Euler constant as a renormalized value of Riemann zeta function at its pole. Rationals related to Dirichlet *L*-functions

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Abstract

This paper has two aims. Since there is more than one way to regularize the value of Riemann zeta function at $s = 1$, I wanted to provide a simple and striking illustration of Euler's constant as being the true renormalized value of Riemann zeta function at its pole. The other aim is to show how sequences of rationals, often the same, arise in computations related to Dirichlet *L*-functions 'at infinity'. A connection with the Liouville function seems to be found and we are led to ask about the possible usefulness of some extensions of this function to rationals.

Keywords: Gamma function, Riemann zeta function, Euler constant, Euler-Lehmer constants, Dirichlet *L*-series, generalized harmonic numbers, Liouville λ function, Thue-Morse sequence

1. Motivation

Euler's constant viewed as a renormalized value of Riemann zeta function at $s = 1$ is still 'a subject of active research' since Stephen Hawking has published in 1977 *Zeta function regularization of path integrals in curved spacetime, (Comm. Math. Phys. 55).*

The paragraphs 2.1 and 2.2 of this paper have mainly an esthetic motivation. We provide a family of expansion formulae none of which is around $z = 1$, and whose summarizing into one single formula is nevertheless much more *illustrative* for this issue than just the convention of taking as a renormalized value the constant term in the Laurent expansion of Riemann zeta function near $s = 1$, which (as pointed out by Jeffrey C. Lagarias) 'also matches the constant term in the Taylor expansion of the digamma function $\psi(z)$ around $z = 1$'.

For the chapters 5-8 to the end, the real motivation is the following: since not only we know the distribution of primes for the first few billions of them, but we know them all, so to say, by their names, it seemed interesting to try to grasp something about primes starting the other way around, i.e., if possible, from infinity. Since, due to the Euler product formula, Riemann zeta function is about primes not just in the critical strip, but virtually anywhere, I tried to compute asymptotic expansions of functions of the form $1/(L(s)-1)$ in the neighborhoods of infinity, where L is either the Riemann zeta function or some other Dirichlet *L*-function:

To explain the irregular features of these expansions, Legendre, Jacobi and Kronecker symbols seemed of no avail. The sequences of irregular signs and some other features of the rationals which arise in a large class of expansions of this type can be described in terms of the Liouville lambda function (adapted to a class of fractions). I formulate a not very obvious connection between these expansions and the Liouville lambda function (known to be directly related both to primes and to Riemann zeta function) taking into account the need of more available data of high accuracy. The studied expansions, where s and the fast growing function $1/(\varsigma(s)-1)$ both go to infinity, are made of either surprisingly small primes or greater numbers with surprisingly small prime factors, very often integer powers of such primes. One can say that the whole study is about the way small numbers are mirrored in an 'infinity' related to the ς function (or to some other Dirichlet *L*-function. The tenuous link between the two themes treated in this paper lies in the possibility to express renormalized values of *L*-functions at $s = 1$ in terms of iterated *L*-functions and of powers of rationals.

2. Expanding lnΓ near the negative poles of the Gamma function

Near 0, lnΓ may be written as

$$\ln\Gamma(x) = -\ln x - \gamma x + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}(-x)^k \qquad (\clubsuit)$$

If $'\zeta(1)' = \gamma$ then it would have been possible to drag the second summand in the RHS under the infinite summation, thus beginning it from k =1.
To make this analogy more convincing, we mean completely unambiguous, one may consider lnΓ near the negative poles of the Gamma function.
We shall now 'discover' a formula where the analogy becomes much more obvious, because it is established not just between Euler's constant and the values of zeta function at greater integer arguments, but between an extended part of a formula, which is indexed on some n and on some k, and where $\varsigma(k)$ appear in company of harmonic numbers of order k while γ appears in company of ordinary harmonic numbers.

One has (for small $x > 0$):

$$\ln\Gamma(-1-x) = -\ln x + (\gamma - 1)x + \sum_{k=2}^{\infty} \frac{\zeta(k) + (-1)^k}{k} x^k \qquad (\spadesuit)$$

In the real domain, one has for all positive integers n:

$$\ln\Gamma\left(-n + (-1)^n x\right) = -\ln x - \sum_{j=1}^{n} \ln j +$$

$$+(-1)^{n+1}(\gamma - H_n)x + \sum_{k=2}^{\infty} (-1)^{(n+1)k} \frac{\zeta(k) + (-1)^k \sum_{m=1}^{n} \frac{1}{m^k}}{k} x^k \qquad (\heartsuit)$$

(which we call 'the second Iohannis formula')
Here again, under the convention $\varsigma(1) = \gamma$, in the RHS of (♥) the third summand might be pulled under the infinite summation, which, doing so, would start with $k = 1$:

$$\ln\Gamma\left(-n + (-1)^n x\right) = -\ln x - \sum_{j=1}^{n} \ln j + \sum_{k=1}^{\infty} (-1)^{(n+1)k} \frac{\zeta(k) + (-1)^k \sum_{m=1}^{n} \frac{1}{m^k}}{k} x^k$$

Note that (♥) is not based on any prior renormalization of Γ or lnΓ at their poles.
Note also that ***in the numerator***, as k runs through the natural integers, the signs of the second summand alternate for any n. But the signs of the terms of the ***whole infinite sum*** alternate only for even n. As we said, the analogy between Euler's constant viewed as '$\varsigma(1)$' and the values of ς at greater integer arguments is strengthened by the analogy between Harmonic numbers and Generalized harmonic numbers of higher orders in the numerator of the fraction under the second summation symbol, let alone by the exponent of -1, indexed also on k and n. (We consider lnΓ only in the real domain, avoiding possible but unnecessary discussions in this context about branches in the complex domain inherited from the logarithmic function.)

