NESTED COLOURINGS OF GRAPHS

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Abstract. A proper vertex colouring of a graph is *nested* if the vertices of each of its colour classes can be ordered by inclusion of their open neighbourhoods. Through a relation to partially ordered sets, we show that the nested chromatic number can be computed in polynomial time.

Clearly, the nested chromatic number is an upper bound for the chromatic number of a graph. We develop multiple distinct bounds on the nested chromatic number using common properties of graphs. We also determine the behaviour of the nested chromatic number under several graph operations, including the direct, Cartesian, strong, and lexicographic product. Moreover, we classify precisely the possible nested chromatic numbers of graphs on a fixed number of vertices with a fixed chromatic number.

1. Introduction

Let G be a finite simple graph on vertex set $V(G)$ and with edge set $E(G)$. A partition $\mathcal{C} = C_1 \cup \cdots \cup C_k$ of the vertices is a proper vertex colouring of G if the C_i are independent sets. The chromatic number $\chi(G)$ is the least cardinality of a proper vertex colouring of G. It is well-known that computing the chromatic number of a graph is NP-complete.

We define a novel colouring of a graph G. In particular, a proper vertex colouring of G is *nested* if the vertices of each of its colour classes can be ordered by inclusion of their open neighbourhoods. The *nested chromatic number* $\chi_N(G)$ is the least cardinality of a nested colouring of G. The nested chromatic number clearly bounds the chromatic number from above. Through a connection to partially ordered sets, we prove that computing the nested chromatic number of a graph can be done in polynomial time (Theorem 2.21).

The concept of a nested colouring is extended to simplicial complexes in [3]. Using Proposition 2.16 it is shown that the nested chromatic number of the underlying graph of a simplicial complex bounds from below the nested chromatic number of the complex itself. Moreover, a new face ideal for a simplicial complex with respect to a colouring is defined therein. It is the nested colourings of simplicial complexes that give rise to ideals which have minimal linear resolutions supported on a cubical complex.

Herein we consider the properties of nested colourings and the nested chromatic number. This note is organised as follows. In Section 2 we introduce the relevant new definitions. In particular, we define the weak duplicate preorder which provides two distinct interpretations of nested colourings (Propositions 2.16 and 2.18). Through the latter interpretation we see that the nested chromatic number is related to the Dilworth number of a graph (Remark 2.20).

In Section 3 we classify the structure of graphs with nested chromatic number 2 (Theorem 3.1). Further, we study the nested chromatic number of regular graphs and diamond- and C_4-free graphs therein. In Section 5 we consider the behaviour of the nested chromatic

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number under many common operations, including: Mycielski's construction, the disjoint union, the join, the direct product, the Cartesian product, the strong product, and the composition or lexicographic product. In Section 6 we provide a classification of the triples $(\#V(G), \chi(G), \chi_N(G))$ that can occur for some graph G (Theorem 6.2).

For standard definitions not given here and for more examples, we refer the reader to any standard graph theory textbook, e.g., [26].

2. Nested colourings

In this section, we introduce three new concepts: nested colourings, the de-duplicate graph, and the weak duplicate preorder.

2.1. Nested colourings & the nested chromatic number.

We first define a nested neighbourhood condition on vertices of a finite simple graph. We use $N_G(u) = \{v\colon \{u,v\} \in E(G)\}$ to denote the open neighbourhood of u in G and $N_G[u] = N_G(u) \cup \{u\}$ to denote the closed neighbourhood of u in G.

Definition 2.1. Let G be a finite simple graph, and let u, v be vertices of G. The vertex u is a *weak duplicate* of v if $N_G(u) \subset N_G(v)$; if equality holds, then u is a *duplicate* of v. Further, a *duplicate-free graph* is a finite simple graph for which no pair of vertices are duplicates. An independent set I of G is *nested* if the vertices of I can be linearly ordered so that $v \leq u$ implies u is a weak duplicate of v.

The order on the vertices of a nested independent set is unique, up to permutations of duplicates. This will be formalised in Section 2.3.

Using this condition on the neighbourhoods, we define a novel proper vertex colouring of a graph.

Definition 2.2. Let G be a finite simple graph, and let $\mathcal{C}$ be a proper vertex k-colouring $C_1 \cup \cdots \cup C_k$ of G. If every colour class of $\mathcal{C}$ is nested, then $\mathcal{C}$ is a *nested colouring* of G. The *nested chromatic number* $\chi_N(G)$ is the least cardinality of a nested colouring of G. Moreover, the graph G is *colour-nested* if $\chi_N(G) = \chi(G)$.

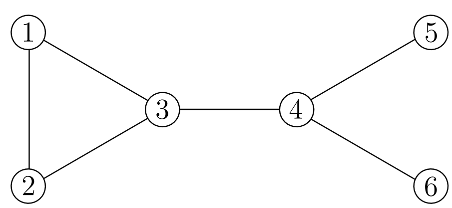


FIGURE 2.1. A graph G with $\chi(G) = 3$ and $\chi_N(G) = 4$.

Example 2.3. Let G be the graph in Figure 2.1. The partition $\{1,4\} \cup \{2\} \cup \{3,5,6\}$ is an optimal proper vertex 3-colouring of G. However, since $N_G(1) = \{2,3\}$ and $N_G(4) = \{3,5,6\}$, i.e., the independent set $\{1,4\}$ is not nested, the 3-colouring is not nested. Indeed, all proper vertex 3-colourings of G are not nested. However, the proper vertex 4-colouring $\{1\} \cup \{2\} \cup \{3,5,6\} \cup \{4\}$ is nested; indeed, $N_G(5) = N_G(6) = \{4\} \subset N_G(3) = \{1,2,4\}$. Notice that the vertices 5 and 6 are duplicates. Finally, as $\chi(G) = 3 < \chi_N(G) = 4$, we see that G is not colour-nested.

We notice that isolated vertices are "ignorable."

Remark 2.4. Isolated vertices are exactly those vertices that have an empty open neighbourhood. Since the empty set is a subset of *every* set, isolated vertices are weak duplicates of *every* vertex of a graph. Thus isolated vertices can be put in to *any* colour class without modifying the nesting of the colour class.

We also have a pair of immediate bounds on the nested chromatic number.

Remark 2.5. Since every nested colouring of a finite simple graph G is a proper colouring of G, we clearly have $\chi(G) \leq \chi_N(G)$. Moreover, $\chi_N(G) \leq \#V(G)$ as the singleton colouring $\biguplus_{v \in V(G)}\{v\}$ is nested.

Graphs with very small or very large chromatic number are colour-nested.

Lemma 2.6. *Let G be a finite simple graph on n vertices. If $\chi(G) \in \{1, n-1, n\}$, where $n \geq 2$, then $\chi_N(G) = \chi(G)$, i.e., G is colour-nested.*

Proof. Suppose that $\chi(G) = 1$. Hence $E(G) = \emptyset$ and every vertex is a duplicate of every other vertex by Remark 2.4. Thus the set $V(G)$ is a nested colouring of G and $\chi_N(G) = \chi(G)$.

Suppose that $\chi(G) = n-1$, where $n \geq 2$. Thus G is K_n with a nonempty subset of the edges connected to some vertex, say, v, removed. Let u be a vertex nonadjacent to v. Thus $N_G(u) = V(G) \setminus \{u, v\}$ contains $N_G(v)$, and v is a weak duplicate of u. Hence $\{u, v\}$ is a nested independent set of G and so $\chi_N(G) \leq n-1$. By the preceding remark we thus have $\chi_N(G) = \chi(G)$.

Suppose that $\chi(G) = n$. By Remark 2.5 we have $\chi(G) = \chi_N(G) = \#V(G)$. $\square$

Moreover, the upper bound in Remark 2.5 is sometimes attained by graphs with small chromatic number.

Example 2.7. Let P be the Petersen graph; see Figure 2.2. It is well-known that $\chi(P) = 3$. However, since no vertex of P is a weak duplicate of another vertex of P, $\chi_N(P) = 10 = \#V(P)$.

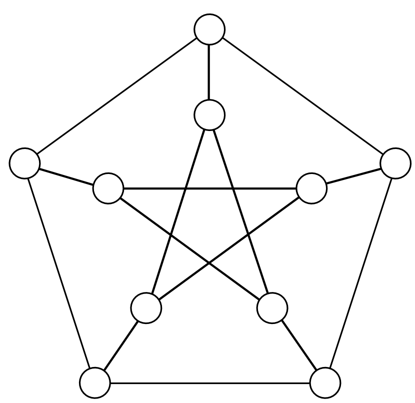


FIGURE 2.2. The Petersen graph.

Remark 2.8. Recall that a Sperner family is a collection of sets in which no set is a subset of another. Thus for a finite simple graph G, $\chi_N(G) = \#V(G)$ if and only if the set of open neighbourhoods of vertices of G forms a Sperner family. For example, the set of open neighbourhoods of vertices of the Petersen graph form a Sperner family.

2.2. **The de-duplicate graph.**

We now define a derivative graph based on the equivalence relation of duplicates.

Definition 2.9. Let G be a finite simple graph. Define $\sim$ to be the equivalence relation of duplicates of G, and let $[\cdot]_\sim$ denote an equivalence class of this relation. The *de-duplicate graph* of G is the graph $G^\star$ with vertices given by the equivalence classes $[v]_\sim$ for $v \in V(G)$ and with an edge between $[u]_\sim$ and $[v]_\sim$ if and only if $(u, v) \in E(G)$.

Notice that $G \cong G^\star$ if and only if G is duplicate-free.

Passing to the de-duplicate graph of G does not change the nested chromatic number.

