May 7, 2014

Exponential convergence to equilibrium in cellular automata asymptotically emulating identity

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Abstract

We consider the problem of finding the density of 1's in a configuration obtained by n iterations of a given cellular automaton (CA) rule, starting from disordered initial condition. While this problems is intractable in full generality for a general CA rule, we argue that for some sufficiently simple classes of rules it is possible to express the density in terms of elementary functions. Rules asymptotically emulating identity are one example of such a class, and density formulae have been previously obtained for several of them. We show how to obtain formulae for density for two further rules in this class, 160 and 168, and postulate likely expression for density for eight other rules. Our results are valid for arbitrary initial density. Finally, we conjecture that the density of 1's for CA rules asymptotically emulating identity always approaches the equilibrium point exponentially fast.

1. Introduction

Cellular automata (CA) are often viewed as computing devices. An initial configuration is taken as an input of the computation, and, after a number of iterations of the CA rule, the resulting final configuration constitutes the output of the computation.

In many practical problems, especially in mathematical modeling, one is not interested in all details of the configuration, but rather in certain aggregate properties, such as, for example, the density of ones. A very common question can then be formulated as follows. Suppose we generated an initial configuration with a given density of ones $p \in [0, 1]$, such that each site is independently set to 1 with probability p and to 0 with probability $1 - p$. We then iterate a given rule n times over this configuration. What is the density of ones in the resulting configuration? Using signal processing terminology, we want to know the "response curve", density of the output as a function of the density of the input.

Numerical studies of the density c_n assuming $p = 0.5$ were first conducted by S. Wolfram. In [1], he presented a table showing c_∞ for all "minimal" CA rules, in many cases postulating exact rational values of c_∞. In [2], H. Fukś obtained formulae for density c_n for many elementary CA rules, starting from initial density $c_0 = 0.5$. Some of these formulae were proved, but most were conjectures based on patterns appearing in sequences of preimage numbers.

In later years, building on the ideas outlined in [2], exact formulae for c_n have been rigorously derived for several CA rules, for example rules 14, 172, 140, and 130 [3, 4, 5, 6]. In the first two cases the forumulae for c_n were proved for $p = 1/2$, while in the last two cases for arbitrary p.

For a given CA rule, the difficulty of finding the density c_n very strongly depends on the rule. Generally, the more complex dynamics of the rule is, the more difficult is to obtain the exact formula for c_n. One exception to this are surjective CA rules (among elementary CA these are rules 15, 30, 45, 51, 60, 90, 105, 106, 150, 154, 170 and 204). Some of them exhibit very complex spatio-temporal behavior, yet it is well known that the symmetric Bernoulli measure ($p = 1/2$) is invariant under the action of a surjective rule, thus for all of them $c_n = 1/2$ for $p = 1/2$ (cf. [7] for a review of this result).

One class of rules for which c_n is easy to obtain are idempotent rules, that is, rules for which the global function F has the property $F^2 = F$ (rule applied twice yields the same result as applied once). One can generalize the notion of idempotence further by considering k-th level *emulators of identity* , for which $F^{k+1} = F^k$ for some k. These are call emulators of identity, because after k iterations further application of the rule is equivalent to application of the identity [8]. And finally, one can introduce the notion of asymptotic emulation of identity, such that F^{k+1} and F^k are not identical, but become closer and closer as $k \to \infty$, as defined in [2]. Rules asymptotically emulating identity will be the main subject of this paper. While the dynamics of these rules is not overly complicated, it is still far from being trivial. In some sense, they resemble finitely-dimensional dynamical systems in the neighbourhood of a hyperbolic fixed point, where orbits starting from the stable manifold converge to the fixed point exponentially fast. In asymptotic emulators of identity, convergence to the equilibrium state is also exponentially fast, as we will subsequently see. For all the above reasons, CA rules asymptotically emulating identity are an ideal testbed for attempts to compute c_n. The goal of this article is to show that the problem of finding c_n for these rules is indeed tractable, and that their formulae for density exhibit remarkable similarity to each other.

2. Preliminaries and definitions

Let $\mathcal{A} = \{0, 1\}$ be called an *alphabet*, or a *symbol set*, and let $X = \mathcal{A}^{\mathbb{Z}}$. A finite sequence of elements of $\mathcal{A}$, $\mathbf{b} = b_1 b_2 \ldots, b_n$, will be called a *block* (or *word*) of length n. Set of all blocks of elements of $\mathcal{A}$ of all possible lengths will be denoted by $\mathcal{A}^\star$.

For $r \in \mathbb{N}$, a mapping $f : \mathcal{A}^{2r+1} \mapsto \mathcal{A}$ will be called *a cellular automaton rule of radius* r. Corresponding to f, we also define a *global mapping* $F : X \to X$ such that $(F(x))_i = f(x_{i-r}, \dots, x_i, \dots, x_{i+r})$ for any $x \in X$.

A *block evolution operator* corresponding to f is a mapping $\mathbf{f} : \mathcal{A}^\star \mapsto \mathcal{A}^\star$ defined as follows. Let $r \in \mathbb{N}$ be the radius of f, and let $\mathbf{a} = a_1 a_2 \dots a_n \in \mathcal{A}^n$ where $n \geq 2r+1$. Then $\mathbf{f}(\mathbf{a})$ is a block of length $n - 2r$ defined as

$$\mathbf{f}(\mathbf{a}) = f(a_1, a_2, \dots, a_{1+2r}) f(a_2, a_3, \dots, a_{2+2r}) \dots f(a_{n-2r}, a_{n-2r+1}, \dots, a_n). \tag{1}$$

For example, let f be a rule of radius 1, and let $\mathbf{b} \in \mathcal{A}^5$, so that $\mathbf{b} = b_1 b_2 b_3 b_4 b_5$. Then $\mathbf{f}(b_1 b_2 b_3 b_4 b_5) = a_1 a_2 a_3$, where $a_1 = f(b_1, b_2, b_3)$, $a_2 = f(b_2, b_3, b_4)$, and $a_3 = f(b_3, b_4, b_5)$. If $\mathbf{f}(\mathbf{b}) = \mathbf{a}$, than we will say that $\mathbf{b}$ is a preimage of $\mathbf{a}$, and write $\mathbf{b} \in \mathbf{f}^{-1}(\mathbf{a})$. Similarly, if $\mathbf{f}^n(\mathbf{b}) = \mathbf{a}$, than we will say that $\mathbf{b}$ is an *n-step preimage* of $\mathbf{a}$, and write $\mathbf{b} \in \mathbf{f}^{-n}(\mathbf{a})$.