2.1. Euler's constant and ζ(2) expressed in terms of Log Γ

In the general case, for non-negative *n*, we have:

$$\lim_{x\to 0}\left\{\frac{\ln(n!x)+\ln\Gamma\left(-n+(-1)^n x\right)}{x}\right\}=(-1)^{n+1}(\gamma-H_n) \qquad (1°)$$

Replacing $(-1)^n$ and ln –Γ by $\ln|\Gamma(x)|$ we'll have

$$\lim_{x\to 0}\left\{\frac{\ln(n!x)+\ln|\Gamma(-n\pm x)|}{x}\right\}=\pm(\gamma-H_n) \qquad (2°)$$

The RHS of this limit identity might be written as

$$\pm\left(''\zeta(1)''-\sum_{k=1}^{n}\frac{1}{k}\right) \qquad (3°)$$

(quotation marks needed, because thinking of Euler's constant as "ζ(1)" is, in some sense, a metaphor. As generally known, the Riemann zeta function has a singularity in 1, which is not removable, and which has residue 1. The philosophical and semiotical problem of metaphors in mathematics, and more generally in science, will not be discussed here.)

One can notice that although the formulae

$$\lim_{x\to 0}\left\{\frac{\ln(n!x)+\ln|\Gamma(-n+x)|}{x}\right\}=-(\gamma-H_n) \qquad (4°)$$

and

$$\lim_{x\to 0}\left\{\frac{\ln(n!x)+\ln|\Gamma(-n-x)|}{x}\right\}=+(\gamma-H_n) \qquad (5°)$$

state convergence to numbers which are identical in absolute value, an arbitrarily chosen sequence of *x* that converges to 0, does not yield identical sequences (in absolute value) in the expressions in the LHS of (4°) and of (5°).

One can effectively compare these sequences adding the expressions that stand under the limit in (4°) and in (5°) and then dividing the result by *x*. (In other words, writing in the numerator the product of the expressions whose logarithm is taken, and writing in the denominator the product of the denominators.) Doing so, for *n*=0, one can find

$$\lim_{x\to 0}\left\{\frac{2\ln x \ +\ln|\Gamma(-x)|+\ln|\Gamma(x)|}{x^2}\right\}=\zeta(2)=\frac{\pi^2}{6} \tag{6°}$$

and, for $n = 1$,

$$\lim_{x\to 0}\left\{\frac{2\ln x \ +\ln|\Gamma(-1-x)|+\ln|\Gamma(-1+x)|}{x^2}\right\}=\zeta(2)+1 \tag{7°}$$

In the general case, one has:

$$\lim_{x\to 0}\left\{\frac{2\ln(n!x)+\ln|\Gamma(-n-x)|+\ln|\Gamma(-n+x)|}{x^2}\right\}=\zeta(2)+\sum_{k=1}^{n}\frac{1}{k^2} \tag{8°}$$

and, therefore,

$$\lim_{\substack{x\to 0\\ n\to\infty}}\left\{\frac{2\ln(n!x)+\ln|\Gamma(-n-x)|+\ln|\Gamma(-n+x)|}{x^2}\right\}=2\zeta(2) \tag{9°}$$

“Comparing” the process of convergence at two consecutive singularities of the gamma function we get:

$$\lim_{x\to 0}\left\{\frac{1}{x^2}\left[1-\frac{2\ln(n!x)+\ln|\Gamma(-n-x)|+\ln|\Gamma(-n+x)|-x^2\sum_{k=1}^{n}\frac{1}{k^2}}{2\ln[(n+1)!x]+\ln|\Gamma(-n-1-x)|+\ln|\Gamma(-n-1+x)|-x^2\sum_{k=1}^{n+1}\frac{1}{k^2}}\right]\right\}$$

$$=\frac{3}{(n+1)^4\pi^2}=\frac{1}{2(n+1)^4\zeta(2)}$$

The following limit formula holds as well:

$$\lim_{x\to 0}\left\{\frac{1}{x^2}\ln\left(\frac{2\ln(n!x)+\ln|\Gamma(-n-x)|+\ln|\Gamma(-n+x)|-x^2\sum_{k=1}^{n}\frac{1}{k^2}}{2\ln[(n+1)!x]+\ln|\Gamma(-n-1-x)|+\ln|\Gamma(-n-1+x)|-x^2\sum_{k=1}^{n+1}\frac{1}{k^2}}\right)\right\}$$

$$=-\frac{1}{2(n+1)^4\zeta(2)}$$

In this limit formula $n!$ and $(n+1)!$ appear as absolute values of the residues in the singularity $-n$ and $-(n+1)$ of the gamma function (see also [3]).

3. Integer powers of ln 2 expressed in terms of lnζ and lnη

$$\lim_{x\to\infty}\left\{\frac{\ln\zeta(\ln\zeta(x)+1)}{x}\right\}=\lim_{x\to\infty}\left\{\frac{\ln\zeta(\zeta(x))}{x}\right\}=\ln 2$$

Let $f_1(x)=\ln\zeta(\ln\zeta(x)+1)$ and $f_2(x)=\ln(\ln\zeta(x))$

then, trivially, for any n

$$\lim_{x\to\infty}\left\{\frac{f_1^n(x)}{x}\right\}=\lim_{x\to\infty}\left\{\frac{f_2^n(x)}{x}\right\}=(\ln 2)^n$$

and, somewhat less trivially,

$$\lim_{x\to\infty}\{2^x[\ln\zeta(\ln\zeta(x)+1)-\ln\zeta(\zeta(x))]\}=\frac{1}{2}$$

The Dirichlet eta function is defined as

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\ldots$$

a series that converges for any complex number with real part greater than 0.

Define $g(x)=|\ln|\ln\eta(x)||$

Then for any $n \geq 0$ $\lim_{x\to\infty}\left\{\frac{g^n(x)}{x}\right\} = (\ln 2)^n$

Define $h(x) = \ln|\zeta(\eta(x))|$

then, again (more or less trivially), for any $n \geq 0$ $\lim_{x\leftarrow\infty}\left\{\frac{h^n(x)}{x}\right\} = (\ln 2)^n$

4. THE ITERATION OF A PAIR OF FUNCTIONS "PRESERVING ASYMPTOTIC PROPORTIONS AT INFINITY"

The nice thing is that for any $N \geq 1$ we have the following general formula, which holds for all iterates of g and h (of strictly positive order):

$$\lim_{x\to\infty}\left\{2^{x(\ln 2)^{n-1}}\left[g^n(x) - h^n(x)\right]\right\} = \gamma - \frac{1}{2} \qquad (✪)$$

$$\lim_{x\to\infty}\left\{2^{x-1}\left[f(x) - h(x)\right]\right\} = \gamma \qquad (✷)$$

but no formula is known to us concerning higher order iterates of f and h, let alone concerning relations that involve higher order iterates of f and g.

