Fibonacci sequence related to a combinatorial problem on binary matrices

Krasimir Yordzhev

Faculty of Mathematics and Natural Sciences, South-West University
2700 Blagoevgrad, Bulgaria
E-mail: yordzhev@yahoo.com

Abstract

We discuss an equivalence relation on the set of square binary matrices with the same number of 1's in each row and each column. Each binary matrix is represented using ordered n-tuples of natural numbers. We give a few starting values of integer sequences related to the discussed problem. The obtained sequences are new and they are not described in the On-Line Encyclopedia of Integer Sequences (OEIS). We show a relationship between some particular values of the parameters and the Fibonacci sequence.

Keyword: binary matrix; equivalence relation; factor-set; Fibonacci number

2010 Mathematics Subject Classification: 05B20; 11B39

1 Introduction

A *binary* (or *boolean*, or (0,1)-*matrix*) is a matrix whose all elements belong to the set $\mathcal{B} = \{0, 1\}$. With $\mathcal{B}_n$ we will denote the set of all $n \times n$ binary matrices.

Let n and k be positive integers. We let Λ_n^k denote the set of all $n \times n$ binary matrices in each row and each column of which there are exactly k in number 1's. Let us denote with $\lambda(n, k) = |\Lambda_n^k|$ the number of all elements of Λ_n^k. There is not any known formula to calculate the $\lambda(n, k)$ for all n and k.

Let $A, B \in \Lambda_n^k$. We will say that $A \sim B$, if A is obtained from B by moving some rows and/or columns. Obviously, the relation defined like that is an equivalence relation. We denote with

$$\mu(n,k) = \left| \Lambda^k_{n/\sim} \right| \tag{1}$$

the number of equivalence classes on the above defined relation.

Problem 1 *Find $\mu(n,k)$ for given integers n and k, $1 \leq k < n$.*

Problem 1 is the subject of discussion in this article.

2 Some values of the integer function $\mu(n,k)$

The task of finding the number of equivalence classes for all integers n and k, $1 \leq k \leq n$ is an open scientific problem. We partially solve this problem by implementing a computer program to find $\mu(n,k)$ for some (not great) values of n and k. Moreover, using bitwise operations, our algorithm received one representative from each equivalence class without examining the whole set Λ_n^k [6].

Let $\mathbb{N}$ be the set of natural numbers and let

$$\mathcal{T}_n = \left\{ \langle x_1, x_2, \ldots, x_n \rangle \mid x_i \in \mathbb{N},\ 0 \leq x_i \leq 2^n - 1,\ i = 1, 2, \ldots, n \right\}. \tag{2}$$

In [5] and [4] we describe an one-to-one correspondence

$$\varphi\ :\ \mathcal{B}_n \cong \mathcal{T}_n \tag{3}$$

which is based on the binary presentation of the natural numbers. If $A \in \mathcal{B}_n$ and $\varphi(A) = \langle x_1, x_2, \ldots x_n \rangle$, then i-th row of A is integer x_i written in binary notation.

In [4] we prove that the representation of the elements of $\mathcal{B}_n$ using ordered n-tuples of natural numbers leads to making a fast and saving memory algorithms.

Let $A \in \mathcal{B}_n$ and let $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle = \varphi(A)$. Then we denote

$$\mathbf{x}^t = \varphi(A^t),$$

where $A^t \in \mathcal{B}_n$ is the transpose of the matrix A .

Definition 1 *Let* $\mathbf{x} = \langle x_1, x_2, \ldots, x_n\rangle \in \mathcal{T}_n$ *and let* $\mathbf{x}^t = \langle y_1, y_2, \ldots, y_n\rangle$. *The element* $\mathbf{x} \in \mathcal{T}_\mathbf{n}$ *we will call* **canonical element**, *if* $x_1 \leq x_2 \leq \cdots \leq x_n$ *and* $y_1 \leq y_2 \leq \cdots \leq y_n$. *The matrix* $A \in \Lambda_m^k$ *we will call* **canonical matrix**, *if* $\varphi(A)$ *is a canonical element in* $\mathcal{T}_n$, *where* φ *is the defined with (3) isomorphism.*

Obviously, when $k = 0$, the zero $n \times n$ matrix is the only matrix in the set Λ_n^0. When $k = n$, there is only one $n \times n$ binary matrix of Λ_n^n, and this is the matrix all elements of which are equal to 1. Therefore

$$\mu(n, 0) = \mu(n, n) = 1. \tag{4}$$

It is easy to prove that for any positive integer n is satisfied

$$\mu(n, 1) = \mu(n, n-1) = 1. \tag{5}$$

When $k = 1$ the only canonical element is $\mathbf{x} = \langle 1, 2, 4, \ldots, 2^{n-1}\rangle \in \mathcal{T}_n$, i.e., if $A \in \Lambda_n^1$ is a canonical matrix, then A is a binary matrix with 1 in the second (not leading) diagonal and 0 elsewhere. For $k = n-1$, if $A \in \Lambda_n^{n-1}$ is a canonical matrix, then A is a binary matrix with 0 in the leading diagonal and 1 elsewhere.