The appropriate mathematical description of an initial distribution of configurations is a probability measure μ on X [9, 10, 7, 11]. Suppose that the initial distribution is a Bernoulli measure μ_p, so and all sites are independently set to 1 or 0, and the probability of finding 1 at a given site is p while the probability of finding 0 is $1-p$. One can then show [4] that the probability $P_n(\mathbf{b})$ of finding a block $\mathbf{b}$ at a given site after n iterations of rule f is given by

$$P_n(\mathbf{b}) = \sum_{\mathbf{a} \in \mathbf{f}^{-n}(\mathbf{b})} P_0(\mathbf{a}). \tag{2}$$

Note that the above probability is site-independent, and this is because the initial measure μ_p is shift-invariant. We will define c_n, the density of 1's, to be the expected value of a site,

$$c_n = P_n(1) \cdot 1 + P_n(0) \cdot 0 = P_n(1). \tag{3}$$

This yields the expression for density

$$c_n = \sum_{\mathbf{a} \in \mathbf{f}^{-n}(1)} P_0(\mathbf{a}). \tag{4}$$

Since the initial distribution is the Bernoulli distribution μ_p, $P_0(\mathbf{a}) = p^{\#_1(\mathbf{a})}(1-p)^{\#_0(\mathbf{a})}$, where $\#_1(\mathbf{a})$ and $\#_0(\mathbf{a})$ denote, respectively, the number of ones (zeros) in $\mathbf{b}$. We then obtain

$$c_n = \sum_{\mathbf{a} \in \mathbf{f}^{-n}(1)} p^{\#_1(\mathbf{a})}(1-p)^{\#_0(\mathbf{a})}. \tag{5}$$

In order to conveniently write the above formula, we will introduce the notion of a *density polynomial*. Let the *density polynomial* associated with a binary string $\mathbf{b} = b_1 b_2 \dots b_n$ be defined as

$$\Psi_{\mathbf{b}}(p, q) = p^{\#_1(\mathbf{b})} q^{\#_0(\mathbf{b})}. \tag{6}$$

If A is a set of binary strings, we define density polynomial associated with A as

$$\Psi_A(p, q) = \sum_{\mathbf{a} \in A} \Psi_{\mathbf{a}}(p, q). \tag{7}$$

Density c_n can thus be written as

$$c_n = \Psi_{\mathbf{f}^{-n}(1)}(p, 1-p) = \Psi_{\mathbf{f}^{-n}(1)}(c_0, 1-c_0). \tag{8}$$

In what follows, we will keep using variables p and q for density polynomials, understanding that in order to obtain c_n, one needs to substitute $q = 1-p$, and that p is the initial density, $p = c_0$.

The problem of finding the density c_n is thus equivalent tot he problem of finding the density polynomial for the set $\mathbf{f}^{-n}(1)$. In order to do this, one has to have detailed knowledge of the structure of $\mathbf{f}^{-n}(1)$, which is usually very difficult to obtain. However, for reasonably simple rules it often possible, as we will shortly see.

3. Asymptotic emulators of identity

We will now define the class of rules we wish to consider, namely rules asymptotically emulating identity. Let f be a CA rule of radius m, g a rule of radius n, and $k = \max\{m, n\}$. Let the distance between rules f and g be defined as

$$d(f, g) = 2^{-2k-1} \sum_{\mathbf{b} \in \mathcal{A}^{2k+1}} |f(\mathbf{b}) - g(\mathbf{b})|, \tag{9}$$

where for $\mathbf{b} = b_1 b_2 \ldots b_{2k+1}$ and rule f of radius r we define $f(\mathbf{b}) = f(b_{k+1-r}, \ldots, b_{k+1+r})$. This simply means that $f(\mathbf{b})$ is the value of the local function on the neighbourhood of the central symbol of $\mathbf{b}$, e.g., for $\mathbf{b} = b_1 b_2 b_3 b_4 b_5 b_6 b_7$ and $r = 1$, $f(\mathbf{b}) = f(b_3, b_4, b_5)$. One can show that the distance defined above is a metric in the space of CA rules [2].

The *composition* $f \circ g$ of two CA rules f and g can be defined in terms of their corresponding global mappings F and G, as a local function of $F \circ G$, where $(F \circ G)(x) = F(G(x))$ for $x \in X$. We note that if f is a rule of radius r, and g of radius s, then $f \circ g$ is a rule of radius $r + s$. For example, the composition of two radius-1 mappings is a radius-2 mapping:

$$(f \circ g)(x_{-2}, x_{-1}, x_0, x_1, x_2) = f(g(x_{-2}, x_{-1}, x_0), g(x_{-1}, x_0, x_1), g(x_0, x_1, x_2)). \tag{10}$$

Multiple composition will be denoted by

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}. \tag{11}$$

We say that a cellular automaton rule f *asymptotically emulates rule* g if

$$\lim_{n \to \infty} d(f^{n+1}, g \circ f^n) = 0. \tag{12}$$

We will be primarily interested in emulators of identity, for which we take as g the local function of identity rule (i.e., rule 204). In [2], it has been found that rules 13, 32, 40, 44, 77, 78, 128, 132, 136, 140, 160, 164, 168, 172, and 232 asymptotically emulate identity. Typical spatio-temporal patterns produced by these rules are shown in Figure 1. All these rules

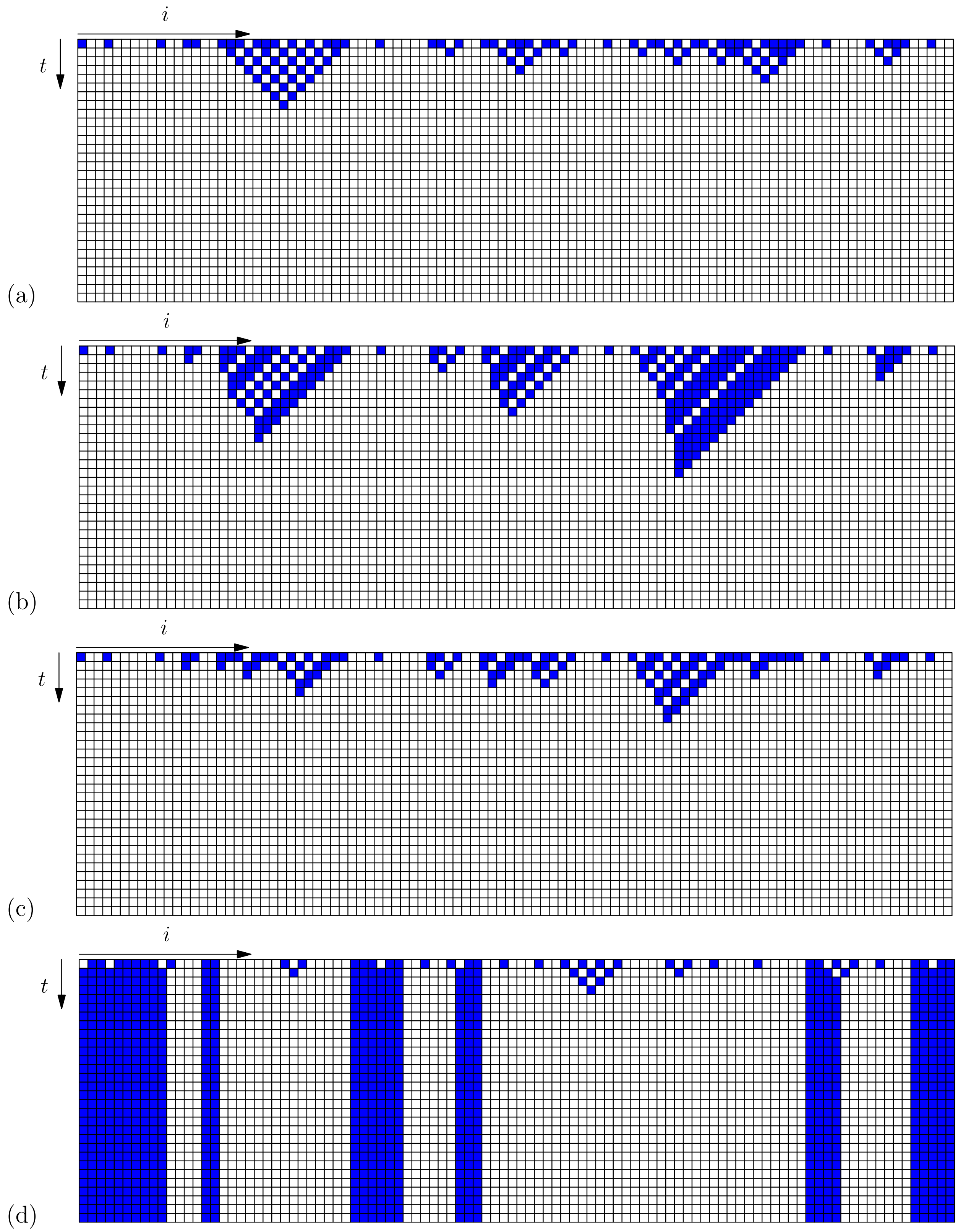


Figure 1: Spatio-temporal pattern produced by rules 160 (a), 168 (b), 40 (c) , and 232 (d), starting with random initial condition.

eventually reach all zero state or a fixed point which corresponds to vertical strips in the spatio-temporal patters (as in the case of rule 232, Figure 1d).

