The 1-box pattern on pattern avoiding permutations

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Abstract

This paper is continuation of the study of the 1-box pattern in permutations introduced by the authors in [7]. We derive a two-variable generating function for the distribution of this pattern on 132-avoiding permutations, and then study some of its coefficients providing a link to the Fibonacci numbers. We also find the number of separable permutations with two and three occurrences of the 1-box pattern.

Keywords: 1-box pattern, 132-avoiding permutations, separable permutations, Fibonacci numbers, Pell numbers, distribution

1 Introduction

In this paper, we study 1-*box patterns*, a particular case of (a, b)-*rectangular patterns* introduced in [7]. That is, let $\sigma = \sigma_1 \cdots \sigma_n$ be a permutation written in one-line notation. Then we will consider the graph of σ, $G(\sigma)$, to be the set of points (i, σ_i) for $i = 1, \ldots, n$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1.

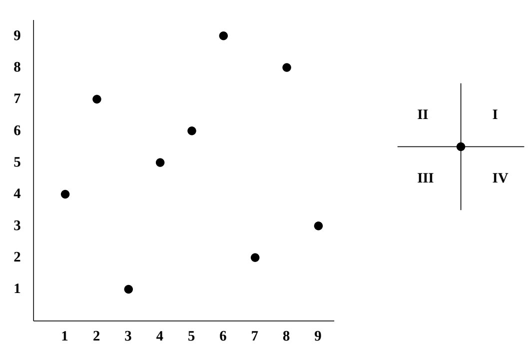

Figure 1: The graph of $\sigma = 471569283$.

Then if we draw a coordinate system centered at a point (i, σ_i), we will be interested in the points that lie in the $2a \times 2b$ rectangle centered at the origin. That is, the (a, b)-rectangle pattern centered at (i, σ_i) equals the set of points $(i \pm r, \sigma_i \pm s)$ such that $r \in \{0, \ldots, a\}$ and $s \in \{0, \ldots, b\}$. Thus σ_i matches the (a, b)-rectangle pattern in σ, if there is at least one point in the $2a \times 2b$-rectangle centered at the point (i, σ_i) in $G(\sigma)$ other than (i, σ_i). For example, when we look for matches of the (2,3)-rectangle patterns, we would look at 4×6 rectangles centered at the point (i, σ_i) as pictured in Figure 2.

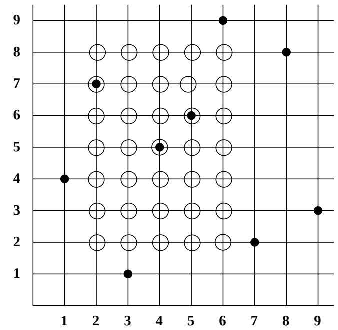

Figure 2: The 4×8-rectangle centered at the point $(4, 5)$ in the graph of $\sigma = 471569283$.

We shall refer to the (k, k)-rectangle pattern as the k-box pattern. For example, if $\sigma = 471569283$, then the 2-box centered at the point $(4, 5)$ in $G(\sigma)$ is the set of circled points pictured in Figure 3. Hence, σ_i matches the k-box pattern in σ, if there is at least one point in the k-box centered at the point (i, σ_i) in $G(\sigma)$ other than (i, σ_i). For example, σ_4 matches the pattern k-box for all $k \geq 1$ in $\sigma = 471569283$ since the point $(5, 6)$ is present in the k-box centered at the point $(4, 5)$ in $G(\sigma)$ for all $k \geq 1$. However, σ_3 only matches the k-box pattern in $\sigma = 471569283$ for $k \geq 3$ since there are no points in 1-box or 2-box centered at $(3, 1)$ in $G(\sigma)$, but the point $(1, 4)$ is in the 3-box centered at $(3, 1)$ in $G(\sigma)$. For $k \geq 1$, we let k-box(σ) denote the set of all i such that σ_i matches the k-box pattern in $\sigma = \sigma_1 \cdots \sigma_n$.

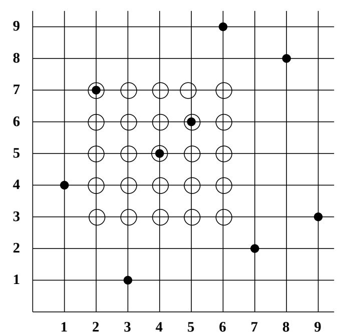

Figure 3: The 2-box centered at the point $(4, 5)$ in the graph of $\sigma = 471569283$.

Note that σ_i matches the 1-box pattern in σ if either $|\sigma_i - \sigma_{i+1}| = 1$ or $|\sigma_{i-1} - \sigma_i| = 1$. For example, the distribution of 1-box(σ) for S_2, S_3, and S_4 is given below, where S_n is the set of all permutations of length n.

σ	1-box(σ)
12	2
21	2

σ	1-box(σ)
123	3
132	2
213	2
231	2
312	2
321	3

σ	1-box(σ)		σ	1-box(σ)
1234	4		2134	4
1243	4		2143	4
1324	2		2314	2
1342	2		2341	3
1423	2		2413	0
1432	3		2431	2
3124	2		4123	3
3142	0		4132	2
3214	3		4213	2
3241	2		4231	2
3412	4		4312	4
3421	4		4321	4

The notion of k-box patterns is related to the *mesh patterns* introduced by Brändén and Claesson [2] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns. This notion was further studied in [1, 4, 5, 8, 9, 10, 12]. In particular, Kitaev and Remmel [5] initiated the systematic study of distribution of marked mesh patterns on permutations, and this study was extended to 132-avoiding permutations by Kitaev, Remmel and Tiefenbruck in [8, 9, 10].

