Linked Partitions and Permutation Tableaux

William Y.C. Chen[1], Lewis H. Liu[2], Carol J. Wang[3]

[1]Center for Applied Mathematics
Tianjin University
Tianjin 300072, P.R. China

[2]Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China

[3]Department of Mathematics
Beijing Technology and Business University
Beijing 100048, P.R. China

[1]chenyc@tju.edu.cn, [2]lewis@cfc.nankai.edu.cn, [3]wang_jian@th.btbu.edu.cn

Abstract

Linked partitions are introduced by Dykema in the study of transforms in free probability theory, whereas permutation tableaux are introduced by Steingrímsson and Williams in the study of totally positive Grassmannian cells. Let $[n] = \{1, 2, \ldots, n\}$. Let $L(n, k)$ denote the set of linked partitions of $[n]$ with k blocks, let $P(n, k)$ denote the set of permutations of $[n]$ with k descents, and let $T(n, k)$ denote the set of permutation tableaux of length n with k rows. Steingrímsson and Williams found a bijection between the set of permutation tableaux of length n with k rows and the set of permutations of $[n]$ with k weak excedances. Corteel and Nadeau gave a bijection from the set of permutation tableaux of length n with k columns to the set of permutations of $[n]$ with k descents. In this paper, we establish a bijection between $L(n, k)$ and $P(n, k-1)$ and a bijection between $L(n, k)$ and $T(n, k)$. Restricting the latter bijection to noncrossing linked partitions, we find that the corresponding permutation tableaux can be characterized by pattern avoidance.

Keywords: linked partition, descent, permutation tableau.

AMS Classifications: 05A05, 05A19.

1 Introduction

The notion of linked partitions was introduced by Dykema [8] in the study of the unsymmetrized T-transform in free probability theory. Let $[n] = \{1, 2, \ldots, n\}$. A linked partition of $[n]$ is a collection of nonempty subsets $B_1, B_2, \ldots, B_k$ of $[n]$, called blocks, such that the union of $B_1, B_2, \ldots, B_k$ is $[n]$ and any two distinct blocks are nearly disjoint. Two blocks B_i and B_j are said to be nearly disjoint if for any $k \in B_i \cap B_j$, one of the following conditions holds:

(a) $k = \min(B_i), |B_i| > 1$ and $k \neq \min(B_j)$, or

(b) $k = \min(B_j), |B_j| > 1$ and $k \neq \min(B_i)$.

The linear representation of a linked partition was introduced by Chen, Wu and Yan [1]. For a linked partition τ of $[n]$, list the n vertices in a horizontal line with labels $1, 2, \ldots, n$. For a block $B = \{i_1, i_2, \ldots, i_k\}$ with $k \geq 2$ and $\min(B) = i_1$, draw an arc from i_1 to i_j for $j = 2, \ldots, k$. For example, the linear representation of the linked partition $\{1,2,4\}\{2,3\}\{3,9\}\{5,6\}\{6,7\}\{8\}$ is illustrated in Figure 1.1.

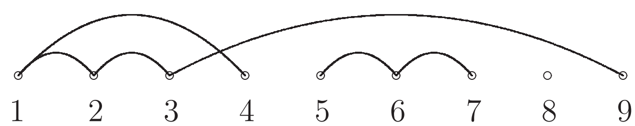

Figure 1.1: The linear representation of a linked partition.

For $i < j$, we use a pair (i, j) to denote an arc from i to j, and we call i and j the left-hand endpoint and the right-hand endpoint of (i, j), respectively. Two arcs (i_1, j_1) and (i_2, j_2) form a crossing if $i_1 < i_2 < j_1 < j_2$, and form a nesting if $i_1 < i_2 < j_2 < j_1$. For the linked partition in Figure 1.1, there is one crossing formed by $(1, 4)$ and $(3, 9)$, while there are three nestings: $(1, 4)$ and $(2, 3)$, $(3, 9)$ and $(5, 6)$, $(3, 9)$ and $(6, 7)$. A linked partition is called noncrossing (resp., nonnesting) if there does not exist any crossing (resp., nesting) in its linear representation. Dykema [8] showed that the number of noncrossing linked partitions of $[n + 1]$ is equal to the n-th large Schröder number. The sequence of the large Schröder numbers is listed as A006318 in OEIS [9]. Chen, Wu and Yan [1] found a bijective proof of this relation and proved that the number of nonnesting linked partitions of $[n]$ also equals the number of noncrossing linked partitions of $[n]$.

