DELICACY OF THE RIEMANN HYPOTHESIS AND CERTAIN SUBSEQUENCES OF SUPERABUNDANT NUMBERS

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arXiv:1306.3434v2 [math.NT] 17 Jun 2013

Abstract. Robin's theorem is one of the ingenious reformulation of the Riemann hypothesis (RH). It states that the RH is true if and only if $\sigma(n) < e^{\gamma} n \log\log n$ for all $n > 5040$ where $\sigma(n)$ is the sum of divisors of n and γ is Euler's constant. In this paper we show that how the RH is delicate in terms of certain subsets of superabundant numbers, namely extremely abundant numbers and some of its specific supersets.

1. Introduction

Let $\sigma(n)$ be the sum of divisors of a positive integer n. Gronwall [3] showed that the order of $\sigma(n)$ is very nearly n. More precisely

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n\log\log n} = e^{\gamma}. \tag{1}$$

In 1984, Robin [7] established an elegant problem equivalent to the RH which is stated in the following theorem.

Theorem 1.1 ([7]). *The Riemann hypothesis is equivalent to*

$$\sigma(n) < e^{\gamma} n \log\log n, \qquad \textit{for all} \;\; n > 5040. \tag{2}$$

where $\gamma \approx 0.57721566$ *is Euler's constant.*

Inequality (2) is called Robin's inequality. He also showed that for all $n \geq 3$

$$\frac{\sigma(n)}{n} \leq e^{\gamma}\log\log n + \frac{0.648214}{\log\log n}, \tag{3}$$

where $0.648214 \approx (\frac{7}{3} - e^{\gamma}\log\log 12)\log\log 12$ with equality for $n = 12$.

A positive integer n is called superabundant number (([2], see also [6])) if

$$\frac{\sigma(n)}{n} > \frac{\sigma(m)}{m}, \qquad \text{for all} \;\; m < n.$$

and it is called colossally abundant, if for some $\varepsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\varepsilon}} \geq \frac{\sigma(m)}{m^{1+\varepsilon}}, \qquad \text{for all } m > 1.$$

Robin also proved that if the RH is not true, then for colossally abundant numbers we have

$$\frac{\sigma(n)}{n\log\log n} = e^{\gamma}(1+\Omega_{\pm}(\frac{1}{\log^{b} n})) \tag{4}$$

where b is any number of the interval $(1-\theta, 1/2)$, and θ being the upper bound of real parts of the zeros of the Riemann zeta function.

In 2009, Akbary and Friggstad [1] proved that to find the first probable counterexample to Robin's inequality it is enough to look for a special subsequence of positive integers. Indeed, If there is any counterexample to (2), then the least such counterexample is a superabundant number (cf.[1]).

During the study of Robin's theorem and looking for the first probable integer which violates (2) and belongs to a proper subset of superabundant numbers, authors in [4] constructed a new subsequence of positive integers (called its elements extremely abundant numbers) via the following definition:

Definition 1.2. A positive integer n is an *extremely abundant* number, if either $n = 10080$ or

$$\forall m \quad \text{s.t.} \quad 10080 \leq m < n, \qquad \frac{\sigma(m)}{m\log\log m} < \frac{\sigma(n)}{n\log\log n}. \tag{5}$$

We denote the following sets of integers by

$$S = \{n : \quad n \text{ is superabundant}\},$$
$$X = \{n : \quad n \text{ is extremely abundant}\}.$$

We also use SA and XA as abbreviations of superabundant and extremely abundant, respectively. It can be shown that $X \subset S$ (see [4]). Combining Gronwall's theorem with Robin's theorem, authors [4] established the following interesting results:

(i) If there is any counterexample to Robin's inequality, then the least one is an XA number.
(ii) The RH is true if and only if X is an infinite set.

The statement (ii) is the first step for showing the delicacy of RH.

