Splittings and Ramsey Properties of Permutation Classes*

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Abstract

We say that a permutation π is *merged* from permutations ρ and τ, if we can color the elements of π red and blue so that the red elements are order-isomorphic to ρ and the blue ones to τ. A *permutation class* is a set of permutations closed under taking subpermutations. A permutation class C is *splittable* if it has two proper subclasses A and B such that every element of C can be obtained by merging an element of A with an element of B.

Several recent papers use splittability as a tool in deriving enumerative results for specific permutation classes. The goal of this paper is to study splittability systematically. As our main results, we show that if σ is a sum-decomposable permutation of order at least four, then the class $\mathrm{Av}(\sigma)$ of all σ-avoiding permutations is splittable, while if σ is a simple permutation, then $\mathrm{Av}(\sigma)$ is unsplittable.

We also show that there is a close connection between splittings of certain permutation classes and colorings of circle graphs of bounded clique size. Indeed, our splittability results can be interpreted as a generalization of a theorem of Gyárfás stating that circle graphs of bounded clique size have bounded chromatic number.

1 Introduction

The study of pattern-avoiding permutations, and more generally, of hereditary permutation classes, is one of the main topics in combinatorics. However, despite considerable effort, many basic questions remain unanswered. For instance, for permutations that avoid the pattern 1324, we still have no useful structural characterization, and no precise asymptotic enumeration either.

Recently, Claesson, Jelínek and Steingrímsson [15] have shown that every permutation π that avoids 1324 can be merged from a permutation avoiding

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132 and a permutation avoiding 213; in other words, the elements of π can be colored red and blue so that there is no red copy of 132 and no blue copy of 213. From this, they deduced that there are at most 16^n 1324-avoiding permutations of order n. They have extended this merging argument to more general patterns, showing in particular that if σ is a layered pattern of size k, then there are at most $(2k)^{2n}$ σ-avoiding permutations of size n. Subsequently, this approach was further developed by Bóna [6, 7], who proved, among other results, that there are at most $(7+4\sqrt{3})^n \simeq 13.93^n$ 1324-avoiders of size n. These results are also based on arguments showing that avoiders of certain patterns can be merged from avoiders of smaller patterns.

Motivated by these results, we address the general problem of identifying when a permutation class C has proper subclasses A and B, such that every element of C can be obtained by merging an element of A with an element of B. We call a class C with this property *splittable*. In this paper, we mostly focus on classes defined by avoidance of a single forbidden pattern, although some of our results are applicable to general hereditary classes as well.

On the negative side, we show that if σ is a simple permutation, then the class $\mathrm{Av}(\sigma)$ of all σ-avoiding permutations is unsplittable. More generally, every wreath-closed permutation class is unsplittable. We also find examples of unsplittable classes that are not wreath-closed, e.g., the class of layered permutations or the class of 132-avoiding permutations.

On the positive side, we show that if σ is a direct sum of two nonempty permutations and has size at least four, then $\mathrm{Av}(\sigma)$ is splittable. This extends previous results of Claesson et al. [15], who address the situation when σ is a direct sum of three permutations, with an extra assumption on one of the three summands.

The concept of splittability is closely related to several other structural properties of classes of relational structures, which have been previously studied in the area of Ramsey theory. We shall briefly mention some of these connections in Subsection 1.2.

We will also establish a less direct, but perhaps more useful, connection between splittability and coloring of circle graphs. Let σ_k be the permutation $1k(k-1)\cdots 32$ of order $k+1$. We will show, as a special case of more general results, that all σ_k-avoiding permutations can be merged from a bounded number, say $f(k)$, of 132-avoiding permutations. Moreover, we prove that the smallest such $f(k)$ is equal to the smallest number of colors needed to properly color every circle graph with no clique of size k. This allows us to turn previous results on circle graphs [21, 25, 26] into results on splittability of σ_k-avoiding permutations, and to subsequently extend these results to more general patterns. We deal with this topic in Subsection 3.4.

1.1 Basic notions

Permutation containment

A *permutation* of order $n \geq 1$ is a sequence π of n distinct numbers from the set $[n] = \{1, 2, \ldots, n\}$. We let $\pi(i)$ denote the i-th element of π. We often represent a permutation π by a *permutation diagram*, which is a set of n points with Cartesian coordinates $(i, \pi(i))$, for $i = 1, \ldots, n$. The set of permutations of order n is denoted by S_n. When writing out short permutations explicitly, we omit all punctuation and write, e.g., 1324 for the permutation 1, 3, 2, 4.

The *complement* of a permutation $\pi \in S_n$ is the permutation $\sigma \in S_n$ satisfying $\sigma(i) = n - \pi(i) + 1$. For a permutation $\pi \in S_n$ and two indices $i, j \in [n]$, we say that the element $\pi(i)$ *covers* $\pi(j)$ if $i < j$ and $\pi(i) < \pi(j)$. If an element $\pi(i)$ is not covered by any other element of π, i.e., if $\pi(i)$ is the smallest element of $\pi(1), \pi(2), \ldots, \pi(i)$, we say that $\pi(i)$ is a *left-to-right minimum*, or just *LR-minimum* of π. If $\pi(i)$ is not an LR-minimum, we say that $\pi(i)$ is a *covered element* of π.

Given two permutations $\sigma \in S_m$ and $\pi \in S_n$, we say that π *contains* σ if there is an m-tuple of indices $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, such that the sequence $\pi(i_1), \ldots, \pi(i_m)$ is *order-isomorphic* to σ, i.e., if for every $j, k \in [m]$ we have $\sigma(j) < \sigma(k) \iff \pi(i_j) < \pi(i_k)$. We then say that $\pi(i_1), \ldots, \pi(i_m)$ is an *occurrence* of σ in π, and the function $f\colon [m] \to [n]$ defined by $f(j) = i_j$ is an *embedding* of σ into π. A permutation that does not contain σ is σ*-avoiding*. We let $\mathrm{Av}(\sigma)$ denote the set of all σ-avoiding permutations, and for a set F of permutations, we let $\mathrm{Av}(F)$ denote the set of permutations that avoid all elements of F.

A set C of permutations is *hereditary* if for every $\pi \in C$ all the permutations contained in π belong to C as well. We use the term *permutation class* to refer to a hereditary set of permutations. It is not hard to see that a set C of permutations is hereditary if and only if there is a (possibly infinite) set F such that $C = \mathrm{Av}(F)$. A *principal* permutation class is a class of the form $\mathrm{Av}(\pi)$ for some permutation π.

Direct sums and inflations

The *direct sum* $\sigma \oplus \pi$ of two permutations $\sigma \in S_m$ and $\pi \in S_n$ is the permutation $\sigma(1), \sigma(2), \ldots, \sigma(m), \pi(1) + m, \pi(2) + m, \ldots, \pi(n) + m \in S_{n+m}$. Similarly, the *skew sum* $\sigma \ominus \pi$ is the permutation $\sigma(1) + n, \sigma(2) + n, \ldots, \sigma(m) + n, \pi(1), \pi(2), \ldots, \pi(n)$ (see Figure 1. A permutation is *decomposable* if it is a direct sum of two nonempty permutations.

Suppose that $\pi \in S_n$ is a permutation, let $\sigma_1, \ldots, \sigma_n$ be an n-tuple of nonempty permutations, and let m_i be the order of σ_i for $i \in [n]$. *The inflation* of π by the sequence $\sigma_1, \ldots, \sigma_n$, denoted by $\pi[\sigma_1, \ldots, \sigma_n]$, is the permutation of order $m_1 + \cdots + m_n$ obtained by concatenating n sequences $\overline{\sigma}_1 \overline{\sigma}_2 \cdots \overline{\sigma}_n$ with these properties (see Figure 2):

- for each $i \in [n]$, $\overline{\sigma}_i$ is order-isomorphic to σ_i, and

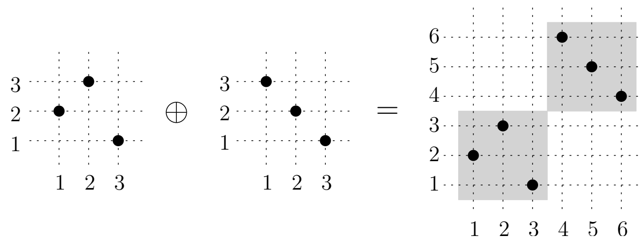

Figure 1: An example of a direct sum: $231 \oplus 321 = 231654$

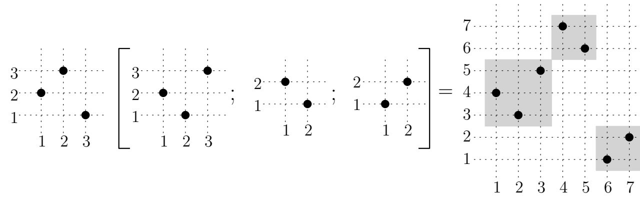

Figure 2: An example of inflation: $231[213, 21, 12] = 4357612$

- for each $i, j \in [n]$, if $\pi(i) < \pi(j)$, then all the elements of $\overline{\sigma}_i$ are smaller than all the elements of $\overline{\sigma}_j$.

We say that a permutation π is *simple* if it cannot be obtained by an inflation, except for the trivial inflations $\pi[1, 1, \dots, 1]$ and $1[\pi]$. Albert and Atkinson [2, Proposition 2] have pointed out that for every permutation ρ there is a unique simple permutation π such that ρ may be obtained by inflating π, and moreover, if π is neither 12 nor 21, then the inflation is determined uniquely.

For two sets A and B of permutations, *the wreath product* of A and B, denoted by $A \wr B$, is the set of all the permutations that may be obtained by inflating an element of A by a sequence of elements of B. If A or B is a singleton set $\{\rho\}$, we just write $\rho \wr B$ or $A \wr \rho$, respectively. Note that if A and B are hereditary, then so is $A \wr B$.