Permutation tableaux are introduced by Steingrímsson and Williams [11] in the study of totally positive Grassmannian cells. They are closely related to the PASEP (partially asymmetric exclusion process) model in statistical physics [2, 4–6]. Permutation tableaux are also in one-to-one correspondence with alternative tableaux introduced by Viennot [12]. More precisely, a permutation tableau is defined by a Ferrers diagram possibly with empty rows such that the cells are filled with 0's and 1's, and

(1) each column contains at least one 1;

(2) there does not exist a 0 with a 1 above (in the same column) and a 1 to the left (in the same row).

The length of a permutation tableau is defined as the number of rows plus the number of columns. A permutation tableau T of length n is labeled by the elements in $[n]$ in increasing order from the top right corner to the bottom left corner. We use (i, j) to denote the cell in row i and column j. For example, Figure 1.2 gives a permutation tableau of length 11 with an empty row.

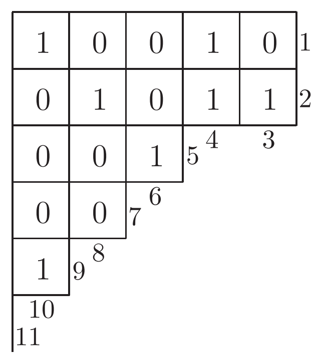

Figure 1.2: A permutation tableau of length 11.

Corteel and Nadeau [3] gave a bijection from permutation tableaux of length n with k columns and permutations of $[n]$ with k descents. Steingrímsson and Williams [11] established a one-to-one correspondence between permutation tableaux of length n with k rows and permutations of $[n]$ with k weak excedances.

The aim of this paper is to demonstrate that linked partitions play a role as an intermediate structure between permutations and permutation tableaux. More precisely, we present two bijections. The first is between linked partitions and permutations, and the second is between linked partitions and permutation tableaux. In fact, the first bijection maps a linked partition of $[n]$ with k blocks to a permutation on $[n]$ with $k - 1$ descents, and the second bijection transforms a linked partition of $[n]$ with k blocks to a permutation tableau of length n with k rows.

Combining the above two bijections, we are led to a one-to-one correspondence between permutations and permutation tableaux, which implies some known properties of permutation tableaux obtained by Corteel and Nadeau [3] and Steingrímsson and Williams [11].

When restricting the second bijection to noncrossing linked partitions, we find that the corresponding permutation tableaux are exactly those that avoid a pattern J_2. Similarly, when restricting this bijection to nonnesting linked partitions, we get permutation tableaux that avoid a pattern I_2. The definitions of the patterns I_2 and J_2 will be given in Section 4.

2 Linked partitions and permutations

In this section, we give a bijection between linked partitions of $[n]$ with k blocks and permutations of $[n]$ with $k-1$ descents.

To describe the construction of this bijection, we give a classification of the vertices in the linear representation of a linked partition. Let τ be a linked partition of $[n]$. A vertex i in the linear representation of τ is called an origin if it is only a left-hand endpoint, or a transient if it is both a left-hand point and a right-hand endpoint, or a singleton if it is an isolated vertex, or a destination if it is only a right-hand endpoint. Figure 2.1 illustrates the four types of vertices.

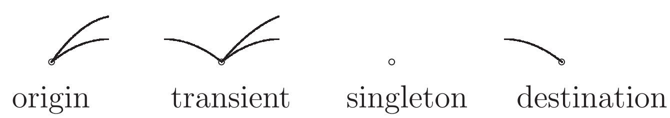

Figure 2.1: Four types of vertices in linked partitions.

Let $L(n,k)$ denote the set of linked partitions of $[n]$ with k blocks. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation on $[n]$. An integer i $(1 \leq i \leq n-1)$ is called a decent (resp., ascent) of π if $\pi_i > \pi_{i+1}$ (resp., $\pi_i < \pi_{i+1}$). Let $P(n,k)$ denote the set of permutations of $[n]$ with k descents, which is counted by the Eulerian number $A(n,k+1)$, see, for example, Stanley [10].