For all these rules, formulae for densities for c_n for $p = 1/2$ have been postulated in [2], and some of these formulae were subsequently proved, as illustrated in Table 1. The general formulae for the density, for arbitrary c_0, have been previously reported for only four of them, rules 128, 132, 136, and 140. For all four cases, proofs of the formulae are known. Below we show these formulae, citing proof source as well.

- **Rule 128** (in [12], c_n has been obtained for rule 254, identical with conjugated and reflected rule 128)
$$c_n = c_0^{2n+1}, \tag{13}$$
- **Rule 132** (in [12], c_n has been obtained for rule 222, identical with conjugated and reflected rule 132)
$$c_n = (1 - c_0)^2 c_0 + \frac{(1 - c_0) c_0{}^3}{1 + c_0} + 2 \frac{c_0}{1 + c_0} c_0{}^{2n+1}, \tag{14}$$
- **Rule 136** (in [12], c_n has been obtained for rule 238, identical with conjugated rule 132)
$$c_n = c_0^{n+1}, \tag{15}$$
- **Rule 140** (in [5], c_n has been obtained for a more general case of the asynchronous version rule 140, here we take the special case of the synchrony rate equal to 1)
$$c_n = c_0(1 - c_0) + c_0^{n+2}. \tag{16}$$

We will show that using the concept of density polynomials, formulae for c_n for arbitrary c_0 can be constructed for many other rules asymptotically emulating identity. In two cases, namely for rules 160 and 168, we give formal proofs for density formulae. For many other cases, we will describe how to "guess" the correct formula for c_n by setting up a recursive equation for density polynomials.

4. Rule 160

The first rule we wish to consider is the rule 160. From now one, we will use subscripts with Wolfram numbers to identify concrete local functions and corresponding block evolution operators, e.g., f_{160} and $\mathbf{f}_{160}$ for rule 160.

Rule 160 is defined by $f_{160}(1, 1, 1) = f_{160}(1, 0, 1) = 1$, and $f_{160}(x_1, x_2, x_3) = 0$ for all other values of x_1, x_2, x_3. This can be simply written as $f(x_1, x_2, x_3) = x_1 x_3$. Rule 160 is one of those few rules for which expressions for f^n can be explicitly given, as the following proposition attests.

Proposition 1 *For elementary CA rule 160 and for any $n \in \mathbb{N}$ we have*
$$f_{160}^n(x_1, x_2, \ldots, x_{2n+1}) = \prod_{i=0}^{n} x_{2i+1}. \tag{17}$$

Rule	c_n	Proof
13	$7/16 - (-2)^{-n-3}$	
32	2^{-1-2n}	[2]
40	2^{-n-1}	
44	$1/6 + \frac{5}{6}2^{-2n}$	
77	$1/2$	[2]
78	$9/16$	
128	2^{-1-2n}	[12]
132	$1/6 + \frac{1}{3}2^{-2n}$	[12]
136	2^{-n-1}	[12]
140	$1/4 + 2^{-n-2}$	[5]
160	2^{-n-1}	this paper
164	$1/12 - \frac{1}{3}4^{-n} + \frac{3}{4}2^{-n}$	
168	$3^n 2^{-2n-1}$	this paper
172	$\frac{1}{8} + \frac{(10-4\sqrt{5})(1-\sqrt{5})^n + (10+4\sqrt{5})(1+\sqrt{5})^n}{40 \cdot 2^{2n}}$	[4]
232	$1/2$	

Table 1: Density of ones c_n for disordered initial state ($c_0 = 0.5$) for elementary rules asymptotically emulating identity. For rules for which the proof is known source of the proof is given. All others formulae are conjectures based on preimage patterns form [2].

Proof. We give proof by induction. For $n = 1$ eq. (17) is obviously true, as remarked above. Suppose now that the formula (17) holds for some n, and let us compute f^{n+1}. We have

$$\begin{aligned} f_{160}^{n+1}(x_1, x_2, \ldots, x_{2n+3}) &= f_{160}\Big(f_{160}^n(x_1, \ldots, x_{2n+1}), f_{160}^n(x_2, \ldots, x_{2n+2}), f_{160}^n(x_3, \ldots, x_{2n+3})\Big) \\ &= f_{160}\left(\prod_{i=0}^{n} x_{2i+1}, \prod_{i=0}^{n} x_{2i+2}, \prod_{i=0}^{n} x_{2i+3}\right) = \prod_{i=0}^{n} x_{2i+1} \prod_{i=0}^{n} x_{2i+3} \\ &= \prod_{i=0}^{n} x_{2i+1} \prod_{i=1}^{n+1} x_{2i+1} = x_1 \left(\prod_{i=1}^{n} x_{2i+1} \prod_{i=1}^{n} x_{2i+1}\right) x_{2n+3} = \prod_{i=0}^{n+1} x_{2i+1}, \end{aligned}$$

where we used the fact that $x_i^2 = x_i$ if $x_i \in \{0, 1\}$. The formula (17) is thus valid for $n + 1$, and this concludes the proof by induction.

The following result is a direct consequence of eq. (17).

Proposition 2 *Block $b_1 b_2 \ldots b_{2n+1}$ is an n-step preimage of 1 under the rule 160 if and only if $b_i = 1$ for every odd i.*

This means that we have $n+1$ ones and n arbitrary symbols in the preimage of 1, therefore,

$$\Psi_{\mathbf{f}_{168}^{-n}(1)}(p, q) = p^{n+1}(p + q)^n. \tag{18}$$

The density of ones $c_n = P_n(1)$ is thus

$$c_n = \Psi_{\mathbf{f}_{168}^{-n}(1)}(c_0, 1 - c_0) = c_0^{n+1}, \tag{19}$$

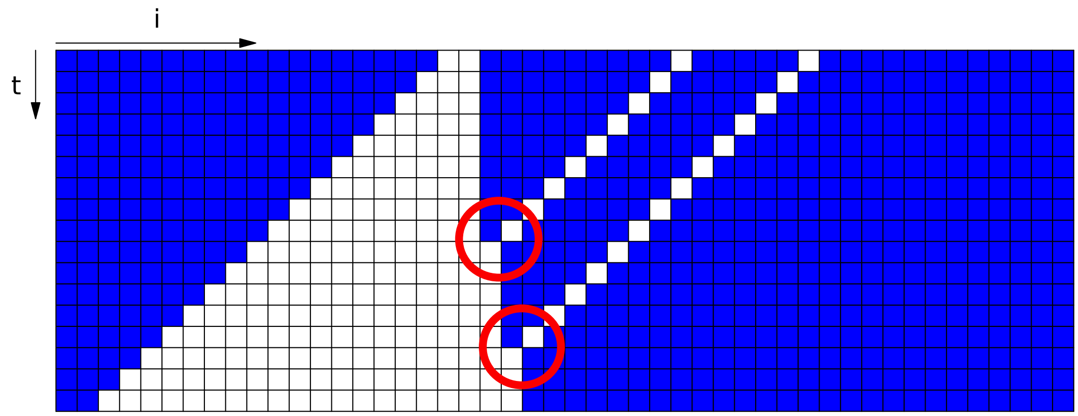


Figure 2: Collision of "defects" in CA rule 168.

and for $c_0 = 1/2$,

$$c_n = 2^{-n-1}. \tag{20}$$

No matter what the initial density, c_n exponentially converges to 0 as $n \to \infty$.

