A generalization of carries processes and Eulerian numbers

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Abstract

We study a generalization of Holte's amazing matrix, the transition probability matrix of the Markov chains of the 'carries' in a *non-standard* numeration system. The stationary distributions are explicitly described by the numbers which can be regarded as a generalization of the Eulerian numbers and the MacMahon numbers. We also show that similar properties hold even for the numeration systems with the negative bases.

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1 Introduction and statements of results

The transition probability matrix so-called 'amazing matrix' of the Markov chain of the 'carries' has very nice properties [5], and has unexpected connection to the Markov chains of riffle shuffles [2, 3]. Diaconis and Fulman [3] studies a variant of the carries process, type B carries process. Novelli and Thibon studies the carries process in terms of noncommutative symmetric functions [7]. This paper studies a generalization of the carries process which includes Diaconis and Fulman's type B carries process as a special case. We study the transition probability matrices of the Markov chains of the carries in the numeration systems with non-standard digit sets. We show that the matrices have the eigenvectors which can be perfectly described by a generalization of Eulerian numbers and the MacMahon numbers [8, 6, 1, 3]. We also show that similar properties hold even for the numeration systems with negative bases.

1.1 Numeration system

Throughout the paper, b denotes a positive integer and $\mathcal{D} = \{d, d+1, \ldots, d+b-1\}$ denotes a set of integers containing 0. Therefore, $-b < d < b$. Then, we have a numeration system $(b, \mathcal{D})$: Suppose that an integer x has a representation of the form,

$$x = (x_k x_{k-1} \cdots x_0)_b \overset{def}{=} x_0 + x_1 b + x_2 b^2 + \cdots + x_k b^k, \quad x_0, x_1, \ldots, x_k \in \mathcal{D}, x_k \neq 0. \tag{1}$$

Then, it can be easily shown that this representation is uniquely determined for x and

$$\left\{ (x_k x_{k-1} \cdots x_0)_b \,\middle|\, k \geq 0, x_0, x_1, \ldots, x_k \in \mathcal{D} \right\} = \begin{cases} \mathbb{Z} & d \neq 0, -b+1, \\ \mathbb{N} & d = 0, \\ -\mathbb{N} & d = -b+1. \end{cases}$$

is closed under the addition, where $\mathbb{N}$ denotes the set of non-negative integers.

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1.2 Carries process

Let $\{X_{i,j}\}_{1\le i\le n, j\ge 0}$ be the set of independent random variables each of which is distributed uniformly over $\mathcal{D}$. Define the two stochastic processes $(A_0, A_1, A_2, \ldots)$, and $(C_0, C_1, C_2, \ldots)$ in the following way: $C_0 = 0$ with probability one. $(A_i)_{i\ge 0}$ is a sequence of $\mathcal{D}$-valued random variables satisfying

$$A_i \equiv C_i + X_{1,i} + \cdots + X_{n,i} \pmod b, \quad i = 0, 1, 2, \ldots.$$

and

$$C_i = \frac{C_{i-1} + X_{1,i-1} + \cdots + X_{n,i-1} - A_{i-1}}{b}, \quad i = 1, 2, 3, \ldots.$$

(See Figure 1.) It is obvious that $(C_0, C_1, C_2, \ldots)$ is a Markov process, which we call the *carries process* with n summands or simply n-*carry process* over $(b, \mathcal{D})$.

	$\cdots$	C_4	C_3	C_2	C_1	C_0
	$\cdots$	$X_{1,4}$	$X_{1,3}$	$X_{1,2}$	$X_{1,1}$	$X_{1,0}$
	$\cdots$	$X_{2,4}$	$X_{2,3}$	$X_{2,2}$	$X_{2,1}$	$X_{2,0}$
	$\cdots$	$X_{3,4}$	$X_{3,3}$	$X_{3,2}$	$X_{3,1}$	$X_{3,0}$
	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
+)	$\cdots$	$X_{n,4}$	$X_{n,3}$	$X_{n,2}$	$X_{n,1}$	$X_{n,0}$
	$\cdots$	A_4	A_3	A_2	A_1	A_0

Figure 1: Carries process

1.3 A generalization of Eulerian numbers

Let $p \ge 1$ be a real number and n a positive integer. Then we define an array of numbers $v_{i,j}^{(p)}(n)$ for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, n+1$ by

$$v_{i,j}^{(p)}(n) = \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} [p(j-r)+1]^{n-i}, \tag{2}$$

and define $v_{i,-1}^{(p)}(n) = 0$. We denote

$$\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle_p = v_{0,j}^{(p)}(n),$$

which can be regarded as a generalization of the Eulerian numbers. In fact, $\left\{ \left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle_p \right\}$ forms the array of the ordinary Eulerian numbers when $p = 1$, and MacMahon numbers [8, 6, 1, 3] when $p = 2$.

1.4 Statement of the result

Throughout the paper, $\Omega = \Omega_n(b, \mathcal{D})$ denotes the state space of the n-carry process over $(b, \mathcal{D})$, that is, the set of possible values of carries, and $p_{i,j}$ denotes the transition probability $\Pr(C_{i+1} = j \mid C_i = i)$ for $i, j \in \Omega_n(b, \mathcal{D})$. For computational convenience, it is desirable for the transition probability matrix to have indices starting from 0. We define $\tilde{p}_{i,j} = p_{i+s,j+s}$, where s is the minimal element of $\Omega_n(b, \mathcal{D})$. Then, we define the matrix P by

$$P = (\tilde{p}_{i,j})_{0\le i,j\le \#\Omega-1}, \tag{3}$$

where $\#\Omega$ denotes the size of the state space, which is explicitly computed in Lemma 1. P is the central object of this paper.