For a set X of permutations, we say that a set of permutations Y is *closed under* $\wr X$ if $Y \wr X \subseteq Y$, and Y is *closed under* $X\wr$ if $X \wr Y \subseteq Y$. Note that Y is closed under $12\wr$ if and only if it is closed under taking direct sums, and it is closed under $21\wr$ if and only if it is closed under skew sums. We say that a set of permutations Y is *wreath-closed* if $Y \wr Y \subseteq Y$. A principal permutation class $\mathrm{Av}(\pi)$ is wreath-closed if and only if π is a simple permutation, and more generally, a permutation class is wreath-closed if and only if it is equal to $\mathrm{Av}(F)$ for a set F that only contains simple permutations [2, Proposition 1]. Simple permutations, inflations and wreath products are crucial concepts in understanding the structure of permutation classes, as demonstrated, e.g., by the work of Brignall [11] or Brignall, Huczynska and Vatter [12].

Merging and splitting

Recall that a permutation π is *merged* from permutations ρ and τ, if it can be partitioned into two disjoint subsequences, one of which is an occurrence of ρ and the other is an occurrence of τ. For two permutation classes A and B, we write $A \odot B$ for the class of all the permutations that can be obtained by merging a (possibly empty) permutation from A with a (possibly empty) permutation from B. Clearly, $A \odot B$ is again a permutation class.

We are interested in finding out which permutation classes C can be merged from finitely many proper subclasses. We say that a multiset $\{P_1, \dots, P_m\}$ of permutation classes forms a *splitting* of a permutation class C if $C \subseteq P_1 \odot P_2 \odot \cdots \odot P_m$. The classes P_i are the *parts* of the splitting. The splitting is *nontrivial* if none of its parts is a superset of C, and the splitting is *irredundant* if no proper submultiset of $\{P_1, \dots, P_m\}$ forms a splitting of C. We say that a class of permutations is *splittable* if it has a nontrivial splitting. To familiarize ourselves with this key notion of the paper, we now provide several equivalent definitions of splittability.

Lemma 1.1. *For a class C of permutations, the following properties are equivalent:*

(a) C is splittable.

(b) C has a nontrivial splitting into two parts.

(c) C has a splitting into two parts, in which each part is a proper subclass of C.

(d) C has a nontrivial splitting into two parts, in which each part is a principal class.

Proof. Suppose that C is splittable, and let $\{P_1, P_2, \dots, P_k\}$ be a nontrivial irredundant splitting of C. Define a class $Q = P_2 \odot \cdots \odot P_k$. By irredundance, the class C is not a subset of Q, and therefore $\{P_1, Q\}$ is a nontrivial splitting of C into two parts. Moreover, $\{P_1 \cap C, Q \cap C\}$ is a splitting into two parts, with each part being a proper subclass of C. This shows that (a) implies (b) and (c).

To prove that (b) implies (d), suppose that $\{P_1, P_2\}$ is a nontrivial splitting of C into two parts. Choose arbitrary permutations $\sigma \in C \setminus P_1$ and $\pi \in C \setminus P_2$. Then $\mathrm{Av}(\sigma)$ is a superset of P_1 but not a superset of C, and similarly for $\mathrm{Av}(\pi)$ and P_2. Thus $\{\mathrm{Av}(\sigma), \mathrm{Av}(\pi)\}$ is a splitting of C that witnesses property (d). Since both (c) and (d) clearly imply (a), the statements (a) through (d) are equivalent. □

We remark that in general, a splittable class need not have a splitting that simultaneously satisfies properties (c) and (d) of Lemma 1.1. For instance, we will later see that the class Av(1342) is splittable, but it cannot be split into its principal proper subclasses, not even if we consider splittings with an arbitrary finite number of parts.

For a multiset $\mathcal{S}$ and an integer k, let $\mathcal{S}^{*k}$ denote the multiset obtained by increasing k-times the multiplicity of each element of $\mathcal{S}$.

Specific permutation classes

We will occasionally refer to several specific permutation classes. One such class is the class $\mathrm{Av}(\{231, 312\})$ whose members are known as *layered permutations*. Layered permutations are exactly the permutations that can be obtained as direct sums of decreasing sequences, or equivalently, they are the elements of $\mathrm{Av}(12) \wr \mathrm{Av}(21)$. Another important class is the class $\mathrm{Av}(\{2413, 3142\}$ of *separable permutations*. Separable permutations are exactly the permutations obtainable from the singleton permutation by iterated direct sums and skew sums. Therefore, they form the smallest nonempty class closed under both direct sums and skew sums.

1.2 Atomicity, amalgamation and Ramseyness

Splittability is related to several previously studied structural properties of permutation classes. These properties, including splittability itself, are not specific to classes of permutations, but are more generally applicable to classes of arbitrary objects that are partially ordered by containment, such as posets, graphs, directed graphs or uniform hypergraphs.

A convenient formalism for such structures is based on the notion of 'relational structure'. A *relational structure* with *signature* $(s_1, \ldots, s_k)$ on a vertex set V is a k-tuple $(R_1, \ldots, R_k)$, where R_i is a relation of arity s_i on V, i.e., R_i is a set of ordered s_i-tuples of (not necessarily distinct) elements of V. For a relational structure, we may define in an obvious way the notion of induced substructure. A *class of relational structures* is a set of isomorphism types of finite relational structures all sharing the same signature and closed under taking induced substructures. A detailed treatment of the topic of relational structures can be found, e.g., in the papers of Cameron [13] or Pouzet [34].

As explained, e.g., in [14], a permutation of order n can be represented as a relational structure formed by two linear orders on a vertex set of size n, and so permutation classes are a particular case of classes of relational structures of signature $(2, 2)$.

A structural property that is directly related to splittability is known as atomicity. A class of relational structures C is *atomic* if it does not have proper subclasses A and B such that $C = A \cup B$. Fraïssé ([19], see also [20]) has proved the following result:

Fact 1.2 (Fraïssé [19]). *For a class C of relational structures, the following properties are equivalent:*

- *C is atomic.*
- *For any two elements $\rho, \tau \in C$ there is a $\sigma \in C$ which contains both ρ and τ as substructures (this is known as* the joint embedding property*, and σ is a* joint embedding *of ρ and τ).*

- *There is a (possibly infinite) relational structure* Γ *such that* C *is the set of all the finite substructures of* Γ *(we then say that* C *is* the age *of* Γ*).*

A permutation class that is not atomic is clearly splittable. Thus, when studying splittability, we mostly focus on atomic classes. For such situation, we may slightly extend Lemma 1.1.

Lemma 1.3. *If* C *is an atomic permutation class, then the following properties are equivalent:*

(a) C *is splittable.*

(b') C *has a nontrivial splitting into two equal parts, i.e., a splitting of the form* $\{P, P\}$.

(c') C *has a splitting of the form* $\{P, P\}$*, where* P *is a proper subclass of* C.

(d') C *has a nontrivial splitting of the form* $\{P, P\}$*, where* P *is a principal class.*

Proof. Let C be a splittable atomic class. By part (d) of Lemma 1.1, C has a nontrivial splitting of the form $\{\mathrm{Av}(\pi), \mathrm{Av}(\pi')\}$. Note that both π and π' belong to C, otherwise the splitting would be trivial. By the joint embedding property, there is a permutation $\sigma \in C$ which contains both π and π'. Consequently, $\mathrm{Av}(\pi)$ and $\mathrm{Av}(\pi')$ are subclasses of $\mathrm{Av}(\sigma)$. This shows that $\{\mathrm{Av}(\sigma), \mathrm{Av}(\sigma)\}$ is a splitting of C witnessing property (d'), and $\{\mathrm{Av}(\sigma) \cap C, \mathrm{Av}(\sigma) \cap C\}$ is witnessing property (c'). The rest of the lemma follows trivially. □

It is not hard to see that any principal class $\mathrm{Av}(\pi)$ of permutations is atomic. Indeed, if π is indecomposable, then $\mathrm{Av}(\pi)$ is closed under direct sums, and otherwise it is closed under skew sums, which in both cases implies the joint embedding property.

Atomic permutation classes have been studied by Atkinson, Murphy and Ruškuc [4, 5, 28]. Among other results, they show (see [4, Theorem 2.2] and [28, Proposition 188]) that every permutation class that is partially well-ordered by containment (equivalently, any class not containing an infinite antichain) is a union of finitely many atomic classes. We remark that such classes of relational structures admit the following Dilworth-like characterization [23, Theorem 1.6]: a class C of relational structures is a union of k atomic classes if and only if it does not contain $k+1$ elements no two of which admit a joint embedding in C. Moreover, a class C that is not a union of finitely many atomic classes contains an infinite sequence of elements, no two of which admit a joint embedding in C.

The joint embedding property can be further strengthened, leading to a concept of amalgamation, which we define here for permutations, though it can be directly extended to other relational structures. Informally speaking, a permutation class C is π-amalgamable if for any two permutations ρ_1 and ρ_2 of C, each having a prescribed occurrence of π, we may find a joint embedding of ρ_1 and ρ_2 in which the two prescribed occurrences of π coincide. Formally, suppose that C is a permutation class, and π is its element. Then C is π-*amalgamable* if for any two permutations $\rho_1, \rho_2 \in C$ and any two mappings f_1 and f_2 such that f_i is an embedding of π into ρ_i, there is a permutation $\sigma \in C$ and two embeddings g_1 and g_2, where g_i is an embedding of ρ_i into σ, with the property that $g_1 \circ f_1 = g_2 \circ f_2$.