Another beautiful formula involving iterates of g and of h is the following one:

Let $$\omega(x,n) = 2^{x(\ln 2)^{n-1}}\left[g^n(x) - h^n(x)\right] \qquad (❖)$$

then again, for any positive integer n, we'll have

$$\lim_{x\to\infty}\left\{\frac{\omega(x,n+1) - \omega(x,n)}{\ln\omega(x,n+1) - \ln\omega(x,n)}\right\} = \gamma - \frac{1}{2} \qquad (❈)$$

5. SEQUENCES OF RATIONALS ARISING IN CONNECTION WITH DIRICHLET *L*-SERIES

Trying to better understand (✪), (✷), (❖) and (❋), I have begun expanding 1/(ς(*x*) -1) for large values.
Having found that, for sufficiently remote *x* values,

$$\frac{1}{\zeta(x)-1}=2^x-\left(\frac{4}{3}\right)^x-1+\left(\frac{8}{9}\right)^x-\left(\frac{4}{5}\right)^x+\left(\frac{2}{3}\right)^x-\left(\frac{16}{27}\right)^x-\left(\frac{4}{7}\right)^x+$$

$$+2\left(\frac{8}{15}\right)^x-2\left(\frac{4}{9}\right)^x+\left(\frac{2}{5}\right)^x+\left(\frac{32}{81}\right)^x+2\left(\frac{8}{21}\right)^x-\left(\frac{4}{11}\right)^x-3\left(\frac{16}{45}\right)^x+\left(\frac{8}{25}\right)^x-$$

$$-\left(\frac{4}{13}\right)^x+3\left(\frac{8}{27}\right)^x+\left(\frac{2}{7}\right)^x-3\left(\frac{4}{15}\right)^x-\left(\frac{64}{243}\right)^x-\dots \qquad (♚)$$

I filled the first five or six of denominators in (♚) as an entry search in OEIS, and learned that this expansion was already known to Benoît Cloitre (A112932), to whom I express my admiration for his entire work. (I just added several more terms to his expansion; for even more terms, see below, page 14 of this paper.)

The derived Cloitre's formula

$$\lim_{x\to\infty}\left\{\zeta(\zeta(x))-2^x+\left(\frac{4}{3}\right)^x+1\right\}=\gamma \qquad (♛)$$

is surprising for who doesn't know (♚). In fact there are uncountable analogues of this limit formula (involving or not Euler's constant).
Our first observation is that several functions have an expansion with, up to the constant term, the same beginning:

$$\frac{1}{\ln\zeta(s)}=2^s-\left(\frac{4}{3}\right)^s-\frac{1}{2}+\left(\frac{8}{9}\right)^s-\left(\frac{4}{5}\right)^s+\left(\frac{2}{3}\right)^s-\left(\frac{16}{27}\right)^s-\left(\frac{4}{7}\right)^s+2\left(\frac{8}{15}\right)^s-\dots$$

$$\frac{1}{\ln(\ln\zeta(x)+1)}=2^x-\left(\frac{4}{3}\right)^x-0+\left(\frac{8}{9}\right)^x-\left(\frac{4}{5}\right)^x+\left(\frac{2}{3}\right)^x-\left(\frac{16}{27}\right)^x-\left(\frac{4}{7}\right)^x+2\left(\frac{8}{15}\right)^x-\dots$$

(with a gain of 1/2 in the constant term for each substitution of *x* by lnς(*x*)+1)
The next example of substitution (which we restrain from writing down) yields the following analogue of (♛):

$$\lim_{x\to\infty}\left\{\zeta[\ln[\ln(\ln\zeta(x)+1)+1]+1]-2^x+\left(\frac{4}{3}\right)^x-\frac{1}{2}\right\}=\gamma$$

The natural question about these expansions is whether other reciprocals of modified Dirichlet L-functions permit similar asymptotic expressions in terms of sums of powers of rationals. Here are but a few examples for the most popular Dirichlet eta, beta end lambda L-functions:

$$\frac{1}{1-\eta(x)}=2^x+\left(\frac{4}{3}\right)^x-1+\left(\frac{8}{9}\right)^x+\left(\frac{4}{5}\right)^x-3\left(\frac{2}{3}\right)^x+\left(\frac{16}{27}\right)^x+$$

$$+\left(\frac{4}{7}\right)^x+2\left(\frac{8}{15}\right)^x-4\left(\frac{4}{9}\right)^x-3\left(\frac{2}{5}\right)^x+\dots \qquad (♔)$$

As one can see, in this peculiar case, the rationals are the same, but the coefficients (and in particular the signs) are not. (Further terms in the expansion of η are not similar to those of ς.) The presence of powers of 2 in the numerators cannot escape our attention, nor did the presence of powers of 3 in the two following examples:

Let β be the Dirichlet beta function, defined as

$$L(s,\chi_{-2})=\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

Then, for large s

$$\frac{1}{1-\beta(s)}=3^s+\left(\frac{9}{5}\right)^s-\left(\frac{9}{7}\right)^s+\left(\frac{27}{25}\right)^s+1-\left(\frac{9}{11}\right)^s-2\left(\frac{27}{35}\right)^s+$$

$$+\left(\frac{9}{13}\right)^s+\left(\frac{81}{125}\right)^s+\left(\frac{3}{5}\right)^s+\left(\frac{27}{49}\right)^s+\left(\frac{9}{17}\right)^s-2\left(\frac{27}{55}\right)^s-\left(\frac{9}{19}\right)^s+\dots \qquad (♕)$$

Let $\quad L(s,\chi_2)=\lambda(s)\equiv\sum_{n=0}^{\infty}\frac{1}{(2n+1)^s}$

Then, for large s

$$\frac{1}{\lambda(s)-1}=3^s-\left(\frac{9}{5}\right)^s-\left(\frac{9}{7}\right)^s+\left(\frac{27}{25}\right)^s-1-\left(\frac{9}{11}\right)^s+2\left(\frac{27}{35}\right)^s-$$

$$-\left(\frac{9}{13}\right)^s-\left(\frac{81}{125}\right)^s+\left(\frac{3}{5}\right)^s+\left(\frac{27}{49}\right)^s-\left(\frac{9}{17}\right)^s+2\left(\frac{27}{55}\right)^s-\left(\frac{9}{19}\right)^s-3\left(\frac{81}{175}\right)^s$$

$$+\left(\frac{3}{7}\right)^s+2\left(\frac{27}{65}\right)^s-\left(\frac{9}{23}\right)^s+\left(\frac{243}{625}\right)^s-2\left(\frac{27}{75}\right)^s+2\left(\frac{27}{77}\right)^s-2\left(\frac{81}{245}\right)-\dots \qquad (♖)$$

