},$$
which is not generated by quasi-bireflections of A. The Hilbert series of A^H is
$$\frac{1 - 3t + 5t^2 - 3t^3 + t^4}{(1-t)^4(1+t^2)^2},$$
so A^H is not cyclotomic Gorenstein.
- Let H be a subgroup $\mathfrak{S}_4$ of order 6. Then H is isomorphic to the symmetric group $\mathfrak{S}_3$, without loss of generality of the form $H = \langle (123), (12)\rangle$. This group is generated by quasi-bireflections, and A^H is a complete intersection (we showed this for $k_{-1}[x_1, x_2, x_3]$ and the extension to A is not difficult).
- Let H be a dihedral group of order 8 (a Sylow-2 subgroup of $\mathfrak{S}_4$). Then H is of the form
$$D_4 = \{1, (1234), (13)(24), (1432), (13), (24), (12)(34), (14)(23)\},$$
so not generated by quasi-bireflections. The Hilbert series of the fixed subring is
$$\begin{aligned} &\frac{1 - 3t + 5t^2 - 5t^3 + 5t^4 - 5t^5 + 5t^6 - 3t^7 + t^8}{(1-t)^4(1+t^4)(1+t^2)^2} \\ = &\frac{(1 - t + t^2)(1 - 2t + 2t^2 - t^3 + 2t^4 - 2t^5 + t^6)}{(1-t)^4(1+t^4)(1+t^2)^2} \end{aligned}$$
so A^H is not cyclotomic Gorenstein.

Note: It might be nice to know degrees of generators and how they compare to $n^2 = 16$.

Question 5.7. For H a subgroup of $\mathfrak{S}_n$, is $k_{-1}[\underline{x}]^H$ a cci if and only if H is generated by quasi-bireflections?

6. Appendix

In this section we find generating functions for the class of restricted partitions having no repeated odd parts and the class having no repeated even parts. It is included since we were unable to find them in the literature.

Let $d_n(k)$ be the number of partitions of k with at most n parts having no repeated odd parts. Make the convention that $d_n(1) = 1$ and $d_n(\ell) = 0$ for $\ell < 0$. Let $D_n(t)$ be the corresponding generating function
$$D_n(t) = \sum_{k=0}^{\infty} d_n(k)t^k.$$
There is only one way to partition k into 1 part, so
$$\begin{aligned} D_1(t) &= 1 + t + t^2 + t^3 + \cdots + t^k + \cdots \\ &= \frac{1}{1-t} \\ &= \frac{1-t^2}{(1-t)(1-t^2)} \end{aligned}$$

We will now try to find a recurrence relation for $d_n(k)$. We will write a partition $\mathcal{P}$ of k having at most n parts as $\mathcal{P} = p_1, p_2, \ldots, p_n$ where $p_1 \geq p_2 \geq \ldots \geq p_n$ and

$k = p_1 + p_2 + \cdots + p_n$. Let $\mathcal{D}_{n,k} = \{\mathcal{P} = p_1, p_2, \ldots, p_n : \text{with no repeated odd parts}\}$. Then we have

$$\mathcal{D}_{n,k} = \{\mathcal{P} : p_n = 0\} \cup_d \{\mathcal{P} : p_n = 1\} \cup_d \{\mathcal{P} : p_n \geq 2\}.$$

- Clearly $|\{\mathcal{P} : p_n = 0\}| = d_{n-1}(k)$.
- If $p_n = 1$, consider the association
$$\mathcal{P} \mapsto \mathcal{P}' = p_1 - 2, p_2 - 2, \ldots, p_{n-1} - 2, 0.$$
Since $p_{n-1} > p_n = 1$, this will be a partition of $k-1-2(n-1) = k-2n+1$. Since parity is preserved there will be no repeated odd parts, and every such partition of $k - 2n + 1$ can occur in this manner. Hence $|\{\mathcal{P} : p_n = 1\}| = d_{n-1}(k - 2n + 1)$.
- If $p_n \geq 2$, consider the association
$$\mathcal{P} \mapsto \mathcal{P}' = p_1 - 2, p_2 - 2, \ldots, p_n - 2.$$
This will be a partition of $k - 2n$ with no repeated odd parts. Once again every such partition can occur in this manner. Hence $|\{\mathcal{P} : p_n \geq 2\}| = d_n(k - 2n)$.