Theorem 2.1. *There is a bijection between $L(n,k)$ and $P(n,k-1)$.*

Proof. We construct a bijection φ between $L(n,k)$ and $P(n,k-1)$ by a recursive procedure. Let $\tau \in L(n,k)$ be a linked partition of $[n]$ with k blocks. It is easily seen that the total number of origins, transients and singletons equals k. Let π denote the permutation $\varphi(\tau)$, which is given as follows.

If $n=1$, that is, $\tau = \{1\}$, then we set $\pi = 1$. We now assume that $n \geq 2$. Let τ' be the linked partition of $[n-1]$ obtained by removing the vertex n along with the arcs associated to n from τ. Assume that τ' has s blocks. Let $i_1 < i_2 < \cdots < i_s$ be the minimal elements of the blocks in τ'. Let $j_1 < j_2 < \cdots < j_t$ be the destinations of τ'. Clearly, $t = n-1-s$. Let $\pi' = \pi_1\pi_2\cdots\pi_{n-1} = \varphi(\tau')$. We assume that the number of descents of π' is one less than the number of blocks in τ'.

We proceed to construct a permutation π by inserting n into π' such that the number of descents of π is one less than the number of blocks of τ. Let k be the minimal element in the block of τ containing n. Here are four cases.

Case 1: $k = i_r$, where $1 \leq r \leq s-1$, that is, there is an arc (i_r, n) in the linear representation of τ. Set

$$\pi = \pi_1\pi_2\cdots\pi_\ell \; n \; \pi_{\ell+1}\cdots\pi_{n-1}, \tag{2.1}$$

where ℓ is the r-th descent of π' from left to right.

Case 2: $k = i_s$. Set $\pi = \pi' \; n$.

Case 3: $k = j_r$, where $1 \leq r \leq t$. Set

$$\pi = \pi_1\pi_2\cdots\pi_\ell \; n \; \pi_{\ell+1}\cdots\pi_{n-1},$$

where ℓ is the r-th ascent in π' from left to right.

Case 4: $k = n$, that is, n is a singleton in τ. Set $\pi = n \; \pi'$.

In any of the above cases, it can be shown that π is a permutation in $P(n, k-1)$. We shall only consider Case 1, since similar arguments apply to other cases. In Case 1, it is easy to check that τ' belongs to $L(n-1, k)$. Hence π' belongs to $P(n-1, k-1)$. By (2.1), we find that π has the same number of descents as π'. It follows that π belongs to $P(n, k-1)$.

It is straightforward to verify that the above procedure is reversible. Hence the map φ is a bijection for any $n \geq 1$. This completes the proof. ∎

For example, Figure 2.2 illustrates the linked partitions of $\{1, 2, 3\}$ and the corresponding permutations.

$\{1,2,3\}$	$\{1,2\}\{2,3\}$	$\{1,3\}\{2\}$	$\{1,2\}\{3\}$	$\{1\}\{2,3\}$	$\{1\}\{2\}\{3\}$
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
123	132	231	312	213	321

Figure 2.2: Linked partitions of $\{1, 2, 3\}$ and the corresponding permutations.

3 Linked partitions and permutation tableaux

The objective of this section is to give a bijection between linked partitions of $[n]$ with k blocks and permutation tableaux of length n with k rows. As consequences, we find some equidistribution properties between linked partitions and permutation tableaux with certain restrictions. For example, the number of permutation tableaux of length n with k rightmost restricted 0's is equal to the number of linked partitions of $[n]$ with k transients. We also show that our bijections can be used to deduce some equidistribution properties of permutation tableaux obtained by Corteel and Nadeau [3] and Steingrímsson and Williams [11].

To construct the bijection between linked partitions and permutation tableaux, we introduce the shape of a linked partition. Let τ be a linked partition of $[n]$. The shape of τ is an integer partition λ of n with empty parts allowed that is defined as follows. Recalling the labeling of a permutation tableau, we assign labels on the boundary of a partition λ of n in increasing order from the top right corner to the bottom left corner, see Figure 1.2. Based on this labeling, we see that λ can be represented by a sequence of labels V and H starting with V, where V stands for the vertical direction and H stands for the horizontal direction.

Using the linear representation of τ, we obtain a sequence $S = s_1 s_2 \cdots s_n$ of directions. If i is a destination, we set $s_i = H$; otherwise, we set $s_i = V$. For example, Figure 3.1 gives the shape of the linked partition in Figure 1.1.