In this paper, we shall study the distribution of the 1-box pattern in 132-avoiding permutations and separable permutations. Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let red(σ) be the permutation found by replacing the i-th largest integer that appears in σ by i. For example, if $\sigma = 2754$, then red(σ) $= 1432$. Given a permutation $\tau = \tau_1 \cdots \tau_j$ in the symmetric group S_j, we say that the pattern τ *occurs* in $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ provided there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. In particular, a permutation σ avoids the pattern 132 if σ does not contain a subsequence of three elements, where the first element is the smallest one, and the second element is the largest one. Let $S_n(\tau)$ denote the set of permutations in S_n which avoid τ. In the theory of permutation patterns (see [3] for a comprehensive introduction to the area), τ is called a *classical pattern*. The results in this paper can be viewed as another contribution to the long line of research in the literature which studies various distributions on pattern-avoiding permutations (e.g. see [3, Chapter 6.1.5] for relevant results).

The outline of this paper is as follows. In Section 2 we shall study the distribution of the 1-box pattern in 132-avoiding permutations. In particular, we shall derive explicit formulas for the generating functions

$$A(t,x) = \sum_{n\geq 0} A_n(x)t^n,$$

$$B(t,x) = \sum_{n\geq 1} B_n(x)t^n \text{ and}$$

$$E(t,x) = \sum_{n\geq 1} E_n(x)t^n$$

where $A_0(x) = 1$ and for $n \geq 1$,

$$\begin{aligned}
A_n(x) &= \sum_{\sigma\in S_n(132)} x^{\text{1-box}(\sigma)} \\
B_n(x) &= \sum_{\sigma=\sigma_1\ldots\sigma_n\in S_n(132),\sigma_1=n} x^{\text{1-box}(\sigma)} \text{ and} \\
E_n(x) &= \sum_{\sigma=\sigma_1\ldots\sigma_n\in S_n(132),\sigma_n=n} x^{\text{1-box}(\sigma)}.
\end{aligned}$$

In Section 3, we shall study the coefficients of x^k in the polynomials $A_n(x)$, $B_n(x)$, and $E_n(x)$ for $k \in \{0,1,2,3,4\}$ as well as the coefficient of the highest power of x in these polynomials. Many of these coefficients can be expressed in terms of the Fibonacci numbers F_n. For example, for $n \geq 2$, the coefficient of x^2 in $A_n(x)$ is F_n and the coefficient of x^2 in $B_n(x)$ and $E_n(x)$ is F_{n-2}. Finally, in Section 4, we shall study the 1-box pattern on *separable permutations.*

2 Distribution of the 1-box pattern on 132-avoiding permutations

In this section, we shall study the generating functions $A(t,x)$, $B(t,x)$, and $E(t,x)$. Clearly, $A_1(x) = B_1(x) = E_1(x) = 1$. One can see from our tables for S_2, S_3, and S_4 that $A_2(x) = 2x^2$, $A_3(x) = 3x^2 + 2x^3$, and $A_4(x) = 5x^2 + 3x^3 + 6x^4$. Similarly, one can check that $B_2(x) = E_2(x) = x^2$, $B_3(x) = E_3(x) = x^2 + x^3$, and $B_4(x) = E_4(x) = 2x^2 + x^3 + 2x^4$.

We shall classify the 132-avoiding permutations $\sigma = \sigma_1 \cdots \sigma_n$ by position of n in σ. That is, let $S_n^{(i)}(132)$ denote the set of $\sigma \in S_n(132)$ such that $\sigma_i = n$. Clearly each $\sigma \in S_n^{(i)}(132)$ has the structure pictured in Figure 4. That is, in the graph of σ, the elements to the left of n, $A_i(\sigma)$, have the structure of a 132-avoiding permutation, the elements to the right of n, $B_i(\sigma)$, have the structure of a 132-avoiding permutation, and all the elements in $A_i(\sigma)$ lie above all the elements in $B_i(\sigma)$. Note that the number of 132-avoiding permutations in S_n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, which is a well-known fact, and the generating function for the C_n's is given by

$$C(t) = \sum_{n\geq 0} C_n t^n = \frac{1-\sqrt{1-4t}}{2t} = \frac{2}{1+\sqrt{1-4t}}.$$

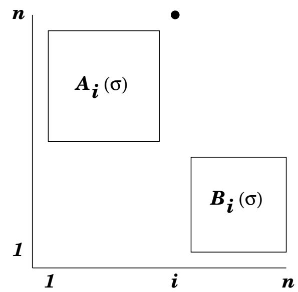

Figure 4: The structure of 132-avoiding permutations.

The following lemma establishes relations between $A_n(x)$, $B_n(x)$, and $E_n(x)$.

Lemma 1. *For all $n \geq 1$, $B_n(x) = E_n(x)$ and for $n \geq 4$,*

$$B_n(x) = x^n + (A_{n-1}(x) - B_{n-1}(x)) + \sum_{i=2}^{n-2} x^{n-i}(A_i(x) - B_i(x)). \tag{1}$$

For $n \geq 2$,

$$A_n(x) = B_n(x) + \sum_{i=2}^{n} B_i(x)A_{n-i}(x). \tag{2}$$

Proof. We begin with deriving relationships for $B_n(x)$ and $E_n(x)$. Any 132-avoiding permutation $\pi = \pi_1 \cdots \pi_n$ beginning with the largest letter n is of one of the three forms described below:

1. the decreasing permutation $n(n-1)\cdots 1$;
2. $n\ell\pi_3\pi_4\cdots\pi_n$ where $\ell < n-1$ and $\ell\pi_3\pi_4\cdots\pi_n$ is a 132-avoiding permutation on $\{1,\ldots,n-1\}$;
3. $n(n-1)\cdots(n-i+1)\ell\pi_{i+2}\pi_{i+3}\cdots\pi_n$, where $2 \leq i \leq n-2$, $\ell < n-i$ and $\ell\pi_{i+2}\pi_{i+3}\cdots\pi_n$ is a 132-avoiding permutation on $\{1,\ldots,n-i\}$.