Let $\binom{\sigma}{\pi}$ denote the set of all occurrences of a permutation π in a permutation σ, and if S is a subsequence of σ, let $\binom{S}{\pi}$ be the set of those occurrences of π that are contained in S. A permutation class C is *π-Ramsey* if for every $\rho \in C$ there is a $\sigma \in C$ such that whenever we color $\binom{\sigma}{\pi}$ by two colors, there is a subsequence $S \in \binom{\sigma}{\rho}$ such that all elements of $\binom{S}{\pi}$ have the same color.

The next lemma follows from the results of Nešetřil [30] (see also [32, Theorem 4.2])

Fact 1.4. *Let C be an atomic permutation class and π an element of C. If C is π-Ramsey then C is also π-amalgamable.*

We say, for $k \in \mathbb{N}$, that a permutation class C is *k-Ramsey* if it is π-Ramsey for every $\pi \in C$ of order at most k, and we say that it is *k-amalgamable* if it is π-amalgamable for every $\pi \in C$ of order at most k. We also say that C is a *Ramsey class* (or *amalgamable class*) if it is k-Ramsey (or k-amalgamable) for every k.

Ramsey classes and amalgamable classes of various types of relational structures have attracted considerable amount of attention, due in part to their connection to so-called Fraïssé limits and homogeneous structures. We shall not explain these concepts here, and refer the interested reader to, e.g., a survey of the field presented in [32].

Cameron [14] has shown that there are only five infinite amalgamable classes of permutations: these are $\mathrm{Av}(12)$, $\mathrm{Av}(21)$, $\mathrm{Av}(231, 312)$ (the class of layered permutations), $\mathrm{Av}(213, 132)$ (the class of complements of layered permutations), and the class of all permutations. Böttcher and Foniok [9] have subsequently proved that all these amalgamable classes are Ramsey. In view of Fact 1.4, there can be no other atomic Ramsey classes of permutations. By suitably adapting Cameron's proof, it is actually possible to deduce that the five nontrivial amalgamable classes are also the only 3-amalgamable permutation classes, that is, any 3-amalgamable class of permutations is already Ramsey.

The above-defined Ramsey properties are closely related to splittability: it is straightforward to observe, by referring to condition (d') of Lemma 1.3, that an atomic class of permutations is unsplittable precisely when it is 1-Ramsey. Consequently, by Fact 1.4, any permutation class that fails to be 1-amalgamable is splittable. To make our exposition self-contained, we prove this simple result here.

Lemma 1.5. *If a permutation class C is not 1-amalgamable, then it is splittable. More precisely, if ρ_1 and ρ_2 are two elements of C that fail to have a 1-amalgamation in C, then C has the splitting $\{Av(\rho_1), Av(\rho_2)\}$.*

Proof. Suppose that C is not 1-amalgamable. That is, there are two permutations $\rho_1 \in C$ and $\rho_2 \in C$ of size n and m respectively, and two embeddings f_1 and

f_2 of the singleton permutation 1 into ρ_1 and ρ_2, such that there does not exist any $\sigma \in C$ with embeddings g_i of ρ_i into σ that would satisfy $g_1 \circ f_1 = g_2 \circ f_2$.

Suppose that $\rho_1(a)$ is the unique element of ρ_1 in the range of f_1, and similarly $\rho_2(b)$ the unique element of ρ_2 in the range of f_2. Let $\sigma \in C$ be arbitrary. Our goal is to color the elements of σ red and blue, so that the red elements avoid ρ_1 and the blue ones avoid ρ_2. To achieve this, we color an element $\sigma(i)$ of σ blue if and only if there is an embedding of ρ_1 into σ which maps $\rho_1(a)$ to $\sigma(i)$. The remaining elements of σ are red.

We see that the red elements of σ do not contain ρ_1, since otherwise there would be an embedding of ρ_1 into σ that maps $\rho_1(a)$ to a red element. Also, the blue elements of σ have no copy of ρ_2, and more generally, in any embedding of ρ_2 into σ, the element $\rho_2(b)$ must map to a red element of σ, since otherwise we would obtain a joint embedding of ρ_1 and ρ_2 identifying $\rho_1(a)$ with $\rho_2(b)$, which is impossible. $\square$

Concepts analogous to splittability and 1-Ramseyness have been previously studied, under various names, in connection to combinatorial structures other than permutations. This line of research dates back at least to the work of Folkman [18], who considered both edge-decompositions and vertex-decompositions of graphs of bounded clique size, and showed, among other results, that the class of graphs avoiding a clique of a given size is 1-Ramsey. Nešetřil and Rödl [31] obtained other examples of 1-Ramsey classes of graphs. Later, Pouzet [33] and El-Zahar and Sauer [17] considered a notion equivalent to splittability in context of atomic classes of relational structures, as part of a hierarchy of several Ramsey-type properties. For further developments in this area, see e.g. the works of Laflamme et al. [27] or Bonato et al. [8]. We will not go into any further details of these results, as they do not seem to be applicable to our problem of identifying splittable classes of permutations.

In Section 2, we will give unsplittability criteria, which will imply, among other results, that any wreath-closed permutation class is unsplittable. Next, in Section 3, we will look for examples of splittable classes. This turns out to be considerably more challenging. Our main result in this direction shows that the class $\mathrm{Av}(\sigma)$ is splittable whenever σ is a decomposable permutation of order at least four. We then describe, in Subsection 3.4, the connection between splittability of permutation classes and the chromatic number of circle graphs. This connection allows us to exploit previous work to give both positive and negative results on the existence of certain permutation splittings.

2 Unsplittable classes and unavoidable patterns

We now focus on the unsplittable permutation classes. By Lemma 1.1, when looking for splittings of a class C, we may restrict our attention to splittings whose parts are principal classes not containing C, i.e., classes of the form $\mathrm{Av}(\pi)$ for some $\pi \in C$. The basic idea of our approach will be to identify a large set of permutations $\pi \in C$ for which we can prove that $\mathrm{Av}(\pi)$ is not a part of any irredundant splitting of C. This motivates our next definition.

Definition 2.1. Let C be a permutation class, and let π be an element of C. We say that π is *unavoidable in* C, if C has no irredundant splitting that contains $\mathrm{Av}(\pi)$ as a part. We let $\mathcal{U}_C$ denote the set of all the unavoidable permutations in C.

The next two observations list several basic properties of unavoidable permutations. These properties follow directly from the definitions or from the arguments used to prove Lemma 1.1. We therefore omit their proofs.

Observation 2.2. *For a permutation class C and a permutation $\pi \in C$, the following statements are equivalent.*

1. *The permutation π is unavoidable in C.*
2. *For any permutation $\tau \in C$, there is a permutation $\sigma \in C$ such that any red-blue coloring of σ has a red copy of τ or a blue copy of π.*
3. *In any irredundant splitting $\{P_1, \dots, P_k\}$ of C, all the parts P_i contain π.*
4. *C has no nontrivial splitting into two parts, where one of the parts is $Av(\pi)$.*

Observation 2.3. *The set $\mathcal{U}_C$ of unavoidable permutations of a nonempty permutation class C has these properties:*

1. *$\mathcal{U}_C$ is a nonempty permutation class, and in particular, it is hereditary.*
2. *$\mathcal{U}_C \subseteq C$.*
3. *If $\{P_1, \dots, P_m\}$ is an irredundant splitting of C, then $\mathcal{U}_C \subseteq P_i$ for each i.*
4. *$\mathcal{U}_C = C$ if and only if C is unsplittable.*

By the last part of the previous observation, to show that a permutation class C is unsplittable, it is enough to prove $\mathcal{U}_C = C$. To achieve this, we will show that certain closure properties of C imply analogous closure properties of $\mathcal{U}_C$.

Lemma 2.4. *Let C be a permutation class. If, for a set of permutations X, the class C is closed under $\wr X$, then $\mathcal{U}_C$ is also closed under $\wr X$, and if C is closed under $X\wr$, then so is $\mathcal{U}_C$. Consequently, if C is wreath-closed, then $\mathcal{U}_C = C$ and C is unsplittable.*

Proof. We first prove that if C is closed under $\wr X$, then so is $\mathcal{U}_C$. Suppose C is closed under $\wr X$. Note that we may assume that X itself is wreath-closed; this is because $\wr$ is associative, so if C is closed under $\wr X$, it is also closed under $\wr(X \wr X)$ and therefore it is closed under $\wr Y$ where Y is the wreath-closure of X.

Choose a permutation $\pi \in \mathcal{U}_C$ of order k, and k permutations $\rho_1, \dots, \rho_k \in X$. We wish to prove that $\pi[\rho_1, \dots, \rho_k]$ belongs to $\mathcal{U}_C$. Without loss of generality, we assume that all ρ_i are equal to a single permutation ρ; if not, we simply put $\rho \in X$ to be a permutation which contains all the ρ_i (such a ρ exists, since X is wreath-closed) and prove the stronger fact that $\pi[\rho, \dots, \rho]$ belongs to $\mathcal{U}_C$.

Let us use the notation $\pi[\rho]$ as shorthand for $\pi[\rho, \dots, \rho]$. So our goal now reduces to showing that $\pi[\rho]$ belongs to $\mathcal{U}_C$ for every $\pi \in \mathcal{U}_C$ and $\rho \in X$. We base our argument on the second equivalent definition from Observation 2.2. Fix a permutation $\tau \in C$. We want to find a permutation $\sigma \in C$ such that each red-blue coloring of σ either contains a red copy of τ or a blue copy of $\pi[\rho]$. We already know that π belongs to $\mathcal{U}_C$, so there is a permutation σ' such that each red-blue coloring of σ' has either a red copy of τ or a blue copy of π. Let ℓ be the order of σ'.

Define σ by $\sigma = \sigma'[\rho]$, and view σ as a concatenation of ℓ blocks, each being a copy of ρ. Fix an arbitrary red-blue coloring of σ. We now define a red-blue coloring of σ' as follows: an element σ'_i of σ' is red if the i-th block in σ has at least one red point, otherwise it is blue. Note that this coloring of σ' has the property that if σ' contains a red copy of any pattern β then σ also contains a red copy of β, and if σ' contains a blue copy of β, then σ has a blue copy of $\beta[\rho]$. In particular, σ either contains a red copy of τ or a blue copy of $\pi[\rho]$. This proves that $\pi[\rho]$ belongs to $\mathcal{U}_C$, as claimed.