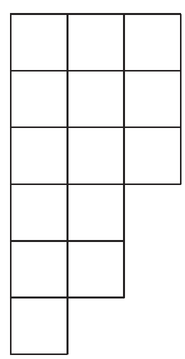

Figure 3.1: The shape of $\{1, 2, 4\}\{2, 3\}\{3, 9\}\{5, 6\}\{6, 7\}\{8\}$.

The construction of the bijection between linked partitions and permutation tableaux also requires a fact proved by Corteel and Nadeau [3] that a permutation tableau is determined by its topmost 1's and rightmost restricted 0's. A 1 in a permutation tableau is called a topmost 1 if there is no 1 above it in the same column. A 0 is said to be restricted if there is a 1 above. The rightmost such 0 in a row is called a rightmost restricted 0. For example, in the permutation tableau in Figure 1.2, the topmost 1's are in the cells $(1, 4)$, $(1, 10)$, $(2, 3)$, $(2, 8)$ and $(5, 6)$, while the rightmost restricted 0's are in the cells $(2, 10)$, $(5, 8)$ and $(7, 8)$. To see that a permutation tableau is uniquely determined by its topmost 1's and rightmost restricted 0's, it suffices to observe the fact that if the positions of topmost 1's are given, then all the cells above them are filled with 0's; if the positions of the rightmost restricted 0's are given, then all the cells to the left of them are filled with 0's; the rest of the cells are filled with 1's.

Let $T(n,k)$ denote the set of permutation tableaux of length n with k rows. Then we have the following correspondence and the explicit construction is given the proof.

Theorem 3.1. *There is a shape preserving bijection between $L(n,k)$ and $T(n,k)$.*

Proof. We construct a shape preserving bijection ϕ between $L(n,k)$ and $T(n,k)$. Let τ be a linked partition in $L(n,k)$. We proceed to construct a permutation tableau $T = \phi(\tau)$.

First, we generate the shape λ of τ, and label the boundary of λ by using the elements of $[n]$ in increasing order from the top right corner to the bottom left corner.

Next, we wish to fill the cells of λ with topmost 1's and rightmost restricted 0's. Let i_1 be the minimum origin in τ, and let j be a destination such that there exists a path $(i_1, i_2, \ldots, i_m, j)$ from i_1 to j with j being maximum in the linear representation of τ. Then, we fill the cell (i_1, j) with 1. For $\ell = 2, \ldots, m$, we fill the cells (i_ℓ, j) with 0.

Let τ' be the linked partition of $[n]$ obtained from τ by removing the arcs in the path $(i_1, i_2, \ldots, i_m, j)$. Repeating the above process for τ', we can fill a column with a topmost 1 and some rightmost restricted 0's until there are no arcs left in the linear representation of τ.

Finally, we define T to be the permutation tableau such that the 1's and 0's filled in λ constitute the topmost 1's and rightmost restricted 0's of T, respectively. As mentioned before, a permutation tableau is uniquely determined by its topmost 1's and rightmost restricted 0's.

The reverse map of ϕ can be described as follows. Let T be a permutation tableau of length n and of shape λ. We can construct a linked partition of $[n]$. We start with the leftmost column. For the column labeled with j, let (i,j) be the cell filled with a topmost 1. If there does not exist any rightmost restricted 0 in column j, then let (i,j) be an arc in the linear representation of τ. Otherwise, let $(i_2, j), (i_3, j), \ldots, (i_m, j)$ be the cells filled with the rightmost restricted 0's. For $\ell = 1, 2, \ldots, m$, let $(i_\ell, i_{\ell+1})$ be the arc in the linear representation of τ, where $i_1 = i$ and $i_{m+1} = j$. Note that there is a unique topmost 1 in each column and a unique rightmost restricted 0 in each row. Thus, for any vertex j in τ, there is at most one arc whose right-hand endpoint is j. This implies that τ is a linked partition of $[n]$. Moreover, it is not hard to check that T and τ have the same shape. This completes the proof. ∎

For example, the permutation tableau corresponding to the linked partition in Figure 1.1 is given in Figure 3.2.