The last expansion formula yields, for example, the two following limit formulae (and many others):

$$-\lim_{x\to\infty}\left\{\Gamma(\lambda(x)-1)-3^{x}+\left(\frac{9}{5}\right)^{x}+\left(\frac{9}{7}\right)^{x}-\left(\frac{27}{25}\right)^{x}+1\right\}=\gamma$$

Since $\quad \lim_{x\to 0}\left\{\lambda(1+x)-\frac{1}{2x}\right\}=\frac{1}{2}(\gamma+\ln 2)$

we'll have, using just the first terms of (♖):

$$\lim_{x\to\infty}\left\{\lambda(\lambda(x))-\frac{1}{2}\left[3^{x}-\left(\frac{9}{5}\right)^{x}-\left(\frac{9}{7}\right)^{x}+\left(\frac{27}{25}\right)^{x}-1\right]\right\}=\frac{1}{2}(\gamma+\ln 2)$$

The RHS of this formula 'happens' to be one of the Euler-Lehmer constants, namely γ(1,2) (see Lehmer's paper, [10]). One can add more terms from (♖) into the brackets in the LHS of this limit formula and that will speed up the convergence. The fraction (1/2) before the brackets derives from the following limit:

$$\lim_{x\to 0}\left\{\lambda(1+x)-\frac{1}{2x}\right\}=\frac{1}{2}(\gamma+\ln 2)$$

(this is the renormalized value of λ at $s=1$)

Now the real question is: given a descending (by absolute value) infinite sequence of rationals $(q)_{n\in N}$, given a sequence of integer coefficients (a_n) and given the corresponding infinite series

$$V(x)=a_1 q_1^{x}+a_2 q_2^{x}+\ldots$$

to what class or subclass of functions belongs $1/V(x)+C$? (where C is a constant, possibly 0)

The class of these functions does not coincide with the Dirichlet *L*-functions. Surprisingly, some irregular series give birth to similar sequences of $(q)_{n\in N}$ as the Riemann zeta function does, although not with same coefficients (a_n).

Here are two examples: $\quad \vartheta(s)=\sum_{p} p^{-s}$

(the prime Zeta function where the sum is taken over all primes.) We have:

$$\frac{1}{\vartheta(s)}=2^{s}-\left(\frac{4}{3}\right)^{s}+\left(\frac{8}{9}\right)^{s}-\left(\frac{4}{5}\right)^{s}-\left(\frac{16}{27}\right)^{s}-\left(\frac{4}{7}\right)^{s}+2\left(\frac{8}{15}\right)^{s}+\left(\frac{32}{81}\right)^{s}$$

$$+2\left(\frac{8}{21}\right)^{s}-\left(\frac{4}{11}\right)^{s}-3\left(\frac{16}{45}\right)^{s}+\left(\frac{8}{25}\right)^{s}-\left(\frac{4}{13}\right)^{s}-\left(\frac{64}{243}\right)^{s}-3\left(\frac{16}{63}\right)^{s}+2\left(\frac{8}{33}\right)^{s}+\ldots \qquad (♞)$$

$$+4\left(\frac{32}{135}\right)^{s}-\left(\frac{4}{17}\right)^{s}+2\left(\frac{8}{35}\right)^{s}-3\left(\frac{16}{75}\right)^{s}-\left(\frac{4}{19}\right)^{s}+2\left(\frac{8}{39}\right)^{s}+\left(\frac{8}{45}\right)^{s}\ldots$$

Comparing (♞) with (♚), one finds among the initial terms of these asymptotic expansions exactly the same summands just with the 3rd, 6th, 10th, 11th, 18th, 19th, 20th, 27th, 31st, 32nd, 33rd… terms skipped, namely -1, (2/3)^*x,* -2(4/9)^*x,* (2/5)^*x,* 3(8/27)^*x,* (2/7)^*x,* -3(4/15)^*x*, etc… The nice thing is that (♞) and (♚) seem to obey to a similar law related to the Liuoville lambda function. This issue will be discussed later.
Here are two more example of irregular series:

let $$\daleth(s)=\sum_{n\in A} n^{-s}-\sum_{m\in B} m^{-s}$$

where A is the set of numbers which correspond to the ranks of zeros in the Thue-Morse sequence, whereas B corresponds to the set of the ranks of 1 in the same sequence.
Then, for large x

$$\frac{1}{1-\daleth(x)}=2^x-\left(\frac{4}{3}\right)^x+1+\left(\frac{8}{9}\right)^x-\left(\frac{4}{5}\right)^x-\left(\frac{2}{3}\right)^x-\left(\frac{16}{27}\right)^x+\left(\frac{4}{7}\right)^x+$$

$$+2\left(\frac{8}{15}\right)^x-\left(\frac{2}{5}\right)^x+\left(\frac{32}{81}\right)^x-2\left(\frac{8}{21}\right)^x+\left(\frac{4}{11}\right)^x-3\left(\frac{16}{45}\right)^x+\left(\frac{8}{25}\right)^x+\left(\frac{4}{13}\right)^x\ldots$$

In comparison to the expansion of 1/(ς(*x*)-1), this one has irregularly changed signs, one missing summand, namely 2(4/9)^*x* and one missing coefficient — in front of (8/27)^*x*.

Let $$\beth(s)=\frac{1}{2^s}+\sum_{\substack{n\in A\\ n>2}} n^{-s}-\sum_{\substack{m\in B\\ m>2}} m^{-s}$$ We have, for large *s*:

$$\frac{1}{\beth(s)}=2^s+\left(\frac{4}{3}\right)^s-1+\left(\frac{8}{9}\right)^s+\left(\frac{4}{5}\right)^s-3\left(\frac{2}{3}\right)^s+\left(\frac{16}{27}\right)^s-\left(\frac{4}{7}\right)^s+2\left(\frac{8}{15}\right)^s+2\left(\frac{1}{2}\right)^s\ldots$$

Another example is provided by 1/(1-1/ς(*s*))= ς(*s*)/(ς(*s*)-1)

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$$ (where μ is the aperiodic Möbius function. The expansion of ς(*s*)/(ς(*s*)-1) is exactly (♚) but without the constant term -1).