We now show that if C is closed under $X\wr$ then so is $\mathcal{U}_C$. Fix a permutation $\rho \in X$ of order k, and a k-tuple $\pi_1, \dots, \pi_k$ of permutations from $\mathcal{U}_C$. We will show that $\rho[\pi_1, \dots, \pi_k]$ belongs to $\mathcal{U}_C$.

Fix a permutation $\tau \in C$. Since π_i is in $\mathcal{U}_C$, there is a permutation $\sigma_i \in C$ whose every red-blue coloring has either a red copy of τ or a blue copy of π_i. Define $\sigma = \rho[\sigma_1, \dots, \sigma_k]$, viewing it as a union of k blocks, with the i-th block being a copy of σ_i. Fix a red-blue coloring of σ. The i-th block of σ either contains a red copy of τ or a blue copy of π_i. Thus, if σ has no red copy of τ, it must have a blue copy of $\rho[\pi_1, \dots, \pi_k]$, showing that $\rho[\pi_1, \dots, \pi_k]$ belongs to $\mathcal{U}_C$.

It remains to show that if C is wreath-closed then $\mathcal{U}_C = C$. But if C is wreath-closed, then it is closed under $\wr C$, so $\mathcal{U}_C$ is also closed under $\wr C$. But since $\mathcal{U}_C$ is a nonempty subclass of C by Observation 2.3, this means that $\mathcal{U}_C = C$. $\square$

Corollary 2.5. *Every wreath-closed permutation class is unsplittable. In particular, if π is a simple permutation then $Av(\pi)$ is unsplittable.*

Not all unsplittable classes are wreath-closed. For instance, the class C of layered permutations is unsplittable; to see this, note that C is closed under $12\wr$ as well as under $\wr 21$, and it is the smallest nonempty class with these properties. Since $\mathcal{U}_C$ has the same closure properties, we must have $\mathcal{U}_C = C$.

There is even an example of a principal class that is unsplittable even though it is not wreath-closed, namely the class $\mathrm{Av}(132)$. To show that this class is indeed unsplittable we need a more elaborate argument.

Lemma 2.6. *Let ρ be an indecomposable permutation. Let C be the class $Av(1 \oplus \rho)$. Then, for any two permutations π, π' such that $\pi \in \mathcal{U}_C$ and $1 \oplus \pi' \in$*

$\mathfrak{U}_C$, *we have* $\pi \oplus \pi' \in \mathfrak{U}_C$.

Proof. Note that if σ and σ' are two permutations avoiding the pattern $1 \oplus \rho$, and if σ'' is obtained by inflating any LR-minimum of σ' by a copy of σ, then σ'' also avoids $1 \oplus \rho$. This follows easily from the fact that ρ is indecomposable.

Let τ be any element of C. Our goal is to find a permutation $\sigma'' \in C$ whose every red-blue coloring has a red copy of τ or a blue copy of $\pi \oplus \pi'$. Since $\pi \in \mathfrak{U}_C$, there is a $\sigma \in C$ such that every red-blue coloring of σ has a red copy of τ or a blue copy of π. Similarly, there is a $\sigma' \in C$ whose every red-blue coloring has a red copy of τ or a blue copy of $1 \oplus \pi'$.

Let σ'' be the permutation obtained by inflating each LR-minimum of σ' by a copy of σ. Fix any red-blue coloring of σ'' that has no red copy of τ. Then every σ-block in σ'' contains a blue copy of π. Consider a two-coloring of σ' in which every LR-minimum is blue, and all the other elements have the same color as the corresponding elements in σ''. This coloring contains a blue copy of $1 \oplus \pi'$. This means that σ'' has a blue copy of π' which is disjoint from all the σ-blocks obtained by inflating the LR-minima of σ'. Combining this blue copy of π' with a blue copy of π in an appropriate σ-block, we get a blue copy of $\pi \oplus \pi'$ in σ''. $\square$

Proposition 2.7. *The class Av*(132) *is unsplittable.*

Proof. Let $C = \mathrm{Av}(132)$. We will show that C is equal to $\mathfrak{U}_C$. Pick a permutation $\pi \in C$, with $n \geq 2$. Let k be the index such that $\pi(k) = n$. If $k = n$, then π can be written as $\pi' \oplus 1$ for some $\pi' \in C$. If $k < n$, then all the elements $\pi(k+1), \dots, \pi(n)$ are smaller than any element in $\pi(1), \dots, \pi(k)$, otherwise we would find an occurrence of 132. Consequently, π can be written as $\pi' \ominus \pi''$ for some $\pi', \pi'' \in C$.

To show that every $\pi \in C$ belongs to $\mathfrak{U}_C$, proceed by induction. For $\pi = 1$ this is clear, for $\pi = 12$, this follows from the fact that C is $\wr 12$-closed. If π is equal to $\pi' \ominus \pi''$, use the fact that C is $\ominus$-closed. If $\pi = \pi' \oplus 1$, use Lemma 2.6 and the fact that $12 \in \mathfrak{U}_C$. $\square$

3 Splittable classes and decomposable patterns

We now focus on splittable permutation classes. We are again mostly interested in principal classes. Let us begin by stating the main result of this section.

Theorem 3.1. *If* π *is a decomposable permutation other than 12, 213 or 132, then Av*(π) *is a splittable class.*

The exclusion of 12, 213 and 132 in the statement of the theorem is necessary, since it follows from the results of the previous section that Av(12), Av(213) and Av(132) are unsplittable.

As the first step towards the proof of Theorem 3.1, we deal with patterns that are decomposable into (at least) three parts.

Proposition 3.2. *Let α, β and γ be three nonempty permutations. The class $Av(\alpha \oplus \beta \oplus \gamma)$ is splittable, and more precisely, it satisfies*

$$Av(\alpha \oplus \beta \oplus \gamma) \subseteq Av(\alpha \oplus \beta) \odot Av(\beta \oplus \gamma).$$

We note that a weaker version of Proposition 3.2 (with an extra assumption that β has the form $\beta' \ominus 1$ for some β') has been recently proved by Claesson, Jelínek and Steingrímsson [15, Theorem 3]. The proof we present below is actually a simple adaptation of the argument from [15].

Proof of Proposition 3.2. Fix a permutation π avoiding the pattern $\alpha \oplus \beta \oplus \gamma$. We will color the elements of π red and blue, so that the red elements will avoid $\alpha \oplus \beta$ and the blue ones will avoid $\beta \oplus \gamma$. We construct the coloring by taking the elements $\pi(1)$ to $\pi(n)$ successively, and having colored the elements $\pi(1), \ldots, \pi(i-1)$ for some i, we determine the color of $\pi(i)$ by these rules:

- If coloring $\pi(i)$ red completes a red occurrence of $\alpha \oplus \beta$, color $\pi(i)$ blue.
- If for some $j < i$ the element $\pi(j)$ is blue and $\pi(j) < \pi(i)$, color $\pi(i)$ blue.
- Otherwise color $\pi(i)$ red.

Clearly, the coloring determined by these rules avoids a red copy of $\alpha \oplus \beta$. We now show that it also avoids a blue copy of $\beta \oplus \gamma$. Suppose for contradiction that π has a blue occurrence of $\beta \oplus \gamma$, and let β_B and γ_B denote the occurrences of β and γ in this blue occurrence of $\beta \oplus \gamma$, with β_B being completely to the left and below γ_B.

Let $\pi(a)$ be the smallest element of β_B. Since $\pi(a)$ is blue, π must have a blue element $\pi(b)$ such that $b \le a$, $\pi(b) \le \pi(a)$, and changing the color of $\pi(b)$ from blue to red would create a red copy of $\alpha \oplus \beta$ in the sequence $\pi(1), \pi(2), \ldots, \pi(b)$. In other words, π has a copy of $\alpha \oplus \beta$ whose rightmost element is $\pi(b)$ and whose remaining elements are all red. Let α_R and β_R denote the copies of α and β in this copy of $\alpha \oplus \beta$, with α_R to the left and below β_R.

Note that every element of α_R is smaller than $\pi(b)$, and therefore every element of α_R is smaller than all the elements of β_B. Let $\pi(c)$ be the leftmost element of β_B. If every element of α_R is to the left of $\pi(c)$, then $\alpha_R \cup \beta_B \cup \gamma_B$ is a copy of $\alpha \oplus \beta \oplus \gamma$ in π, which is impossible. Thus at least one element of α_R is to the right of $\pi(c)$, and consequently, all the elements of β_R are to the right of $\pi(c)$. It follows that all the red elements in β_R are smaller than $\pi(c)$, for otherwise they would be colored blue by the second rule of our coloring. This means that $\alpha_R \cup \beta_R \cup \gamma_B$ is a copy of $\alpha \oplus \beta \oplus \gamma$, a contradiction. $\square$

Corollary 3.3. *If α and β are permutations of order at least two, then the class $Av(\alpha \oplus \beta)$ is splittable.*

Proof. We see that $\mathrm{Av}(\alpha \oplus \beta) \subseteq \mathrm{Av}(\alpha \oplus 1 \oplus \beta)$, and Proposition 3.2 shows that

$$\mathrm{Av}(\alpha \oplus 1 \oplus \beta) \subseteq \mathrm{Av}(\alpha \oplus 1) \odot \mathrm{Av}(1 \oplus \beta).$$

Therefore, $\mathrm{Av}(\alpha \oplus \beta)$ admits the splitting $\{\mathrm{Av}(\alpha \oplus 1), \mathrm{Av}(1 \oplus \beta)\}$. $\square$

To prove Theorem 3.1, it remains to deal with the classes $\mathrm{Av}(\alpha \oplus \beta)$, where α or β has order one. As the two cases are symmetric, we may assume that $\alpha = 1$. We may also assume that β is indecomposable, because otherwise $\mathrm{Av}(1 \oplus \beta)$ is splittable by Proposition 3.2. Finally, we may assume that β has order at least three, since we already know that Av(12) and Av(132) are unsplittable. Let us therefore focus on the classes $\mathrm{Av}(1 \oplus \sigma)$, where σ is an indecomposable permutation of order at least three.