Remark 1. *As we will show in the later sections, this matrix P which we regard as a generalization of Holte's amazing matrix is determined only by b, n and p, and therefore these amazing matrices with same n and p form a commutative family. Holte's amazing matrix corresponds to the case when $p = 1$ and Diaconis and Fulman's type B carries process corresponds to the case when $p = 2$.*

Theorem 1. *Let $\Omega = \Omega_n(b, \mathcal{D})$ be the state space of the n-carry process over $(b, \mathcal{D})$, and $m = \#\Omega_n(b, \mathcal{D})$. let p be defined by*

$$p = \begin{cases} \frac{1}{\{(n-1)(-l)\}} & (n-1)l \notin \mathbb{Z}, \\ 1 & (n-1)l \in \mathbb{Z}, \end{cases} \tag{4}$$

where $l = d/(b-1)$ and $\{x\} = x - \lfloor x \rfloor$, and let $V = \left(v_{i,j}^{(p)}(n) \right)_{0 \le i,j \le m-1}$. Then, the transition probability matrix P is diagonalized by V:

$$VPV^{-1} = \operatorname{diag}\left(1, b^{-1}, \ldots, b^{-(m-1)}\right).$$

In particular, by Lemma 5*, the probability vector π of the stationary distribution of the carries process is*

$$\pi = (\pi(s), \pi(s+1), \ldots, \pi(s+m-1)) = \frac{1}{p^n n!} \left(\left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle_p, \ldots, \left\langle \begin{matrix} n \\ m-1 \end{matrix} \right\rangle_p \right)$$

Remark 2. *It is remarkable that our amazing matrix has the eigenvalues of the same form $1, 1/b, 1/b^2, \ldots$ as those of Holte's amazing matrix.*

Corollary 1. *Let S_n be the sum of n independent random variables each of which is distributed uniformly over the unit interval $[0, 1]$. Then, for all positive real numbers $p \ge 1$ and integers k, the probability of S_n being in the interval $\frac{1}{p} + [k-1, k]$ is*

$$\Pr\left(S_n \in \frac{1}{p} + [k-1, k] \right) = \frac{1}{p^n n!} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_p. \tag{5}$$

Remark 3. *This corollary can be derived directly from the formula of the distribution of sums of independent uniform random variables in* [4]*, and it is shown for the case $p = 2$ in* [3]*.*

Example 1. *Let $p \ge 1$ be a real number. As will be shown in the later section, the array of generalized Eulerian numbers satisfies the following recursive relations*

$$\left\langle \begin{matrix} n+1 \\ k \end{matrix} \right\rangle_p = (pk+1) \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_p + (p(n+1-k)-1) \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle_p.$$

and the boundary conditions

$$\left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle_p = 1, \quad \text{and} \quad \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_p = 0 \ \text{ for } \ k > n.$$

(See Figure 2.)

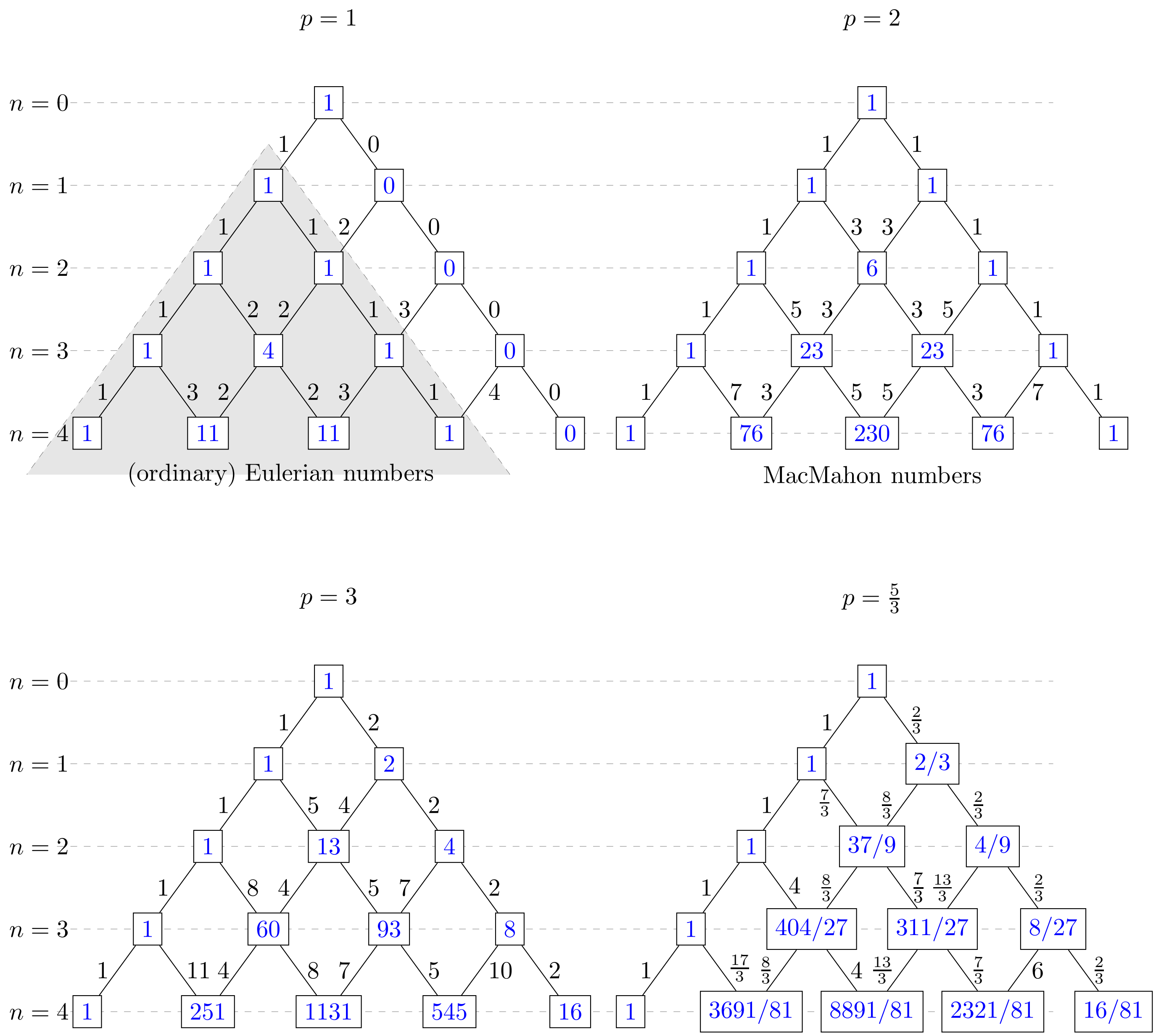
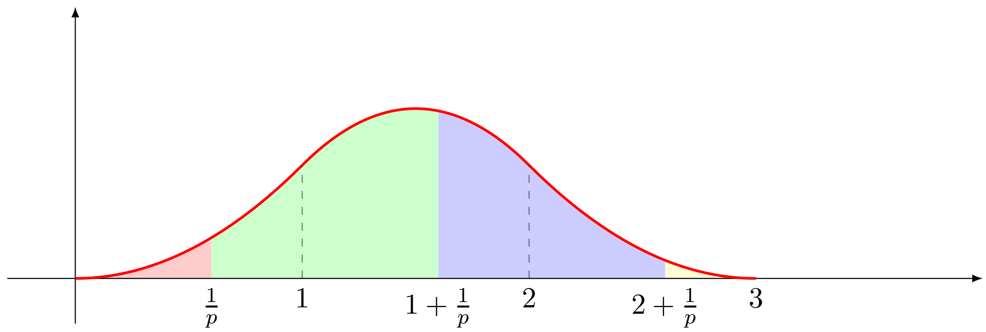