This yields the following recurrence relation

$$d_n(k) = d_{n-1}(k) + d_{n-1}(k - 2n + 1) + d_n(k - 2n).$$

In terms of generating functions we have

$$D_n(t) \quad = \quad D_{n-1}(t) + D_{n-1}(t)t^{2n-1} + D_n(t)t^{2n}.$$

This gives the recurrence

$$\begin{aligned} D_n(t) \quad &= \quad D_{n-1}(t)\frac{(1 + t^{2n-1})}{(1 - t^{2n})} \\ &= \quad D_{n-1}(t)\frac{(1 - t^{4n-2})}{(1 - t^{2n-1})(1 - t^{2n})}. \end{aligned}$$

Using this last recurrence relation a simple induction argument proves the following Proposition.

Proposition 6.1. *The generating function $D_n(t)$ for the number of partitions with at most n parts having no repeated odd parts is given by*

$$D_n(t) \quad = \quad \frac{(1 - t^2)(1 - t^6)(1 - t^{10}) \cdots (1 - t^{4n-2})}{(1 - t)(1 - t^2)(1 - t^3) \cdots (1 - t^{2n-1})(1 - t^{2n})}.$$

Remark 6.2. We note using the

Online Encyclopedia of Integer Sequences (http://oeis.org/)

for specific values of n we found that $D_n(t)$, the Hilbert series of $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$, is also the Hilbert series of the invariants of $\mathcal{A} = k[y_1 \ldots, y_n] \otimes E(e_1, \ldots, e_n)$ under the action of $\mathfrak{S}_n$, where k is any field of characteristic not equal to two, the degree of each $y_i = 2$, $E(e_1, \ldots, e_n)$ is the exterior algebra on elements e_i of degree 1, and $\mathfrak{S}_n$ acts on both $k[y_1 \ldots, y_n]$ and $E(e_1, \ldots, e_n)$ by permutations. (See [AM, pp. 110-11]). We note that one can filter $k_{-1}[\underline{x}]$ by letting I be the ideal generated by $\{x_1^2, \ldots, x_n^2\}$. Then the associated graded algebra

$$\mathrm{gr}(k_{-1}[\underline{x}]) = k_{-1}[\underline{x}]/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots \oplus I^m/I^{m+1} \oplus \cdots$$

is isomorphic as a graded algebra to $\mathcal{A}$ under the map that associates $y_i \mapsto x_i^2 + I^2$ and $e_i \mapsto x_i + I$. Further the action of $\mathfrak{S}_n$ on $k_{-1}[\underline{x}]$ extends to an action on $\mathrm{gr}(k_{-1}[\underline{x}])$, and

$$\mathcal{A}^{\mathfrak{S}_n} \cong \mathrm{gr}(k_{-1}[\underline{x}])^{\mathfrak{S}_n} \cong \mathrm{gr}(k_{-1}[\underline{x}]^{\mathfrak{S}_n}).$$

Since $\mathrm{gr}(k_{-1}[\underline{x}]^{\mathfrak{S}_n})$ has the same Hilbert series as $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$, it follows that $D_n(t)$ is the Hilbert series of $k_{-1}[\underline{x}]^{\mathfrak{S}_n}$.

Let $s_n(k)$ be the number of partitions of k with at most n parts having no repeated even parts (not even repeated 0 parts), and let $S_n(t)$ be the corresponding generating function. The purpose of this section is to find $S_n(t)$.