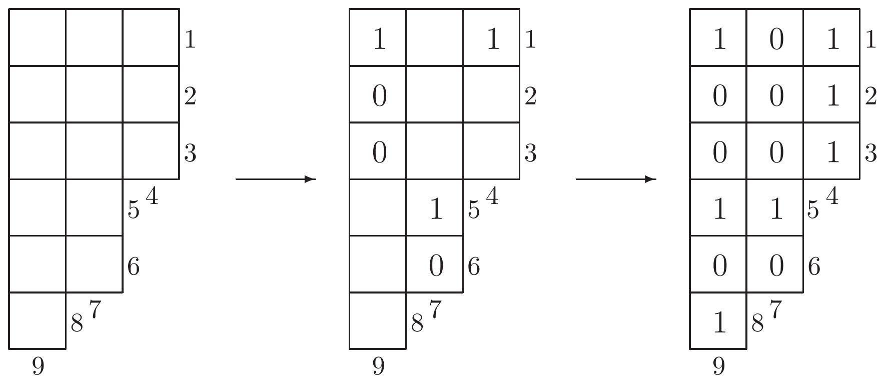

Figure 3.2: From a linked partition to a permutation tableau.

For the permutation tableau given in Figure 1.2, the corresponding linked partition is given in Figure 3.3.

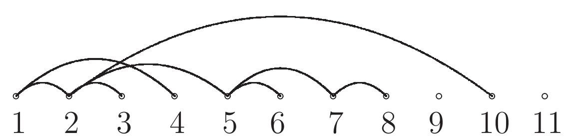

Figure 3.3: The linked partition corresponding to the permutation tableau in Figure 1.2.

It is easy to see that the bijection ϕ has the following properties.

Corollary 3.2. *For $n \geq 1$, let τ be a linked partition of $[n]$. Then the number of arcs in the linear representation of τ is equal to the total number of topmost 1's and rightmost restricted 0's in $T = \phi(\tau)$.*

Corollary 3.3. *Assume that $n \geq 1$.*

(1) For $0 \leq k \leq n-2$, the number of linked partitions of $[n]$ with k transients equals the number of permutation tableaux of length n with k rightmost restricted 0's;

(2) For $0 \leq k \leq n$, the number of linked partitions of $[n]$ with k singletons equals the number of permutation tableaux of length n with k rows that do not contain any topmost 1 or restricted 0.

To conclude this section, we remark that our bijections φ and ϕ can be used to deduce some results on permutation tableaux obtained by Corteel and Nadeau [3] and Steingrímsson and Williams [11]. Corteel and Nadeau [3] showed that the number of permutation tableaux of length n with k columns is equal to the number of permutations of $[n]$ with k descents. This fact follows from bijections φ and ϕ. Noting that the number of permutations of n with $k-1$ descents is equal to the number of permutations of $[n]$ with $n-k$ descents, we see that the number of permutation tableaux of length n with k rows is equal to the number of permutations with $n-k$ descents, which is equivalent to the result of Corteel and Nadeau [3].

On the other hand, the number of permutations of $[n]$ with $k-1$ descents is equal to the number of permutations of $[n]$ with k weak excedances, see, for example, Stanley [10, Chapter 1]. This leads to the fact that the number of permutation tableaux of length n with k rows is equal to the number of permutations of $[n]$ with k weak excedances, as proved by Steingrímsson and Williams [11].

4 Pattern avoiding permutation tableaux

In this section, we discuss restrictions of the bijection ϕ in Section 3 to noncrossing linked partitions and nonnesting linked partitions, and characterize the corresponding permutation tableaux by pattern avoidance.

We introduce two patterns I_2 and J_2 as given in Figure 4.1, where a dot means a topmost 1 or a rightmost restricted 0. We use I_2 and J_2 to denote these two patterns because similar notation has been used in the context of fillings of Ferrers diagrams, see de Mier [7].

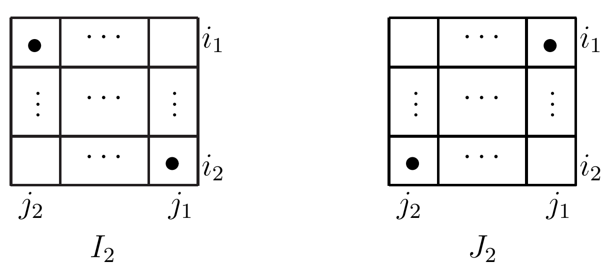

Figure 4.1: Patterns I_2 and J_2.