Turning back to Dirichlet L-series, define the function:

$$\chi_{-3}(n)=\begin{cases}1 & \iff \quad n\equiv 1 \pmod 3\\ 0 & \iff \quad n\equiv 0 \pmod 3\\ -1 & \iff \quad n\equiv 2 \pmod 3\end{cases}$$

Let $$L(s,\chi_{-3})=\sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s}=1-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{5^s}+\ldots$$

Then, for large *s*

$$\frac{1}{1-L(s,\chi_{-3})}=2^s+1-\left(\frac{4}{5}\right)^s+\left(\frac{4}{7}\right)^s-\left(\frac{2}{5}\right)^s-\left(\frac{4}{11}\right)^s+\left(\frac{8}{25}\right)^s+$$
$$+\left(\frac{4}{13}\right)^s+\left(\frac{2}{7}\right)^s-\left(\frac{4}{17}\right)^s-2\left(\frac{8}{35}\right)^s+\left(\frac{4}{19}\right)^s-\left(\frac{2}{11}\right)^s-\left(\frac{4}{23}\right)^s$$
$$+\left(\frac{8}{49}\right)^s+2\left(\frac{4}{25}\right)^s+\dots$$ (♝)

It seems that in the RHS of (♝) there are no denominators ≡ 0 (mod 3)… Those precisely which display this feature in (♚), are skipped in (♝).
One finds for

$$L(s,\chi_{-5})=\frac{1}{1^s}-\frac{1}{2^s}-\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{6^s}-\frac{1}{7^s}-\frac{1}{8^s}+\frac{1}{9^s}+\frac{1}{11^s}-\dots$$

the asymptotical expansion

$$\frac{1}{1-L(s,\chi_{-5})}=2^s-\left(\frac{4}{3}\right)^s+1+\left(\frac{8}{9}\right)^s-\left(\frac{2}{3}\right)^s-\left(\frac{16}{27}\right)^s-\left(\frac{4}{7}\right)^s+2\left(\frac{4}{9}\right)^s+$$
$$+\left(\frac{32}{81}\right)^s+2\left(\frac{8}{21}\right)^s+\left(\frac{4}{11}\right)^s-\left(\frac{4}{13}\right)^s+3\left(\frac{8}{27}\right)^s-\dots$$ (◒)

Here again, there are no denominators divisible by 5.It seems that coefficients larger than 1 in absolute value show up often 1) when the denominators have at least two distinct factors 2) when the numerator and denominator are the same power of two different primes.
We have, for the two Dirichlet characters modulo 6, the beginnings of the two following expansions.

Let $$L(s,\chi_6)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\dots$$

Then, for large *s*

$$\frac{1}{L(s,\chi_6)-1}=5^s-\left(\frac{25}{7}\right)^s+\left(\frac{125}{49}\right)^s-\left(\frac{25}{11}\right)^s-\left(\frac{25}{13}\right)^s-\left(\frac{625}{343}\right)^s+$$
$$+2\left(\frac{125}{77}\right)^s-\left(\frac{25}{17}\right)^s+2\left(\frac{125}{91}\right)^s-\left(\frac{25}{19}\right)^s+\left(\frac{3125}{2401}\right)^s-3\left(\frac{625}{539}\right)^s-$$
$$-\left(\frac{25}{23}\right)^s+2\left(\frac{125}{119}\right)^s+\left(\frac{125}{121}\right)^s-1-3\left(\frac{625}{637}\right)^s\dots$$ (♟)

Let $$L(s,\chi_{-6}) = 1 - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{11^s} + \frac{1}{13^s} - \dots$$

Then, for large s

$$\frac{1}{1-L(s,\chi_{-6})} = 5^s + \left(\frac{25}{7}\right)^s + \left(\frac{125}{49}\right)^s - \left(\frac{25}{11}\right)^s + \left(\frac{25}{13}\right)^s + \left(\frac{625}{343}\right)^s -$$

$$-2\left(\frac{125}{77}\right)^s - \left(\frac{25}{17}\right)^s + 2\left(\frac{125}{91}\right)^s + \left(\frac{25}{19}\right)^s + \left(\frac{3125}{2401}\right)^s - 3\left(\frac{625}{539}\right)^s -$$

$$-\left(\frac{25}{23}\right)^s - 2\left(\frac{125}{119}\right)^s + \left(\frac{125}{121}\right)^s + 1 + 3\left(\frac{625}{637}\right)^s - \dots$$ (♙)

As one can see, (♟) and (♙) are similar up to some signs. The rationals the expansions (♟) and (♙) are made of are products of the small primes (with small exponents) which appear in the Dirichlet L-series $L(s,\chi_6)$ and $L(s,\chi_{-6})$: 7, 11, 13, 17, 19, etc.
Due to the structure of χ, the numerators are always powers of 5; in turn, even as a factor, 3 does not appear in the denominators, where one can find the next primes: 7, 11, 13, 17, 19, etc. either as a factor or as an integer power (e.g. $343=7^3$ and $2401=7^4$). Examples of factors: $539=7^2*11$, $637=7^2*13$). The way new bigger primes appear in the denominators and then generate the next ones might be an interesting process. For example, the summands of rank 6, 7, 8 and 9 are products of the second summand — (25/7)^s — by, respectively, the summands of rank 3, 4, 5 divided by 5. Etc. etc.
I will finish this chapter with the following remark: in (♚), which is about Riemann zeta function, the integer coefficients greater than 1 appear only near fractions that either have a composite denominator with at least two distinct prime factors or near a fraction whose numerator and denominator are exactly the same power (greater than 1) of 2 and, respectively, 3. In the known terms of (♚), if one neglects the constant term, the (odd) primes appear for the first time in the denominators exactly with their own density multiplied by 3/2. As will be shown in the next chapter after 13, this frequency slows down.

6. TWO COMPARISONS

For the two real Dirichlet characters modulo 10, the rationals related to $1/(L_{10}—1)$ are easily computable (up to a certain point).