Proposition 2.10. *If G is a finite simple graph, then $\chi_N(G) = \chi_N(G^\star)$.*

Proof. Suppose $\mathcal{C}$ is the nested colouring $C_1 \uplus \cdots \uplus C_k$ of G. For $1 \leq i \leq k$, let

$$C'_i = \{[v]_\sim \colon\ v \in C_i\} \setminus \cup_{j=1}^{i-1} C'_j.$$

By construction, $C'_1 \uplus \cdots \uplus C'_k$ is a partition $\mathcal{C}'$ of $V(G^\star)$. Since C_i is an independent set, C'_i is as well. Hence $\mathcal{C}'$ is a proper colouring of $G^\star$. Moreover, since C_i is nested, C'_i is nested with the order on the vertices of C'_i induced by the order of the vertices of C_i, and so $\mathcal{C}'$ is a nested colouring of $G^\star$. Thus $\chi_N(G) \geq \chi_N(G^\star)$.

On the other hand, suppose $\mathcal{D}$ is the nested colouring $D_1 \uplus \cdots \uplus D_r$ of $G^\star$. For $1 \leq i \leq r$, let $D'_i = \{v \colon\ v \in V(G) \text{ and } [v]_\sim \in D_i\}$. Since $\mathcal{D}$ is a partition of $V(G^\star)$, $D'_1 \uplus \cdots \uplus D'_r$ is a partition $\mathcal{D}'$ of $V(G)$. Moreover, since D_i is an independent set, D'_i is as well. Hence $\mathcal{D}'$ is a proper colouring of G. Since each D_i is nested, D'_i is nested with the order on the vertices of D'_i induced by the order of the vertices of D_i, where the order on duplicate vertices is arbitrary. Hence $\mathcal{D}'$ is a nested colouring of G. Thus $\chi_N(G^\star) \geq \chi_N(G)$. ☐

Since a complete graph is the de-duplicate of a complete multipartite graph and the Turán graph, then its nested chromatic number is simple to compute.

Corollary 2.11. *If $n_1, \ldots, n_r$ are positive integers, then $\chi_N(K_{n_1,\ldots,n_r}) = \chi(K_{n_1,\ldots,n_r}) = r$.*

Corollary 2.12. *If $T_{n,r}$ is the Turán graph on n vertices that is r-partite, then $\chi_N(T_{n,r}) = \chi(T_{n,r}) = r$.*

Moreover, duplicate-free graphs have been studied under various other names.

Remark 2.13. Duplicate-free graphs were studied by Sumner [25] as "point-determining graphs." Sumner showed that every connected point-determining graph has at least two vertices that can each be removed leaving point-determining induced subgraphs.

They were also studied as "mating graphs" or "M-graphs" by Bull and Pease [1] in order to understand mating-type systems. In this case, vertices are identified with individuals in a population, and edges correspond to compatibility in mating. Thus duplicate vertices correspond to individuals with identical mating compatibilities and so need not be represented.

Kilibarda [14] proved a bijection between unlabeled (connected) mating graphs on n vertices with unlabeled (connected) graphs without endpoints on n vertices. Thus [22, A004110] and [22, A004108] enumerate the number of unlabeled (connected) duplicate-free graphs on n vertices. We note that Kilibarda called the de-duplicate graph $G^\star$ the "reduction of G.'

Finally, duplicate-free graphs were used by McSorley [20] as "neighbourhood distinct graphs" to classify the neighbourhood anti-Sperner graphs, a related but distinct set of graphs. A graph is *neighbourhood anti-Sperner*, or *NAS*, if every vertex is weakly duplicated by some other vertex. Porter [23] introduced the concept of NAS graphs, and showed that every NAS graph has a pair of duplicate vertices. Porter and Yucas [24] established more properties of NAS graphs.

2.3. **The weak duplicate preorder.**

We next define a preorder on the vertices of a graph using the concept of weak duplicates. It is particularly important to notice that the preorder is in the *reverse* order of containment.

Definition 2.14. Let G be a finite simple graph. The *weak duplicate preorder* on $V(G)$ is the preorder defined by $v \leq u$ if u is a weak duplicate of v.

Exchanging a vertex of a clique for a lesser vertex in the preorder generates another clique of the graph.

Lemma 2.15. *Let G be a finite simple graph, and let C be a clique of G. If u is a vertex of C, and $v \leq u$ under the weak duplicate preorder on $V(G)$, then $(C \uplus \{v\}) \setminus \{u\}$ is a clique of G.*

Proof. Since $N_G(u) \subset N_G(v)$, $C \subset N_G(u)$ implies $C \subset N_G(v)$. Thus $(C \uplus \{v\}) \setminus \{u\}$ is a clique of G. □

This gives an alternate condition on a partition of the vertices that is equivalent to being a nested colouring.

Proposition 2.16. *Let G be a finite simple graph. If $\mathcal{C} = C_1 \uplus \cdots \uplus C_k$ is a partition of $V(G)$, then the following conditions are equivalent:*

(i) *$\mathcal{C}$ is nested;*
(ii) *there is an ordering on the vertices of each colour class C_i such that if v is less than u in that order, and K is a clique of G containing u, then $(K \uplus \{v\}) \setminus \{u\}$ is a clique of G; and*
(iii) *there is an ordering on the vertices of each colour class C_i such that if v is less than u in that order, and $\{u, w\}$ is an edge of G, then $\{v, w\}$ is an edge of G.*

Proof. Suppose that condition (i) holds. Since each independent set C_i is nested, the vertices of C_i are comparable under the weak duplicate preorder. If we arbitrarily order the duplicates in C_i, then the induced order on C_i is the desired order for condition (ii) by Lemma 2.15.

Clearly, condition (ii) implies condition (iii), since edges are cliques of G.

Suppose now that condition (iii) holds. Since $\{u, w\} \in E(G)$ implies that $\{v, w\} \in E(G)$, $N_G(u)$ is a subset of $N_G(v)$. That is, the order on the vertices of C_i respects the weak duplicate preorder, and so C_i is nested. In particular, condition (i) holds. □

When the graph is duplicate-free, the preorder is a partial order.

Definition 2.17. Let G be a duplicate-free finite simple graph. The weak duplicate preorder on G is then a partial order, and we write P_G for the poset on $V(G)$ under the weak duplicate partial order induced by G.

The key observation is that, when G is duplicate-free, the chain covers of P_G are in bijection with the nested colourings of G.

Proposition 2.18. *Let G be a duplicate-free finite simple graph. A partition $C_1 \uplus \cdots \uplus C_k$ of $V(G)$ is a nested colouring of G if and only if it is a chain cover of P_G.*

Proof. This follows from the definitions of a nested independent set and the weak duplicate partial order. In particular, $N_G(u) \subset N_G(v)$ if and only if $v \leq u$, and in both cases the sets of vertices form a partition of $V(G)$. $\square$

Dilworth [5, Theorem 1.1] proved that the width (or Dilworth number) of a poset P, i.e., the maximum cardinality of an antichain of P, is precisely the minimum cardinality of a chain cover of P. Hence the nested chromatic number of a graph is the width of the poset of the de-duplicate of the graph.

Corollary 2.19. *If G is a finite simple graph, then $\chi_N(G)$ is the width of $P_{G^\star}$.*

Remark 2.20. Let G be a finite simple graph. A vertex v of G *dominates* a vertex u of G if $N_G(u) \subset N_G[v]$. Notice the subtle difference between domination and weak duplication, namely, u and v may be adjacent in the former. The *Dilworth number of G* is the cardinality of the largest set of vertices of G such that no vertex dominates any other in the set.

Following Felsner, Raghavan, and Spinrad [9], we partially order the vertices of a duplicate-free graph G by $v \leq u$ if v dominates u. The width of this partial order is precisely the Dilworth number of the graph G. This partial order is in the *reverse* order of containment, as in the weak duplicate partial order.

The Dilworth number of a graph is *not* the nested chromatic number of the graph despite the similarities. Recall that threshold graphs are precisely the graphs with Dilworth number 1. In Corollary 5.8 we classify the nested chromatic number of threshold graphs as one more than the number of domination steps in the construction of the graph.

As a consequence, the nested chromatic number can be computed in polynomial time.

Theorem 2.21. *The nested chromatic number of a finite simple graph on n vertices can be computed in $O(n^3)$ time.*

Proof. Fulkerson [10] proved that computing the width of a poset is equivalent to computing the cardinality of a maximum matching of a related bipartite graph. Hopcroft and Karp [11] proved that computing the latter can be done in $O(n^{5/2})$ time.

Computing the relations between the n vertices corresponds to computing $\binom{n}{2}$ subset containments, where each subset has size at most $O(n)$. Hence computing the poset structure on $P_{G^\star}$ takes $O(n^3)$ time. Thus computing the nested chromatic number of a finite simple graph on n vertices via the width of the weak duplicate partial order takes $O(n^3)$ time. $\square$

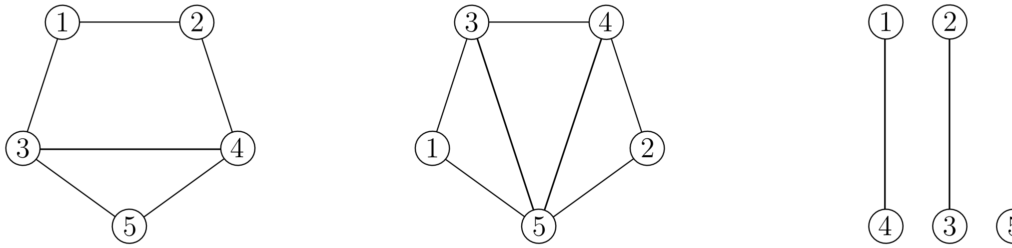
Remark 2.22. Since the nested chromatic number of a graph is the width of an associated poset, existing tools can be used to compute the value for specific cases. Indeed, the computer algebra system *Macaulay2* [18] handles posets with the package *Posets* [4], which can compute the width of a poset. Furthermore, using the package *Nauty* [2], one can generate all the graphs on a small number of vertices (with specific restrictions, e.g., bipartite only, if desired). The latter package uses the software `nauty` [19] at its core.