5. Rule 168

Rule 168 is defined by $f_{168}(1,1,1) = f_{168}(1,0,1) = f_{168}(0,1,1) = 1$, and $f_{168}(x_1, x_2, x_3) = 0$ for all other values of x_1, x_2, x_3. Its dynamics and preimage structure is considerably more complex that for rule 160. Nevertheless, upon careful examination of preimages of 1, it is possible to discover an interesting pattern in these preimages, described in the following proposition.

Proposition 3 *Let A_n be a set of all strings of length $2n+1$ ending with 1 such that, counting from the right, the first pair of zeros begins at k-th position from the right, and the number of isolated zeros in the substring to the right of this pair of zeros is m, satisfying $m < k - n - 1$. Moreover, let B_n be the set of all strings of length $2n+1$ ending with 1 which do not contain 00. Block $\mathbf{b} \in \mathcal{A}^{2n+1}$ is an n-step preimage of 1 under the rule 168 if and only if $\mathbf{b} \in A_n \cup B_n$.*

In lieu of a formal proof, we will present discussion of spatio-temporal dynamics of rule 168 and explain how it leads to the above result. First of all, let us note that $\mathbf{f}_{168}^{-1}(1) = \{011, 101, 111\}$. This means that if a block $\mathbf{b}$ ends with 1, its preimage must also end with 1, and, by induction, its n-step preimage must end with 1 as well. This explains that ending with 1 is a necessary condition for being a preimage of 1, and elements of both A_n and B_n have that property.

Next, let us note that one can consider a block $\mathbf{b}$ as consisting of blocks of zeros of various lengths separated by blocks of ones of various length. Suppose that a given block contains one isolated zero and to the left of it a pair of adjacent zeros, like in Figure 2. When the rule is iterated, the block 00 will increase its length by moving its left boundary to the left, while its right boundary will remain in place. The isolated zero, on the other hand, simply moves to the left, as illustrated in Figure 2. When the boundary of the growing cluster of zeros collides with the isolated zero, the isolated zero is annihilated, and the boundary of the cluster of zeros jumps one unit to the right. Two such collisions as shown in Figure 2, marked by circles.

Armed with this information, we can now attempt to describe conditions which a block must satisfy in order to be an n-step preimage of 1. If a block of length $2n+1$ is an n-step preimage of 1, then either it contains a block of two or more zeros or not. If it does not, and ends with 1, then it necessarily is a preimage of 1. This is because when the rule is iterated, all isolated zeros move to the left, and after n iterations we obtain 1, as shown in Figure 3 (left). Blocks of this type constitute elements of B_n.

If, on the other hand, there is at least one cluster of adjacent zeros in a block of length $2n+1$, then everything depends on the number of isolated zeros to the right of the rightmost cluster of zeros. Clearly, if there are not too many isolated zeros, and the rightmost cluster of zeros is not too far to the right, then the collisions of isolated zeros with the boundary of the cluster of zeros will not be able to move the boundary sufficiently far to change the final outcome, which will remain 1. This situation is illustrated in Figure 3 (center). Blocks of this type are elements of A_n.

Obviously, the balance of clusters of zeros and individual zeros is a delicate one, and if there are too many isolated zeros, they may change the final outcome to 0, as in Figure 3 (right).

The question is then, what is the condition for this balance? To find this out, suppose that we have a string $\mathbf{b} \in \mathcal{A}^{2n+1}$ and the first pair of zeros begins at k-th position from the right. If there are no isolated zeros in the substring to the right of this pair, then we want the end of the rightmost cluster of zeros to be not further than just to the right of the center of $\mathbf{b}$. Since the center of $\mathbf{b}$ is at the $n+1$-th position from the right, we want $k > n+1$.

If the are m isolated zeros in the substring to the right of this pair of zeros, we must push the boundary of the rightmost cluster of zeros m units to the left, because these isolated zeros, after colliding with the rightmost cluster of zeros, will move the boundary to the right. The condition should, therefore, be in this case $k > n+1+m$, or, equivalently, $m < k-n-1$, as required for elements of A_n. $\square$

With the above proposition, we can construct density polynomials associated with both A_n and B_n. The following lemma will be useful for this purpose. It can be proved by well known methods described in a typical book on enumerative combinatorics [13].

Lemma 1 *The number of binary strings $a_1 a_2 \ldots a_l$ such that $a_1 = a_l = 1$ and having only m isolated zeros is*

$$\binom{l-m-1}{m}. \tag{21}$$

Now note that elements of the set A_n described in Proposition 3 have the structure

$$\underbrace{\star\ldots\star}_{2n-k} 00 a_1 a_2 \ldots a_{k-1}, \tag{22}$$

where the string $a_1 a_2 \ldots a_{k-1}$ has only isolated zeros and $a_1 = a_{k-1} = 1$. Moreover,

$$k \in \{n+2, n+3, \ldots, 2n\}.$$

```
101101101101011101101    101111001111011101111    101111001111011101011
 1101101101011101101      1111000111011101111      1111000111011101011
  01101101011101101        11000011011101111        11000011011101011
   101101011101101          000001011101111          000001011101011
    1101011101101            0000011101111            0000011101011
     01011101101              00001101111              00001101011
      011101101                000101111                000101011
       1101101                  0001111                  0001011
        01101                    00111                    00011
         101                      011                      001
          1                        1                        0
```

Figure 3: Examples of blocks of length 21 for which 10 iterations of $\mathbf{f}_{168}$ produce 1 (left and center) and 0 (right).

Furthermore, the number of isolated zeros m must satisfy

$$m < k - n - 1,$$

meaning that

$$m \in \{0, 1, \ldots, k - n - 2\}. \tag{23}$$

Using our lemma, the density polynomial of the set of strings of type (22) with fixed k and m is therefore

$$(p+q)^{2n-k} q^2 \binom{k-1-m-1}{m} q^m p^{k-m-1} = (p+q)^{2n-k} q^2 \binom{k-m-2}{m} q^m p^{k-m-1}. \tag{24}$$

This yields the density polynomial associated with the set A_n,

$$\Psi_{A_n}(p,q) = \sum_{k=n+2}^{2n} \sum_{m=0}^{k-n-2} (p+q)^{2n-k} \binom{k-m-2}{m} q^{m+2} p^{k-m-1}, \tag{25}$$

which, by changing index j to $k = n + j + 2$, becomes

$$\Psi_{A_n}(p,q) = \sum_{j=0}^{n-2} \sum_{m=0}^{j} (p+q)^{n-j-2} \binom{n+j-m}{m} q^{m+2} p^{n+j-m+1}. \tag{26}$$

For the set B_n, the associated density polynomial is

$$\Psi_{B_n}(p,q) = \sum_{m=0}^{n} \binom{2n+1-m}{m} q^m p^{2n+1-m}. \tag{27}$$

The resulting density polynomial for n-step preimages of 1 is, therefore,

$$\Psi_{A_n \cup B_n}(p,q) = \Psi_{\mathbf{f}_{168}^{-n}(1)}(p,q) = \sum_{j=0}^{n-2} \sum_{m=0}^{j} (p+q)^{n-j-2} \binom{n+j-m}{m} q^{m+2} p^{n+j-m+1} \tag{28}$$

$$+ \sum_{m=0}^{n} \binom{2n+1-m}{m} q^m p^{2n+1-m}. \tag{29}$$

This expression, while complicated, can be written in a closed form. One can namely show by induction (we omit the proof) that it sums to

$$\Psi_{\mathbf{f}_{168}^{-n}(1)}(p,q) = p^{n+1}(p+2q)^n. \tag{30}$$

If the initial density is $p = c_0$, $q = 1 - c_0$, we obtain

$$c_n = \Psi_{\mathbf{f}_{168}^{-n}(1)}(c_0, 1-c_0) = c_0^{n+1}(c_0 + 2 - 2c_0)^n = c_0^{n+1}(2 - c_0)^n. \tag{31}$$

For the symmetric case, $c_0 = 1/2$,

$$c_n = \Psi_{\mathbf{f}_{168}^{-n}(1)}(1/2, 1/2) = \frac{3^n}{2^{2n+1}}. \tag{32}$$

As in the case of rule 160, the density exponentially converges to 0 as $n \to \infty$.