This structural observation implies immediately (1). Indeed, in the decreasing permutation each element is an occurrence of the 1-box pattern thus giving a contribution of x^n to the function $B_n(x)$. Also, in the second case, n is not an occurrence of the 1-box pattern in π and it does not effect whether any of the remaining elements in π are occurrences of the 1-box pattern in π. Thus, in this case we have a contribution of $(A_{n-1}(x)-B_{n-1}(x))$ to $B_n(x)$. Finally, in the last case, for any i, $2 \leq i \leq n-2$, each of the elements $n-i+1, n-i+2, \ldots, n$ is an occurrence of the 1-box pattern in π and these elements do not effect whether any of the remaining elements in π are occurrences of the 1-box pattern in π. Thus, in this case we have a contribution of $\sum_{i=2}^{n-2} x^{n-i}(A_i(x) - B_i(x))$ to $B_n(x)$.

We can use similar methods to prove that for all $n \geq 4$,

$$E_n(x) = x^n + (A_{n-1}(x) - E_{n-1}(x)) + \sum_{i=2}^{n-2} x^{n-i}(A_i(x) - E_i(x)). \tag{3}$$

That is, if π is a 132-avoiding permutation in S_n that ends in n, we have the following three cases:

1. π is the increasing permutation $1 \cdots n$;

2. $\pi = \pi_1 \cdots \pi_{n-2} \ell n$ where $\ell < n-1$ and $\pi_1 \cdots \pi_{n-2}\ell$ is a 132-avoiding permutation on $\{1, \ldots, n-1\}$;

3. $\pi_1 \cdots \pi_{n-i-1}\ell(n-i+1)(n-i+2)\cdots n$, where $2 \leq i \leq n-2$, $\ell < n-i$ and $\pi_1 \cdots \pi_{n-i-1}\ell$ is a 132-avoiding permutation on $\{1, \ldots, n-i\}$.

Arguing as above, we see that the identity permutations contributes x^n to $E_n(x)$, the elements in case (2) contribute $A_{n-1}(x) - E_{n-1}(x)$ to $E_n(x)$, and the elements in case (3) contribute $\sum_{i=2}^{n-2} x^{n-i}(A_i(x) - E_i(x))$ to $E_n(x)$.

Given that we have computed that $B_n(x) = E_n(x)$ for $1 \leq n \leq 3$, one can easily use (1) and (3) to prove that $B_n(x) = E_n(x)$ for all $n \geq 1$ by induction.

To prove (2), note that $S_n(132) = S_n^{(1)}(132) \cup S_n^{(n)}(132) \cup_{2 \leq i \leq n-1} S_n^{(i)}(132)$. Clearly, the permutations in $S_n^{(1)}(132)$ contribute $B_n(x)$ to $A_n(x)$ and the permutations in $S_n^{(n)}(132)$ contribute $E_n(x)$ to $A_n(x)$. Now suppose that $2 \leq i \leq n$ and $\pi = \pi_1 \cdots \pi_n \in S_n^{(i)}(132)$. Then all the elements in $\pi_1 \cdots \pi_{i-1}$ are strictly greater than all the elements in $\pi_{i+1} \cdots \pi_n$. It follows that $\pi_{i+1} \leq n-2$. Hence the elements $\pi_1 \cdots \pi_{i-1} n$ have no effect as to whether any of the elements in $\pi_{i+1} \cdots \pi_n$ are occurrences of the 1-box pattern in π. Hence the elements $S_n^{(i)}(132)$ contribute $E_i(x)A_{n-i}(x)$ to $A_n(x)$. Thus for all $n \geq 2$,

$$A_n(x) = B_n(x) + E_n(x) + \sum_{i=2}^{n} E_i(x)A_{n-i}(x). \tag{4}$$

It is easy to see that since $B_n(x) = E_n(x)$ for all $n \geq 1$, (4) implies (2). ☐

The following theorem gives the generating function for the entire distribution of the 1-box pattern over 132-avoiding permutations.

Theorem 2. *We have*

$$A(t,x) = \frac{1 + t + t^2 - tx - t^2x - t^3x + t^3x^2 - \sqrt{F(t,x)}}{2(t(1-xt) + x^2t^2)} \tag{5}$$

where $F(t,x) = (1+t+t^2-tx-t^2x-t^3x+t^3x^2)^2 + 4((1+t)(1-xt)+x^2t^2)(t(1-xt)+x^2t^2)$. *Also,*

$$B(t,x) = E(t,x) = \frac{t(1-xt) + x^2t^2}{(1+t)(1-xt) + x^2t^2} A(t,x).$$

Proof. Multiplying both parts of (2) by t^n and summing over all $n \geq 2$ we obtain

$$A(t,x) - (1+t) = (B(t,x) - t) + (B(t,x) - t)A(t,x).$$

Solving for $A(t,x)$, we obtain that

$$A(t,x)=\frac{1+B(t,x)}{1+t-B(t,x)}. \tag{6}$$

Now multiplying both parts of (1) by t^n and summing over all $n\geq 2$ we obtain

$$B(t,x)-(t+x^2t^2+(x^2+x^3)t^3)=\frac{x^4t^4}{1-xt}+t(A(t,x)-(1+t+2x^2t^2))$$

$$-t(B(t,x)-(t+x^2t^2))+\frac{x^2t^2}{1-xt}\left((A(t,x)-(1+t))-(B(t,x)-t)\right).$$