Figure 2: Arrays of generalized Eulerian numbers for $p = 1$ (upper left), $p = 2$ (upper right), $p = 3$ (lower left), and $p = \frac{5}{3}$ (lower right)

The probability density function of S_3 described in Theorem 1 is

$$f(x) = \begin{cases} x^2/2 & \text{if } 0 \le x < 1, \\ -\left(x - \frac{3}{2}\right)^2 + \frac{3}{4} & \text{if } 1 \le x < 2, \\ (x-3)^2/2 & \text{if } 2 \le x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$



p	$\Pr(S_3 \in 1/p + [-1, 0])$	$\Pr(S_3 \in 1/p + [0, 1])$	$\Pr(S_3 \in 1/p + [1, 2])$	$\Pr(S_3 \in 1/p + [2, 3])$
1	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	0
2	$\frac{1}{48}$	$\frac{23}{48}$	$\frac{23}{48}$	$\frac{1}{48}$
3	$\frac{1}{162}$	$\frac{60}{162}$	$\frac{93}{162}$	$\frac{8}{162}$
5/3	$\frac{27}{750}$	$\frac{404}{750}$	$\frac{311}{750}$	$\frac{8}{750}$

The probability vectors appear in the third rows of the triangles of the generalized Eulerian numbers (Figure 2).

2 Proof

2.1 State space and transition probability

Lemma 1. *Let $\Omega = \Omega_n(b, \mathcal{D})$ be the state space of the n-carry process over the numeration system $(b, \mathcal{D})$. Then, $\Omega = \{s, s+1, \ldots, t\}$ with*

$$s = -\lceil (n-1)(-l) \rceil = \lfloor (n-1)l \rfloor, \quad t = \lceil (n-1)(l+1) \rceil,$$

where $l = d/(b-1)$. Therefore, the size of the state space Ω is

$$\#\Omega = \begin{cases} n+1 & (n-1)l \notin \mathbb{Z}, \\ n & (n-1)l \in \mathbb{Z}. \end{cases}$$

Proof. Suppose that we add n numbers,

$$(x_{1,\nu}x_{1,\nu-1}\cdots x_{1,1}x_{1,0})_b, (x_{2,\nu}x_{2,\nu-1}\cdots x_{2,1}x_{2,0})_b, \ldots, (x_{n,\nu}x_{n,\nu-1}\cdots x_{n,1}x_{n,0})_b,$$

and get the sum $(a_{\nu+1}, a_\nu, \ldots, a_1, a_0)_b$ and the carries $c_0, c_1, c_2, \ldots$. That is, $c_0 = 0$ and

$$c_{i+1} = \frac{c_i + x_{1,i} + \cdots + x_{n,i} - a_i}{b},$$

where $a_i \in \mathcal{D}$ and $a_i \equiv c_i + x_{1,i} + \cdots + x_{n,i} \pmod{b}$. Let F be defined by

$$F = \{x_1 b^{-1} + x_2 b^{-2} + \cdots + x_m b^{-m} \mid x_i \in \mathcal{D}_b, m > 0 \in \mathbb{Z}\}.$$

Then, F is a dense subset of the interval $(l, l+1)$, where $l = d/(b-1)$.

$$\begin{aligned} c_i = c &\iff (x_{1,i-1}\cdots x_{1,0})_b + \cdots + (x_{n,i-1}\cdots x_{n,0})_b = cb^i + (a_{i-1}\cdots a_0)_b \\ &\iff \frac{(x_{1,i-1}\cdots x_{1,0})_b}{b^i} + \cdots + \frac{(x_{n,i-1}\cdots x_{n,0})_b}{b^i} = c + \frac{(a_{i-1}\cdots a_0)_b}{b^i} \end{aligned}$$

Since $\frac{(x^1_{i-1}\cdots x^1_0)_b}{b^i}, \ldots, \frac{(x^n_{i-1}\cdots x^n_0)_b}{b^i}, \frac{(a_{i-1}\cdots a_0)_b}{b^i} \in F$, we have $c \in nF - F \subset ((n-1)l-1, (n-1)(l+1)+1)$. Conversely, if $c \in ((n-1)l-1, (n-1)(l+1)+1) \cap \mathbb{Z}$, then $c + F \subset nF$. Therefore s is the smallest integer strictly greater than $(n-1)l-1$ and t is the greatest integer strictly smaller than $(n-1)(l+1)+1$, that is,

$$s = -\lceil (n-1)(-l) \rceil, \quad t = \lceil (n-1)(l+1) \rceil.$$

□

Theorem 2.