Definition 1.3. Let $n_1 = 10080$. We find n_2 such that

$$\frac{\sigma(n_2)/n_2}{\sigma(n_1)/n_1} > 1 + \frac{\log n_2/n_1}{\log n_2 \log\log n_1}.$$

Now we find n_3 such that

$$\frac{\sigma(n_3)/n_3}{\sigma(n_2)/n_2} > 1 + \frac{\log n_3/n_2}{\log n_3 \log\log n_2},$$

and so on. We define X' to be the set of all $n_1,\ n_2,\ n_3, \ldots$.

$$X \subset X' \subset S. \tag{6}$$

Lemma 1.4. *If $m \in X'$, then there exists $n > m$ such that*

$$\frac{\sigma(n)/n}{\sigma(m)/m} > 1 + \frac{\log n/m}{\log n \log\log m}. \tag{7}$$

Proof. Given $m \in X'$. Then by (3)

$$\frac{\sigma(m)}{m} \leq \left(e^\gamma + \frac{0.648214}{(\log\log m)^2}\right) \log\log m, \tag{8}$$

Since

$$\frac{\log\log m}{\log\log m'} \left(1 + \frac{\log m'/m}{\log m' \log\log m}\right) < 1$$

and decreasing for $m' > m$ and tends to 0 as m' goes to infinity, then for some $m' > m$ we have

$$\frac{\log\log m}{\log\log m'} \left(1 + \frac{\log m'/m}{\log m' \log\log m}\right) \left(e^\gamma + \frac{0.648214}{(\log\log m)^2}\right) = e^\gamma - \varepsilon, \tag{9}$$

where $\varepsilon > 0$. Hence by Gronwall's theorem there is $n \geq m'$ such that

$$\begin{aligned} \frac{\sigma(n)}{n} &> (e^\gamma - \varepsilon) \log\log n \\ &= \frac{\log\log m}{\log\log m'} \left(1 + \frac{\log m'/m}{\log m' \log\log m}\right) \left(e^\gamma + \frac{0.648214}{(\log\log m)^2}\right) \log\log n \\ &\geq \left(1 + \frac{\log n/m}{\log n \log\log m}\right) \frac{\sigma(m)}{m}, \end{aligned}$$

where the last inequality holds by (8) and (9). □

Now we are going to state the main theorem of this paper which is the second step towards the delicacy of the RH, i.e.,

Theorem 1.5. *The set X' has infinite number of elements.*

Proof. If the RH is true, then the set X' has infinite elements by (6). If RH is not true, then there exists $m_0 \geq 10080$ such that

$$\frac{\sigma(m_0)/m_0}{\sigma(m)/m} > \frac{\log\log m_0}{\log\log m}, \qquad \text{for all} \quad m \geq 10080.$$

By Lemma 7 there exists $m' > m_0$ such that m' satisfies the inequality

$$\frac{\sigma(m')/m'}{\sigma(m_0)/m_0} > 1 + \frac{\log m'/m_0}{\log m' \log\log m_0}.$$

Let n be the first number greater than m_0 which satisfies
$$\frac{\sigma(n)/n}{\sigma(m_0)/m_0} > 1 + \frac{\log n/m_0}{\log n \log\log m_0}.$$
Then $n \in X'$. □

Definition 1.6. Let $n_1 = 10080$. We find n_2 such that
$$\frac{\sigma(n_2)/n_2}{\sigma(n_1)/n_1} > 1 + \frac{2\log n_2/n_1}{(\log n_2 + \log n_1)\log\log n_1}.$$
Now we find n_3 such that
$$\frac{\sigma(n_3)/n_3}{\sigma(n_2)/n_2} > 1 + \frac{2\log n_3/n_2}{(\log n_3 + \log n_2)\log\log n_2},$$
and so on. We define X'' to be the set of all $n_1,\ n_2,\ n_3, \ldots$.

Note that
$$\#X = 8150, \qquad \#X'' = 8187, \qquad \#X' = 8378$$
up to the $250\,000^{th}$ element of S (we used the list of SA numbers tabulated in [5]) and
$$\#(X''\backslash X) = 37, \qquad \#(X'\backslash X) = 228.$$

Acknowledgement

We would like to express our thanks to Professor Jean-Louis Nicolas for his suggestions in the initial version. We would like to thank Muhammad Ali Khan and Melinda Oroszlányová for their comments and suggestions which rather improved the English presentation of the paper.

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