To handle these 'hard' cases of Theorem 3.1, we will introduce the notion of ordered matchings, and study the splittings of hereditary classes of matchings. Matchings are more general structures than permutations, in the sense that the containment poset of permutations is a subposet of the containment poset of matchings. Moreover, the arguments we use in our proof can be more naturally presented in the terminology of matchings, rather than permutations. The downside is that we will have to introduce a lot of basic terminology related to matchings.

In Subsection 3.1, we will introduce matchings and describe how they relate to permutations. Next, in Subsection 3.2, we present the proof of Theorem 3.1. Although the proof is constructive, the splittings we construct in the proof involve classes defined by avoidance of rather large patterns. Such splittings do not seem to reveal much information about the classes being split. For this reason, Subsection 3.3 gives another splitting algorithm, which is less general, but which provides more natural splittings involving avoiders of small patterns. Using this alternative splitting approach, we establish, in Subsection 3.4, an equivalence between the existence of certain permutation splittings and the colorability of circle graphs of bounded clique size. This allows us to use previous bounds on the chromatic number of clique-avoiding circle graphs to deduce both positive and negative results about existence of permutation splittings.

3.1 Ordered matchings

Let $P = \{x_1 < x_2 < \cdots < x_{2n}\}$ be a set of $2n$ real numbers, represented by points on the real line. A *matching* (or, more properly, an *ordered perfect matching*) on the point set P is a set M of pairs of points from P, such that every point of P belongs to exactly one pair from M. The elements of M are the *arcs* of the matching M, while the elements of P are the *endpoints* of M.

We represent an arc α of a matching M by an ordered pair (a, b) of points, and we make the convention that a is left of b. The points a and b are referred to as the *left endpoint* and *right endpoint* of α, respectively, and denoted by $\mathrm{left}(\alpha)$ and $\mathrm{right}(\alpha)$. We visualize the arcs as half-circles connecting the two endpoints, and situated in the upper half-plane above the real line.

If M is a matching on a point set $\{x_1 < x_2 < \cdots < x_{2n}\}$ and N is a matching on a point set $P' = \{y_1 < y_2 < \cdots < y_{2n}\}$, we say that M and N are *isomorphic*, written as $M \cong N$, if for every i, j we have the equivalence $(x_i, x_j) \in M \iff (y_i, y_j) \in N$. A matching M *contains* a matching N, if it has a subset $M' \subseteq M$ such that the matching M' is isomorphic to N. If M does not contain N, we say that M *avoids* N. Let $\mathrm{MAv}(N)$ be the set of the

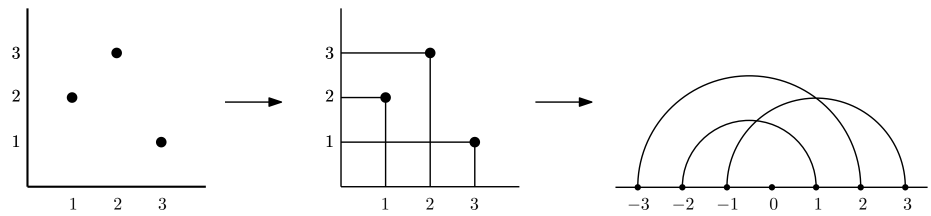

Figure 3: Transforming a permutation $\pi = 231$ into the matching $\widehat{\pi}$.

isomorphism classes of matchings that avoid N.

Let M be a matching, and let $\alpha = (a, b)$ and $\beta = (c, d)$ be two arcs of M. We say that α *crosses* β *from the left* if $a < c < b < d$, and we say that α *is nested below* β if $c < a < b < d$. We say that a point $x \in \mathbb{R}$ is *nested below the arc* α if $a < x < b$ (note that x does not have to be an endpoint of M). We say that two arcs are *in series* if they neither cross nor nest, which means that one of them is completely to the left of the other. We say that an arc α of a matching M is *short* if its endpoints are adjacent, i.e., if there is no endpoint of M nested below α. An arc that is not short is *long*.

We define the *directed intersection graph* of M, denoted by $\overrightarrow{G_M}$, to be the graph whose vertices are the arcs of M, and $\overrightarrow{G_M}$ has a directed edge from α to β if α crosses β from the left. The *intersection graph* of M, denoted by G_M, is the graph obtained from $\overrightarrow{G_M}$ by omitting the orientation of its edges. Note that two arcs of M are adjacent in G_M if and only if the half-circles representing the two arcs intersect. We say that a matching is *connected* if its intersection graph is a connected graph.

Let us remark that the graphs that arise as intersection graphs of matchings are known as *circle graphs* in graph theory literature. We refer the reader to the surveys [10, 36] for more information on this graph class.

Let M and N be matchings with m and n arcs, respectively. We let $M \uplus N$ denote the matching Q which is a disjoint union of two matchings Q_1 and Q_2, such that $Q_1 \cong M$, $Q_2 \cong N$, and any endpoint of Q_1 is to the left of any endpoint of Q_2. This determines $M \uplus N$ uniquely up to isomorphism. We say that a matching M is $\uplus$*-indecomposable* if it cannot be written as $M \cong M_1 \uplus M_2$ for some nonempty matchings M_1 and M_2. Note that a connected matching is $\uplus$-indecomposable, but the converse is not true in general. Any matching M can be uniquely written as $M \cong M_1 \uplus M_2 \uplus \cdots \uplus M_k$ where each M_i is a nonempty $\uplus$-indecomposable matching. We call the matchings M_i the *blocks* of M.

We say that a matching M is *merged* from two matchings M_1 and M_2, if the arcs of M can be colored red and blue so that the red arcs form a matching isomorphic to M_1 and the blue ones are isomorphic to M_2. Given this concept of merging, we may speak of splittability of matching classes in the same way as we do in the case of permutations.

We now introduce two distinct ways in which a permutation can be encoded

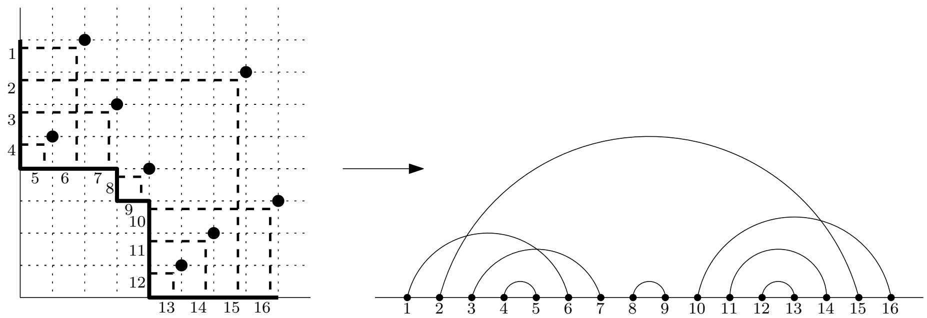

Figure 4: The diagram of the permutation $\pi = 58641273$, together with its envelope P, represented as the thick line with numbered steps. The dashed lines represent the arcs of the envelope matching $E(\pi)$.

by a matching. Let π be a permutation. We let $\widehat{\pi}$ denote the matching on the point set $\{-n, -n+1, \dots, -1\} \cup \{1, 2, \dots, n\}$ which contains the arc of the form $(-\pi(i), i)$ for each $i \in [n]$. As shown in Figure 3, we may visualize the permutation matching $\widehat{\pi}$ by taking the diagram of π, connecting each point of the diagram to the two coordinate axes by a horizontal and vertical segment, and then deforming the figure so that the pairs of segments become half-circles. Clearly, the matching $\widehat{\pi}$ is $\uplus$-indecomposable. It can also be easily verified that $\widehat{\pi}$ is connected if and only if the permutation π is indecomposable. If C is a set of permutations, we let $\widehat{C}$ denote the set $\{\widehat{\pi} \mid \pi \in C\}$.

We say that a matching M is a *permutation matching* if $M \cong \widehat{\pi}$ for a permutation π.

Observation 3.4. *For a matching M, the following properties are equivalent:*

1. *M is a permutation matching.*
2. *Any left endpoint of M is to the right of any right endpoint.*
3. *There is a point $x \in \mathbb{R}$ nested below all the arcs of M.*
4. *M has no two arcs in series.*

Permutation matchings form a hereditary class within the class of all matchings.

Observation 3.5. *A permutation π contains a permutation σ if and only if the matching $\widehat{\pi}$ contains $\widehat{\sigma}$. A permutation π can be merged from permutations $\sigma_1, \dots, \sigma_k$ if and only if the matching $\widehat{\pi}$ can be merged from matchings $\widehat{\sigma}_1, \dots, \widehat{\sigma}_k$. Hence, $Av(\pi)$ is splittable if and only if $\widehat{Av(\pi)}$ is.*

We now introduce another, less straightforward way to encode a permutation by a matching.

Definition 3.6. Let π be a permutation of order n. The *envelope* of π is a directed lattice path P_π with these properties (see Fig. 4):

- P_π connects the point $(0, n)$ to the point $(n, 0)$,
- every step of P_π is either a *down-step* connecting a point (i, j) to a point $(i, j-1)$, or a *right-step* connecting (i, j) with $(i+1, j)$, and
- for every $i \in [n]$, the path P_π contains the right step $(i-1, j) \to (i, j)$ where $j = \min\{\pi(1), \ldots, \pi(i)\} - 1$.