For χ(1)=1, χ(3)=1, χ(7)=1, χ(9)=1 (otherwise 0), we have

$$\frac{1}{L(s,\chi_{10})-1} = 3^s - \left(\frac{9}{7}\right)^s - 1 - \left(\frac{9}{11}\right)^s - \left(\frac{9}{13}\right)^s + \left(\frac{27}{49}\right)^s - \left(\frac{9}{17}\right)^s$$

$$-\left(\frac{9}{19}\right)^s + \left(\frac{3}{7}\right)^s - \left(\frac{9}{23}\right)^s + 2\left(\frac{27}{77}\right)^s - \left(\frac{9}{29}\right)^s + 2\left(\frac{27}{91}\right)^s - \left(\frac{9}{31}\right)^s$$

$$+\left(\frac{3}{11}\right)^s - \left(\frac{9}{37}\right)^s - \left(\frac{81}{343}\right)^s + \left(\frac{3}{13}\right)^s + 2\left(\frac{27}{119}\right)^s + \left(\frac{27}{121}\right)^s - \left(\frac{9}{41}\right)^s$$

$$-\left(\frac{9}{43}\right)^s + 2\left(\frac{27}{133}\right)^s - \left(\frac{9}{47}\right)^s + 2\left(\frac{27}{143}\right)^s - 2\left(\frac{9}{49}\right)^s + \left(\frac{3}{17}\right)^s - \left(\frac{9}{53}\right)^s$$

$$+2\left(\frac{27}{161}\right)^s + \left(\frac{27}{169}\right)^s + \left(\frac{3}{19}\right)^s - \left(\frac{9}{59}\right)^s - 3\left(\frac{81}{539}\right)^s + 2\left(\frac{27}{187}\right)^s + \left(\frac{1}{7}\right)^s$$

$$+2\left(\frac{27}{203}\right)^s + 2\left(\frac{3}{23}\right)^s + 2\left(\frac{27}{209}\right)^s - 3\left(\frac{81}{637}\right)^s + 2\left(\frac{27}{217}\right)^s + 2\left(\frac{27}{221}\right)^s - 2\left(\frac{9}{77}\right)^s$$

$$+\left(\frac{1}{9}\right)^s + 2\left(\frac{27}{247}\right)^s + 2\left(\frac{27}{253}\right)^s + 2\left(\frac{27}{259}\right)^s + 2\left(\frac{3}{29}\right)^s \ldots \qquad \textbf{(Ψ)}$$

For the other real character modulo 10 (with χ(3) = χ(7) = -1 and χ(1) = χ(9) = 1)

$$\frac{1}{1-L(s,\chi_{-10})} = 3^s - \left(\frac{9}{7}\right)^s + 1 + \left(\frac{9}{11}\right)^s - \left(\frac{9}{13}\right)^s + \left(\frac{27}{49}\right)^s - \left(\frac{9}{17}\right)^s$$

$$+\left(\frac{9}{19}\right)^s - \left(\frac{3}{7}\right)^s - \left(\frac{9}{23}\right)^s - 2\left(\frac{27}{77}\right)^s + \left(\frac{9}{29}\right)^s + 2\left(\frac{27}{91}\right)^s + \left(\frac{9}{31}\right)^s$$

$$+\left(\frac{3}{11}\right)^s - \left(\frac{9}{37}\right)^s - \left(\frac{81}{343}\right)^s - \left(\frac{3}{13}\right)^s + 2\left(\frac{27}{119}\right)^s + \left(\frac{27}{121}\right)^s + \left(\frac{9}{41}\right)^s$$

$$-\left(\frac{9}{43}\right)^s - 2\left(\frac{27}{133}\right)^s - \left(\frac{9}{47}\right)^s - 2\left(\frac{27}{143}\right)^s + 2\left(\frac{9}{49}\right)^s - \left(\frac{3}{17}\right)^s - \left(\frac{9}{53}\right)^s$$

$$+2\left(\frac{27}{161}\right)^s + \left(\frac{27}{169}\right)^s + \left(\frac{3}{19}\right)^s + \left(\frac{9}{59}\right)^s + 3\left(\frac{81}{539}\right)^s - 2\left(\frac{27}{187}\right)^s + \left(\frac{1}{7}\right)^s$$

$$-2\left(\frac{27}{203}\right)^s - 2\left(\frac{3}{23}\right)^s + 2\left(\frac{27}{209}\right)^s - 3\left(\frac{81}{637}\right)^s - 2\left(\frac{27}{217}\right)^s + 2\left(\frac{27}{221}\right)^s - 2\left(\frac{9}{77}\right)^s \ldots$$

$$-\left(\frac{1}{9}\right)^s - 2\left(\frac{27}{247}\right)^s - 2\left(\frac{27}{253}\right)^s + 2\left(\frac{27}{259}\right)^s + 2\left(\frac{3}{29}\right)^s \ldots \qquad \textbf{(☬)}$$

Time and again, we see integer powers of 3 in the numerators, with two unexpected exponent 0 in the numerators of 1/7 and 1/9.
One can notice the beginnings of an interrupted sequence of the primes < 60 in the denominators of fractions whose numerator is 9, and another sequence of ordered primes in fractions whose numerator is 3.
We never encounter numbers divisible by 5 in the denominator: one should remember that $\chi(5) = 0$.
All composite numbers in the denominators which aren't integer powers of primes brings about a coefficient equal to the numbers of its factors (distinct or not).
An integer power of a prime does not give place to coefficients unless it coincides with the exponent in the numerator: e.g. 2(9/49), -2(3/23), 2(3/29)
The changes of signs obey a not at all obvious rule. In turn, the presence of coefficients greater than 1 (in absolute value) might be decrypted adding to the aforementioned rules the following one if the number of distinct factors in the denominator is the power of 3 in the numerator, one sees a coefficient, e.g. -2(9/77)
The second comparison concerns the Riemann zeta function and the Dirichlet eta function. They belong to different moduli, the latter being the alternate version of the other.

For the Riemann zeta function, the sequence of rationals is (with the coefficients written in parenthesis):
2, -4/3, -1, 8/9, -4/5, 2/3, -16/27, -4/7, (2) 8/15, (2) -4/9, 2/5, 32/81, (2) 8/21, -4/11, (3) -16/45, 8/25, -4/13, (3) 8/27, 2/7, (3) -4/15, -64/243, (3) -16/63, (2) 8/33, (4) 32/135, -4/17, (2) 8/35, 2/9, (3) -16/75, -4/19, (2) 8/39, (4) -16/81, (3) -4/21, 2/11, (7) 8/45,… (These may be found in (♚), here - with several terms added.)