Solving for $B(t,x)$, we obtain that

$$B(t,x)=\frac{t(1-xt)+x^2t^2}{(1+t)(1-xt)+x^2t^2}A(t,x). \tag{7}$$

Combining (6) and (7), we see that $A(t,x)$ satisfies the following quadratic equation

$$(t(1-xt)+x^2t^2)A^2(t,x)-(1+t+t^2-tx-t^2x-t^3x+t^3x^2)A(t,x)+(1+t)(1-xt)+x^2t^2=0$$

which can be solved to yield (5). □

We used Mathematica to find the first few terms of $A(t,x)$ and $B(t,x)=E(t,x)$. That is, we have that

$$\begin{aligned}A(t,x) \;=\; & 1+t+2x^2t^2+x^2(3+2x)t^3+x^2\left(5+3x+6x^2\right)t^4+x^2\left(8+5x+19x^2+10x^3\right)t^5+\\ & x^2\left(13+8x+50x^2+35x^3+26x^4\right)t^6+x^2\left(21+13x+119x^2+95x^3+127x^4+54x^5\right)t^7+\\ & x^2\left(34+21x+265x^2+230x^3+451x^4+295x^5+134x^6\right)t^8+\\ & x^2\left(55+34x+564x^2+517x^3+1373x^4+1118x^5+895x^6+306x^7\right)t^9+\\ & x^2\left(89+55x+1160x^2+1107x^3+3790x^4+3548x^5+4010x^6+2283x^7+754x^8\right)t^{10}+\cdots.\end{aligned}$$

and

$$\begin{aligned}B(t,x) \;=\; & E(t,x)\\ =\; & t+x^2t^2+x^2(1+x)t^3+x^2\left(2+x+2x^2\right)t^4+\\ & x^2\left(3+2x+6x^2+3x^3\right)t^5+x^2\left(5+3x+16x^2+11x^3+7x^4\right)t^6+\\ & x^2\left(8+5x+39x^2+30x^3+36x^4+14x^5\right)t^7+\\ & x^2\left(13+8x+88x^2+75x^3+131x^4+81x^5+33x^6\right)t^8+\\ & x^2\left(21+13x+190x^2+171x^3+410x^4+319x^5+233x^6+73x^7\right)t^9+\\ & x^2\left(34+21x+395x^2+372x^3+1156x^4+1044x^5+1087x^6+579x^7+174x^8\right)t^{10}+\cdots.\end{aligned}$$

3 Properties of coefficients of $A_n(x)$ and $B_n(x) = E_n(x)$

In this section, we shall explain several of the coefficients of the polynomials $A_n(x)$ and $B_n(x) = E_n(x)$ and show their connections with the Fibonacci numbers.

In Subsection 3.1, we study the coefficients of x^k in the the polynomials $A_n(x)$ and $B_n(x) = E_n(x)$ for $k \in \{0, 1, 2, 3, 4\}$ and, in Subsection 3.2, we derive the generating functions for the highest coefficients for these polynomials.

3.1 The four smallest coefficients and the Fibonacci numbers

Clearly the coefficient of x in either $A_n(x)$, $B_n(x)$, or $E_n(x)$ is 0 by the definition of an occurrence of the 1-box pattern. The following theorem states that for $n \geq 2$, each 132-avoiding permutation of length n has at least two occurrences of the 1-box pattern. In what follows, we need the notion of the celebrated n-th *Fibonacci number* F_n defined as $F_0 = F_1 = 1$ and, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Also, for a polynomial $P(x)$, we let $P(x)|_{x^m}$ denote the coefficient of x^m.

Theorem 3. *For $n \geq 2$, $A_n(x)|_{x^0} = B_n(x)|_{x^0} = E_n(x)|_{x^0} = 0$.*

Proof. Clearly, it is enough to prove the claim for $A_n(x)$. We proceed by induction on n. The claim is clearly true for $n = 2$. Next suppose that $n \geq 3$ and $\sigma = S_n(132)$. From the structure of 132-avoiding permutations presented in Figure 4, either $A_i(\sigma)$ is empty in which case $B_i(\sigma)$ has at least two elements and it contains an occurrence of the 1-box pattern by the induction hypothesis, or $A_i(\sigma)$ has a single element $n-1$ leading to two occurrence of the pattern formed by n and $n-1$, or $A_i(\sigma)$ has at least two elements and we apply the induction hypothesis to it. □

Theorem 4. *For $n \geq 2$, $A_n(x)|_{x^2} = F_n$ and $B_n(x)|_{x^2} = E_n(x)|_{x^2} = F_{n-2}$.*

Proof. We proceed by induction on n. Note that $A_2(x)|_{x^2} = 2 = F_2$ and $B_2(x)|_{x^2} = E_2(x)|_{x^2} = 1 = F_0$. Similarly, $A_3(x)|_{x^2} = 3 = F_3$ and $B_3(x)|_{x^2} = E_3(x)|_{x^2} = 1 = F_1$. Thus our claim holds for $n = 2$ and $n = 3$.

For $n \geq 4$, it follows from (1) and Theorem 3 that

$$\begin{aligned} B_n(x)|_{x^2} &= x^n|_{x^2} + (A_{n-1}(x)|_{x^2} - B_{n-1}(x)|_{x^2}) + \sum_{i=2}^{n-2} (x^{n-i}(A_i(x) - B_i(x)))|_{x^2} \\ &= A_{n-1}(x)|_{x^2} - B_{n-1}(x)|_{x^2} + (A_{n-2}(x) - B_{n-2}(x))|_{x^0} \\ &= F_{n-1} - F_{n-3} = F_{n-2}. \end{aligned}$$

But then by (2), we have that

$$A_n(x)|_{x^2} = B_n(x)|_{x^2} + \sum_{i=2}^{n} (B_i(x)A_{n-i}(x))|_{x^2}. \tag{8}$$

Note that since $n \geq 4$ and $2 \leq i \leq n$

$$\begin{aligned}(B_i(x)A_{n-i}(x))|_{x^2} &= (B_i(x)|_{x^0})(A_{n-i}(x))|_{x^2}) + (B_i(x)|_{x^1})(A_{n-i}(x))|_{x^1}) + \\ &\quad (B_i(x)|_{x^2})(A_{n-i}(x))|_{x^0}) \\ &= (B_i(x)|_{x^2})(A_{n-i}(x))|_{x^0})\end{aligned}$$

since $B_i(x)|_{x^1} = A_{n-i}(x)|_{x^1} = 0$ for $i \geq 1$ and $B_i(x)|_{x^0} = 0$ for $i \geq 2$. But then since $A_i(x)|_{x^0} = 0$ for $i \geq 2$ and $A_i(x)|_{x^0} = 1$ for $i = 0, 1$, it follows that (8) reduces to

$$\begin{aligned}A_n(x)|_{x^2} &= B_n(x)|_{x^2} + B_n(x)|_{x^2} + B_{n-1}(x)|_{x^2} \\ &= F_{n-2} + F_{n-2} + F_{n-3} = F_{n-2} + F_{n-1} = F_n.\end{aligned}$$