Note that all the points of the form $(i, \pi(i))$ are strictly above the envelope P_π, and P_π is the highest non-increasing lattice path with this property. For any $i \in [n]$, the *down-step in row* i of P_π is the (unique) down-step of the form $(k, i) \to (k, i-1)$ for some k, and the *right-step in column* i is the right-step of the form $(i-1, k) \to (i, k)$.

Note also that P_π contains a down-step $(i-1, j) \to (i-1, j-1)$ followed by a right-step $(i-1, j-1) \to (i, j-1)$ if and only if $\pi(i)$ is an LR-minimum of π and is equal to j.

Definition 3.7. Let π be a permutation with envelope P_π. The *envelope matching* of π, denoted by $E(\pi)$, is the matching on the point set $[2n]$ determined by these rules:

1. Label the steps of P_π by $\{1, 2, \ldots, 2n\}$, in the order in which they are encountered when P_π is traversed from $(0, n)$ to $(n, 0)$.
2. For every i and j such that $\pi(i) = j$, suppose that the down-step in row j has label a and the right-step in column i has label b. Then the arc (a, b) belongs to $E(\pi)$. (Note that we must have $a < b$, since the point (i, j) is above the path P_π.)

Note that each arc of $E(\pi)$ corresponds in an obvious way to an element of π, and the short arcs of $E(\pi)$ correspond precisely to the LR-minima of π.

Observation 3.8. *Suppose that $\pi \in S_n$ is a permutation. Let $\pi(i)$ and $\pi(j)$ be two elements of π, and let α_i and α_j be the two corresponding arcs of $E(\pi)$. Then right$(\alpha_i) <$ right(α_j) if and only if $i < j$, and left$(\alpha_i) <$ left(α_j) if and only if $\pi(i) > \pi(j)$. It follows that α_i is nested below α_j if and only if $\pi(i)$ covers $\pi(j)$.*

Moreover, if $i < j$ and $\pi(i) > \pi(j)$, then the arcs α_i and α_j are crossing if and only if π has an LR-minimum that covers both $\pi(i)$ and $\pi(j)$, otherwise they are in series.

Observation 3.8 implies that $E(\pi)$ determines π uniquely, and if σ and π are permutations such that $E(\sigma)$ contains $E(\pi)$, then σ contains π. Note however, that the converse of this last fact does not hold in general: the permutation 132 contains 21, while $E(132) = \{(1, 5), (2, 6), (3, 4)\}$ does not contain $E(21) = \{(1, 2), (3, 4)\}$.

Lemma 3.9. *For a matching M on the point set $[2n]$, the following statements are equivalent.*

1. *There is a permutation π such that $M = E(\pi)$.*
2. *For any left endpoint a of M, if $a+1$ is a right endpoint of M, then $(a, a+1)$ is an arc of M.*
3. *If (a,b) and (c,d) are two arcs of M such that $a < c < b < d$, then $b \neq c+1$.*

Proof. One may easily observe that the second and third property are equivalent, and that the first property implies the second one. We now show that the second property implies the first. Let M be a matching satisfying the second property. Construct a lattice path P from $(0,n)$ to $(n,0)$, consisting of down-steps and right-steps of unit length, where the a-th step of P is a down-step if and only if the point a is a left endpoint of the matching M. We label the steps of P from 1 to $2n$ in the order in which they appear on P. In this way, the endpoint a of M corresponds naturally to the a-labelled step of P.

We construct a permutation π as follows: for every arc (a,b) of M, let i be the column containing the b-labelled step of P (which is a right-step) and let j be the row containing the a-labelled step of P (which is a down-step). We then define $\pi(i) = j$.

By construction, all elements of π are above the path P. Moreover, since M satisfies the second property, we see that whenever a down-step labelled a is directly followed by a right-step labelled $a+1$, then $(a, a+1)$ is an arc, and the permutation π has an LR-minimum in the column containing step $a+1$ and row containing step a. Hence P is the envelope of π, and it follows that $M = E(\pi)$. □

Any matching satisfying the conditions of Lemma 3.9 will be referred to as an *envelope matching*. Notice that in an envelope matching, every long arc has a short arc nested below it.

Let $R(\pi)$ denote the submatching of $E(\pi)$ formed by the long arcs of $E(\pi)$. The matching $R(\pi)$ no longer determines the permutation π uniquely (e.g., both $R(12)$ and $R(213)$ consist of a single arc). We call $R(\pi)$ the *reduced envelope matching* of π. The next lemma shows how these concepts are related to avoidance of patterns of the form $1 \oplus \sigma$.

Lemma 3.10. *For any permutations σ and π, the matching $R(\pi)$ contains $\widehat{\sigma}$ if and only if π contains $1 \oplus \sigma$. Moreover, suppose that $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is a multiset of permutations such that*

$$R(\pi) \in MAv(\widehat{\sigma}_1) \odot MAv(\widehat{\sigma}_2) \odot \cdots \odot MAv(\widehat{\sigma}_k).$$

Then

$$\pi \in Av(1 \oplus \sigma_1) \odot Av(1 \oplus \sigma_2) \odot \cdots \odot Av(1 \oplus \sigma_k).$$

In particular, if $MAv(\widehat{\sigma})$ has a splitting $\{MAv(\widehat{\sigma}_1), \ldots, MAv(\widehat{\sigma}_k)\}$, then $Av(1 \oplus \sigma)$ has a splitting $\{Av(1 \oplus \sigma_1), \ldots, Av(1 \oplus \sigma_k)\}$.

Proof. Suppose that $R(\pi)$ contains $\widehat{\sigma}$, and let $M \subseteq R(\pi)$ be a submatching of $R(\pi)$ isomorphic to $\widehat{\sigma}$. From Observation 3.8 and Lemma 3.9, we may easily deduce that $E(\pi)$ has a short arc β nested below all the arcs of M. It follows that $M \cup \{\beta\}$ is a copy of $\widehat{1 \oplus \sigma}$ in $E(\pi)$, and hence π contains $1 \oplus \sigma$.

To prove the converse, suppose that π has a subsequence $S = s_0, s_1, \dots, s_n$ order-isomorphic to $1 \oplus \sigma$. Let α_i denote the arc of $E(\pi)$ representing the element s_i of π. By Observation 3.8, α_0 is nested below all the arcs $\alpha_1, \dots, \alpha_n$. Therefore, the arcs $\alpha_1, \dots, \alpha_n$ all belong to $R(\pi)$, and no two of them are in series. Thus, $\alpha_1, \dots, \alpha_n$ form a copy of $\widehat{\sigma}$ in $R(\pi)$.

Suppose now that π is a permutation such that $R(\pi) \in \mathrm{MAv}(\widehat{\sigma}_1) \odot \mathrm{MAv}(\widehat{\sigma}_2) \odot \cdots \odot \mathrm{MAv}(\widehat{\sigma}_k)$. Color the arcs of $R(\pi)$ by k colors $c_1, \dots, c_k$ in such a way that the arcs colored by c_i avoid $\widehat{\sigma}_i$. We transfer this k-coloring to the covered elements of π, by assigning to each covered element the color of the corresponding arc of $R(\pi)$ (recall that a *covered element* of a permutation is an element which is not a LR-minimum).

Let π_i denote the subpermutation of π formed by all the LR-minima of π together with all the covered elements whose color is c_i. Note that $R(\pi_i)$ is isomorphic to the submatching of $R(\pi)$ consisting of the arcs of color c_i. Hence $R(\pi_i)$ avoids $\widehat{\sigma}_i$, and by first part of the lemma, π_i avoids $1 \oplus \sigma_i$. We deduce that

$$\pi \in \mathrm{Av}(1 \oplus \sigma_1) \odot \mathrm{Av}(1 \oplus \sigma_2) \odot \cdots \odot \mathrm{Av}(1 \oplus \sigma_k).$$

From this, the last claim of the lemma follows easily. □

3.2 Proof of Theorem 3.1

We are ready to prove Theorem 3.1. Recall that the goal is to show that if π is a decomposable pattern different from 12, 132 or 213, then $\mathrm{Av}(\pi)$ is splittable. From Corollary 3.3, we already know that we may restrict our attention to the cases when π has the form $1 \oplus \sigma$ for an indecomposable permutation σ of order at least three.

As we have seen, a permutation σ may be uniquely represented by an envelope matching $E(\sigma)$. However, we have also seen that the containment order of permutations does not coincide with the containment order of the corresponding envelope matchings. Our first goal will be to describe permutation containment in terms of envelope matchings.

Definition 3.11. Let M be a matching with n arcs, and let I be an open interval on the real line. The *tangling of M in I* is an operation which produces a matching N with $n+1$ arcs, defined as follows. First, we reorder the endpoints of M belonging I in such a way that all the left endpoints in I appear to the left of all the right endpoints, while the relative position of the left endpoints as well as the relative position of the right endpoints remains the same. Next, we create a new short arc α whose endpoints belong to I, and which is nested below all the other arcs that have at least one endpoint in I. Let N be the resulting matching.