$$p_{i,j} = \frac{1}{b^n} \sum_{k=0}^{j - \left\lfloor \frac{d(n-1)+i}{b} \right\rfloor} (-1)^k \binom{n+1}{k} \binom{n + b(j+1-k) - d(n-1) - i - 1}{n}, \qquad i, j \in \Omega_n(b, \mathcal{D}).$$

Proof. This proof is essentially the same as that of Holte [5] for the case when $\mathcal{D} = \{0, 1, \ldots, b-1\}$. $p_{i,j}$ is the probability of $C_{k+1} = j$ under $C_k = i$ for some $k > 0$. $C_{k+1} = j$ and $C_k = i$ implies there exists a number $a \in \mathcal{D}$, such that,

$$j = \frac{i + X_{1,k} + X_{2,k} + \cdots + X_{n,k} - a}{b}.$$

We count the number N of the solutions $(x_1, x_2, \ldots, x_n, a) \in \mathcal{D}^{n+1}$ of the equation

$$bj + a = i + x_1 + x_2 + \cdots + x_n.$$

This is equal to the number of solutions $(x_1, x_2, \ldots, x_n, y) \in \mathcal{D}^n \times \{0, 1, \ldots, b-1\}$ for the equation

$$bj + d + b - 1 - i = x_1 + x_2 + \cdots + x_n + y.$$

By adding d to the both sides, N is equal to the number of solutions $(x_1, \ldots, x_n, z) \in \mathcal{D}^{n+1}$ for

$$b(j+1) + 2d - 1 - i = x_1 + x_2 + \cdots + x_n + z.$$

Thus, N is the coefficient of $x^{b(j+1)+2d-1-i}$ in $(x^d + x^{d+1} + \cdots + x^{d+b-1})^{n+1}$. Since

$$\begin{aligned} (x^d + x^{d+1} + \cdots + x^{d+b-1})^{n+1} &= \left\{ x^d (1 + x + \cdots + x^{b-1}) \right\}^{n+1} \\ &= x^{d(n+1)} \left(\frac{1 - x^b}{1 - x} \right)^{n+1} \\ &= x^{d(n+1)} \left(\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} x^{bk} \right) \left(\sum_{r=0}^{\infty} \binom{n+r}{n} x^r \right), \end{aligned}$$

we have

$$
\begin{aligned}
N &= \sum_{\substack{r,\,k\,\geq\,0 \\ d(n+1)+bk+r\,=\,b(j+1)+2d-1-i}} (-1)^k \binom{n+1}{k}\binom{n+r}{n} x^r \\
&= \sum_{\substack{k\,\geq\,0 \\ b(j+1)+2d-1-i-d(n+1)-bk\,\geq\,0}} (-1)^k \binom{n+1}{k}\binom{n+b(j+1)+2d-1-i-d(n+1)-bk}{n} x^r \\
&= \sum_{k=0}^{j+1+\left\lfloor -\frac{1+i+d(n-1)}{b}\right\rfloor} (-1)^k \binom{n+1}{k}\binom{n+b(j+1)-1-i-d(n-1)-bk}{n} x^r.
\end{aligned}
$$

$j+1+\left\lfloor -\frac{1+i+d(n-1)}{b}\right\rfloor = j-\left\lfloor \frac{i+d(n-1)}{b}\right\rfloor$, $p_{i,j}=\frac{N}{b^n}$ and the theorem follows. □

Lemma 2. *Let $P=(\tilde{p}_{i,j})$ be the matrix defined by (3) and let p be defined by (4). Then,*

$$
\tilde{p}_{i,j} = \frac{1}{b^n}\sum_{r=0}^{j}(-1)^r\binom{n+1}{r}\binom{n+b(j-r)+\frac{b-1}{p}-i}{n}. \tag{6}
$$

Proof. Let s be the minimal element of the state space $\Omega_n(b,\mathcal{D})$ of the n-carry process. Then, recall that

$$
s = -\left\lceil (n-1)\frac{-d}{b-1}\right\rceil = \left\lfloor (n-1)\frac{d}{b-1}\right\rfloor.
$$

Therefore,

$$
\tilde{p}_{i,j} = p_{i+s,j+s} = \frac{1}{b^n}\sum_{r\geq 0}(-1)^r\binom{n+1}{r}\binom{n+b(j+s+1-r)-d(n-1)-i-s-1}{n}
$$

$$
\begin{aligned}
& n+b(j+s+1-r)-d(n-1)-i-s-1 \\
=\ & n+(b-1)s+b(j+1-r)-d(n-1)-i-1 \\
=\ & n+(b-1)\left\lfloor (n-1)\frac{-d}{b-1}\right\rfloor + b(j+1-r)-d(n-1)-i-1 \\
=\ & n+(b-1)\frac{(n-1)d-(n-1)d \pmod{b-1}}{b-1} + b(j+1-r)-d(n-1)-i-1 \\
=\ & n+(b-1)-(n-1)d \pmod{b-1} + b(j-r)-i \\
=\ & n+(b-1)\frac{(b-1)-(n-1)d \pmod{b-1}}{b-1} + b(j-r)-i \\
=\ & \begin{cases} n+(b-1)\left\{\frac{(n-1)(-d)}{b-1}\right\}+b(j-r)-i & \frac{(n-1)(-d)}{b-1}\in\mathbb{Z}, \\ n+(b-1)+b(j-r)-i & \frac{(n-1)(-d)}{b-1}\notin\mathbb{Z}. \end{cases}
\end{aligned}
$$

Here, $x \pmod N$ denotes the integer $y\in\{0,1,\ldots,N-1\}$ such that $x-y\in N\mathbb{Z}$. Then, we calculate a common upper bound of the range of the summation in (6):

$$
n+b(j-r)+\frac{b-1}{p}-i \geq n \Longleftrightarrow r \leq j+\frac{b-1}{pb}-\frac{i}{p}.
$$

Since $p\geq 1$ and $b>1$, $j+\frac{b-1}{pb}-\frac{i}{p}\leq j$. □

2.2 Generalized Eulerian numbers

Lemma 3.