□

Corollary 1. *For $n \geq 2$, the number of 132-avoiding permutations of length n that do not begin (resp. end) with n and contain exactly two occurrences of the 1-box pattern is F_{n-1}.*

Proof. A proof is straightforward from Theorem 4, since

$$A_n(x)|_{x^2} - B_n(x)|_{x^2} = A_n(x)|_{x^2} - E_n(x)|_{x^2} = F_{n-1}.$$

□

Theorem 5. *For $n \geq 3$, $A_n(x)|_{x^3} = F_{n-1}$ and $B_n(x)|_{x^3} = E_n(x)|_{x^3} = F_{n-3}$.*

Proof. We proceed by induction on n, the length of permutations, and the formulas (1) and (2). Note that we have computed that $A_3(x)|_{x^2} = 2 = F_2$, $A_4(x)|_{x^2} = 3 = F_3$, $B_3(x)|_{x^2} = E_3(x)|_{x^2} = 1 = F_0$, and $B_4(x)|_{x^2} = E_4(x)|_{x^2} = 1 = F_1$. Thus our claim holds for $n = 3$ and $n = 4$.

For $n \geq 5$, it follows from (1) and Theorem 3 that

$$\begin{aligned}B_n(x)|_{x^3} &= x^n|_{x^3} + (A_{n-1}(x)|_{x^3} - B_{n-1}(x)|_{x^3}) + \sum_{i=2}^{n-2}(x^{n-i}(A_i(x) - B_i(x)))|_{x^3} \\ &= A_{n-1}(x)|_{x^3} - B_{n-1}(x)|_{x^3} + (A_{n-2}(x) - B_{n-2}(x))|_{x^1} + (A_{n-3}(x) - B_{n-3}(x))|_{x^0} \\ &= F_{n-2} - F_{n-4} = F_{n-3}.\end{aligned}$$

But then by (2), we have that

$$A_n(x)|_{x^3} = B_n(x)|_{x^3} + \sum_{i=2}^{n}(B_i(x)A_{n-i}(x))|_{x^3}. \tag{9}$$

Note that since $n \geq 5$ and $2 \leq i \leq n$,

$$\begin{aligned}(B_i(x)A_{n-i}(x))|_{x^3} &= (B_i(x)|_{x^0}) + (A_{n-i}(x)|_{x^3})(B_i(x)|_{x^1})(A_{n-i}(x)|_{x^2}) + \\ &\quad (B_i(x)|_{x^2})(A_{n-i}(x)|_{x^1}) + (B_i(x)|_{x^3})(A_{n-i}(x)|_{x^0}) \\ &= (B_i(x)|_{x^3})(A_{n-i}(x)|_{x^0})\end{aligned}$$

since $B_i(x)|_{x^1} = A_{n-i}(x)|_{x^1} = 0$ for $i \geq 1$ and $B_i(x)|_{x^0} = 0$ for $i \geq 2$. But then since $A_i(x)|_{x^0} = 0$ for $i \geq 2$ and $A_i(x)|_{x^0} = 1$ for $i = 0, 1$, it follows that (9) reduces to

$$\begin{aligned} A_n(x)|_{x^3} &= B_n(x)|_{x^3} + B_n(x)|_{x^3} + B_{n-1}(x)|_{x^3} \\ &= F_{n-3} + F_{n-3} + F_{n-4} = F_{n-3} + F_{n-2} = F_{n-1}. \end{aligned}$$

□

Corollary 2. *For $n \geq 3$, the number of 132-avoiding permutations of length n that do not begin (resp. end) with n and contain exactly three occurrences of the 1-box pattern is F_{n-2}.*

Proof. A proof is straightforward from Theorem 5, since

$$A_n(x)|_{x^3} - B_n(x)|_{x^3} = A_n(x)|_{x^3} - E_n(x)|_{x^3} = F_{n-2}.$$

□

Regarding the number of 132-avoiding permutations with exactly four occurrences of the 1-box pattern, we can derive the following recurrence relations involving the Fibonacci numbers.

Theorem 6. *We have that for $n \leq 3$, $A_n(x)|_{x^4} = B_n(x)|_{x^4} = E_n(x)|_{x^4} = 0$, $B_4(x)|_{x^4} = 2$, $B_5(x)|_{x^4} = 6$, and for $n \geq 4$,*

$$A_n(x)|_{x^4} = 2B_n(x)|_{x^4} + B_{n-1}(x)|_{x^4} + \sum_{i=2}^{n-2} F_{i-2}F_{n-i}; \tag{10}$$

while for $n \geq 6$,

$$B_n(x)|_{x^4} = B_{n-1}(x)|_{x^4} + B_{n-2}(x)|_{x^4} + F_{n-1} + \sum_{i=4}^{n-3} F_{i-2}F_{n-1-i}. \tag{11}$$

Proof. The initial conditions follow from the expansions of $A(t,x)$ and $B(t,x)$ given above.