As an interesting additional remark, note that by substituting $p = q = 1$ to $\Psi_{\mathbf{f}_{168}^{-n}(1)}(p,q)$ we obtain $\operatorname{card} \mathbf{f}_{168}^{-n}(1)$, thus

$$\operatorname{card} \mathbf{f}_{168}^{-n}(1) = \operatorname{card} A_n + \operatorname{card} B_n = \Psi_{\mathbf{f}_{168}^{-n}(1)}(1,1) = 3^n. \tag{33}$$

Density polynomials are thus useful not only for determining densities, but also to enumerate n-step preimages in CA. The above result, $\operatorname{card} \mathbf{f}_{168}^{-n}(1) = 3^n$, has been observed in [2], but no proof was given.

6. Rule 40

In the previous two examples (rule 160 and 168), we were able to gain detailed understanding of the structure of preimages of 1, and therefore also compute the density of ones in a rigorous way. In the next example this will not be the case, but we will show that even then one can often conjecture what the expressions for c_n are. The conjecture will based on patters present in density polynomials. Such patters can often be readily observed when a first few density polynomials are generated with the help of a computer program.

Let us now consider the rule 40, for which $f_{40}(0,1,1) = f_{160}(1,0,1) = 1$, and $f_{40}(x_1, x_2, x_3) = 0$ for all other values of x_1, x_2, x_3. The first 10 density polynomials for preimages of 1, generated by a computer program, are

$$\begin{aligned}
\Psi_{\mathbf{f}_{40}^{-1}(1)}(p,q) =& 2p^2q,\\
\Psi_{\mathbf{f}_{40}^{-2}(1)}(p,q) =& p^4q+3p^3q^2,\\
\Psi_{\mathbf{f}_{40}^{-3}(1)}(p,q) =& 3p^5q^2+5p^4q^3,\\
\Psi_{\mathbf{f}_{40}^{-4}(1)}(p,q) =& p^7q^2+7p^6q^3+8p^5q^4,\\
\Psi_{\mathbf{f}_{40}^{-5}(1)}(p,q) =& 4p^8q^3+15p^7q^4+13p^6q^5,\\
\Psi_{\mathbf{f}_{40}^{-6}(1)}(p,q) =& p^{10}q^3+12p^9q^4+30p^8q^5+21p^7q^6,\\
\Psi_{\mathbf{f}_{40}^{-7}(1)}(p,q) =& 5p^{11}q^4+31p^{10}q^5+58p^9q^6+34p^8q^7,\\
\Psi_{\mathbf{f}_{40}^{-8}(1)}(p,q) =& p^{13}q^4+18p^{12}q^5+73p^{11}q^6+109p^{10}q^7+55p^9q^8,\\
\Psi_{\mathbf{f}_{40}^{-9}(1)}(p,q) =& 6p^{14}q^5+54p^{13}q^6+162p^{12}q^7+201p^{11}q^8+89p^{10}q^9,\\
\Psi_{\mathbf{f}_{40}^{-10}(1)}(p,q) =& p^{16}q^5+25p^{15}q^6+145p^{14}q^7+344p^{13}q^8+365p^{12}q^9+144p^{11}q^{10},\\
\Psi_{\mathbf{f}_{40}^{-11}(1)}(p,q) =& 7p^{17}q^6+85p^{16}q^7+361p^{15}q^8+707p^{14}q^9+655p^{13}q^{10}+233p^{12}q^{11}.
\end{aligned}$$

Upon closer inspection of these polynomials, one can suspect that they can perhaps be recursively generated. Denoting for simplicity $U_n(p,q) = \Psi_{\mathbf{f}_{40}^{-n}(1)}(p,q)$, suppose that they satisfy second-order difference equation,

$$U_n(p,q) = \alpha(p,q)U_{n-2} + \beta(p,q)U_{n-1}, \tag{34}$$

where $\alpha(p,q)$ and $\beta(p,q)$ are some unknown functions. Polynomials satisfying such a relation are known as generalized Lucas polynomials.

Knowing our first four polynomials, we can write the above equation for $n = 3$ and $n = 4$,

$$\begin{aligned}
U_3(p,q) &= \alpha(p,q)U_1 + \beta(p,q)U_2,\\
U_4(p,q) &= \alpha(p,q)U_2 + \beta(p,q)U_3.
\end{aligned} \tag{35}$$

This constitutes a system of two linear equations with two unknowns $\alpha(p,q)$ and $\beta(p,q)$. Solving this system one obtains $\alpha(p,q) = p^2q(p+q)$ and $\beta(p,q) = pq$, meaning that the recurrence equation (34) takes the form

$$U_n(p,q) = p^2q(p+q)U_{n-2} + pqU_{n-1}, \tag{36}$$

where $U_0(p,q) = p$, $U_1(p,q) = 2p^2q$. We verified that eq. (36) holds for up to $n = 12$, thus one can strongly suspect that it is valid for any n.

Assuming, therefore, that the linear difference equation (36) is valid for any n, we can now solve it by standard methods. The solution is

$$\begin{aligned}
U_n(p,q) = &-\frac{pq\left(-2\,p-q+\sqrt{5\,q^2+4\,pq}\right)}{\sqrt{5q^2+4pq}\left(q+\sqrt{5q^2+4pq}\right)}\left(-\frac{2\,p^2q+2\,pq^2}{q+\sqrt{5\,q^2+4\,pq}}\right)^n\\
&-\frac{pq\left(2\,p+q+\sqrt{5\,q^2+4\,pq}\right)}{\sqrt{5\,q^2+4\,pq}\left(q-\sqrt{5\,q^2+4\,pq}\right)}\left(-\frac{2\,p^2q+2\,pq^2}{q-\sqrt{5\,q^2+4\,pq}}\right)^n.
\end{aligned} \tag{37}$$

The density c_n can now be computed by taking $c_n = U_n(c_0, 1-c_0)$, after simplification and rationalization yielding

$$c_n = \left(\frac{1}{2}\,c_0 - \frac{3}{2}\,\frac{c_0\sqrt{5-6\,c_0+{c_0}^2}}{c_0-5}\right)\left(\frac{1}{2}\left(1-c_0+\sqrt{5-6\,c_0+{c_0}^2}\right)c_0\right)^n + \left(\frac{1}{2}\,c_0 + \frac{3}{2}\,\frac{c_0\sqrt{5-6\,c_0+{c_0}^2}}{c_0-5}\right)\left(\frac{1}{2}\left(1-c_0-\sqrt{5-6\,c_0+{c_0}^2}\right)c_0\right)^n. \quad (38)$$