The ease of computing the nested chromatic number on all graphs of small size is very helpful when proving results such as Theorem 6.2.

The poset P_G need not be unique; see Figure 2.3.

Furthermore, the poset need not be ranked; see Figure 2.4.

However, the height of the poset, i.e., the length of the longest chain, is restricted to at most half the number of vertices.



(i) The graph G. *(ii) The graph H.* *(iii) The poset $P_G = P_H$.*

FIGURE 2.3. The graphs G and H are non-isomorphic, but $P_G = P_H$.

(i) The graph G. *(ii) The poset P_G.*

FIGURE 2.4. The weak duplicate poset P_G need not be ranked.

Proposition 2.23. *If G is a duplicate-free finite simple graph on n vertices, then the height of P_G is at most $\left\lfloor \frac{n-1}{2} \right\rfloor$. That is, at most half of the vertices of G can be in a nested independent set of G.*

Proof. Suppose the height of P_G is h, i.e., there exist $h+1$ vertices $v_0, \dots, v_h$ of G such that $N_G(v_h) \subsetneq \cdots \subsetneq N_G(v_0)$. This implies $N_G(v_0) \geq h$, and since $v_i \notin N_G(v_0)$ for $0 \leq i \leq h$, $n \geq 2h+1$. That is, $\frac{n-1}{2} \geq h$. $\square$

Example 2.24. The preceding proposition implies that a poset with a large height relative to the number of vertices, e.g., a chain, cannot be the poset associated to a finite simple graph under the weak duplicate partial order. However, there exist posets with small height that are also not associated to a finite simple graph.

Consider the poset P on $\{1, 2, 3, 4\}$ with covering relations $1 < 2$, $1 < 3$, and $1 < 4$. This poset is not associated to a graph, as determined by a search of the 11 graphs on 4 vertices. Notice, however, that the dual of P is the poset associated to the graph $K_3 \cup K_1$.

Question 2.25. What posets are isomorphic to some P_G, where G is a duplicate-free finite simple graph?

3. FAMILIES OF GRAPHS

We now look at three families of graphs with well-behaved nested chromatic numbers.

3.1. Bipartite graphs.

Due to the structure of bipartite graphs it is possible to classify the graphs with nested chromatic number 2, i.e., colour-nested bipartite graphs.

Theorem 3.1. *Let r and s be positive integers, and let $1 \leq a_r \leq \cdots \leq a_1 \leq s$ be a sequence of nonnegative integers. Construct the graph $G = G_{a_1,\dots,a_r;s}$ on the vertex set*

$\{u_1, \ldots, u_r, v_1, \ldots, v_s\}$ with an edge between u_i and v_j if and only if $j \leq a_i$. Then the following statements are true:

(i) *$\{u_1, \ldots, u_r\} \cupdot \{v_1, \ldots, v_s\}$ is a nested colouring of G,*
(ii) *$\chi_N(G) = \chi(G) = 2$, i.e., G is colour-nested, and*
(iii) *every nontrivial finite simple bipartite graph that is colour-nested arises this way.*

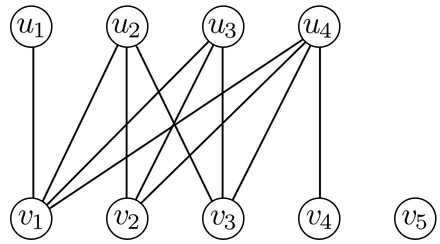


FIGURE 3.1. The colour-nested bipartite graph $G_{4,3,3,1;5}$.

Proof. Let $U = \{u_1, \ldots, u_r\}$ and $V = \{v_1, \ldots, v_s\}$.

By construction $N_G(u_i) = \{v_1, \ldots, v_{a_i}\}$ and hence $N_G(v_j) = \{u_i\colon\ a_i \geq j\}$. Thus $U \cupdot V$ is a colouring of G. Since $a_1 \geq 1$, u_1 and v_1 are adjacent, hence $\chi(G) = 2$. Since $a_{i+1} \leq a_i$, $N_G(u_{i+1})$ is a subset of $N_G(u_i)$ and $N_G(v_{i+1})$ is a subset of $N_G(v_i)$. Thus $U \cupdot V$ is a nested colouring of G, and so $\chi_N(G) = 2$. This completes parts (i) and (ii).

Now let G be any nontrivial finite simple bipartite graph that is colour-nested. Suppose $\{u_1, \ldots, u_r\} \cupdot \{v_1, \ldots v_s\}$ is a nested colouring of G, such that $N_G(u_{i+1}) \subset N_G(u_i)$ and $N_G(v_{i+1}) \subset N_G(v_i)$. Furthermore, by Remark 2.4 we may assume without loss of generality that any isolated vertices of G are in $\{v_1, \ldots v_s\}$. Set $a_i = \max\{j\colon\ v_j \in N_G(u_i)\}$. Since $N_G(u_{i+1}) \subset N_G(u_i)$, $1 \leq a_r \leq \cdots \leq a_1 \leq s$. Thus G arises as in the construction above, completing part (iii). □

The connected and duplicate-free colour-nested bipartite graphs have a simple classification.

Corollary 3.2. *Let r and s be positive integers, and let $1 \leq a_r \leq \cdots \leq a_1 \leq s$ be a sequence of nonnegative integers. The following statements are true.*

(i) *$G_{a_1,\ldots,a_r;s}$ is connected if and only if $a_1 = s$, and*
(ii) *$G_{a_1,\ldots,a_r;s}$ is connected and duplicate-free if and only if $r = s$ and $a_i = i$, for $1 \leq i \leq r$.*

Proof. Let $G = G_{a_1,\ldots,a_r;s}$, $U = \{u_1, \ldots, u_r\}$, and $V = \{v_1, \ldots, v_s\}$.

Since $a_r \geq 1$, we have $N_G(v_1) = U$. As $U \cupdot V$ is a nested colouring of G, G is connected if and only if $N_G(u_1) = V$, i.e., $a_1 = s$. This completes part (i).

Furthermore, $N_G(u_i) = N_G(u_j)$ if and only if $a_i = a_j$, and $N_G(v_i) = N_G(v_j)$ if and only if $\max\{k\colon\ a_k \geq i\} = \max\{k\colon\ a_k \geq j\}$, i.e., there exists a k such that $a_k - a_{k+1} \geq 2$. This completes part (ii). □

Further, this permits an enumeration of certain colour-nested bipartite graphs.

Corollary 3.3. *If $n \geq 3$ is an odd integer, then there are precisely 2^{n-3} unique connected colour-nested bipartite graphs with n vertices.*

Furthermore, there exists a unique duplicate-free colour-nested bipartite graph with n vertices, where $n \geq 2$ is an integer.

Proof. By Theorem 3.1 and Corollary 3.2, we need only look at graphs $G_{a_1,\dots,a_r;s}$ where $1 \leq a_r \leq \cdots \leq a_1 = s$ such that $r + s = n$.

For fixed r and s, $G_{a_1,\dots,a_r;s} = G_{a'_1,\dots,a'_r;s}$ if and only if $a_i = a'_i$; otherwise the vertex degree sequences of the u_i would differ. Hence there are $\binom{n-2}{r-1}$ graphs (we choose with repetition the $r-1$ values $a_1, \dots, a_{r-1}$ among the s options) with the specified r and s. Thus among all choices of r and s there are $\sum_{r=1}^{n-1} \binom{n-2}{r-1} = 2^{n-2}$ possible graphs. However, if $r > s$, then $G_{a_1,\dots,a_r;s} = G_{b_1,\dots,b_s;r}$ where $b_i = \max\{j\colon\ a_j \geq i\}$ for $1 \leq i \leq s$. Thus we have double counted in our enumeration, and so there are 2^{n-3} unique graphs.

The second statement follows by Corollary 3.2(ii). In particular, if $n = 2m$, then $G_{1,\dots,m;m}$ is the unique duplicate-free graph, and if $n = 2m+1$, then $G_{1,\dots,m;m+1}$ is the unique duplicate-free graph. □

3.2. **Regular graphs.**

The nested chromatic number of a graph is the same as the number of vertices if and only if the graph is duplicate-free. Moreover, large girth can force a regular graph to be duplicate-free.

Proposition 3.4. *Let G be a d-regular finite simple graph, for $d \geq 1$. The graph G is duplicate-free if and only if $\chi_N(G) = \#V(G)$.*

In particular, if the girth of G is at least 5, then $\chi_N(G) = \#V(G)$.

Proof. Let G be a d-regular finite simple graph. Since $\#N_G(u) = d$ for all vertices u of G, u is a weak duplicate of v if and only if u and v are duplicates.

Now suppose G has girth at least 5. If u and v are distinct duplicates, then u and v have at least two common neighbours, say, $\{w, x\}$. Thus either $\{u, v, w, x\}$ induces a 4-cycle or $\{v, w, x\}$ induces a 3-cycle, i.e., the girth of G is at most 4. This contradicts the girth of G being at least 5, and so G is duplicate-free. □

As an immediate consequence, we can compute the nested chromatic number of snarks and Kneser graphs. See Example 2.7 for the Petersen graph, which is both a snark and the Kneser graph $KG_{5,2}$.

Corollary 3.5. *If G is a snark, then $\chi_N(G) = \#V(G)$.*

Proof. Snarks are 3-regular and have girth at least 5. □

Corollary 3.6. *If n and k are positive integers so that $n \geq 2k$, then the nested chromatic number of the Kneser graph $KG_{n,k}$ is $\chi_N(KG_{n,k}) = \#V(KG_{n,k}) = \binom{n}{k}$.*

Proof. Recall that the vertices of the Kneser graph $KG_{n,k}$ are the k-subsets of $\{1, \dots, n\}$, and a pair of vertices are adjacent if the corresponding sets are disjoint. This implies that no two vertices are duplicates, otherwise they would be the same k-subset. By Proposition 3.4, $KG_{n,k}$ being duplicate-free implies that $\chi_N(KG_{n,k}) = \#V(KG_{n,k})$. □

Let $\overline{G}$ denote the complement of the finite simple graph G. The nested chromatic number of the n-cycle C_n and the n-anticycle $\overline{C_n}$ are simple expressions, for large n.