By (2), we have that

$$A_n(x)|_{x^4} = B_n(x)|_{x^4} + \sum_{i=2}^{n} (B_i(x)A_{n-i}(x))|_{x^4}. \tag{12}$$

Note that since $n \geq 4$ and $2 \leq i \leq n$,

$$\begin{aligned} (B_i(x)A_{n-i}(x))|_{x^4} &= (B_i(x)|_{x^0})(A_{n-i}(x)|_{x^4}) + (B_i(x)|_{x^1})(A_{n-i}(x)|_{x^3}) + \\ &\quad (B_i(x)|_{x^2})(A_{n-i}(x)|_{x^2}) + (B_i(x)|_{x^3})(A_{n-i}(x)|_{x^1}) + \\ &\quad (B_i(x)|_{x^4})(A_{n-i}(x)|_{x^0}) \\ &= (B_i(x)|_{x^2})(A_{n-i}(x)|_{x^2}) + (B_i(x)|_{x^4})(A_{n-i}(x)|_{x^0}) \end{aligned}$$

since $B_i(x)|_{x^1} = A_{n-i}(x)|_{x^1} = 0$ for $i \geq 1$ and $B_i(x)|_{x^0} = 0$ for $i \geq 2$. But then since $A_i(x)|_{x^0} = 0$ for $i \geq 2$ and $A_i(x)|_{x^0} = 1$ for $i = 0, 1$, (12) reduces to

$$A_n(x)|_{x^4} = B_n(x)|_{x^4} + B_n(x)|_{x^4} + B_{n-1}(x)|_{x^4} + \sum_{i=2}^{n-2} \left(B_i(x)|_{x^2}\right) \left(A_{n-i}(x)|_{x^2}\right).$$

Then we can apply Theorem 4 to obtain (10).

Let $n \geq 6$. From (1),

$$B_n(x)|_{x^4} = \left(A_{n-1}(x)|_{x^4} - B_{n-1}(x)|_{x^4}\right) + \left(A_{n-2}(x)|_{x^2} - B_{n-2}(x)|_{x^2}\right),$$

since only the term corresponding to $i = n-2$ from the sum contributes to x^4. Applying (10) and Theorem 4, we obtain

$$\begin{aligned} B_n(x)|_{x^4} &= \left(2B_{n-1}(x)|_{x^4} + B_{n-2}(x)|_{x^4} + \sum_{i=2}^{n-3} F_{i-2}F_{n-1-i}\right) - B_{n-1}(x)|_{x^4} + F_{n-2} - F_{n-4} \\ &= B_{n-1}(x)|_{x^4} + B_{n-2}(x)|_{x^4} + F_{n-3} + F_{n-4} + F_{n-2} - F_{n-4} + \sum_{i=4}^{n-3} F_{i-2}F_{n-1-i} \\ &= B_{n-1}(x)|_{x^4} + B_{n-2}(x)|_{x^4} + F_{n-1} + \sum_{i=4}^{n-3} F_{i-2}F_{n-1-i}. \end{aligned}$$

Note that $B_5(x)|_{x^4} = 6$, $B_4(x)|_{x^4} = 2$, $B_3(x)|_{x^4} = 0$, and $F_4 = 5$ so that (11) does not hold for $n = 5$. □

We can use Theorem 6 to find the generating functions for $A_n(x)|_{x^4}$ and $B_n(x)|_{x^4}$. That is, let

$$\mathbb{A}_4(t) = \sum_{n \geq 4} (A_n(x)|_{x^4}) t^n$$

and

$$\mathbb{B}_4(t) = \sum_{n \geq 4} (B_n(x)|_{x^4}) t^n.$$

Then we have the following theorem.

Theorem 7.

$$\mathbb{A}_4(t) = \frac{t^4(6 + t - 7t^2 - t^3 + 3t^4 + t^5)}{(1 - t - t^2)^3} \tag{13}$$

and

$$\mathbb{B}_4(t) = \frac{t^4(2 - t^2 + t^3 + t^4)}{(1 - t - t^2)^3}. \tag{14}$$

Proof. First observe that

$$\begin{aligned}
\sum_{n\geq 7}\left(\sum_{i=4}^{n-3} F_{i-2}F_{n-1-i}\right)t^n &= t^3\sum_{n\geq 7}\left(\sum_{j=2}^{n-5} F_jF_{n-3-j}\right)t^{n-3}\\
&= t^3\sum_{n\geq 4}\left(\sum_{j=2}^{n-2} F_jF_{n-j}\right)t^n\\
&= t^3\left(\sum_{j\geq 2} F_jt^j\right)^2.
\end{aligned}$$

Using the fact that $\sum_{n\geq 0} F_nt^n = \frac{1}{1-t-t^2}$, it follows that

$$\begin{aligned}
\sum_{n\geq 7}\left(\sum_{i=4}^{n-3} F_{i-2}F_{n-1-i}\right)t^n &= t^3\left(\frac{1}{1-t-t^2}-(1+t)\right)^2\\
&= t^3\frac{(t^2(2+t))^2}{(1-t-t^2)^2} = \frac{(2+t)^2t^7}{(1-t-t^2)^2}.
\end{aligned}$$

Next observe that

$$\sum_{n\geq 6} F_{n-1}t^n = t\left(\frac{1}{1-t-t^2}-(1+t+2t^2+3t^3+5t^4)\right) = \frac{(8+5t)t^5}{1-t-t^2}.$$

Thus

$$\begin{aligned}
H(t) &= \sum_{n\geq 6} H_nt^n\\
&= \sum_{n\geq 6} F_{n-1}t^n + \sum_{n\geq 7}\left(\sum_{i=4}^{n-3} F_{i-2}F_{n-1-i}\right)t^n\\
&= \frac{(2+t)^2t^7}{(1-t-t^2)^2} + \frac{(8+5t)t^5}{1-t-t^2}\\
&= \frac{(8+t-9t^2-4t^3)t^6}{(1-t-t^2)^2}.
\end{aligned}$$

Here we use Mathematica to simplify the last expression.