In the symmetric case $c_0 = 1/2$ we obtain, after simplification,

$$c_n = 2^{-n-1}. \quad (39)$$

For the symmetric case $c_0 = 1/2$, it is also possible to obtain the above expression for c_n by a different method. One can show (we omit the proof here) that the generalized Lucas polynomials $U_n(p,q)$ defined by eq. (36) can be written in the form

$$U_n(p,q) = \Psi_{\mathbf{f}_{40}^{-n}(1)}(p,q) = \sum_{k=1}^{n+1} T_{n+1,k}\, p^{2n+2-k} q^{k-1}, \quad (40)$$

where the values of $T_{n,k}$ form the triangle

$$\begin{array}{c}
0,2\\
0,1,3\\
0,0,3,5\\
0,0,1,7,8\\
0,0,0,4,15,13\\
0,0,0,1,12,30,21\\
0,0,0,0,5,31,58,34\\
0,0,0,0,1,18,73,109,55\\
0,0,0,0,0,6,54,162,201,89\\
0,0,0,0,0,1,25,145,344,365,144.
\end{array}$$

The above triangle is known as the skew triangle associated with the Fibonacci numbers [14]. The coefficients $T_{n,k}$ can be generated by the recursive procedure [14],

$$\begin{aligned}
&T_{n,k} = T_{n-1,k-1} + T_{n-2,k-1} + T_{n-2,k-2}, \quad (41)\\
&T_{n,k} = 0 \ \text{ if } \ k < 0 \ \text{ or } \ k > n,\\
&T_{0,0} = 1, \quad T_{2,1} = 0.
\end{aligned}$$

Let us now compute c_n for the symmetric initial condition $c_0 = 1/2$,

$$c_n = \Psi_{\mathbf{f}_{40}^{-n}(1)}(1/2, 1/2) = 2^{-2n-1} \sum_{k=1}^{n+1} T_{n+1,k}. \quad (42)$$

Define now

$$S_n = \sum_{k=1}^{n} T_{n,k}, \tag{43}$$

so that

$$c_n = 2^{-2n-1} S_{n+1}. \tag{44}$$

Using the recursion definition of T, we obtain

$$\sum_{k=1}^{n} T_{n,k} = \sum_{k=1}^{n} T_{n-1,k-1} + \sum_{k=1}^{n} T_{n-2,k-1} + \sum_{k=1}^{n+1} T_{n-2,k-2}, \tag{45}$$

hence

$$S_n = S_{n-1} + 2S_{n-2}. \tag{46}$$

From the definition of $T(n, k)$ we know that $S_1 = 1$ and $S_2 = 2$, and therefore the solution of the above second-order difference equation is $S_n = 2^n$, hence

$$c_n = 2^{-2n-1} \cdot 2^n = 2^{-n-1}, \tag{47}$$

the same as in eq. (39), as expected.

7. Rules 232, 13, 32, 77, 78, 172, and 44

Elementary CA rule 232 is a special case of the "majority voting rule" with radius 1, defined as

$$f_{232}(x_1, x_2, x_3) = \text{majority}\{x_1, x_2, x_3\}, \tag{48}$$

or, more explicitly, $f_{232}(1,1,1) = f_{232}(1,1,0) = f_{232}(1,0,1) = f_{232}(0,1,1) = 1$, and for all other values of x_1, x_2, x_3, $f_{232}(x_1, x_2, x_3) = 0$.

We proceed in a similar fashion as in the case of rule 40. The first few density polynomials are

$$\begin{aligned}
\Psi_{\mathbf{f}_{40}^{-1}(1)}(p,q) =& 3\,qp^2 + p^3, \\
\Psi_{\mathbf{f}_{40}^{-2}(1)}(p,q) =& p^5 + 5\,p^4q + 8\,p^3q^2 + 2\,p^2q^3, \\
\Psi_{\mathbf{f}_{40}^{-3}(1)}(p,q) =& p^7 + 7\,p^6q + 19\,p^5q^2 + 24\,p^4q^3 + 11\,p^3q^4 + 2\,p^2q^5, \\
\Psi_{\mathbf{f}_{40}^{-4}(1)}(p,q) =& p^9 + 9\,p^8q + 34\,p^7q^2 + 69\,p^6q^3 + 79\,p^5q^4 + 47\,p^4q^5 + 15\,p^3q^6 + 2\,p^2q^7, \\
\Psi_{\mathbf{f}_{40}^{-5}(1)}(p,q) =& p^{11} + 11\,p^{10}q + 53\,p^9q^2 + 146\,p^8q^3 + 251\,p^7q^4 \\
& + 275\,p^6q^5 + 187\,p^5q^6 + 79\,p^4q^7 + 19\,p^3q^8 + 2\,p^2q^9, \\
& \ldots,
\end{aligned}$$

and again, upon closer inspection it turns out that that they are generalized Lucas polynomials. Denoting $U_n(p,q) = \Psi_{\mathbf{f}_{132}^{-n}(1)}(p,q)$, these polynomials satisfy

$$U_n(p,q) = -pq(p+q)^2 U_{n-2}(p,q) + (p^2 + 3pq + q^2) U_{n-1}(p,q). \tag{49}$$

Solution of the above equation is

$$U_n(p,q) = \frac{p^2\,(p+2\,q)\,(p^2+2\,pq+q^2)^n}{p^2+pq+q^2} - \frac{(p-q)\,(pq)^{n+1}}{p^2+pq+q^2}. \tag{50}$$

The density c_n can now be computed by taking $c_n = U_n(c_0, 1-c_0)$, yielding

$$c_n = \frac{{c_0}^2\,(2-c_0)}{{c_0}^2-c_0+1} + \frac{(2\,c_0-1)\,c_0\,(c_0-1)\,(c_0\,(1-c_0))^n}{{c_0}^2-c_0+1}. \tag{51}$$

We can see that c_n exponentially converges to c_∞, where

$$c_\infty = \frac{{c_0}^2\,(2-c_0)}{{c_0}^2-c_0+1}. \tag{52}$$

For $c_0 = 1/2$, the second term in eq. (51) vanishes and $c_\infty = 1/2$, thus we obtain $c_n = 1/2$, in agreement with Table 1.

There are six other rules for which we were able to obtain expressions for c_n in the same way as above, except that the order of the difference equation for density polynomials was not always 2, like in eq. (49), but it was sometimes lower or (most of the time) higher. For these rules, which are 13, 32, 77, 78, 172, and 44, we give below the recurrence formula for the density polynomial, followed by the expression for c_n obtained by solving that recurrence equation.