For the Dirichlet eta function, the sequence of rationals is:

$$\frac{1}{1-\eta(x)} = 2^x + \left(\frac{4}{3}\right)^x - 1 + \left(\frac{8}{9}\right)^x + \left(\frac{4}{5}\right)^x - 3\left(\frac{2}{3}\right)^x + \left(\frac{16}{27}\right)^x + \left(\frac{4}{7}\right)^x$$

$$+2\left(\frac{8}{15}\right)^x - 4\left(\frac{4}{9}\right)^x - 3\left(\frac{2}{5}\right)^x + \left(\frac{32}{81}\right)^x + 2\left(\frac{8}{21}\right)^x + \left(\frac{4}{11}\right)^x + 3\left(\frac{16}{45}\right)^x + 2\left(\frac{1}{3}\right)^x$$

$$+\left(\frac{8}{25}\right)^x + \left(\frac{4}{13}\right)^x - 5\left(\frac{8}{27}\right)^x - 3\left(\frac{2}{7}\right)^x - 9\left(\frac{4}{15}\right)^x + \left(\frac{64}{243}\right)^x + 3\left(\frac{16}{63}\right)^x$$

$$+2\left(\frac{8}{33}\right)^x + 4\left(\frac{32}{135}\right)^x + \left(\frac{4}{17}\right)^x + 2\left(\frac{8}{35}\right)^x + 5\left(\frac{2}{9}\right)^x + 3\left(\frac{16}{75}\right)^x + \left(\frac{4}{19}\right)^x$$

$$+2\left(\frac{8}{39}\right)^x + 2\left(\frac{1}{5}\right)^x - 6\left(\frac{16}{81}\right)^x - 9\left(\frac{4}{21}\right)^x - 3\left(\frac{2}{11}\right)^x - 17\left(\frac{8}{45}\right)^x$$

$$+\left(\frac{4}{23}\right)^x + \left(\frac{128}{729}\right)^x + 4\left(\frac{32}{189}\right)^x \ldots \qquad (♔)$$

In spite of all the similitudes, in these expansions there are a lot of new items in comparison to the expansion of 1/(ς(*s*) - 1). The summand 2(1/3)^*x* does not appear in the zeta expansion at all. Nor does 2(1/5)^*x*.

Coefficients are not the same as in (♚).

Interesting seems to be, in the eta Dirichlet asymptotic expansion, the presence of summands of the form

$$-(k+2)\left(\frac{2^k}{3^k}\right)^s$$

If the denominator of a fraction is a composite number with at least two distinct factors, then the coefficient of that fraction is greater than 1. Yet it is not clear wether and in which way it depends on the number of factors (distinct or not) in the denominator and/or on the power in the numerator. If the sign of summand is +, then coefficients coincide with the number of factors of the denominator, provided there exist at least two distinct factors
If the number of factors (distinct or not) in the denominator matches the power or 2 in the numerator, then a coefficient > 1 (in absolute value) appears.
It is not clear whether any odd number divides at least one denominator in the presumably infinite (possibly displaying farther some fractional coefficients!) with sequence of fractions attached to a Dirichlet *L*-function.
It is not clear if the sets of denominators are or not closed under multiplication. Nor if the following weaker statement holds or not: for any denominators *n* and *m* in the set $D_{L(s,\,\chi)}$ of the denominators in the expansion of 1/(*L*(*s*, *χ*)-1) there is a *k* in $D_{L(s,\,\chi)}$ so that *nm* divides *k.*
Statistics and dynamics of the sets of these rationals might reveal interesting.
An informal question: suppose you wake up tomorrow morning in a world where the correctness and the proof of GRH is not anymore a matter of discussion. How slightly an ‘artificially’ constructed or altered function may deviate from having all zeros on its critical line? And which would be the criteria of ‘slightness’?

7. Rationals and Hurwitz zeta function

Hurwitz zeta function is defined as

$$\zeta(s,q)=\sum_{n=0}^{\infty}\frac{1}{(q+n)^s}$$

We’ll just give two special cases in order to show the possibility of the study of rationals in this broader context:

$$\frac{1}{\zeta\left(x,\frac{3}{2}\right)}=\left(\frac{3}{2}\right)^x-\left(\frac{9}{10}\right)^x-\left(\frac{9}{14}\right)^x+\ldots \qquad \text{(for large } x\text{)}$$

$$\frac{1}{\zeta\left(\frac{\pi}{2},x\right)}=\left(\frac{\pi}{2}\right)^x-\left(\frac{\pi^2}{2\pi+4}\right)^x+\ldots \qquad \text{(also for large } x \text{ values)}$$

One can of course define Hurwitz analogues of Dirichlet functions, although these would lose some of their essential properties (as does the Riemann zeta function itself).
For example the Hurwitz lambda function would be:

$$\lambda(s,q)=\sum_{n=0}^{\infty}\frac{1}{(2n+q)^{s}}$$

One has for large values of the first argument

$$\frac{1}{\lambda\left(x,\frac{3}{2}\right)}=\left(\frac{3}{2}\right)^{x}-\left(\frac{9}{22}\right)^{x}-\dots$$

and

$$\frac{1}{\lambda\left(x,\frac{\pi}{2}\right)}=\left(\frac{\pi}{2}\right)^{x}-\left(\frac{\pi^{2}}{2\pi+16}\right)^{x}-\dots$$

8. ON A CONNECTION WITH THE LIOUVILLE FUNCTION

It would be nice if a connection with the Liouville function could be established at least for the asymptotic expansion of Dirichlet L-functions whose asymptotic part of the graph for real arguments lies above the line $y = 1$
The Liouville function, denoted by $\lambda(n)$, is defined, for positive integers, as

$$\lambda(n)=(-1)^{\Omega(n)}$$

where $\Omega(n)$ is the number of prime factors of n, counted with multiplicity (sequence A008836 in OEIS)
The Liouville function for rationals of the first kind, denoted by $\lambda_V(p/q)$, might be defined as follows:

$$\lambda_V\left(\frac{p}{q}\right)=\begin{cases}\lambda(q) & \text{if } \left|\Omega(p)-\Omega(q)\right|=1\\ -\lambda(q) & \text{if } \Omega(p)=\Omega(q)\\ 1 & \text{if } p=1 \text{ and } q\neq 1 \text{ (regardless of their } \Omega \text{ values)}\end{cases} \qquad (※)$$

In fact, at least in (♚), (♟), (♆), (♖) (i.e. in the asymptotic expansions for Riemann zeta function, Dirichlet lambda, Dirichlet $L(s,\chi_6)$ and Dirichlet $L(s,\chi_{10})$ functions), the signs of the summands might (conjecturally) be rewritten exactly in terms of $\lambda_V(p/q)$.