First we briefly consider a slight variation. Let $w_n(k)$ be the number of partitions of k with exactly n nonzero parts having no repeated even parts, and let $W_n(t)$ be the corresponding generating function. Let $\mathcal{P}$ be such a partition. Correspond to $\mathcal{P}$ the partition $\mathcal{P} \mapsto \mathcal{P}' = p_1 - 1, p_2 - 1, \ldots, p_n - 1$. This will be a partition of $k - n$ with at most n parts having no repeated odd parts, and any such partition can occur in this manner. Hence $w_n(k) = d_n(k - n)$, and $W_n(t) = t^n D_n(t)$.

Let $\mathcal{S}_{n,k}$ be the collection of all partitions of k with at most n parts having no repeated even parts. Then we have

$$\mathcal{S}_{n,k} = \{\mathcal{P} : p_n = 0\} \cup_d \{\mathcal{P} : p_n = 1\} \cup_d \{\mathcal{P} : p_n \geq 2\}.$$

Since there are no repeated empty parts, the partitions in the first set will be partitions having exactly $n-1$ nonzero parts and $|\{\mathcal{P} : p_n = 0\}| = w_{n-1}(k)$. For each partition $\mathcal{P}$ in the second set we correspond $\mathcal{P} \mapsto \mathcal{P}' = p_1, p_2, \ldots, p_{n-1}, 0$, which will be a partition of $k - 1$ with exactly $n - 1$ nonzero parts and no repeating even parts. Since all such occur in this manner, we have $|\{\mathcal{P} : p_n = 1\}| = w_{n-1}(k - 1)$. Similar to the no repeated odd case we see that $|\{\mathcal{P} : p_n \geq 2\}| = s_n(k - 2n)$. This gives the recurrence relation

$$s_n(k) = w_{n-1}(k) + w_{n-1}(k - 1) + s_n(k - 2n)$$

In terms of generating functions we have

$$S_n(t) = W_{n-1}(t) + W_{n-1}(t)t + S_n(t)t^{2n},$$

and

$$S_n(t) = W_{n-1}(t)\frac{(1 + t)}{(1 - t^{2n})} = D_{n-1}(t)\frac{(1 + t)t^{n-1})}{(1 - t^{2n})}. \tag{E6.2.1}$$

Summarizing we have the following Proposition.

Proposition 6.3. *If $S_n(t)$ is the generating function for the number of partitions having at most n parts with no repeated even parts, then*

$$S_n(t) \quad = \quad \frac{(1 - t^2)(1 - t^6)(1 - t^{10}) \cdots (1 - t^{4n-2})t^{n-1}(1 + t)}{(1 - t)(1 - t^2)(1 - t^3) \cdots (1 - t^{2n-1})(1 - t^{2n})(1 + t^{2n-1})}.$$

and

$$S_n(t) = D_n(t)\frac{t^{n-1}(1 + t)}{(1 + t^{2n-1})}.$$

Proof. From (E6.2.1) we have

$$\begin{aligned}
S_n(t) &= D_{n-1}(t)\frac{t^{n-1}(1+t)}{(1-t^{2n})}\\
&= \left(\frac{(1-t^2)(1-t^6)\cdots(1-t^{4n-6})}{(1-t)(1-t^2)\cdots(1-t^{2n-2})}\right)\left(\frac{t^{n-1}(1+t)}{(1-t^{2n})}\right)\\
&= \left(\frac{(1-t^2)\cdots(1-t^{4n-6})}{(1-t)\cdots(1-t^{2n-2})}\right)\left(\frac{t^{n-1}(1+t)}{(1-t^{2n})}\right)\frac{(1-t^{2n-1})(1+t^{2n-1})}{(1-t^{2n-1})(1+t^{2n-1})}\\
&= \frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-2})t^{n-1}(1+t)}{(1-t)(1-t^2)(1-t^3)\cdots(1-t^{2n-1})(1-t^{2n})(1+t^{2n-1})}\\
&= D_n(t)\frac{t^{n-1}(1+t)}{(1+t^{2n-1})}.
\end{aligned}$$

□

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