- **Rule 13:**

$$U_n(p,q) = qp\,(q+p)^4\,U_{n-3}(p,q) + \left(q^2+pq+p^2\right)(q+p)^2\,U_{n-2}(p,q), \tag{53}$$

$$c_n = \frac{(1-c_0)^3\,(-1+c_0)^n}{c_0-2} + \frac{{c_0}^2\,({c_0}^2-2\,c_0+2)\,(-c_0)^n}{c_0+1} + \frac{({c_0}^3-2\,{c_0}^2+c_0)^2-1}{(c_0-2)\,(c_0+1)}. \tag{54}$$

- **Rule 32:**

$$U_n(p,q) = pqU_{n-1}(p,q), \tag{55}$$

$$c_n = c_0^{n+1}(1-c_0)^n. \tag{56}$$

- **Rule 77:**

$$\begin{aligned} U_n(p,q) = {} & (p+q)^2\,q^2p^2U_{n-3}(p,q) \\ & + \left(p^4+2qp^3+q^2p^2+2\,q^3p+q^4\right)U_{n-2}(p,q) + 2pqU_{n-1}(p,q), \end{aligned} \tag{57}$$

$$c_n = \frac{{c_0}^3\,(-{c_0}^2)^n}{{c_0}^2+1} - \frac{(1-c_0)^3\left(-(1-c_0)^2\right)^n}{{c_0}^2-2\,c_0+2} - \frac{{c_0}^5-3\,{c_0}^4+3\,{c_0}^3-2\,{c_0}^2+c_0-1}{({c_0}^2+1)\,({c_0}^2-2\,c_0+2)}. \tag{58}$$

- **Rule 78:**

$$U_n(p,q) = (p+q)^6 q^2 p^2 U_{n-5}(p,q) - (p+q)^4 q^2 p^2 U_{n-4}(p,q) - (p^2+q^2)(p+q)^4 U_{n-3}(p,q) + (p^2+q^2)(p+q)^2 U_{n-2}(p,q) + (p+q)^2 U_{n-1}(p,q), \quad (59)$$

$$c_n = \frac{1 + c_0 - {c_0}^2 + {c_0}^4 - 2\,{c_0}^5 + {c_0}^6}{(c_0+1)\,(2-c_0)} + \frac{1}{2}\,\frac{(2\,{c_0}^2 + 1 - 2\,c_0)\,c_0\,(1-c_0)\,(c_0-1)^n}{2-c_0} - \frac{1}{2}\,(2\,c_0-1)\,{c_0}^2{c_0}^n - \frac{1}{2}\,\frac{(1-c_0)\,{c_0}^2\,(-c_0)^n}{c_0+1} + \frac{1}{2}\,(1-c_0)\,(2\,c_0-1)\,(1-c_0)^n. \quad (60)$$

The above is valid for $n > 1$.

- **Rule 172:**

$$U_n(p,q) = -pq\,(q+p)^4\,U_{n-3}(p,q) - (q+p)^2\,p^2 U_{n-2}(p,q) + (q+p)\,(q+2\,p)\,U_{n-1}(p,q), \quad (61)$$

$$c_n = (c_0-1)^2\,c_0 - \frac{\left(3\,c_0 - 4 + \sqrt{4c_0 - 3c_0^2}\right)\left(c_0 - 2 + \sqrt{4c_0-3c_0^2}\right)c_0\left(\frac{1}{2}\,c_0 - \frac{1}{2}\,\sqrt{4c_0-3c_0^2}\right)^n}{12\,c_0 - 16} + \frac{\left(3\,c_0 - 4 - \sqrt{4c_0-3c_0^2}\right)\left(-c_0 + 2 + \sqrt{4c_0-3c_0^2}\right)c_0\left(\frac{1}{2}\,c_0 + \frac{1}{2}\,\sqrt{4c_0-3c_0^2}\right)^n}{12\,c_0-16}. \quad (62)$$

- **Rule 44:**

$$U_n(p,q) = -(p+q)^2 q^2 p^4 U_{n-4}(p,q) + q^2 p^4 U_{n-3}(p,q) + (p+q)^2 U_{n-1}(p,q), \quad (63)$$

$$c_n = \frac{({c_0}^2 - c_0 + 1)\,c_0\,(c_0-1)}{{c_0}^3 - {c_0}^2 - 1} - \frac{1}{3}\,\frac{c_0}{1 + {c_0}^2\,(1-c_0)}\Big(\alpha\lambda_1^n + (\beta + i\,\gamma)\lambda_2^n + (\beta - i\,\gamma)\lambda_3^n\Big), \quad (64)$$

where

$$\lambda_1 = {c_0}^{4/3}\,(1-c_0)^{2/3}, \qquad \lambda_{2,3} = \mp\frac{1}{2}\,{c_0}^{4/3}\,(1-c_0)^{2/3}\left(\pm 1 + i\sqrt{3}\right),$$

and

$$\alpha = -\,(1+c_0)\left(1 + c_0 - {c_0}^2\right) - \frac{\sqrt[3]{1-c_0}}{{c_0}^{2/3}}\Delta,$$
$$\beta = -\,(1+c_0)\left(1 + c_0 - {c_0}^2\right) + \frac{1}{2}\,\frac{\sqrt[3]{1-c_0}}{{c_0}^{2/3}}\Delta,$$
$$\gamma = -\frac{\sqrt{3}}{2}\,\frac{\sqrt[3]{1-c_0}\,\left(\Delta - 2\,\sqrt[3]{1-c_0}\,(2-c_0)\,(1+{c_0}^2)\right)}{{c_0}^{2/3}},$$
$$\Delta = \sqrt[3]{c_0}\left(2 - {c_0}^3\right) - \sqrt[3]{1-c_0}\,(c_0-2)\left(1 + {c_0}^2\right).$$

Rule	c_n	Proof/conjecture
13	eq. (54)	conj.
32	eq. (56)	conj.
40	eq. (38)	conj.
44	eq. (64)	conj.
77	eq. (58)	conj.
78	eq. (60)	conj.
128	eq. (13)	proof [12]
132	eq. (14)	proof [12]
136	eq. (15)	proof [12]
140	eq. (16)	proof [5]
160	eq. (19)	proof
164	unknown	
168	eq. (31)	proof
172	eq. (62)	conj.
232	eq. (51)	conj.

Table 2: Density of ones c_n for arbitrary initial density for elementary rules asymptotically emulating identity.

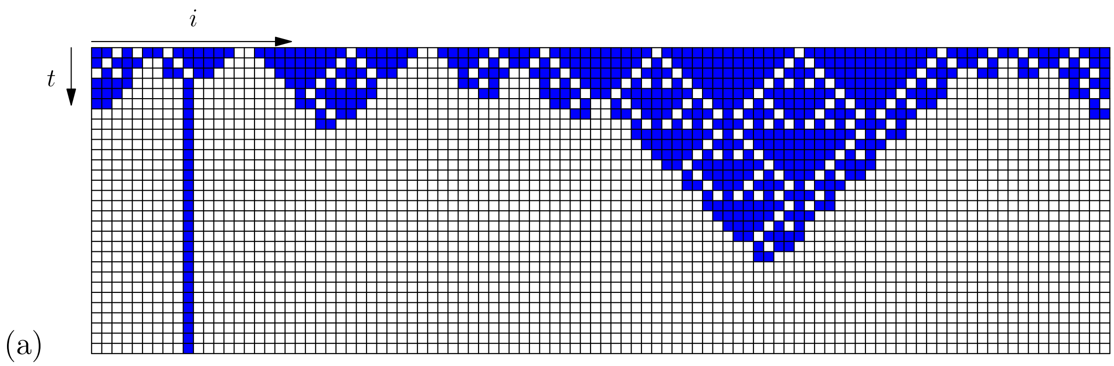
8. The remaining rule

Among 15 CA rules asymptotically emulating identity, we either proved or conjectured general expressions for c_n for 14 of them. In all cases, exponential convergence to c_∞ can be observed. What remains is only rule the 164 for which we were not able to find a closed form expression for density polynomials. We have attempted to find recurrence equations up to 6-th order for this rule, to no avail. One suspects that the reason for this is dynamics of rule 164, far more complicated than for other rule considered in this paper. In Figure 4(a), one can clearly see that in the spatio-temporal pattern generated by this rule exhibits the characteristic triangles of varying size. Similar triangles are frequently observed in complex "chaotic" rules.

Nevertheless, we have studied behaviour of c_n numerically. Figure 4(b) shows semi-logarithmic plots of $|c_n - c_\infty|$ as a function of n, obtained by averaging 100 runs of simulations using a lattice with 10^5 sites. The value of c_∞ in each case has been taken as the steady-state value, that is, the final value of c_n which was no longer changing. From this plots it is clear that the graphs of $|c_n - c_\infty|$ vs. n closely follow straight lines in all cases, strongly suggesting that the approach to the fixed point is also exponential, just like for the other 14 rules.