$$v^{(p)}_{i,n+1}(n) = 0.$$

Proof.

$$v^{(p)}_{i,n+1}(n) = \sum_{r=0}^{n+1}(-1)^r \binom{n+1}{r}[p(n+1-r)+1]^{n-i}$$

This is a linear combination of

$$\sum_{r=0}^{n+1}(-1)^r \binom{n+1}{r} r^k = 0, \qquad k = 0, 1, \ldots, n.$$

□

Lemma 4.

$$v^{(p)}_{i,j}(n) = [p(n+1-j)-1]v^{(p)}_{i,j-1}(n-1) + (pj+1)v^{(p)}_{i,j}(n-1). \tag{7}$$

Proof. The first term T_1 of right hand side of (7) can be rewritten as

$$T_1 = \sum_{k=1}^{j}(-1)^{k-1}\binom{n}{k-1}[p(n+1-j)-1][p(j-k)+1]^{n-1-i},$$

and the second term T_2

$$T_2 = \sum_{k=1}^{j}(-1)^k \binom{n}{k}(pj+1)[p(j-k)+1]^{n-1-i} + (pj+1)(pj+1)^{n-1-i}.$$

Thus the right hand side of (7) is

$$\begin{aligned} T_1 + T_2 &= \sum_{k=1}^{j}(-1)^k \left\{-\binom{n}{k-1}[p(n+1-j)-1] + \binom{n}{k}(pj+1)\right\}[p(j-k)+1]^{n-1-i} \\ &\qquad +(pj+1)^{n-i} \\ &= \sum_{k=1}^{j}(-1)^k \binom{n+1}{k}[p(j-k)+1]^{n-i} + (pj+1)^{n-i} \\ &= v^{(p)}_{i,j}(n). \end{aligned}$$

□

Lemma 5.

$$\sum_{j=0}^{n} v^{(p)}_{i,j}(n) = \begin{cases} p^n n! & \textit{if } i = 0, \\ 0 & \textit{if } i > 0. \end{cases}$$

Proof. By Lemma 4, we have

$$\begin{aligned}
\sum_{j=0}^{n} v_{i,j}^{(p)}(n) &= \sum_{j=0}^{n} \Big\{ [p(n+1-j)-1] v_{i,j-1}^{(p)}(n-1) + (pj+1) v_{i,j}^{(p)}(n-1) \Big\} \\
&= \sum_{j=0}^{n} [p(n+1-j)-1] v_{i,j-1}^{(p)}(n-1) + \sum_{j=0}^{n} (pj+1) v_{i,j}^{(p)}(n-1) \\
&= \sum_{j=0}^{n-1} [p(n-j)-1] v_{i,j}^{(p)}(n-1) + \sum_{j=0}^{n-1} (pj+1) v_{i,j}^{(p)}(n-1) \\
&= \sum_{j=0}^{n-1} pm v_{i,j}^{(p)}(n-1) \\
&= pn \sum_{j=0}^{n-1} v_{i,j}^{(p)}(n-1) \\
&= \begin{cases} p^n n! & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}
\end{aligned}$$

□

The following Proposition 1 shows a symmetry of the generalized Eulerian number.

Proposition 1. *Let n be a positive integer. Then*

$$v_{i,n-1-j}^{(1)} = (-1)^i v_{i,j}^{(1)}(n) \quad \textit{for } 0 \le j \le n-1.$$

Let $p > 1$ and p^* be the real number satisfying

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

Then,

$$v_{i,n-j}^{(p^*)}(n) = (-1)^i \left(\frac{p^*}{p} \right)^{n-i} v_{i,j}^{(p)}(n) \quad \textit{for } 0 \le j \le n.$$

Proof. We show the proof only for the second part. The first part can be proved in the same manner.

If $p > 1$ then $p^* = p/(p-1)$.

$$\begin{aligned}
v^{(p^*)}_{i,n-j}(n) &= \sum_{k=0}^{n-j}(-1)^k\binom{n+1}{k}(p^*(n-j-k)+1)^{n-i} \\
&= \sum_{k=n+1-j}^{n+1}(-1)^k\binom{n+1}{k}(p^*(n-j-k)+1)^{n-i} \\
&= -\sum_{k'=0}^{j}(-1)^{n+1-k'}\binom{n+1}{n+1-k'}(p^*(n-j-(n+1-k'))+1)^{n-i} \\
&= -\sum_{k'=0}^{j}(-1)^{n+1-k'}\binom{n+1}{k'}\left(\frac{p}{p-1}(k'-j-1)+1\right)^{n-i} \\
&= -\sum_{k=0}^{j}(-1)^{n+1-k}\binom{n+1}{k}\left(\frac{-1}{p-1}\right)^{n-i}(p(j-k+1)-p+1)^{n-i} \\
&= (-1)^n\left(\frac{-1}{p-1}\right)^{n-i}\sum_{k=0}^{j}(-1)^k\binom{n+1}{k}(p(j-k)+1)^{n-i} \\
&= \frac{(-1)^i}{(p-1)^{n-i}}\sum_{k=0}^{j}(-1)^k\binom{n+1}{k}(p(j-k)+1)^{n-i}
\end{aligned}$$

□

2.3 Left eigenvectors

Proof of Theorem 1. The proof is essentially the same as that of Holte [5]. It suffices to show that

$$\sum_{k=0}^{m-1} v^{(p)}_{i,k}(n)\tilde{p}_{k,j} = \frac{1}{b^i}v^{(p)}_{i,j}(n).$$

We prove the theorem for the case in which $p \neq 1$, i.e., $m = n+1$, and the other case can be proved in the same manner. By Lemma 2 we have

$$\tilde{p}_{k,j} = \frac{1}{b^n}\sum_{r=0}^{j}(-1)^r\binom{n+1}{r}\binom{n+K(j,r)-k}{n},$$

where we put $K(j,r) = b(j-r)+\frac{b-1}{p}$ for the simplicity of the notation.