An algorithm for finding all canonical elements of $\mathcal{T}_n$ is described in detail in [6]. For $k = 2, 3, 4$ and $k = 5$, we will display the first elements of the sequences $\{\mu(n, k)\}_{n=k}^{\infty}$ for some values of the parameter n. Using a computer program [6] we obtained the following sequences

$$\{\mu(n, 2)\}_{n=2}^{\infty} = \{1,\ 1,\ 2,\ 5,\ 13,\ 42,\ 155,\ 636,\ 2\,889,\ 14\,321,\ \ldots\} \tag{6}$$

$$\{\mu(n, 3)\}_{n=3}^{\infty} = \{1,\ 1,\ 3,\ 25,\ 272,\ 4\,070,\ 79\,221,\ \ldots\} \tag{7}$$

$$\{\mu(n, 4)\}_{n=4}^{\infty} = \{1,\ 1,\ 5,\ 161,\ 7\,776,\ 626\,649,\ \ldots\} \tag{8}$$

$$\{\mu(n, 5)\}_{n=5}^{\infty} = \{1,\ 1,\ 8,\ 1\,112,\ 287\,311,\ \ldots\} \tag{9}$$

The obtained integer sequences (6) $\div$ (9) are not described in the On-Line Encyclopedia of Integer Sequences (OEIS)[1].

3 The function $\mu(n, k)$ and Fibonacci numbers

The sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers is defined by the recurrence relation (see for example [2] or [3])

$$f_0 = f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n = 2, 3, \dots \tag{10}$$

In this section, we will prove that the sequence $\{\mu(k+2,k)\}_{k=0}^{\infty}$ coincides with the Fibonacci sequence (10).

Lemma 1 *If $A = (\alpha_{i\,j}) \in \Lambda_n^k$ is a canonical matrix then*

$$\alpha_{1\,1} = \alpha_{1\,2} = \cdots = \alpha_{1\,n-k} = 0, \quad \alpha_{1\,n-k+1} = \alpha_{1\,n-k+2} = \cdots = \alpha_{1\,n} = 1,$$

$$\alpha_{1\,1} = \alpha_{2\,1} = \cdots = \alpha_{n-k\,1} = 0, \quad \alpha_{n-k+1\,1} = \alpha_{n-k+2\,1} = \cdots = \alpha_{n\,1} = 1,$$

Proof. Immediately.

□

Corollary 1 *If $\mathbf{x} = \langle x_1, x_2, \dots, x_n\rangle \in \mathcal{T}_n$ is a canonical element then $x_1 = 2^k - 1$.*

□

Theorem 1 *Let the sequence $\{\mu(k+2,k)\}_{k=0}^{\infty}$ is defined by (1) where $n = k+2$, and let $\{f_k\}_{k=0}^{\infty}$ be the Fibonacci sequence (10). Then for every integer $k = 0, 1, 2, 3, \dots$ the equality*

$$\mu(k+2,k) = f_k$$

is true.

Proof. When $k = 0$ the assertion follows from (10) and (4). When $k = 1$ the assertion follows from (10) and (5). When $k = 2$ there are two canonical elements in $\mathcal{T}_4$ and these are $\mathbf{x}_1 = \langle 3, 3, 12, 12\rangle$ and $\mathbf{x}_2 = \langle 3, 5, 10, 12\rangle$ (see (6) and [6]). Therefore

$$\mu(2,0) = f_0, \quad \mu(3,1) = f_1 \quad \text{and} \quad \mu(4,2) = f_2$$

Let k be an arbitrary positive integer such that $k \geq 3$ and let $A = (\alpha_{i\,j}) \in \Lambda_{k+2}^k$, $1 \leq i, j \leq k+2$ be a canonical matrix. Then, according to Lemma 1 $\alpha_{1\,1} = \alpha_{1\,2} = \alpha_{2\,1} = 0$ and $\alpha_{1\,3} = \alpha_{1\,4} = \ldots = \alpha_{1\,n} = \alpha_{3\,1} = \alpha_{4\,1} = \ldots = \alpha_{n\,1} = 1$. Therefore, the following two cases are possible:

i) $\alpha_{2\,2} = 0$, i.e., A is of the form

$$A = \begin{pmatrix} 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 1 & 1 & & & \\ \vdots & \vdots & & B & \\ 1 & 1 & & & \end{pmatrix}$$

We denote by $\mathcal{M}_1$ the set of all canonical matrices of this kind. Let A be an arbitrary matrix of $\mathcal{M}_1$. In A, we remove the first and second rows and the first and second columns. We obtain the matrix $B \in \Lambda_k^{k-2}$. It is easy to see that B is the canonical matrix.