Corollary 3.7. *Let $n \geq 3$ be an integer. The following statements are true:*

(i) $\chi_N(C_3) = 3$ *and* $\chi_N(\overline{C_3}) = 1$,

(ii) $\chi_N(C_4) = 2$ *and* $\chi_N(\overline{C_4}) = 4$, *and*

(iii) $\chi_N(C_n) = n = \chi_N(\overline{C_n})$, *for* $n \geq 5$.

Proof. Parts (i) and (ii) are easy to verify. Since C_n has girth n and is 2-regular, by Proposition 3.4, $\chi_N(C_n) = n$ for $n \geq 5$.

Let $n \geq 5$. The n-anticycle $\overline{C_n}$ is $(n-3)$-regular. Suppose u and v are distinct vertices of $\overline{C_n}$ such that u is a duplicate of v. This implies that there is a vertex w, distinct from u and v, that is nonadjacent to u and v. Hence $\{u, v, w\}$ is an independent set in $\overline{C_n}$ and so induces a 3-cycle in C_n, which is absurd. Thus $\overline{C_n}$ is duplicate-free and $\chi_N(\overline{C_n}) = n$ by Proposition 3.4. □

This further emphasises the distinction between the chromatic number and the nested chromatic number.

Remark 3.8. Since C_n is a planar graph, this shows that planar graphs can have arbitrarily large nested chromatic number. This contrasts the chromatic number for planar graphs, which is bounded by 4. See Proposition 6.3 for more about the nested chromatic number and planar graphs.

Let G be a finite simple graph, and let $\overline{G}$ denote the complement of G. In this case, $\chi(G) + \chi(\overline{G}) \leq \#V(G) + 1$. However, the nested chromatic number can break this bound. Indeed, by the previous lemma, we have $\chi_N(C_n) + \chi_N(\overline{C_n}) = 2n = 2\#V(C_n)$ for $n \geq 5$. On the other hand, $\chi_N(P_4) + \chi_N(\overline{P_4}) = \#V(P_4) = 4$, since $\overline{P_4} \cong P_4$.

We offer a conjecture suggested by the preceding remark.

Conjecture 3.9. *If G is a finite simple graph, then $\chi_N(G) + \chi_N(\overline{G}) \geq \#V(G)$.*

3.3. Diamond- and C_4-free graphs.

Let the *diamond graph* be K_4 with any edge removed; see Figure 3.2(i). If G is both diamond- and C_4-free, then only the presence of leaves, i.e., degree 1 vertices, can reduce the nested chromatic number from $\#V(G)$.

(i) The diamond graph. *(ii) The 4-cycle C_4.*

FIGURE 3.2. The forbidden graphs in Section 3.3.

Theorem 3.10. *Let G be a connected finite simple graph that is both diamond- and C_4-free. If G has ℓ leaves, then $\#V(G) - \ell \leq \chi_N(G) \leq \#V(G)$. Furthermore, equality holds in the upper bound if and only if either $\ell = 0$ or $G = K_2$.*

In particular, if the minimum degree of a vertex $\delta(G)$ is at least 2, then $\chi_N(G) = \#V(G)$.

Proof. Suppose G is a connected finite simple graph that is both diamond- and C_4-free, and further suppose G has ℓ leaves.

Let u and v be distinct vertices of G. If u and v have two neighbours in common, say, w and x, then $\{u, v, w, x\}$ must be a 4-clique of G since it cannot be a diamond or a C_4; thus u and v must be adjacent. Hence if u is a weak duplicate of v, then u and v must have exactly one neighbour in common, and so $\#N_G(u) = 1$, i.e., u is a leaf. Thus at most one element of each nested independent set is a non-leaf, and so $\chi_N(G)$ is at least the number of non-leaves, i.e., $\#V(G) - \ell$.

Clearly, if G has no leaves, then $\chi_N(G) = \#V(G)$. Suppose G is not K_2, and G has at least one leaf, say, u. Let v be the unique neighbour of v. As G is connected and not K_2, v must have at least one neighbour not u, say, w. Hence $N_G(u) = \{v\} \subset N_G(w)$, and u is a weak duplicate of w. Thus $\{u, w\}$ is a nested independent set of G, and so $\chi_N(G) < \#V(G)$. □

Clearly, graphs with girth at least 5 are diamond- and C_4-free. Since d-regular graphs have no leaves, if $d \geq 2$, then this recovers the second part of Proposition 3.4, as well as Corollary 3.5 and Corollary 3.7(iii).

Trees, which have infinite girth, are diamond- and C_4-free graphs.

Corollary 3.11. *Let G be a finite simple tree with at least three vertices. If G has ℓ leaves, then $\#V(G) - \ell \leq \chi_N(G) < \#V(G)$.*

This immediately gives the nested chromatic number for path graphs.

Corollary 3.12. *Let P_n be the path graph on n vertices. The nested chromatic number of P_n is*

$$\chi_N(P_n) = \begin{cases} 2 & \text{if } 2 \leq n \leq 4, \\ 4 & \text{if } n = 5, \text{ and} \\ n-2 & \text{if } n \geq 6. \end{cases}$$

Proof. If $2 \leq n \leq 5$, then it is simple to verify the claim.

Suppose $n \geq 6$, and without loss of generality assume the edges of P_n are $\{\{1, 2\}, \ldots, \{n-1, n\}\}$. In this case, $N_{P_n}(1) = \{2\} \subset N_{P_n}(3) = \{2, 4\}$ and $N_{P_n}(n) = \{n-1\} \subset N_{P_n}(n-2) = \{n-3, n-1\}$. Since $n \geq 6$, $n - 2 \neq 3$, and so

$$\{1, 3\} \cupdot \{n-2, n\} \cupdot \{2\} \cupdot \{4\} \cupdot \cdots \cupdot \{n-3\} \cupdot \{n-1\}$$

is a nested colouring of P_n. Hence $\chi_N(P_n) \leq n - 2$, and so equality holds by Corollary 3.11. □

We close with some comments about the class of diamond- and C_4-free graphs.

Remark 3.13. The class of diamond- and C_4-free graphs has been studied in the more general setting of diamond- and even-cycle-free graphs by Kloks, Müller, and Vušković [15]. Some of their results specify to the case of diamond- and C_4-free graphs.

In a more focused case, Eschen, Hoàng, Spinrad, and Srithavan [8] studied structural results on this class of graphs. Moreover, they provide a polynomial-time recognition algorithm. They make use of an alternate classification of diamond- and C_4-free graphs: they are precisely the graphs such that every nonadjacent pair of vertices has at most one common neighbour.

We further note that diamond- and C_4-free graphs were called *weakly geodetic graphs* in the past; see, e.g., [12].

4. Induced subgraphs

A first natural operation to consider is that of taking induced subgraphs.

4.1. **Induced subgraphs.**

The nested chromatic number behaves the same as the chromatic number under taking induced subgraphs.

Proposition 4.1. *Let G be a finite simple graph, and let H be an induced subgraph of G. If $C_1 \cupdot \cdots \cupdot C_k$ is a nested colouring $\mathcal{C}$ of G, then $(C_1 \cap V(H)) \cupdot \cdots \cupdot (C_k \cap V(H))$ is a nested colouring $\mathcal{C}'$ of H.*

In particular, $\chi_N(H) \leq \chi_N(G)$.

Proof. It is already known that $\mathcal{C}'$ is a proper colouring of H, since $\mathcal{C}$ is a proper colouring of G. Moreover, since $N_H(v) = N_G(v) \cap V(H)$ for $v \in V(H)$, the nesting of C_i implies the nesting of $C_i \cap V(H)$. □

Together with Corollary 3.7, the preceding proposition implies that the maximum length of an induced cycle, if one exists and is big enough, forms an effective lower bound for the nested chromatic number.

Corollary 4.2. *Let G be a finite simple graph which has at least one induced cycle. If the maximum length of an induced cycle c is at least 5, then $\chi_N(G) \geq c$.*

Proof. This follows from Proposition 4.1 and Corollary 3.7. □

Remark 4.3. We offer a pair of comments about the preceding corollary.

(i) If the girth of a graph is finite and at least 5, then it is a lower bound for the nested chromatic number of the graph.

(ii) Suppose the longest induced *odd* cycle of G has length $2k-1$. Erdős and Hajnal [7, Theorem 7.7] proved that $\chi(G) \leq 2k$. On the other hand, the preceding result shows that $\chi_N(G) \geq 2k-1$, if $k \geq 3$. See [13] for further results bounding $\chi(G)$ using the length of the longest induced odd cycle.

Let G be a finite simple graph, and let v be a vertex of G. The *vertex deletion of G by v* is the induced subgraph $G - v$ of G on vertex set $V(G) \setminus \{v\}$. The chromatic number is reduced by at most one after vertex deletion. The nested chromatic number is reduced by at most one more than the degree of the vertex that was deleted.

Proposition 4.4. *Let G be a finite simple graph. If v is any vertex of G, then*

$$\chi_N(G) - \#N_G(v) - 1 \leq \chi_N(G - v) \leq \chi_N(G).$$

Proof. The upper bound follows immediately from Proposition 4.1.

Let $C_1 \cupdot \cdots \cupdot C_k$ be a nested colouring of $G - v$. This implies that $C_1' \cupdot \cdots \cupdot C_k'$, where $C_i' = C_i \setminus N_G(v)$, together with $\{v\}$ and the singleton sets containing each neighbour of v is a nested colouring of G. This follows as the presence of v only affects the neighbourhoods of its neighbours. Hence $k + \#N_G(v) + 1 \geq \chi_N(G)$, and so $k \geq \chi_N(G) - \#N_G(v) - 1$. □

Both bounds in the preceding proposition are achievable.