$$\begin{aligned}
\sum_{k=0}^{n} v^{(p)}_{i,k}(n)\tilde{p}_{k,j} &= \sum_{k=0}^{n}\frac{1}{b^n}\sum_{r=0}^{j}(-1)^r\binom{n+1}{r}\binom{n+K(j,r)-k}{n}v^{(p)}_{i,k}(n) \\
&= \frac{1}{b^n}\sum_{r=0}^{j}(-1)^r\binom{n+1}{r}\sum_{k=0}^{K(j,r)}\binom{n+K(j,r)-k}{n}v^{(p)}_{i,k}(n) \\
&= \frac{1}{b^n}\sum_{r=0}^{j}(-1)^r\binom{n+1}{r}\{pK(j,r)+1\}^{n-i}.
\end{aligned}$$

The third equality in the above transformation is derived as follows: First recall that $v_{i,k}^{(p)}(n)$ is the coefficient of x^k in

$$\left(\sum_{\nu=0}^{n+1}(-1)^{\nu}\binom{n+1}{\nu}x^{\nu}\right)\left(\sum_{\mu=0}^{\infty}(p\mu+1)^{n-i}x^{\mu}\right)=(1-x)^{n+1}\left(\sum_{\mu=0}^{\infty}(p\mu+1)^{n-i}x^{\mu}\right),$$

and

$$\frac{1}{(1-x)^{n+1}}=\sum_{k=0}^{\infty}\binom{n+k}{n}x^k.$$

Therefore, $\sum_{k=0}^{K(j,r)}\binom{n+K(j,r)-k}{n}v_{i,k}^{(p)}(n)$ is the coefficient of $x^{K(j,r)}$ in $\sum_{\mu=0}^{\infty}(p\mu+1)^{n-i}x^{\mu}$.

It can be easily confirmed that $pK(j,r)+1=b(p(j-r)+1)$ which completes the proof. □

Theorem 1 gives a way of finding a numeration system $(b,\mathcal{D})$ whose n-carry process has the stationary distribution of the form

$$\pi=(\pi(s),\pi(s+1),\dots,\pi(s+m-1))=\frac{1}{p^n n!}\left(\left\langle {n \atop 0} \right\rangle_p,\left\langle {n \atop 1} \right\rangle_p,\dots,\left\langle {n \atop m-1} \right\rangle_p\right)$$

for any given n and rational $p\geq 1$. For instance, if $p=\frac{K}{L}$ where K and L are coprime positive integers such that $K\geq L$, then we can choose b and d as

$$b=(n-1)K+1,\qquad d=-L. \tag{8}$$

Example 2. *We construct numeration systems for $p=2$ and $5/3$.*

n	b	$\mathcal{D}$	P	V
2	3	$\{-1,0,1\}$	$\frac{1}{3^2}\begin{pmatrix}3&6&0\\1&7&1\\0&6&3\end{pmatrix}$	$\begin{pmatrix}1&6&1\\1&0&-1\\1&-2&1\end{pmatrix}$
3	5	$\{-1,0,1,2,3\}$	$\frac{1}{5^3}\begin{pmatrix}10&80&35&0\\4&68&52&1\\1&52&68&4\\0&35&80&10\end{pmatrix}$	$\begin{pmatrix}1&23&23&1\\1&5&-5&-1\\1&-1&-1&1\\1&-3&3&-1\end{pmatrix}$
4	7	$\{-1,0,1,2,3,4,5\}$	$\frac{1}{7^4}\begin{pmatrix}35&826&1330&210&0\\15&640&1420&325&1\\5&470&1451&470&5\\1&325&1420&640&15\\0&210&1330&826&35\end{pmatrix}$	$\begin{pmatrix}1&76&230&76&1\\1&22&0&-22&-1\\1&4&-10&4&1\\1&-2&0&2&-1\\1&-4&6&-4&1\end{pmatrix}$

Table 1: Amazing matrices with $p=2$

n	b	$\mathcal{D}$	P	V
2	6	$\{-3,-2,\ldots,2\}$	$\frac{1}{6^2}\begin{pmatrix}10 & 25 & 1\\ 6 & 27 & 3\\ 3 & 27 & 6\end{pmatrix}$	$\begin{pmatrix}1 & 37/9 & 4/9\\ 1 & -1/3 & -2/3\\ 1 & -2 & 1\end{pmatrix}$
3	11	$\{-3,-2,\ldots,7\}$	$\frac{1}{11^4}\begin{pmatrix}84 & 804 & 439 & 4\\ 56 & 745 & 520 & 10\\ 35 & 676 & 600 & 20\\ 20 & 600 & 676 & 35\end{pmatrix}$	$\begin{pmatrix}1 & 404/27 & 311/27 & 8/27\\ 1 & 28/9 & -11/3 & -4/9\\ 1 & -4/3 & -1/3 & 2/3\\ 1 & -3 & 3 & -1\end{pmatrix}$
4	16	$\{-3,-2,\ldots,12\}$	$\frac{1}{16^4}\begin{pmatrix}715 & 20176 & 37390 & 7240 & 15\\ 495 & 18000 & 38326 & 8680 & 35\\ 330 & 15900 & 38960 & 10276 & 70\\ 210 & 13900 & 39280 & 12020 & 126\\ 126 & 12020 & 39280 & 13900 & 210\end{pmatrix}$	$\begin{pmatrix}1 & 3691/81 & 8891/81 & 2321/81 & 16/81\\ 1 & 377/27 & -31/9 & -101/9 & -8/27\\ 1 & 19/9 & -61/9 & 29/9 & 4/9\\ 1 & -7/3 & 1 & 1 & -2/3\\ 1 & -4 & 6 & -4 & 1\end{pmatrix}$

Table 2: Amazing matrices with $p = \frac{5}{3}$

2.4 Sum of independent uniform random variables

Proof of Corollary 1. Let $X_{i,j} (i = 1, 2, \ldots, n, j = 1, 2, \ldots)$ be independent random variables each distributed uniformly on $\mathcal{D}$. Then, for each integer $k \geq 1$, the random variables

$$X_i^{(k)} = \frac{X_{i,1}}{b} + \frac{X_{i,2}}{b^2} + \cdots + \frac{X_{i,k}}{b^k}, \qquad i = 1, 2, \ldots, n$$

are independent random variables uniformly distributed over the set

$$R_k = \left\{ \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots + \frac{x_k}{b^k} \;\middle|\; x_i \in \mathcal{D} \right\}.$$