9. Conclusions

We have demonstrated that density polynomials are useful for computing density of ones after n iterations of a CA rule starting from a Bernoulli distribution. In many CA rules, patterns in density polynomials can be detected, and then formally proved, such as in the



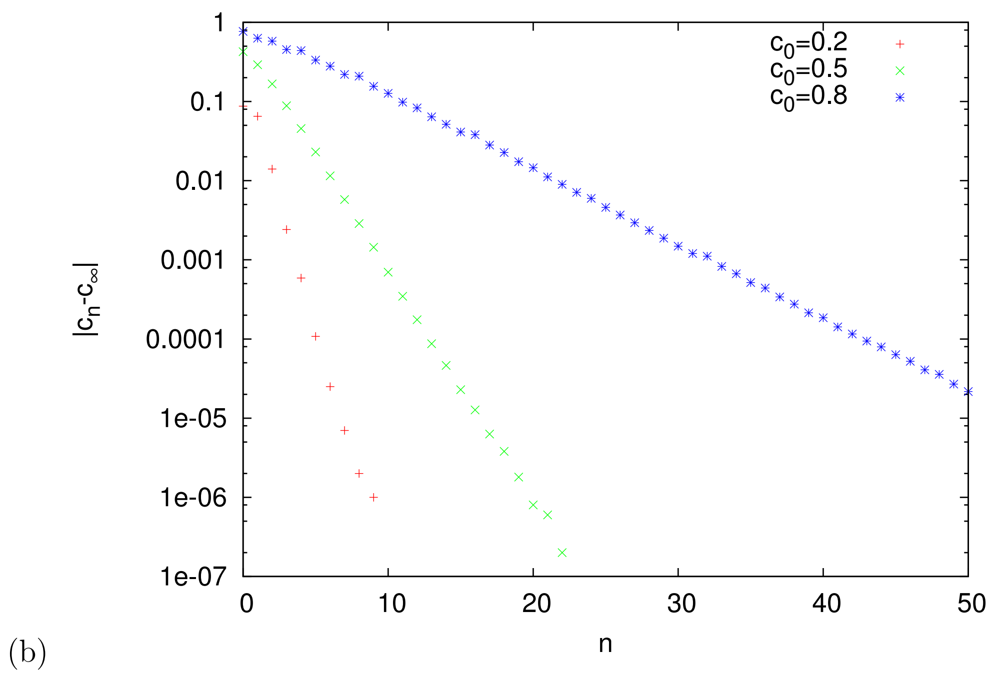


Figure 4: (a) Spatio-temporal pattern for rule 164, starting from random initial condition with density 0.85. (b) Density c_n as a function of n for rule 164. Lattice with 10^5 sites and periodic configurations was used. Each points corresponds to average of 100 experiments.

case of rule 160 and 168. In other cases, one can recognize in density polynomials known polynomial classes, such as generalized Lucas polynomials, and then conjecture closed-form expressions for c_n. Our results are summarized in Table 2. While at the moment we do not have formal proofs of the conjecture formulas, it is hoped that such proofs can eventually be constructed using methods similar to those presented here (for rules 160 and 168) or in [4]. Finally, inspection of Tables 1 and 2 and results we obtained for rules considered in this paper suggests an interesting possible conjecture.

Conjecture 1 *For any CA rule asymptotically emulating identity, the density of 1's after n iterations, starting from a Bernoulli distribution, is always in the form*

$$c_n \sim \sum_{i=1}^{k} a_n \lambda_i^n, \tag{65}$$

where a_i, λ_i *are constants which may only depend on the initial density* c_0*, and* $|\lambda_i| \leq 1$.

Note that some of the λ_i's can be complex, and then they come in conjugate pairs, like in rule 44 (eq. 64). When one of the λ_i's is equal to 1, then $c_\infty > 0$, otherwise $c_\infty = 0$.

Such behavior of c_n strongly resembles hyperbolicity in finitely-dimensional dynamical systems. Hyperbolic fixed points are common type of fixed points in dynamical systems. If the initial value is near the fixed point and lies on the stable manifold, the orbit of the dynamical system converges to the fixed point exponentially fast. One could argue that the exponential convergence to equilibrium observed in CA described in this paper is somewhat related to finitely-dimensional hyperbolicity. We suspect that the the finite-dimensional map, known as the local structure map [15], which approximates dynamics of a given CA, should posses a stable hyperbolic fixed point for every CA asymptotically emulating identity. This hypothesis is currently under investigation and will be discussed elsewhere.

Acknowledgments
H. Fukś acknowledges financial support from the Natural Sciences and Engineering Research Council of Canada (NSERC) in the form of Discovery Grant, and J.M. Gómez Soto acknowledges financial support from Research Council of México (CONACYT) and Research Council of Zacatecas (COZYT). This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca) and Compute/Calcul Canada.

References

[1] S. Wolfram, *Cellular Automata and Complexity: Collected Papers.* Addison-Wesley, Reading, Mass., 1994.

[2] H. Fukś, "Sequences of preimages in elementary cellular automata," *Complex systems* **14** (2003) 29–43, `arXiv:1305.6533`.

[3] H. Fukś and J. Haroutunian, "Catalan numbers and power laws in cellular automaton rule 14," *Journal of cellular automata* **4** (2009) 99–110, `arXiv:0711.1338`.

[4] H. Fukś, "Probabilistic initial value problem for cellular automaton rule 172," *DMTCS proc.* **AL** (2010) 31–44, `arXiv:1007.1026`.

[5] H. Fukś and A. Skelton, "Orbits of Bernoulli measure in asynchronous cellular automata," *Dis. Math. Theor. Comp. Science* **AP** (2011) 95–112.

[6] H. Fukś and A. Skelton, "Response curves for cellular automata in one and two dimensions – an example of rigorous calculations," *International Journal of Natural Computing Research* **1** (2010) 85–99, arXiv:1108.1987.

[7] M. Pivato, "Ergodic theory of cellular automata," in *Encyclopedia of Complexity and System Science*, R. A. Meyers, ed. Springer, 2009.

[8] T. Rogers and C. Want, "Emulation and subshifts of finite type in cellular automata," *Physica D* **70** (1994) 396–414.

[9] P. Kůrka and A. Maass, "Limit sets of cellular automata associated to probability measures," *Journal of Statistical Physics* **100** (2000) 1031–1047.

[10] P. Kůrka, "On the measure attractor of a cellular automaton," *Discrete and Continuous Dynamical Systems* (2005) 524 – 535.

[11] E. Formenti and P. Kůrka, "Dynamics of cellular automata in non-compact spaces," in *Encyclopedia of Complexity and System Science*, R. A. Meyers, ed. Springer, 2009.

[12] N. Boccara and H. Fukś, "Modeling diffusion of innovations with probabilistic cellular automata," in *Cellular Automata: A Parallel Model*, M. Delorme and J. Mazoyer, eds. Kluwer Academic Publishers, Dordrecht, 1998. arXiv:adap-org/9705004.

[13] R. P. Stanley, *Enumerative combinatorics*. Wadsworth Publ. Co., Belmont, CA, 1986.

[14] OEIS Foundation Inc., "The on-line encyclopedia of integer sequences." http://oeis.org/A084938.

[15] H. Fukś, "Construction of local structure maps for cellular automata," *J. of Cellular Automata* **7** (2013) 455–488, arXiv:1304.8035.