Example 4.5. Let $n \geq 3$. Notice that $C_n - v = P_{n-1}$ and $\#N_{C_n}(v) = 2$ for any vertex v of C_n. Combining Corollaries 3.7 and 3.12, we have that $\chi_N(C_n - v) = \chi_N(P_{n-1}) = \chi_N(C_n) - \#N_{C_n}(v) - 1$ if $n \geq 5$ and $n \neq 6$.

On the other hand, let G be as in Figure 4.1. We have $\chi_N(G) = \chi_N(G-5) = \chi_N(G-6) = 4$ despite $\#N_G(5) = \#N_G(6) = 4$.

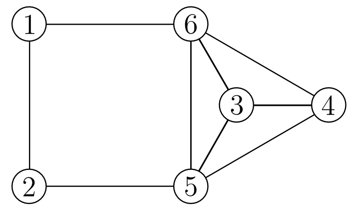


FIGURE 4.1. A graph G such that $\chi_N(G) = \chi_N(G-5) = \chi_N(G-6) = 4$.

4.2. **Criticality.**

Recall that a vertex v of a finite simple graph G is a *critical vertex* if $\chi(G-v) = \chi(G)-1$. Further, if every vertex of G is a critical vertex, then G is *vertex-critical* (or *vertex-colour-critical*). Critical vertices are never weak duplicates of other vertices in G.

Lemma 4.6. *Let G be a finite simple graph. If v is a critical vertex of G, and v is a weak duplicate of $w \in G$, then $w = v$.*

Proof. Suppose $\chi(G) = k$, and let $C_1 \uplus \cdots \uplus C_{k-1}$ be an optimal colouring of $G-v$. Assume $w \neq v$ and $w \in C_1$, without loss of generality. Since $N_G(v) \subset N_G(w)$, v is independent of the vertices in C_1. Hence $(C_1 \uplus \{v\}) \uplus C_2 \uplus \cdots \uplus C_{k-1}$ is a colouring of G, contradicting $\chi(G) = k$. Thus $w = v$. □

This implies that the number of critical vertices provides a lower bound for the nested chromatic number.

Corollary 4.7. *Let G be a finite simple graph. If c is the number of critical vertices of G, then $\chi_N(G) \geq c$.*

In particular, if G is vertex-critical, then $\chi_N(G) = \#V(G)$.

Proof. By Lemma 4.6, every critical vertex of G must be at the top of its own nested colour class, which immediately implies the bound. □

We define a concept of criticality for the nested chromatic number.

Definition 4.8. A finite simple graph G is *nested-critical* if the deletion of any vertex reduces the nested chromatic number of G.

Graphs with large nested chromatic number are nested-critical.

Proposition 4.9. *Let G be a finite simple graph. If $\chi_N(G) = \#V(G)$, then G is nested-critical.*

In particular, if G is vertex-critical, then G is nested-critical.

Proof. This follows immediately since $\chi_N(G-v) \leq \#V(G-v) < \chi_N(G)$.

The second claim follows from Corollary 4.7. □

Being nested-critical does not imply being vertex-critical. For example, P_n is nested-critical for $n \geq 7$ but is never vertex-critical.

However, if G is colour-nested, then being nested-critical is equivalent to being vertex-critical.

Proposition 4.10. *Let G be a finite simple graph. If G is colour-nested, then the following conditions are equivalent:*

(i) *G is nested-critical,*
(ii) *G is vertex-critical,*
(iii) *$\chi_N(G) = \#V(G)$, and*
(iv) *$G = K_{\#V(G)}$.*

Proof. Since $\chi(G) = \chi_N(G)$, and $\chi(H) \leq \chi_N(H)$ in general, we clearly have condition (i) implying condition (ii), and the latter implies condition (iii) by Corollary 4.7. Condition (iii) implies $\chi(G) = \#V(G)$, which is equivalent to $G = K_{\#V(G)}$. Finally, $K_{\#V(G)}$ is nested-critical by Proposition 4.9. □

On the other hand, if G is colour-nested, then $G - v$ need not be colour-nested. The graph G in Figure 4.1 is colour-nested, though $G - 5$ and $G - 6$ are not colour-nested.

4.3. **A topological remark.**

Consider the *homomorphism complex* $\mathrm{Hom}(H, G)$ coming from the homomorphisms from H to G, where G and H are finite simple graphs; see [16, Definition 3.2]. Kozlov showed [16, Theorem 3.3] that $\mathrm{Hom}(H, G)$ collapses onto $\mathrm{Hom}(H, G - u)$ if u is a weak duplicate of some other vertex v in G. Thus if $C_1 \uplus \cdots \uplus C_k$ is a nested k-colouring of G, and G' is an induced subgraph of G with vertex set $\{v_1, \ldots, v_k\}$, where v_i is a minimal element of C_i under the weak duplicate preorder, then $\mathrm{Hom}(H, G)$ collapses onto $\mathrm{Hom}(H, G')$, and so $\mathrm{Hom}(H, G)$ and $\mathrm{Hom}(H, G')$ have the same simple homotopy type.

This is particularly interesting as the neighbourhood complex of G, i.e., the simplicial complex of subsets of $V(G)$ which have a common neighbour, is homotopy equivalent to $\mathrm{Hom}(K_2, G)$. Thus Lovász's lower bound on the chromatic number [17, Theorem 2] can be interpreted as $\mathrm{conn}\,\mathrm{Hom}(K_2, G) \leq \chi(G) - 3$, where $\mathrm{conn}\, X$ is the connectivity of the complex X. We note that this lower bound is strict in the case of Kneser graphs.

Hence we see that there exists an induced subgraph G' of G on $\chi_N(G)$ vertices such that $\mathrm{Hom}(K_2, G') \leq \chi(G) - 3$. Thus if $\chi_N(G) < \#V(G)$, then G has more redundancy than necessary for such topological bounds on the chromatic number to be useful. Indeed, it is this redundant and recursive nature that is exploited in the associated algebras studied in [3].

Further, recall that the *independence complex* $\mathrm{Ind}(G)$ of a finite simple graph G is the simplicial complex with faces given by the independent sets of G. Engström showed [6, Lemma 3.2] that $\mathrm{Ind}(G)$ collapses onto $\mathrm{Ind}(G - v)$ if v is weakly duplicated by some other vertex u. More generally, this implies that if $C_1 \uplus \cdots \uplus C_k$ is a nested k-colouring of G, and G'' is an induced subgraph of G with vertex set $\{u_1, \ldots, u_k\}$, where u_i is a maximal element of C_i under the weak duplicate preorder, then $\mathrm{Ind}(G)$ collapses onto $\mathrm{Ind}(G'')$, and so $\mathrm{Ind}(G)$ and $\mathrm{Ind}(G'')$ have the same simple homotopy type. Again, this emphasises the redundant structure present in graphs with $\chi_N(G) < \#V(G)$.

We note that the de-duplicate graph $G^\star$ of G can be constructed in both of these fashions. In particular, let $C_1 \uplus \cdots \uplus C_k$ be the nested k-colouring such that each colour-class is an equivalence class of duplicates. Selecting an induced subgraph of G with precisely one element from each colour class generates a graph isomorphic to $G^\star$. Hence $\mathrm{Hom}(H, G)$ and $\mathrm{Hom}(H, G^\star)$ have the same simple homotopy type, as do $\mathrm{Ind}(G)$ and $\mathrm{Ind}(G^\star)$.

5. Behaviour of the nested chromatic number

Now we consider the behaviour of the nested chromatic number under various graph operations.

5.1. **Mycielski's construction.**

Let G be a simple graph on $V(G) = \{u_1, \dots, u_n\}$. The *Mycielski graph of* G is the graph $\mu(G)$ with vertex set $\{u_1, \dots, u_n, v_1, \dots, v_n, w\}$ with edge set

$$E(\mu(G)) = E(G) \uplus \{\{u_i, v_j\} \colon\ u_j \in N_G(u_i)\} \uplus \{\{w, v_i\} \colon\ 1 \le i \le n\}.$$

This construction was first described by Mycielski [21], wherein he proved that $\chi(\mu(G)) = \chi(G) + 1$, and further that $\mu(G)$ is triangle-free if G is triangle-free. We further note that G is an induced subgraph of $\mu(G)$.

Unlike the chromatic number, which only increases by one, the nested chromatic number doubles and increases by one under the Mycielski construction.

Proposition 5.1. *If G is a finite simple graph, then $\chi_N(\mu(G)) = 2\chi_N(G) + 1$.*

Proof. The vertices of $\mu(G)$ have the open neighbourhoods: $N_{\mu(G)}(u_i) = \{u_j, v_j \colon\ u_j \in N_G(u_i)\}$, $N_{\mu(G)}(v_i) = N_G(u_i)$, and $N_{\mu(G)}(w) = \{v_1, \dots, v_n\}$.

Let $\mathcal{C}$ be any nested colouring $C_1 \uplus \cdots \uplus C_k$ of G. Each C_i remains a nested independent set in $\mu(G)$. Moreover, substituting v_j for u_j in each C_i generates a nested independent set C_i' in $\mu(G)$. Thus $C_1 \uplus \cdots \uplus C_k \uplus C_1' \uplus \cdots \uplus C_k' \uplus \{w\}$ is a nested colouring of $\mu(G)$, and so $\chi_N(\mu(G)) \le 2\chi_N(G) + 1$.