Therefore

$$\lim_{k\to\infty} \Pr(X_i^{(k)} \in [a, b]) = b - a, \qquad \text{for } a \geq b \in [l, l+1] \text{ and } i = 1, 2, \ldots, n.$$

Let $X_1, X_2, \ldots, X_n$ be independent random variables each of which is distributed uniformly over $[l, l+1]$. Then, for any integer $c \in \Omega$,

$$\lim_{k\to\infty} \Pr(X_1^{(k)} + X_2^{(k)} + \cdots + X_n^{(k)} \in c + [l, l+1]) = \Pr(X_1 + X_2 + \cdots X_n \in c + [l, l+1]).$$

Since

$$\lim_{k\to\infty} \Pr(X_1^{(k)} + X_2^{(k)} + \cdots + X_n^{(k)} \in c + [l, l+1]) = \pi(c),$$

we have

$$\pi(c) = \Pr(X_1 + X_2 + \cdots X_n \in c + [l, l+1]).$$

Let $p > 1$ be a non-integral rational number, and suppose that we choose b and d so that

$$p = \frac{1}{\{(n-1)(-l)\}}$$

holds, which is always possible by (8). Then, we have

$$\frac{1}{p} = (n-1)(-l) - \lfloor (n-1)(-l) \rfloor \iff nl + \frac{1}{p} = l - \lfloor (n-1)(-l) \rfloor = l + (1 - \lceil (n-1)(-l) \rceil)$$
$$\iff nl + \frac{1}{p} + [k-1,k] = (s+k) + [l, l+1].$$

Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables each uniformly distributed on the unit interval $[0,1]$.

$$\begin{aligned}
\Pr(Y_1 + Y_2 + \ldots + Y_n \in \frac{1}{p} + [k-1,k]) &= \Pr((Y_1 + l) + \ldots + (Y_n + l) \in nl + \frac{1}{p} + [k-1,k]) \\
&= \Pr(X_1 + X_2 + \cdots + X_l \in nl + \frac{1}{p} + [k-1,k]) \\
&= \Pr(X_1 + X_2 + \cdots + X_l \in (s+k) + [l, l+1]) \\
&= \pi(s+k) \\
&= \frac{1}{p^n n!} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_p .
\end{aligned}$$

When p is an integer, the statement is proved by a similar argument. Since both sides of the equation (5) are continuous function of p, the statement of the theorem holds when p is irrational. □

Remark 4. *Corollary* 1 *gives a different proof and a new interpretation for* Proposition 1 *for the case* $i = 0$.

3 Negative base

In this section, we consider carries processes over the numeration systems with the negative bases. Let $b > 1$ be an integer and $\mathcal{D} = \{d, d+1, \ldots, d+b-1\}$ a set of integers containing 0. Suppose an integer x can be represented in the form:

$$x = (x_l x_{l-1} \cdots x_0)_{-b} = x_l(-b)^l + x_{l-1}(-b)^{l-1} + \cdots + x_1(-b) + x_0,$$

where l is a non-negative integer and $x_l \neq 0$. Then this representation is unique and the set

$$\{(x_l x_{l-1} \ldots x_0)_{-b} \,|\, l \geq 0, x_k \in \mathcal{D}\}.$$

is closed under the addition. We can define the n-carry process over the numeration system $(-b, \mathcal{D})$ in the same manner as the positive base case. Let $\{X_{i,j}\}_{1 \leq i \leq n, j \geq 0}$ be a set of i.i.d. random variables each distributed uniformly on $\mathcal{D}$. Then the carries process $(C_0, C_1, C_2, \ldots)$ is defined as follows: $\Pr(C_0 = 0) = 1$ and

$$C_i = \frac{C_{i-1} + X_{1,i-1} + \cdots + X_{n,i-1} - A_{i-1}}{-b} \qquad \text{for } i > 0,$$

where A_j is $\mathcal{D}$-valued. These carries processes have the properties similar to those of the positive base cases. The proofs of the following Lemma 6, Theorem 3, and Lemma 7 are similar to those of Lemma 1,Theorem 2, and Lemma 2. The proof of Theorem 4 needs an additional combinatorial argument.

Lemma 6. *Let $\Omega = \Omega_n(-b, \mathcal{D})$ be the state space of the n-carry process over the numeration system $(-b, \mathcal{D})$. Then, $\Omega = \{s, s+1, \dots, t\}$ with*

$$s = -\lceil (n-1)(-l) \rceil = \lfloor (n-1)l \rfloor, \quad t = \lceil (n-1)(l+1) \rceil,$$

where $l = (-d-b)/(b+1)$. Therefore, the size of the state space Ω is

$$\#\Omega = \begin{cases} n+1 & (n-1)l \notin \mathbb{Z}, \\ n & (n-1)l \in \mathbb{Z}. \end{cases}$$

Theorem 3. *Let $i, j \in \Omega(-b, \mathcal{D})$. Then the transition probability $p_{i,j} = \Pr(C_{t+1} = j \,|\, C_t = i)$ for $t > 0$ is*

$$p_{i,j} = \frac{1}{b^n} \sum_{r=0}^{-j+1+\left\lfloor \frac{-i-1+(1-n)d}{b} \right\rfloor} (-1)^r \binom{n+1}{r} \binom{n - b(j-1+r) - i - 1 + (1-n)d}{n}.$$

We denote

$$\tilde{p}_{i,j} = p_{i+s, j+s}.$$

where s is the minimal element of the state space $\Omega(-b, \mathcal{D})$ of the n-carry process over $(-b, \mathcal{D})$.