We can now rewrite (11) as

$$B_n(x)|x^4 = B_{n-1}(x)|x^4 + B_{n-2}(x)|x^4 + H_n \tag{15}$$

for $n \geq 6$. Multiplying both sides of (15) by t^n and summing for $n \geq 6$, we see that

$$\mathbb{B}_4(t) - 2t^4 - 6t^5 = t(\mathbb{B}_4(t) - 2t^4) + t^2\mathbb{B}_4(t) + H(t).$$

Solving for $\mathbb{B}_4(t)$ and using Mathematica, we obtain that

$$\mathbb{B}_4(t) = \frac{t^4(2-t^2+t^3+t^4)}{(1-t-t^2)^3}.$$

Next observe that

$$\begin{aligned}
\sum_{n\geq 4}\left(\sum_{i=2}^{n-2} F_{i-2}F_{n-i}\right)t^n &= \sum_{n\geq 4}\left(\sum_{j=0}^{n-4} F_jF_{n-2-j}\right)t^n \\
&= t^2\sum_{n\geq 4}\left(\sum_{j=0}^{n-4} F_jF_{n-2-j}\right)t^{n-2} \\
&= t^2\left(\sum_{j\geq 0} F_jt^j\right)\left(\sum_{j\geq 0} F_jt^j-(1+t)\right) \\
&= \frac{(2+t)t^4}{(1-t-t^2)^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
G(t) &= \sum_{n\geq 5} G_nt^n = \sum_{n\geq 5}\left(\sum_{i=2}^{n-2} F_{i-2}F_{n-i}\right)t^n \\
&= \frac{(2+t)t^4}{(1-t-t^2)^2} - 2t^4 \\
&= \frac{(5+2t-4t^2-2t^3)t^5}{(1-t-t^2)^2}.
\end{aligned}$$

We can now rewrite (10) as

$$A_n(x)|_{x^4} = 2A_n(x)|_{x^4} + B_{n-1}(x)|_{x^4} + G_n \tag{16}$$

for $n \geq 5$. Multiplying both sides of (16) by t^n and summing for $n \geq 5$, we obtain that

$$\mathbb{A}_4(t) - 6t^4 = 2(\mathbb{B}_4(t) - 2t^4) + t\mathbb{B}_4(t) + G(t).$$

Solving for $\mathbb{A}_4(t)$ then gives

$$\mathbb{A}_4(t) = \frac{t^4(6+t-7t^2-t^3+3t^4+t^5)}{(1-t-t^2)^3}.$$

□

3.2 The highest coefficient of x in $A(t,x)$ and $B(t,x) = E(t,x)$

Let $a_n = A_n(x)|_{x^n}$, $b_n = B_n(x)|_{x^n}$, and $e_n = E_n(x)|_{x^n}$. Thus, for example, a_n is the number of permutations $\pi \in S_n(132)$ such that every element of π is an occurrence of the 1-box pattern in π. The identity element in S_n and its reverse show that a_n, b_n, and e_n are nonzero for all $n \geq 1$. Moreover, the fact that $B_n(x) = E_n(x)$ for all $n \geq 1$ implies $b_n = e_n$ for all $n \geq 1$. In this section, we shall compute the generating functions

$$A(t) = \sum_{n\geq 0} a_n t^n \text{ and } B(t) = \sum_{n\geq 1} b_n t^n.$$

Theorem 8.

$$A(t) = \frac{1 - t + 2t^3 - \sqrt{1 - 2t - 3t^2 + 4t^3 - 4t^4}}{2t^2}$$

and

$$B(t) = \frac{1 + t - 2t^2 + 2t^3 - \sqrt{1 - 2t - 3t^2 + 4t^3 - 4t^4}}{2(1 - t + t^2)}.$$

The initial values for a_n are

$$1, 1, 2, 2, 6, 10, 26, 54, 134, 306, 754, \ldots$$

and the initial values for b_n are

$$0, 1, 1, 1, 2, 3, 7, 14, 33, 73, 174, \ldots.$$

Proof. Our proof of the theorem is very similar to the proofs of Lemma 1 and Theorem 2.

First we claim that for $n \geq 4$,

$$b_n = 1 + \sum_{k=2}^{n-2} (a_k - b_k). \tag{17}$$

Here 1 corresponds to the decreasing permutation $n(n-1)\cdots 1$, and the sum counts permutations of the form $\pi_1 \cdots \pi_{n-k-1}\ell(n-k+1)(n-k+2)\cdots n$, where $2 \leq k \leq n-2$, $\ell < n-k$ and $\pi_1 \cdots \pi_{n-k-1}\ell$ is a 132-avoiding permutation on $\{1, \ldots, n-k\}$ with the maximum number of occurrences of the 1-box pattern. There are no other permutations counted by b_n. Multiplying both parts of (17) by t^n, summing over all $n \geq 4$, and using the fact that $b_1 = b_2 = b_3 = 1$, we obtain

$$B(t) - (t + t^2 + t^3) = \frac{t^4}{1-t} + \frac{t^2}{1-t}\left((A(t) - (1+t)) - (B(t) - t)\right),$$

from where we get

$$B(t) = \frac{t - t^2 + t^2 A(t)}{1 - t + t^2}. \tag{18}$$

Using the fact that $S_n(132) = S_n^{(1)}(132) \cup S_n^{(n)}(132) \cup_{2\leq i\leq n-1} S_n^{(i)}(132)$, it is easy to see that for $n \geq 4$,

$$a_n = b_n + e_n + \sum_{k=2}^{n-2} e_k a_{n-k} = 2b_n + \sum_{k=2}^{n-2} b_k a_{n-k}. \tag{19}$$

Multiplying both sides of (19) by t^n and using the facts that $a_0 = a_1 = 1$ and $a_2 = a_3 = 2$, we see that

$$A(t) - (1 + t + 2t^2 + 2t^3) = 2(B(t) - (t + t^2 + t^3)) + (B(t) - t)(A(t) - (1 + t)).$$

This leads to

$$A(t) = \frac{1 + t^2 + (1 - t)B(t)}{1 + t - B(t)}. \tag{20}$$

Solving the system of equations given by (18) and (20) for $A(t)$ and $B(t)$ we get the desired result. □

4 The 1-box pattern on separable permutations

In this section we enumerate separable permutations with m, $0 \leq m \leq 3$, occurrences of the 1-box pattern.

For two non-empty words, A and B, we write $A < B$ to indicate that any element in A is less than each element in B. We say that $\pi' = \pi_i\pi_{i+1}\cdots\pi_j$ is an *interval* in a permutation $\pi_1\cdots\pi_n$ if π' is a permutation of $\{k, k+1, \ldots, k+j-i\}$ for some k, that is, if π' consists of consecutive values.