Isolated vertices of G remain isolated in $\mu(G)$, and none of the v_j nor w can be isolated. If u_i is not an isolated vertex of G, then u_i is not a weak duplicate of any v_j since the v_j are not adjacent to any other v_t in $\mu(G)$, and every v_j is not a weak duplicate of u_i since only the v_j are adjacent to w. Moreover, with the exception of isolated vertices, w is not a weak duplicate of or weakly duplicated by any other vertex. Thus any nontrivial nested independent set of $\mu(G)$ that does not contain isolated vertices is contained exclusively in one of $\{u_1, \dots, u_n\}$, $\{v_1, \dots, v_n\}$, and $\{w\}$.

Let $\mathcal{C}$ be any nested colouring $C_1 \uplus \cdots \uplus C_k$ of $\mu(G)$. By the preceding paragraph, we may assume without loss of generality that $C_1 \uplus \cdots \uplus C_i = \{u_1, \dots, u_n\}$, $C_{i+1} \uplus \cdots \uplus C_{k-1} = \{v_1, \dots, v_n\}$, and $C_k = \{w\}$. Thus $C_1 \uplus \cdots \uplus C_i$ induces a nested colouring on G, and so $i \ge \chi_N(G)$. Similarly, substituting u_j for v_j in $C_{i+1} \uplus \cdots \uplus C_{k-1}$, we have another nested colouring of G, and so $k - i - 1 \ge \chi_N(G)$. Hence $k \ge 2\chi_N(G) + 1$. □

Example 5.2. In [21], Mycielski presented the family M_i recursively defined by $M_2 = K_2$ and $M_{k+1} = \mu(M_k)$, for $k \ge 2$. Since M_2 is a triangle-free graph with $\chi(M_2) = 2$, M_k is a triangle-free graph with $\chi(M_k) = k$. For $2 \le k \le 4$, M_k is the triangle-free graph with fewest vertices such that $\chi(M_k) = k$.

The nested chromatic number of $M_2 = K_2$ is $\#V(M_2) = 2$. By Proposition 5.1, it follows that $\chi_N(M_k) = \#V(M_k) = 2^{k-2} \cdot 3 - 1$.

5.2. **Disjoint union.**

The chromatic number of a graph is the maximum of the chromatic numbers of the components of the graph. The nested chromatic number, on the other hand, is additive along the components.

Proposition 5.3. *Let G be a finite simple graph without isolated vertices, and let $G_1, \dots, G_t$ be the components of G. The partition $\mathcal{C}$ of $V(G)$ is a nested colouring of G if and only if $\mathcal{C} = \mathcal{C}_1 \uplus \cdots \uplus \mathcal{C}_t$, where $\mathcal{C}_1, \dots, \mathcal{C}_t$ are nested colourings of $G_1, \dots, G_t$, respectively.*

In particular, $\chi_N(G) = \chi_N(G_1) + \cdots + \chi_N(G_t)$.

Proof. If $v \in G_i$, then $N_{G_i}(v) = N_G(v)$. Since there are no isolated vertices, none of these neighbourhoods are empty. Hence no colour class can contain vertices from two separate components of G. Furthermore, since the neighbourhoods do not change, the nesting of a colour class does not change when it is considered in G or a component. □

Hence the disjoint union is also additive.

Corollary 5.4. *If $G_1, \dots, G_t$ are finite simple graphs without isolated vertices, then*

$$\chi_N(G_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} G_t) = \chi_N(G_1) + \cdots + \chi_N(G_t).$$

In particular, if G is a nontrivial finite simple graph and $t \geq 1$, then $\chi_N(tG) = t\chi_N(G)$.

Moreover, a graph with small nested chromatic number and no isolated vertices is connected.

Corollary 5.5. *Let G be a finite simple graph with no isolated vertices. If $\chi_N(G) \leq 3$, then G is connected.*

5.3. **Join.**

Let G and H be finite simple graphs. The *join of G and H* is the graph $G \vee H$ with vertex set $V(G) \mathbin{\dot\cup} V(H)$, where all edges of G and H are preserved and every vertex in $V(G)$ is adjacent to every vertex in $V(H)$. In particular, the open neighbourhoods of g in $V(G)$ and h in $V(H)$ are

$$N_{G\vee H}(g) = N_G(g) \mathbin{\dot\cup} V(H) \quad \text{and} \quad N_{G\vee H}(h) = N_H(h) \mathbin{\dot\cup} V(G),$$

respectively.

Both the chromatic number and the nested chromatic number are additive across joins.

Proposition 5.6. *Let G and H be finite simple graphs. The partition $\mathcal{C}$ of $V(G) \mathbin{\dot\cup} V(H)$ is a nested colouring of $G \vee H$ if and only if $\mathcal{C} = \mathcal{C}_G \mathbin{\dot\cup} \mathcal{C}_H$, where $\mathcal{C}_G$ and $\mathcal{C}_H$ are nested colourings of G and H, respectively.*

In particular, $\chi_N(G \vee H) = \chi_N(G) + \chi_N(H)$.

Proof. As every vertex of G is adjacent to every vertex of H, no colour class can contain vertices from both G and H. Moreover, since all the vertices of G (resp., H) have their neighbourhoods modified in a uniform way, nesting is not changed. □

This implies, in particular, that adding a dominating vertex, i.e., a vertex adjacent to every other vertex, to a graph increases the nested chromatic number by precisely 1.

Example 5.7. Many common families of graphs are constructed by adding a dominating vertex to another common graph. Consider the following examples.

(i) The star graph S_n is the trivial graph on n vertices with a dominating vertex added. Hence $\chi_N(S_n) = 2$ for $n \geq 1$.

(ii) The windmill graph $Wd_{k,n}$ is nK_k with a dominating vertex added. Hence

$$\chi_N(Wd_{k,n}) = \chi_N(nK_k) + 1 = n\chi_N(K_k) + 1 = nk + 1.$$

(iii) The wheel graph W_n is the cycle graph C_n with a dominating vertex added. Hence $\chi_N(W_n) = n + 1$ for $n = 3$ and $n \geq 5$, and $\chi_N(W_4) = 3$, by Corollary 3.7.

Further, threshold graphs are colour-nested. Recall that a threshold graph is a graph that can be constructed from a single isolated vertex by repeatedly adding a new isolated vertex or a new dominating vertex.

Corollary 5.8. *If G is a threshold graph constructed with d dominating steps, then $\chi_N(G) = \chi(G) = d+1$.*

Proof. Adding isolated vertices does not change the nested chromatic number as seen in Remark 2.4. By Proposition 5.6, adding a dominating vertex increases the nested chromatic number by 1. Hence the nested chromatic number of G is one more than the number of dominating steps. That is $\chi_N(G) = d+1$. Moreover, after d dominating steps, the clique number of G, $\omega(G)$, is $d+1$. Since $\omega(G) \leq \chi(G) \leq \chi_N(G)$, we have $\chi(G) = d+1$. □

5.4. **Direct product.**

Let G and H be finite simple graphs. The *direct* (or *tensor*) *product of G and H* is the graph $G \times H$ with vertex set $V(G) \times V(H)$, where (g, h) is adjacent to (g', h') if and only if g is adjacent to g' in G and h is adjacent to h' in H. In particular, the open neighbourhood of (g, h) in $G \times H$ is

$$N_{G\times H}(g, h) = N_G(g) \times N_H(h).$$

Notice that $G \times K_1 \cong \overline{K_n}$, so $\chi_N(G \times K_1) = 1$. Moreover, $(G \cup G') \times H = (G \times H) \cup (G' \times H)$. Thus following Section 5.2, we need only consider finite simple graphs G that are connected and have at least two vertices.

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. The *direct product of P and Q* is the poset $(P \times Q, \leq_{P\times Q})$, where $(p, q) \leq_{P\times Q} (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$. The weak duplicate poset of the direct product of two graphs is the direct product of the weak duplicate posets of the graphs.

Lemma 5.9. *If G and H are duplicate-free connected finite simple graphs, neither of which is K_1, then $P_{G\times H} = P_G \times P_H$.*

Proof. Since $N_{G\times H}(g, h) = N_G(g) \times N_H(h)$, it is immediate that (g, h) is a weak duplicate of (g', h') in $G \times H$ if and only if g is a weak duplicate of g' in G and h is a weak duplicate of h' in H. The claim follows immediately. □

As a consequence, the nested chromatic number of the direct product of graphs is bounded below by the product of the nested chromatic numbers of the factors. We note that the chromatic number of the direct product of graphs is bounded above by the minimum of the chromatic numbers of the factors (Hedetniemi's conjecture says equality holds).

Proposition 5.10. *If G and H are connected finite simple graphs, neither of which is K_1, then*

$$\chi_N(G) \cdot \chi_N(H) \leq \chi_N(G \times H) \leq \min\{\#V(G) \cdot \chi_N(H), \chi_N(G) \cdot \#V(H)\}.$$

In particular, if $\chi_N(H) = \#V(H)$, then $\chi_N(G \times H) = \chi_N(G) \cdot \chi_N(H)$.

Proof. By Proposition 2.10 we may assume G and H are duplicate-free. Thus by Lemma 5.9 we have that $P_{G\times H} = P_G \times P_H$. Hence $\chi_N(G \times H)$ is the width of $P_G \times P_H$, by Corollary 2.19.

Clearly, if A and B are antichains of P_G and P_H, respectively, then $A \times B$ is an antichain of $P_{G\times H}$. Hence the width of $P_{G\times H}$ is at least the product of the widths of P_G and P_H, i.e., $\chi_N(G \times H) \geq \chi_N(G) \cdot \chi_N(H)$.

On the other hand, let A be any antichain of $P_{G\times H}$. For each g in P_G, let $A_g = \{h \in P_H : (g, h) \in A\}$. By construction, A_g must be an antichain of P_H for all $g \in P_G$. This implies that $\#A_g \leq \chi_N(H)$ and so $\chi_N(G \times H) \leq \#V(G) \cdot \chi_N(H)$. As the graph direct product is commutative, we also then have $\chi_N(G \times H) \leq \chi_N(G) \cdot \#V(H)$ by symmetry. □

Both bounds are achievable.