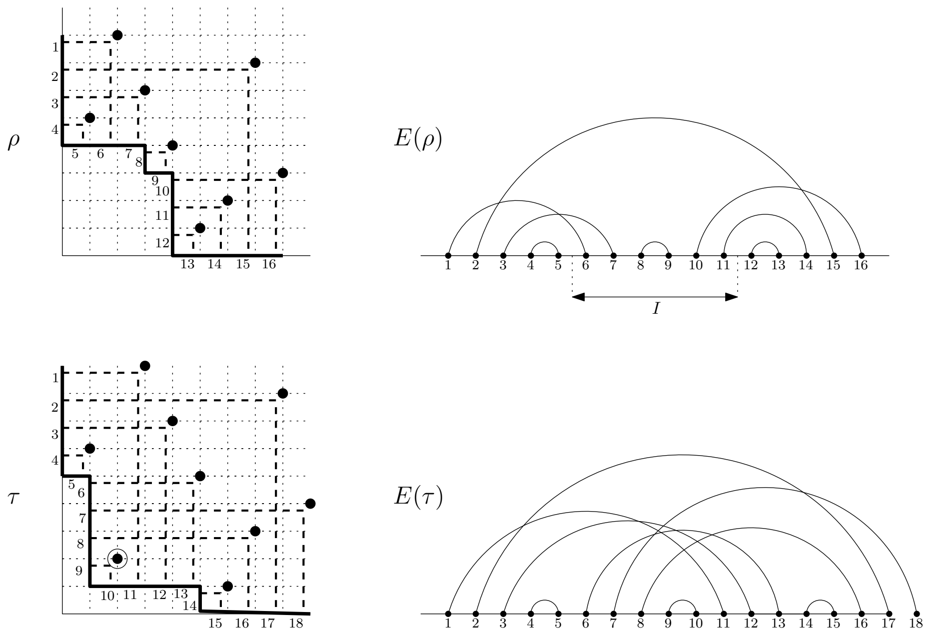

Figure 5: Illustration of Lemma 3.12: a permutation τ is obtained by a inserting a new LR-minimum $\tau(2)$ into a permutation ρ (the new element is circled in the permutation diagram of τ). The envelope matching $E(\tau)$ is then obtained from $E(\rho)$ by tangling in the interval I.

See the right part of Figure 5 for an example of a matching $E(\tau)$ obtained by tangling a matching $E(\rho)$.

Note that if M is an envelope matching, then any tangling of M is again an envelope matching.

Lemma 3.12. *Let $\tau \in S_n$ and $\rho \in S_{n-1}$ be two permutations. Suppose that τ contains ρ, and fix an index $a \in [n]$ such that ρ is order-isomorphic to $\tau \setminus \{\tau(a)\}$. If $\tau(a)$ is not an LR-minimum of τ, then the matching $E(\rho)$ is contained in $E(\tau)$, and more precisely, $E(\rho)$ is obtained from $E(\tau)$ by removing the long arc representing $\tau(a)$. If, on the other hand, $\tau(a)$ is an LR-minimum of τ, then $E(\tau)$ can be created from $E(\rho)$ by tangling in such a way that the short arc inserted by the tangling operation represents $\tau(a)$.*

Proof. The case when $\tau(a)$ is not an LR-minimum follows directly from Observation 3.8. Suppose that $\tau(a)$ is an LR-minimum of τ (see Figure 5). Let γ be the arc of $E(\tau)$ representing $\tau(a)$. Let $\bar{\rho} = \bar{\rho}(1)\bar{\rho}(2)\cdots\bar{\rho}(n-1)$ denote the sequence $\tau \setminus \{\tau(a)\}$. That is, $\bar{\rho}(j) = \tau(j)$ for $j < a$ and $\bar{\rho}(j) = \tau(j+1)$ otherwise. By assumption, $\bar{\rho}$ is order-isomorphic to ρ.

For each $i \in [n-1]$, let α_i be the arc representing $\rho(i)$ in $E(\rho)$, and let β_i be the arc representing $\bar{\rho}(i)$ in $E(\tau)$. Thus, $E(\tau)$ is equal to $\{\beta_1, \ldots, \beta_{n-1}\} \cup \{\gamma\}$.

Let us compare, for some $1 \le i < j \le n-1$, the mutual position of α_i and α_j with the mutual position of β_i and β_j. By Observation 3.8, we see that $\mathrm{right}(\beta_i) < \mathrm{right}(\beta_j)$ and $\mathrm{right}(\alpha_i) < \mathrm{right}(\alpha_j)$. We also see that $\mathrm{left}(\alpha_i) < \mathrm{left}(\alpha_j)$ if and only if $\mathrm{left}(\beta_i) < \mathrm{left}(\beta_j)$.

Observation 3.8 also shows that if α_i and α_j are nested, then β_i and β_j are nested as well, and if α_i and α_j are crossing, then so are β_i and β_j. Thus, the only situation when the relative position of α_i and α_j can differ from the relative position of β_i and β_j is when α_i and α_j are in series, while β_i and β_j are crossing. This happens when $\tau(a)$ covers both $\bar{\rho}(i)$ and $\bar{\rho}(j)$ in τ, but no LR-minimum of ρ covers both $\rho(i)$ and $\rho(j)$.

We partition the set $[n-1]$ into three (possibly empty) disjoint parts L, M and R by defining $L = \{1, \dots, a-1\}$, $R = \{i; \bar{\rho}(i) < \tau(a)\}$, and $M = [n-1] \setminus (L \cup R)$. We will use the shorthand α_L to denote the set $\{\alpha_i; i \in L\}$, and similarly for β_M, $\bar{\rho}_R$, etc.

Note that $\bar{\rho}_M$ is precisely the set of elements of $\bar{\rho}$ that are covered by $\tau(a)$ in τ. Therefore, the matching $\alpha_L \cup \alpha_R$ is isomorphic to $\beta_L \cup \beta_R$, and the arc γ is nested below all the arcs in β_M.

Let x be the rightmost endpoint of an arc in α_L. Note that an arc α_i belongs to α_L if and only if its right endpoint is in the interval $(-\infty, x]$. Symmetrically, let y be the leftmost endpoint of an arc in α_R, and note that α_i belongs to α_R if and only if its left endpoint is in $[y, \infty)$. Let I be the open interval (x, y), and note that an arc α_i belongs to α_M if and only if it either has at least one endpoint in I or if I is nested below it.

Combining the above facts, we see that $E(\tau)$ can be obtained by tangling $E(\rho)$ in the interval I. $\square$

It follows from Lemma 3.12 that if τ is a permutation that contains a pattern ρ, then $E(\tau)$ may be obtained from $E(\rho)$ by a sequence of tanglings and insertions of long arcs.

For the rest of this subsection, fix a permutation π of the form $1 \oplus \sigma$, where σ is an indecomposable permutation of order at least three.

Note that the matching $E(\pi)$ is in fact isomorphic to $\widehat{\pi}$, and that $R(\pi)$ is isomorphic to $\widehat{\sigma}$. Note also that $\widehat{\sigma}$ is connected, since σ is indecomposable.

The proof of Theorem 3.1 relies on two technical lemmas. We first state the two lemmas and prove that they imply Theorem 3.1, and then prove the two lemmas themselves.

Lemma 3.13. *Let σ be an indecomposable permutation of order at least three. Then the class $MAv(\widehat{\sigma})$ is splittable. Furthermore, there exist two connected $\widehat{\sigma}$-avoiding matchings M_1 and M_2, such that the class $MAv(\widehat{\sigma})$ admits the splitting $\{MAv(M_1), MAv(M_2)\}^{*2}$.*

Lemma 3.14. *Suppose that σ is an indecomposable permutation of order at least two, π is the permutation $1 \oplus \sigma$, and N is a $\widehat{\sigma}$-avoiding matching. There is a π-avoiding permutation $\tau \equiv \tau(N)$, such that if ρ is any π-avoiding permutation whose reduced envelope matching avoids N, then ρ avoids τ.*

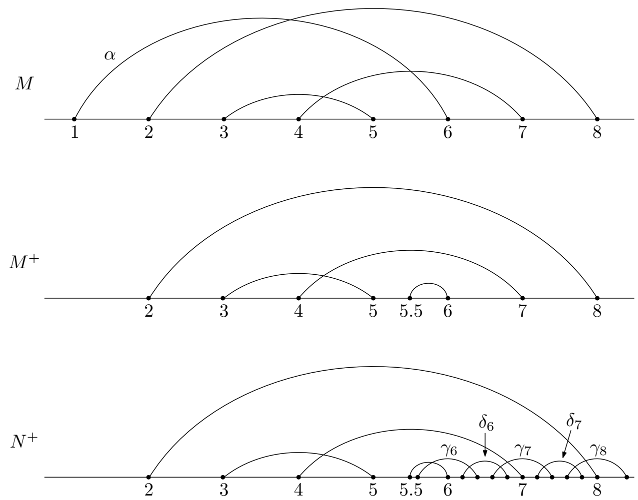

Figure 6: The matching M representing the permutation $\sigma = 2413$ (top), and the corresponding matchings M^+ (middle) and N^+ (bottom) used in the proof of Lemma 3.13.

Let us show how the two lemmas imply Theorem 3.1. Let M_1 and M_2 be the two matchings from Lemma 3.13. For each M_i, Lemma 3.14 provides a π-avoiding pattern $\tau_i \equiv \tau(M_i)$. We claim that $\mathrm{Av}(\pi)$ has a splitting $\{\mathrm{Av}(\tau_1), \mathrm{Av}(\tau_2)\}^{*2}$.

To see this, let ρ be a π-avoiding permutation. Consider its reduced envelope matching $R = R(\rho)$. By Lemma 3.10, R avoids $\widehat{\sigma}$, so by Lemma 3.13, it can be merged from four matching R_1, R_2, R_3, and R_4, where R_1 and R_2 avoid M_1, while R_3 and R_4 avoid M_2.

We now define four permutations $\rho_1, \dots, \rho_4$, all of which are subpermutations of ρ. The permutation ρ_i consists of those elements of ρ which are either LR-minima of ρ or which are covered and correspond to arcs of R_i. Note that R_i is then precisely the reduced envelope matching of ρ_i. Since R_1 and R_2 avoid M_1, Lemma 3.14 shows that ρ_1 and ρ_2 avoid τ_1. Similarly, ρ_3 and ρ_4 avoid τ_2. Thus ρ admits the splitting $\{\mathrm{Av}(\tau_1), \mathrm{Av}(\tau_2)\}^{*2}$, as claimed.