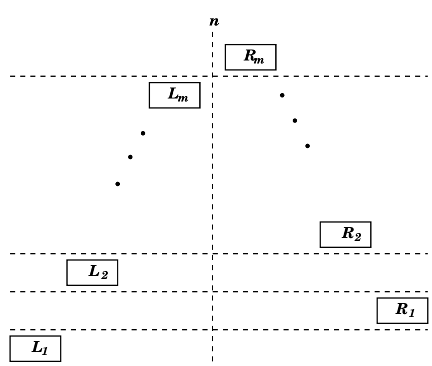

Figure 5: The structure of a separable permutation.

A permutation is *separable* if it avoids simultaneously the patterns 2413 and 3142. It is known and is not difficult to see that any separable permutation π of length n has the following structure (also illustrated in Figure 5):

$$\pi = L_1L_2\cdots L_mnR_mR_{m-1}\cdots R_1 \tag{21}$$

where

- for $1 \leq i \leq m$, L_i and R_i are non-empty, with possible exception of L_1 and R_m, separable permutations which are intervals in π, and
- $L_1 < R_1 < L_2 < R_2 < \cdots < L_m < R_m$. In particular, L_1, if it is non-empty, contains the element 1.

For example, if $\pi = 215643$ then $L_1 = 21$, $L_2 = 5$ $R_1 = 43$ and $R_2 = \emptyset$.
The following theorem is similar to the case of 132-avoiding permutations.

Theorem 9. *Apart from the empty permutation and the permutation* 1*, there are no separable permutations avoiding the* 1*-box pattern.*

Proof. Our proof is straightforward by induction on n, the length of permutations and is similar to the proof of Theorem 3. Indeed, the base cases for $n \leq 2$ are easy to check. Now assume that $n \geq 3$ and R_n is non-empty (the case when R_n is empty can be considered similarly substituting R_n with L_n in our arguments). If R_n has only one element, $n-1$, then n and $n-1$ give two occurrences of the 1-box pattern; otherwise, R_n contains an occurrence of the pattern by the inductive hypothesis. □

By definition of an occurrence of the 1-box pattern, we cannot have any permutations with exactly one occurrence of the 1-box pattern.

Theorem 10. *The number c_n of separable permutations of length n with exactly two occurrences of the* 1*-box pattern is given by $c_0 = c_1 = 0$, $c_2 = 2$, and for $n \geq 3$, $c_n = 2c_{n-1} + c_{n-2}$. The generating function for this sequence is*

$$\sum_{n\geq 0} c_n t^n = \frac{2t^2}{1-2t-t^2}.$$

The initial values for c_ns, for $n \geq 0$, are $0, 0, 2, 4, 10, 24, 58, 140, 338, 816, 1970, \ldots$, and this is essentially the sequence A052542 in [11]. Apart from the initial 0*s, the sequence of c_ns is simply twice the* Pell numbers*.*

Proof. Suppose that $n \geq 3$ and π is a separable permutation in S_n which is counted by c_n. Thus π either contains a consecutive sequence of the form $a(a+1)$ or $(a+1)a$. If we remove a from π and decrease all the elements that are greater than or equal to $a+1$ by one, we will obtain a separable permutation π' in S_{n-1}. By Theorem 9, we must have at least two occurrences of the pattern in the obtained permutation π'. In fact, it is easy to see that we will either get two occurrences or three occurrences of the 1-box pattern in π'.

By Theorem 11 below the number of possibilities to get π' with three occurrences of the 1-box pattern (necessarily formed by either a consecutive subword of the form $a(a+1)(a+2)$ or by $(a+2)(a+1)a$) is given by c_{n-2}. This is indeed the case because we can reverse removing the element in this case by turning $a(a+1)(a+2)$ to $a(a+2)(a+1)(a+3)$ or $(a+2)(a+1)a$ to $(a+3)(a+1)(a+2)a$ and increasing by 1 each element of π that is larger than $(a+2)$. On the other hand, the number of possibilities to get π' with two occurrences of the 1-box pattern (formed by either a consecutive elements of the form

$a(a+1)$ or by $(a+1)a)$ is given by $2c_{n-1}$. Indeed, to reverse removing the element in this case we need either to turn $a(a+1)$ to either $(a+1)a(a+2)$ or to $a(a+2)(a+1)$, or to turn $(a+1)a$ to either $(a+2)a(a+1)$ or to $(a+1)(a+2)a$. In each of these cases the suggested substitutions create, in an injective way, separable permutations with exactly two occurrences of the 1-box pattern.

Our considerations above justify the recursion $c_n = 2c_{n-1} + c_{n-2}$ (the initial values for it are easy to see). Finally, using the standard technique, it is straightforward to derive the generating function based on the recursion above. □

Theorem 11. *For $n \geq 1$, the number of separable permutations of length n with exactly three occurrences of the 1-box pattern is equal to the number of separable permutations of length $n-1$ with exactly two occurrences of this pattern.*

Proof. It is easy to see that if a separable permutation has exactly three occurrences of the 1-box pattern, then these occurrences are necessarily formed by either a consecutive subword of the form $a(a+1)(a+2)$ or by $(a+2)(a+1)a$. In either case, removing the middle element and reducing by 1 all elements that are larger than $(a+1)$, we get a separable permutation with exactly two occurrences of the 1-box pattern. This operation is obviously reversible. □

Even though we were not deriving formulas for separable permutations with other number of occurrences of the 1-box pattern, we provide initial values for the number of separable permutations with exactly four occurrences of the 1-box pattern (not in [11]):

$$0, 0, 0, 0, 8, 42, 178, 664, 2288, \ldots,$$

and with the maximum number of occurrences of this pattern on separable permutations (again, not in [11]):

$$0, 0, 2, 2, 8, 14, 54, 128, 466, \ldots.$$

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