Lemma 7. *Let p be defined by*

$$p = \begin{cases} \frac{1}{\{(n-1)l\}} & (n-1)l \notin \mathbb{Z} \quad (\Leftrightarrow m = n+1), \\ 1 & (n-1)l \in \mathbb{Z} \quad (\Leftrightarrow m = n). \end{cases} \tag{9}$$

where $l = (-d-b)/(b+1)$ and $m = \#\Omega_n(-b, \mathcal{D})$. Then,

$$\tilde{p}_{i,j} = \frac{1}{b^n} \sum_{r=0}^{n-j} (-1)^r \binom{n+1}{r} \binom{n + b(n+1-j-r) - \frac{b+1}{p} - i}{n}.$$

Theorem 4. *Let $P = (\tilde{p}_{i,j})_{0 \le i,j \le m-1}$ be the transition probability matrix of the n-carry process over the numeration system $(-b, \mathcal{D})$,and $m = \#\Omega(-b, \mathcal{D})$ be the size of the state space. Let p be defined by*

$$p = \begin{cases} \frac{1}{\{(n-1)l\}} & (n-1)l \notin \mathbb{Z} \quad (\Leftrightarrow m = n+1), \\ 1 & (n-1)l \in \mathbb{Z} \quad (\Leftrightarrow m = n). \end{cases}$$

where $l = (-d-b)/(b+1)$ and $\{x\} = x - \lfloor x \rfloor$. Let $V = (v_{i,j}^{(p)})_{0 \le i,j \le m-1}$. Then, we have

$$VPV^{-1} = \mathrm{diag}\left(1, (-b)^{-1}, \dots, (-b)^{m-1}\right).$$

In particular, the n-carry process over $(-b, \mathcal{D})$ has the stationary distribution

$$\pi = (\pi(s), \pi(s+1), \dots, \pi(s+m-1)) = \frac{1}{p^n n!} \left(\left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle_p, \left\langle \begin{matrix} n \\ 1 \end{matrix} \right\rangle_p, \dots, \left\langle \begin{matrix} n \\ m-1 \end{matrix} \right\rangle_p \right).$$

Proof. It suffices to show that

$$\sum_{k=0}^{m-1} v_{i,k}^{(p)}(n) \tilde{p}_{k,j} = \frac{1}{(-b)^i} v_{i,j}^{(p)}(n). \tag{10}$$

Recall that

$$\tilde{p}_{k,j} = \frac{1}{b^n}\sum_{r=0}^{n-j}(-1)^r\binom{n+1}{r}\binom{n+K(j,r)-k}{n},$$

where, we put $K(j,r) = b(n+1-j-r) - \frac{b+1}{p}$ for the simplicity of the notation. Therefore,

$$\begin{aligned}
\text{L.H.S. of (10)} &= \sum_{k=0}^{m}\frac{1}{b^n}\sum_{r=0}^{n-j}(-1)^r\binom{n+1}{r}\binom{n+K(j,r)-k}{n}v_{i,k}^{(p)}(n)\\
&= \frac{1}{b^n}\sum_{r=0}^{n-j}\sum_{k=0}^{K(j,r)}(-1)^r\binom{n+1}{r}\binom{n+K(j,r)-k}{n}v_{i,k}^{(p)}(n)\\
&= \frac{1}{b^n}\sum_{r=0}^{n-j}(-1)^r\binom{n+1}{r}\sum_{k=0}^{K(j,r)}\binom{n+K(j,r)-k}{n}v_{i,k}^{(p)}(n)\\
&= \frac{1}{b^n}\sum_{r=0}^{n-j}(-1)^r\binom{n+1}{r}[pK(j,r)+1]^{n-i}\\
&\qquad \text{(We use the same argument as in the proof of Theorem 2.)}\\
&= \frac{1}{b^n}\sum_{r'=j+1}^{n+1}(-1)^{n+1-r'}\binom{n+1}{n+1-r'}\left[pK(j,n+1-r')+1\right]^{n-i}\\
&\qquad \big(\text{We use the transformation } r' = n+1-r.\big)\\
&= \frac{1}{b^n}\sum_{r=j+1}^{n+1}(-1)^{n+1-r}\binom{n+1}{r}\left[-b(p(j-r')+1)\right]^{n-i}.
\end{aligned}$$

□

4 Concluding remarks

Many natural questions arise.

In the forthcoming paper, we will show a formula for the right eigenvectors, which involves Stirling numbers.

Our theorems hold only for the numeration systems $(b,\mathcal{D})$, where $\mathcal{D}$ consists of consecutive integers containing 0. For example, the 2-carry process over $(3,\{-1,0,4\})$ has rather large state space

$$\Omega_2(3,\{-1,0,4\}) = \{-5,-4,\ldots,4\},$$

and the transition probability matrix

$$P = \frac{1}{9}\begin{pmatrix}
1&2&0&3&0&2&1&0&0&0\\
2&0&0&2&1&4&0&0&0&0\\
1&0&2&0&3&2&0&1&0&0\\
0&1&2&0&3&0&2&1&0&0\\
0&2&0&0&2&1&4&0&0&0\\
0&1&0&2&0&3&2&0&1&0\\
0&0&1&2&0&3&0&2&1&0\\
0&0&2&0&0&2&1&4&0&0\\
0&0&1&0&2&0&3&2&0&1\\
0&0&0&1&2&0&3&0&2&1
\end{pmatrix},$$

whose characteristic polynomial $\det(xI - P)$ is

$$(x-1)(3x-1)(9x-1)(531441x^7 - 19683x^5 + 5103x^4 - 1944x^3 - 297x^2 + 24x + 2).$$

Although there are eigenvalues of the form $1, 1/3, 1/3^2$, we have no knowledge on the rest of the eigenvalues. The difficulty comes from the geometric structure of the fundamental domain $\{(x_l x_{l-1} \dots x_0)_b \,|\, l \geq 0, x_k \in \mathcal{D}\}$.

Diaconis and Fulman [2, 3] shows the relation between carries processes and shufflings for the case when $p = 1$ and 2. We do not know whether there exist some shufflings corresponding to the cases with $p \neq 1, 2$.

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