Example 5.11. Let $G = P_4$, and let H be the graph on $V(H) = \{1, 2, 3, 4\}$ with edge set $E(H) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\}$. In this case, $\chi_N(G) = 2$ and $\chi_N(H) = 3$. However, $\chi_N(G \times H) = 8 = \chi_N(G) \cdot \#V(H)$.

On the other hand, equality holds with the lower bound for the bipartite double cover $G \times K_2$ of G. In particular, this implies that the crown graph on $2n$ vertices, i.e., $K_n \times K_2$, has nested chromatic number $2n$.

5.5. **Cartesian product.**

Let G and H be finite simple graphs. The *Cartesian product of* G *and* H is the graph $G \,\square\, H$ with vertex set $V(G) \times V(H)$, where (g, h) is adjacent to (g', h') if and only if either $g = g'$ and h is adjacent to h' in H or $h = h'$ and g is adjacent to g' in G. In particular, the open neighbourhood of (g, h) in $G \,\square\, H$ is

$$N_{G \square H}(g, h) = \{g\} \times N_H(h) \cup N_G(g) \times \{h\}.$$

Notice that $G \,\square\, K_1$ is isomorphic to G, so $\chi_N(G \,\square\, K_1) = \chi_N(G)$. Moreover, $(G \cup G') \,\square\, H = (G \,\square\, H) \cup (G' \,\square\, H)$. Thus following Section 5.2, we need only consider finite simple graphs G that are connected and have at least two vertices.

The weak duplicates generated in the Cartesian product come from leaves.

Lemma 5.12. *Let G and H be connected finite simple graphs, neither of which is K_1. The vertex (g, h) is a weak duplicate of the distinct vertex (g', h') in $G \,\square\, H$ if and only if $N_G(g) = \{g'\}$ and $N_H(h) = \{h'\}$.*

Proof. By definition, (g, h) is a weak duplicate of (g', h') if and only if

$$N_{G \square H}(g, h) = \{g\} \times N_H(h) \cup N_G(g) \times \{h\} \subset N_{G \square H}(g', h') = \{g'\} \times N_H(h') \cup N_G(g') \times \{h'\}.$$

If $g = g'$, then (g, h) being a weak duplicate of (g', h') forces $h = h'$, since $N_G(g)$ is nonempty and open. Hence we may assume $g \neq g'$ and $h \neq h'$. In this case, (g, h) is a weak duplicate of (g', h') if and only if $\{g\} \times N_H(h) \subset N_G(g') \times \{h'\}$ and $N_G(g) \times \{h\} \subset \{g'\} \times N_H(h')$, since $N_G(g)$ and $N_H(h)$ are nonempty and open. The latter is equivalent to $N_H(h) = \{h'\}$ and $N_G(g) = \{g'\}$, again since the neighbourhoods are nonempty. $\square$

Thus except $K_2 \,\square\, K_2 = C_4$, all Cartesian products of connected graphs are duplicate-free.

Corollary 5.13. *Let G and H be connected finite simple graphs, neither of which is K_1. The graph $G \,\square\, H$ is duplicate-free if and only if $G \neq K_2$ or $H \neq K_2$.*

Proof. By Lemma 5.12, (g, h) is a duplicate of the distinct vertex (g', h') if and only if $N_G(g) = \{g'\}$, $N_G(g') = \{g\}$, $N_H(h) = \{h'\}$, and $N_H(h') = \{h\}$, i.e., $G = H = K_2$. $\square$

Moreover, we can compute the nested chromatic number of Cartesian products of connected graphs. Whereas the chromatic number of the Cartesian product of graphs is the maximum of the chromatic numbers of the factors, the nested chromatic number is close to the product of the nested chromatic numbers of the factors. We recall that $[v]_\sim$ is the equivalence class of duplicate vertices in G, defined in Definition 2.9.

Proposition 5.14. *Let G and H be connected finite simple graphs, neither of which is K_1. If $G \neq K_2$ or $H \neq K_2$, then*

$$\chi_N(G \,\square\, H) = \#V(G \,\square\, H) - \ell'(G) \cdot \ell'(H),$$

where $\ell'(L) = \#\{[v]_\sim \colon v \text{ is a leaf of } L\}$ for a graph L.

In particular, if G or H has minimum vertex degree of at least 2, then $\chi_N(G \,\square\, H) = \#V(G \,\square\, H)$.

Proof. By Lemma 5.12, $\{(g,h),(g',h')\}$ is a nested independent set of $G \,\square\, H$ if and only if $N_G(g) = \{g'\}$ and $N_H(h) = \{h'\}$. Since $G \,\square\, H$ is duplicate-free by Corollary 5.13, no colour class can contain more than two vertices.

Let L be the set of all colour classes of two vertices. Every nested colouring of $G \,\square\, H$ consists of subset of L of pairwise disjoint elements together with singleton sets of the remaining vertices. In particular, $\chi_N(G \,\square\, H)$ is $\#V(G \,\square\, H)$ minus the largest subset of L that consists of pairwise disjoint elements.

Since the weak duplicate vertex of each element of L is unique, selection of a subset of L of pairwise disjoint elements depends only on the weakly duplicated vertex of each element of L. In particular, if (g,h) is the weak duplicate vertex of an element of L, then no other element of L with weak duplicate vertex (i,j) such that $[g]_\sim = [i]_\sim$ and $[h]_\sim = [j]_\sim$ can be in such a disjoint set. Thus the largest subset of L that consists of pairwise disjoint elements is of size $\ell'(G) \cdot \ell'(H)$. $\square$

Example 5.15. The cube graph Q_n is defined recursively by $Q_1 = K_2$ and $Q_n = Q_{n-1} \,\square\, K_2$, so Q_n has no leaves for $n \geq 2$. Hence $\chi_N(Q_n) = 2^n$ if $n \neq 2$ and $\chi_N(Q_2) = 2$. This also follows by Proposition 3.4 since Q_n is n regular and duplicate-free for $n \neq 2$ by Corollary 5.13.

5.6. **Strong product.**

Let G and H be finite simple graphs. The *strong product of G and H* is the graph $G \boxtimes H$ with vertex set $V(G) \times V(H)$, where (g,h) is adjacent to the distinct vertex (g',h') if and only if $g = g'$ or g is adjacent to g' in G, and $h = h'$ and h is adjacent to h' in H. In particular, the open neighbourhood of (g,h) in $G \boxtimes H$ is

$$N_{G \boxtimes H}(g,h) = N_G[g] \times N_H[h] \setminus \{(g,h)\}.$$

Notice that $G \boxtimes K_1$ is isomorphic to G, so $\chi_N(G \boxtimes K_1) = \chi_N(G)$. Moreover, $(G \cup G') \boxtimes H = (G \boxtimes H) \cup (G' \boxtimes H)$. Thus following Section 5.2, we need only consider finite simple graphs G that are connected and have at least two vertices.

With the exception of $G \boxtimes K_1$, the strong product of connected graphs has no weak duplicate vertices.

Lemma 5.16. *Let G and H be connected finite simple graphs, neither of which is K_1. The vertices (g,h) and (g',h') are weak duplicates in $G \boxtimes H$ if and only if $(g,h) = (g',h')$.*

Proof. Since G and H are connected, $N_G(g) \neq \emptyset \neq N_H(h)$.

Suppose (g,h) is a weak duplicate of (g',h'). This implies that $\{g\} \times N_H(h) \subset N_{G \boxtimes H}(g',h')$, and so $g \in N_G[g']$, i.e., $g' \in N_G[g]$. By symmetry, we also have $h' \in N_H[h]$. If $(g,h) \neq (g',h')$, then $(g',h') \in N_{G \boxtimes H}(g,h) \subset N_{G \boxtimes H}(g',h')$, which is absurd. $\square$

Thus the nested chromatic number of the strong product is the number of vertices of the product.

Proposition 5.17. *If G and H are connected finite simple graphs, neither of which is K_1, then $\chi_N(G \boxtimes H) = \#V(G) \cdot \#V(H)$.*

5.7. **Composition.**

Let G and H be finite simple graphs. The *composition* (or *lexicographic product*) *of* G *and* H is the graph $G[H]$ with vertex set $V(G) \times V(H)$, where (g, h) is adjacent to (g', h') if and only if either g is adjacent to g' in G or $g = g'$ and h is adjacent to h' in H. In particular, the open neighbourhood of (g, h) in $G[H]$ is

$$N_{G[H]}(g, h) = N_G(g) \times V(H) \cup \{g\} \times N_H(h).$$

Clearly, composition is non-commutative, in general.

The weak duplicates in the composition come from weak duplicates of the operands.

Lemma 5.18. *Let G and H be finite simple graphs. The vertex (g, h) is a weak duplicate of the distinct vertex (g', h') in $G[H]$ if and only if either $g = g'$ and h is a weak duplicate of h' in H or g is a weak duplicate of g' in G and h is an isolated vertex in H.*

Proof. Suppose $g = g'$. This implies that (g, h) is a weak duplicate of (g', h') if and only if $N_H(h) \subset N_H(h')$, i.e., h is a weak duplicate of h' in H.

Assume $g \neq g'$. Further suppose h is an isolated vertex in H. Thus $N_{G[H]}(g, h) = N_G(g) \times V(H)$, and so (g, h) is a weak duplicate of (g', h') if and only if $N_G(g) \subset N_G(g')$, i.e., g is a weak duplicate of g' in G.

Now suppose h is not an isolated vertex in H, and suppose (g, h) is a weak duplicate of (g', h'). This implies that g is adjacent to g'; hence $\{g'\} \times V(H) \subset N_{G[H]}(g', h')$, i.e., $V(H) \subset N_H(h')$, which is absurd. $\square$

From this we can derive conditions classifying which compositions are duplicate-free.