One can find additional heuristic confirmation in the beginning of the following asymptotic expansion:

$$\frac{1}{L(s,\chi_{+3})-1} = 2^s - 1 - \left(\frac{4}{5}\right)^s - \left(\frac{4}{7}\right)^s + \left(\frac{2}{5}\right)^s - \left(\frac{4}{11}\right)^s + \left(\frac{8}{25}\right)^s -$$

$$\left(\frac{4}{13}\right)^s + \left(\frac{2}{7}\right)^s - \left(\frac{4}{17}\right)^s + 2\left(\frac{8}{35}\right)^s - \left(\frac{4}{19}\right)^s + \left(\frac{2}{11}\right)^s - \left(\frac{4}{23}\right)^s +$$

$$+\left(\frac{8}{49}\right)^s - 2\left(\frac{4}{25}\right)^s + \left(\frac{2}{13}\right)^s + 2\left(\frac{8}{55}\right)^s + \dots$$

The signs in the expansions of Dirichlet L-functions (♕), (♝), (◒), (♙), (☬), whose real asymptotic part of the graph lies under the line $y = 1$ (with the important exception of the Dirichlet eta function) seem to obey a completely different law: the signs seem to be entirely determined by the residues of the denominators taken with respect to the Dirichlet character modulus.

The signs of the summands in the expansion (♔) related to the Dirichlet eta function deserves a separate discussion:

We define the Liouville function for rationals (of the second kind) as follows:

$$\lambda_A\left(\frac{p}{q}\right) = \begin{cases} 1 & \text{if } |\Omega(p) - \Omega(q)| = 1 \\ -1 & \text{if } \Omega(p) = \Omega(q) \end{cases} \qquad (⨸)$$

where $\Omega(p)$ denotes again the number of prime factors of p counted with multiplicity.

NOTE

Both (※) and (⨸) suffice to characterize the signs of the summands we were able to compute with sufficient accuracy (for more summands of see also page 14). Both these formulations are liable to be refined and/or completed when new terms of the expansions of $1/(\varsigma(x)$-1) and of 1/(1- $\eta(1)$) will be properly calculated. They have the signs predicted by, respectively, (※) and (⨸), but we have to b cautious: for example, the case $|\Omega(p) - \Omega(q)| > 1$ remains unclear: till now, the only encountered rational with $|\Omega(p) - \Omega(q)| = 2$ is (1/9) in **(Ψ)**, which in that expansion has the same sign as (1/7).

It is not clear for the time being whether the summands +2(1/3)^s and +2(1/5)^s are positive because $|\Omega(1) - \Omega(5)| = 1$ and $|\Omega(1) - \Omega(3)| = 1$ or just because, as in (♚), it should be supposed, as for the Riemann zeta function (see (※)), that $\lambda_A(1/q) = 1$ for all q. At least heuristically, these questions will rapidly find their answers, provided one might be interested in further computation and study of these expansions.

Liouville lambda function is related to Riemann zeta function by the well-known expansion formula:

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$$ (✤) (by the way, another aperiodic character)

Interestingly, the expansion is made of exactly the same summands as the corresponding to Riemann zeta function expansion (♚) up to some signs.

$$\frac{1}{1-\frac{\zeta(2s)}{\zeta(s)}} = \frac{\zeta(s)}{\zeta(s)-\zeta(2s)} = 2^s - \left(\frac{4}{3}\right)^s + 1 + \left(\frac{8}{9}\right)^s - \left(\frac{4}{5}\right)^s - \left(\frac{2}{3}\right)^s - \left(\frac{16}{27}\right)^x$$

$$-\left(\frac{4}{7}\right)^s + 2\left(\frac{8}{15}\right)^s + 2\left(\frac{4}{9}\right)^s - \left(\frac{2}{5}\right)^s + \left(\frac{32}{81}\right)^s + 2\left(\frac{8}{21}\right)^s - \left(\frac{4}{11}\right)^s$$

$$-3\left(\frac{16}{45}\right)^s + \left(\frac{8}{25}\right)^s - \left(\frac{4}{13}\right)^s - 3\left(\frac{8}{27}\right)^s - \left(\frac{2}{7}\right)^s + 3\left(\frac{4}{15}\right)^s - \left(\frac{64}{243}\right)^s$$

$$-3\left(\frac{16}{63}\right)^s + 2\left(\frac{8}{33}\right)^s + 4\left(\frac{32}{135}\right)^s - \left(\frac{4}{17}\right)^s + 2\left(\frac{8}{35}\right)^s + \left(\frac{2}{9}\right)^s - 3\left(\frac{16}{75}\right)^s$$

$$-\left(\frac{4}{19}\right)^s + 2\left(\frac{8}{39}\right)^s + 4\left(\frac{16}{81}\right)^s + 3\left(\frac{4}{21}\right)^s - \left(\frac{2}{11}\right)^s - 7\left(\frac{8}{45}\right)^s \dots$$ **(⚛)**

The signs of this asymptotic expansion (⚛) are directly related to the Liouville λ-function applied only to the denominators — just as in (✤) — regardless of the number of prime factors (i.e. regardless of the exponent of 2) in the numerator.
Yet, (✤) holds for any $s > 1$, while the asymptotical (⚛) holds only in some neighborhood of infinity.
This is a naive but not necessarily false way of thinking that primes 'behave at infinity' basically in the way they behave anywhere. Anyhow, since s tends to infinity, tiny primes and prime factors of larger numbers (e.g. $243 = 3^5$, $135 = 5*3^3$) are mirrored in the 'neighborhoods of infinity' through a function closely related to the Riemann zeta function.

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