Conversely, let $B = (\beta_{i\,j}) \in \Lambda_k^{k-2}$ $(k \geq 3)$ and let B be a canonical matrix. From B we obtain the matrix $A = (\alpha_{i\,j}) \in \Lambda_{k+2}^k$ as follows: $\alpha_{1\,1} = \alpha_{1\,2} = \alpha_{2\,1} = \alpha_{2\,2} = 0$, $\alpha_{1\,j} = \alpha_{2\,j} = 1$, $3 \leq j \leq k+2$ and $\alpha_{i\,1} = \alpha_{i\,2} = 1$, $3 \leq i \leq k+2$. For each $i, j \in \{3, 4, \ldots, k+2\}$ we assume $\alpha_{i\,j} = \beta_{i-2\,j-2}$. It is easy to see that the so obtained matrix A is a canonical matrix.

Therefore, $|\mathcal{M}_1| = \mu(k, k-2)$ for any integer $k \geq 3$.

ii) $\alpha_{2\,2} = 1$, i.e., A is of the form

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 0 & & & & \\ 1 & 1 & & & & \\ \vdots & \vdots & & & & \\ 1 & 1 & & & & \end{pmatrix}$$

Let $\mathcal{M}_2$ be the set of all canonical matrices of this kind and let $A = (\alpha_{i\,j})$, $\alpha_{2\,2} = 1$ be an arbitrary matrix of $\mathcal{M}_2$. We change $\alpha_{2\,2}$ from 1 to 0 and remove the first row and the first column of A. In this way we obtain a matrix of Λ_{k+1}^{k-1}, which is easy to see that it is canonical.

Conversely, let $B = (\beta_{i\,j}) \in \Lambda_{k+1}^{k-1}$ and let B be a canonical matrix. According to Lemma 1 $\beta_{1\,1} = \beta_{1\,2} = \beta_{2\,1} = 0$. We change $\beta_{1\,1}$ from 0 to 1. In B, we add a first row and a first column and get the matrix $A = (\alpha_{i\,j}) \in \Lambda_{k+2}^k$, such that $\alpha_{1\,1} = \alpha_{1\,2} = \alpha_{2\,1} = 0$, $\alpha_{1\,j} = 1$ for $j = 3, 4, \ldots, k+2$, $\alpha_{i\,1} = 1$ for $i = 3, 4, \ldots, k+2$ and $\alpha_{s+1\,t+1} = \beta_{s\,t}$ for $s, t \in \{1, 2, \ldots, k+1\}$. It is easy to see that the resulting matrix A is canonical and $A \in \mathcal{M}_2$.

Therefore, $|\mathcal{M}_2| = \mu(k+1, k-1)$ for every integer $k \geq 3$.

If $\mathcal{M}$ is the set of all canonical matrices, $\mathcal{M} \subseteq \Lambda_{k+2}^k$, then obviously

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset \quad \text{and} \quad \mathcal{M}_1 \cup \mathcal{M}_2 = \mathcal{M}.$$

Therefore

$$\mu(k+2,k) = |\mathcal{M}| = |\mathcal{M}_1| + |\mathcal{M}_2| = \mu(k,k-2) + \mu(k+1,k-1)$$

for all integers $k \geq 3$, which proves the theorem. $\square$

References

[1] On-line encyclopedia of integer sequences. http://oeis.org/.

[2] K. Atanassov, V. Atanassova, A. Shannon, and J. Tumer. *New Visual Perspectives on Fibonacci Num.* World Scientific, 2002.

[3] T. Koshy. *Fibonacci and Lucas Numbers with Applications.* John Wiley & Sons, 2011.

[4] H. Kostadinova and K. Yordzhev. A representation of binary matrices. *Mathematics and education in mathematics*, 39:198–206, 2010.

[5] K. Yordzhev. An example for the use of bitwise operations in programming. *Mathematics and education in mathematics*, 38:196–202, 2009.

[6] K. Yordzhev. Bitwise operations related to a combinatorial problem on binary matrices. *I.J.Modern Education and Computer Science*, (4):19–24, 2013.