More precisely, let T be a permutation tableau of length n, and let T' be the permutation tableau obtained from T by replacing the topmost 1's and rightmost restricted 0's by dots and removing all other 1's and 0's. We say that T avoids the pattern I_2 if T'

does not contain four cells (i_1, j_1), (i_1, j_2), (i_2, j_1) and (i_2, j_2), where $i_1 < i_2 < j_1 < j_2$, such that the cells (i_1, j_2) and (i_2, j_1) are filled with dots, while the cell (i_2, j_2) is empty. Similarly, we say that T avoids the pattern J_2 if T' does not contain four cells (i_1, j_1), (i_1, j_2), (i_2, j_1) and (i_2, j_2), where $i_1 < i_2 < j_1 < j_2$, such that the cells (i_1, j_1) and (i_2, j_2) are filled with dots, while the cell (i_2, j_1) is empty. For example, Figure 4.2 illustrates a permutation tableau avoiding the pattern I_2. Note that this permutation tableau does not avoid the pattern J_2, because the cells $(1, 3)$, $(1, 6)$, $(2, 3)$ and $(2, 6)$ form pattern J_2.

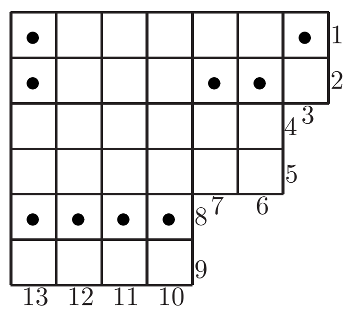

Figure 4.2: A permutation tableau avoiding I_2.

By the construction of the bijection ϕ in Section 3, we obtain characterizations of permutation tableaux corresponding to noncrossing linked partitions and nonnesting linked partitions. To be more specific, we have the following correspondences.

Theorem 4.1. *There is a bijection between noncrossing linked partitions of* $[n]$ *and* J_2*-avoiding permutation tableaux of length* n*, and there is a bijection between nonnesting linked partitions of* $[n]$ *and* I_2*-avoiding permutation tableaux of length* n*.*

Since the number of noncrossing linked partitions of $[n]$ equals the number of nonnesting linked partitions of $[n]$, see Chen, Wu and Yan [1], one sees that the number of J_2-avoiding permutation tableaux of length n equals the number of I_2-avoiding permutation tableaux of length n.

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the National Science Foundation of China, Beijing Natural Science Foundation, and Beijing Commission of Education.

References

[1] W.Y.C. Chen, S.Y.J. Wu and C.H. Yan, Linked partitions and linked cycles, European J. Combin. 29 (2008) 1408–1426.

[2] S. Corteel, R. Brak, A. Rechnitzer and J. Essam, A combinatorial derivation of the PASEP stationary state, Electron. J. Combin. 13 (2006) #R108.

[3] S. Corteel and P. Nadeau, Bijections for permutation tableaux, European J. Combin. 30 (2009) 295–310.

[4] S. Corteel and L.K. Williams, Tableaux combinatorics for the symmetric exclusion process I, Adv. Appl. Math. 37 (2007) 293–310.

[5] S. Corteel and L.K. Williams, A Markov chain on permutations which projects to the PASEP, Int. Math. Res. Not. (2007) Article ID rnm055.

[6] S. Corteel and L.K. Williams, Tableaux combinatorics for the asymmetric exclusion process and Askey-Wilson polynomials, Duke Math. J. 159 (2011) 385–415.

[7] A. de Mier, k-Noncrossing and k-nonnesting graphs and fillings of ferrers diagrams, Combinatorica 27 (2007) 699–720.

[8] K.J. Dykema, Multilinear function series and transforms in free probability theory, Adv. Math. 208 (2007) 351–407.

[9] The OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[10] R.P. Stanley, Enumerative Combinatorics, Vol. I, Cambridge University Press, Cambridge, 1997.

[11] E. Steingrímsson and L. Williams, Permutation tableaux and permutation patterns, J. Combin. Theory Ser. A 114 (2007) 211–234.

[12] X. Viennot, Alternative tableaux, permutations and partially asymmetric exclusion process, Isaac Newton Institute, April 2007.