Corollary 5.19. *Let G and H be finite simple graphs. The graph $G[H]$ is duplicate-free if and only if H is duplicate-free and either*

(i) *H has no isolated vertices, or*
(ii) *H has an isolated vertex and G is duplicate-free.*

Proof. Let (g, h) and (g', h') be distinct vertices of $G[H]$.

Suppose $g = g'$. By Lemma 5.18, (g, h) and (g', h') are duplicates in $G[H]$ if and only if h and h' are duplicates in H.

Now suppose $g \neq g'$. By Lemma 5.18, (g, h) and (g', h') are duplicates in $G[H]$ if and only if h and h' are isolated vertices in H and g and g' are duplicates in G. $\square$

Further, we can bound the nested chromatic number of a graph composition, and equality holds when the secondary graph has no isolated vertices.

Proposition 5.20. *If G and H are finite simple graphs, then*

$$\chi_N(G[H]) \leq \#V(G) \cdot \chi_N(H).$$

Moreover, equality holds if H has no isolated vertices.

Proof. Let $\mathcal{C}$ be a nested colouring $C_1 \cup \cdots \cup C_k$ of H. For each $g \in V(G)$ and for $1 \leq i \leq k$, set $C_{i,g} = \{g\} \times C_i$. By Lemma 5.18, $C_{i,g}$ is a nested independent set of $G[H]$, and so the family $C_{i,g}$ forms a nested colouring of $G[H]$. Hence $\chi_N(G[H]) \leq \#V(G) \cdot \chi_N(H)$.

Assume H has no isolated vertices. If $\{(g_1, h_1), \ldots, (g_t, h_t)\}$ is a nested independent set of $G[H]$, then by Lemma 5.18 $g_1 = \cdots = g_t$ and $\{h_1, \ldots, h_t\}$ forms a nested independent set of H. Thus any nested colouring of $G[H]$ is of the form described in the first paragraph, and so $\chi_N(G[H]) = \#V(G) \cdot \chi_N(H)$. $\square$

We suspect the following question has a negative answer.

Question 5.21. Does there exist a pair of finite simple graphs G and H such that $\chi_N(G[H]) < \#V(G) \cdot \chi_N(H)$?

5.8. **Monotonicity.**

Recall that a graph property is *monotone decreasing* (*monotone increasing*, respectively) if it is preserved under deletion (respectively, addition) of edges. For example, removing an edge can only decrease the chromatic number of a graph, so being k-colourable is a monotone decreasing graph property. However, having a nested k-colouring is neither monotone decreasing or increasing. To see this, we use three of the graph products discussed above.

Let G and H be finite simple graphs, and suppose $\chi_N(H) < \#V(H)$. By construction, we know that $G \times H$ is $G \boxtimes H$ with edges removed and $G \boxtimes H$ is likewise $G[H]$ with edges removed. That is,

$$E(G \times H) \subset E(G \boxtimes H) \subset E(G[H]) \subset E(K_{\#V(G)\cdot\#V(H)}).$$

However, by Propositions 5.10 and 5.20, both $\chi_N(G\times H)$ and $\chi_N(G[H])$ are at most $\#V(G)\cdot \chi_N(H) < \#V(G) \cdot \#V(H)$. Hence using Proposition 5.17 we have that

$$\chi_N(G \times H) < \chi_N(G \boxtimes H) > \chi_N(G[H]) < \chi_N(K_{\#V(G)\cdot\#V(H)}).$$

6. On the existence of graphs

Given integers c and n such that $1 \le c \le n$, it is known that there exists a finite simple graph G on n vertices with $\chi(G) = c$. We show that if we are also given an integer s such that $1 \le c \le s \le n$, then G can be chosen so that $\chi_N(G) = s$ for all but a few specific cases.

For fixed $n \ge 2$, the case when $c \in \{1, n-1, n\}$ was handled in Lemma 2.6. The one other infinite case is that there does not exist a bipartite graph with nested chromatic number 3.

Lemma 6.1. *If G is a bipartite graph, then $\chi_N(G) \ne 3$.*

Proof. Let G be a bipartite graph, and suppose, without loss of generality (see Remark 2.4), that G has no isolated vertices. Suppose $\chi_N(G) \le 3$. Hence by Corollary 5.5 we may assume G is connected, and so G has a unique proper 2-colouring $B \cup W$.

We may assume without loss of generality that W is a nested independent set of G. Let $w_1, \dots, w_t$ be the elements of W ordered such that $N_G(w_{i+1}) \subset N_G(w_i)$. Thus for each $b \in B$ there is a k such that $b \in N_G(w_i)$ if and only if $1 \le i \le k$, i.e., $N_G(b) = \{w_1, \dots, w_k\}$. Hence B is also nested, and $\chi_N(G) = 2$. $\square$

We are ready to give the classification.

Theorem 6.2. *Let c, s, and n be integers such that $1 \le c \le s \le n$. There does not exist a finite simple graph G on n vertices with $\chi(G) = c$ and $\chi_N(G) = s$ if and only if one of the following conditions holds:*

(i) $c = 1$ *and* $s > 1$,
(ii) $c = 2$ *and* $s = 3$,
(iii) $c = 2$ *and* (n, s) *is one of* $(4, 4)$, $(5, 5)$, $(6, 5)$, *and* $(7, 7)$, *or*
(iv) $c = n - 1$ *and* $s = n$.

Moreover, if such a graph G exists, then it may be chosen to be connected.

Proof. If $c = s = n = 1$, then $G = K_1$. Suppose $n \geq 2$. If $c = 1$, $c = n-1$, or $c = n$, then by Lemma 2.6 there exists a finite simple graph G on n vertices with $\chi(G) = c$ and $\chi_N(G) = s$ if and only if $s = c$. Hence we may also suppose $2 \leq c \leq n-2$.

By Lemma 6.1 if $c = 2$, then $s \neq 3$. Moreover, checking the 143 bipartite graphs with between 4 and 7 vertices shows that if (n, s) is one of $(4, 4)$, $(5, 5)$, $(6, 5)$, and $(7, 7)$, then there is no finite simple graph G on n vertices with $\chi(G) = c$ and $\chi_N(G) = s$. Thus the conditions (i)–(iv) each imply the absence of the desired graph.

Moreover, checking the 1251 simple graphs with between 2 and 7 vertices, we see that, except for the conditions (i)–(iv), the desired (connected) simple graphs do indeed exist.

To show the presence of the desired graphs in the remaining case, we proceed by induction on the number of vertices n.

Base case: Suppose $n = 8$. Checking the 12346 (11117 of which are connected) simple graphs on 8 vertices, we see that there exists a (connected) simple graph with $\chi(G) = c$ and $\chi_N(G) = s$ for $2 \leq c \leq s \leq 6$, with the exception of $c = 2$ and $s = 3$.

Inductive step: Suppose $n \geq 9$. By induction, there exists a connected simple graph G on $n-1$ vertices with $\chi(G) = c$ and $\chi_N(G) = s$ for $2 \leq c \leq s \leq n-1$, except for $(c, s) = (2, 3)$ and $(c, s) = (n-2, n-1)$. If we duplicate any vertex of G, then the resulting connected graph G' has n vertices, $\chi(G') = \chi(G)$, and $\chi_N(G') = \chi(G)$ since the duplicate vertex can always be put in the same colour class as the duplicated vertex. If we add a dominating vertex to G, then the resulting connected graph G'' has n vertices and $\chi(G'') = \chi(G) + 1$. Moreover, $\chi_N(G'') = \chi(G) + 1$ by Proposition 5.6. Together these two operations generate the desired (connected) graph for all relevant c and s, except $c = 2$ and $s = n$.

If n is even, then $\chi(C_n) = 2$ and $\chi_N(C_n) = n$, by Corollary 3.7. If n is odd, then consider the graph H found by adding the vertex 0 and the edges $\{0, 1\}$ and $\{0, 5\}$ to C_{n-1}. Clearly, H is a connected simple graph on n vertices. Moreover, $\chi(H) = 2$, since the partition of the vertices into even and odd vertices is a proper 2-colouring of H. Further still, the neighbourhoods of H are: $N_H(0) = \{1, 5\}$, $N_H(1) = \{0, n-1, 2\}$, $N_H(5) = \{0, 4, 6\}$, $N_H(n-1) = \{1, n-2\}$, and $N_H(i) = \{i-1, i+1\}$ for $1 < i < n-1$ and $i \neq 5$. Thus no two vertices of H are weak duplicates, and so $\chi_N(H) = n$. $\square$

See Remark 2.22 for comments about using computer algebra systems to determine the nested chromatic number of a finite simple graph.

In Remark 3.8, we noted that the nested chromatic number for a planar graph need not be bound above by four, as is the chromatic number. Indeed, we show that every possible nested chromatic number can occur for a connected planar graph.

Proposition 6.3. *Let $n \geq 2$. For $2 \leq k \leq n$, there exists a connected planar simple graph G on n vertices with $\chi_N(G) = k$.*

Proof. Let G be the graph K_k if $2 \leq k \leq 4$, otherwise let G be the graph C_k if $k \geq 5$. Then clearly G is a connected planar graph with $\chi_N(G) = k$ by Lemma 2.6 or Corollary 3.7, respectively.

Without loss of generality, let $V(G) = \{1, \ldots, k\}$, and suppose $k-1$ and k are adjacent. Modify G by adding $n-k$ new vertices $\{k+1, \ldots, n\}$ and $n-k$ new edges $\{k-1, i\}$, where $k+1 \leq i \leq n$, to create the graph G'. Clearly, G' is a connected planar graph, as the new vertices are all leaves on the planar graph G. Further still, $\{1\} \uplus \cdots \uplus \{k-1\} \uplus \{k, \ldots, n\}$ is a nested colouring of G'. Thus $\chi_N(G') = \chi_N(G) = k$, by Proposition 4.1. $\square$

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