Proof of Lemma 3.13. Let m be the order of σ. By our assumption on σ, we know that $m \geq 3$. Let M be the matching isomorphic to $\widehat{\sigma}$ on the set of endpoints $\{1, 2, \dots, 2m\}$. Note that $\{1, 2, \dots, m\}$ are left endpoints of M, and $\{m+1, \dots, 2m\}$ are right endpoints.

Define two new matchings M^+ and M^- as follows. Let α be the arc of M incident to the leftmost endpoint of M, i.e., α is the arc of the form $(1, x)$ for some $x \in \{m+1, m+2, \dots, 2m\}$. Let M^+ be the matching obtained from M by removing the arc α and inserting a new arc with endpoints $(x - 0.5, x)$ (see Figure 6). Symmetrically, define the matching M^- by considering the arc $\beta = (y, 2m) \in M$ incident to the rightmost endpoint of M, and replacing β with an arc $(y, y+0.5)$.

We now show that it is possible to insert new arcs to M^+ in order to obtain a connected matching N^+ that avoids M. If M has exactly three arcs, we easily check by hand that this is possible, so let us assume that $m \geq 4$.

Let γ_i denote the arc $(i-0.4, i+0.4)$ and let δ_j denote the arc $(j+0.2, j+0.8)$. Define sets of arcs $\Gamma = \{\gamma_i;\ x \leq i \leq 2m\}$ and $\Delta = \{\delta_j;\ x \leq j < 2m\}$, and consider the matching $N^+ = M^+ \cup \Gamma \cup \Delta$. We claim that N^+ is a connected M-avoiding matching.

To see that N^+ is connected, notice first that since M is connected, every connected component of M^+ has at least one arc incident to one of the endpoints $\{x, \dots, 2m\}$. Since any such arc is crossed by an arc of Γ, and since $\Gamma \cup \Delta$ induce a connected matching, we see that N^+ is connected.

Let us argue that N^+ avoids M. Suppose for contradiction that N^+ has a submatching $\overline{M} \subseteq N^+$ isomorphic to M. Note that $\overline{M}$ is a permutation matching, i.e., it has no two arcs in series. In particular, $\overline{M}$ contains at most one arc from Γ and at most one arc from Δ. Since $\overline{M}$ is connected and has at most one arc of Γ, it may not contain arcs from two distinct components of M^+. On the other hand, $\overline{M}$ must contain at least one arc $\gamma_i \in \Gamma$, otherwise it would be a subset of a single component of M^+, which is impossible. Therefore, $\overline{M}$ has no arc whose right endpoint is to the left of x, since any such arc is in series with all the arcs of Γ, and $\overline{M}$ does not contain the arc $(x-0.5, x)$ since this arc does not belong to any connected permutation submatching of N^+ having more than two arcs.

Let $\mathrm{cr}(M)$ denote the number of crossings in the matching M, i.e., the number of edges in the intersection graph of M. We see that $\mathrm{cr}(M^+) = \mathrm{cr}(M) - (2m - x)$. Moreover, $\mathrm{cr}(M) = \mathrm{cr}(\overline{M}) \leq \mathrm{cr}(M^+) + 2$, since $\overline{M}$ is obtained by adding at most one arc of γ and a most one arc of Δ to a submatching of M^+. This shows that $x \geq 2m-2$. We see that $x \neq 2m$, because M is connected. Also, $x \neq 2m-1$, because $\overline{M}$ has at least four arcs. This leaves the possibility $x = 2m-2$ and $\overline{M}$ contains the two arcs of M with endpoints $2m-1$ and $2m$, together with the two arcs γ_{2m-1} and δ_{2m-2}. Since M is connected, this only leaves the possibility when M is the matching $\{(1,6),(2,8),(3,5),(4,7)\}$, but in this case we easily check that N^+ has no copy of M.

We conclude that N^+ is a connected matching containing M^+ but not M. By a symmetric argument, we obtain a connected matching N^- containing M^- but not M. We now show that Lemma 3.13 holds with $M_1 = N^+$ and $M_2 = N^-$, i.e., we show that the class of matchings $\mathrm{MAv}(M) = \mathrm{MAv}(\widehat{\sigma})$ admits the splitting $\mathcal{S} = \{\mathrm{MAv}(N^+), \mathrm{MAv}(N^-)\}^{*2}$.

Let R be any $\widehat{\sigma}$-avoiding matching. Note that a disjoint union of N^+-avoiding matchings is again N^+-avoiding, and the same is true for N^-, since both N^+ and N^- are connected. In particular, if each connected component of R admits the splitting $\mathcal{S}$, then so does R. We may therefore assume that R is connected.

Recall that G_R is the intersection graph of R. Let $\alpha_0 \in R$ be the arc that contains the leftmost endpoint of R. We partition the arcs of R into sets $L_0, L_1, L_2, \dots,$ where an arc β belongs to L_i if and only if the shortest path between α_0 and β in G_R has length i. We refer to the elements of L_i as *arcs of level* i. In particular, α_0 is the only arc of level 0. Since R is connected, the level of each arc is well defined.

Clearly, an arc of level i may only cross arcs of level $i-1$, i or $i+1$. In particular, if each level L_i admits the splitting $\mathcal{S}' = \{\mathrm{MAv}(N^+), \mathrm{MAv}(N^-)\}$, then the union $L_0 \cup L_2 \cup L_4 \cup \cdots$ of all the even levels admits this splitting as well, and the same is true for the union of the odd layers. It then follows that the matching R admits the splitting $\mathcal{S}'^{*2} = \mathcal{S}$.

It is thus enough to show, for each fixed i, that L_i admits the splitting $\mathcal{S}'$. Since L_0 only contains the arc α_0, we focus on the remaining levels, and fix $i \geq 1$ arbitrarily. Note that each arc of level i crosses at least one arc of level $i-1$. For an arc $\beta \in L_i$, let $\nu(\beta)$ be an arbitrary arc of level $i-1$ crossed by β. We now partition L_i into two sets L_i^+ and L_i^- as follows:

$$L_i^+ = \{\beta \in L_i \colon \nu(\beta) \text{ crosses } \beta \text{ from the left}\}, \text{ and}$$
$$L_i^- = \{\beta \in L_i \colon \nu(\beta) \text{ crosses } \beta \text{ from the right}\}.$$

Our goal is to show that L_i^+ avoids N^+ and L_i^- avoids N^-.

Recall that a block of a matching is a maximal $\uplus$-indecomposable submatching. The rest of the proof of Lemma 3.13 is based on two claims.

Claim 1. *Let B be a block of the matching L_i^+, let x be the leftmost endpoint of B, and let β be and arc of B. Then the left endpoint of $\nu(\beta)$ is to the left of x. Similarly, if B' is a block of L_i^-, x' is the rightmost endpoint of B', and β' is an arc of B', then the right endpoint of $\nu(\beta')$ is to the right of x'.*

Proof of Claim 1. We prove the claim for L_i^+, the case of L_i^- is analogous. If $i = 1$, then $\nu(\beta)$ is the arc α_0, and the claim holds. Suppose now that $i > 1$. Let y be the rightmost endpoint of B. Note that any arc of R that has exactly one endpoint in the interval $[x, y]$ must cross at least one arc of B, since B is $\uplus$-indecomposable. In particular, any arc having exactly one endpoint in $[x, y]$ must have level at least $i-1$.

Let $P = (\alpha_0, \alpha_1, \ldots, \alpha_{i-1}, \beta)$ be a shortest path in G_R between α_0 and β, chosen in such a way that $\alpha_{i-1} = \nu(\beta)$. Since α_0 contains the leftmost endpoint of R, both endpoints of α_0 are outside of $[x, y]$, and the same is true for every α_j with $j < i-1$. Consequently, the arc $\nu(\beta)$ must have at least one endpoint outside of $[x, y]$, and since $\nu(\beta)$ crosses β from the left, the left endpoint of $\nu(\beta)$ is to the left of x, and the claim is proved. $\square$

Claim 2. *Every block of the matching L_i^+ avoids M^+, and every block of L_i^- avoids M^-.*

Proof of Claim 2. Let us prove the claim for L_i^+, the other part is symmetric. Let B be a block of L_i^+, and suppose that it contains a matching $\overline{M^+}$ isomorphic to M^+. Let ζ be the arc of $\overline{M^+}$ that corresponds to the arc $(x-0.5, x) \in M^+$ via the isomorphism between $\overline{M^+}$ and M^+. By Claim 1, the left endpoint of $\nu(\zeta)$ is to the left of any endpoint of B, and in particular, it is to the left of any endpoint of $\overline{M^+}$. But this means that the matching $\overline{M^+} \cup \{\nu(\zeta)\} \setminus \{\zeta\}$ is isomorphic to M, contradicting the assumption that R is M-avoiding. $\square$

Since every block of L_i^+ avoids M^+, it also avoids N^+. This means that L_i^+ avoids N^+ as well. Similarly, L_i^- avoids N^-, showing that L_i admits the splitting $\mathcal{S}'$. This means that R admits the splitting $\mathcal{S}$, and Lemma 3.13 is proved. $\square$

Proof of Lemma 3.14. Consider a $\widehat{\sigma}$-avoiding matching N. Suppose that its point set is the set $[2n]$. Our goal is to construct a π-avoiding permutation τ whose envelope matching T has the property that any sequence of tanglings and long arc insertions performed on T will create a matching whose long arcs contain a copy of N or of $\widehat{\sigma}$. It will then be easy to verify that such a permutation τ has the properties stated in the lemma.

Let $m \geq 2$ be the order of σ. Let M be the matching isomorphic to $\widehat{\sigma}$ on the point set $[2m]$. By assumption, σ is indecomposable, so M is connected. Let $i \geq m+1$ be the endpoint connected to m by an arc of M, and let $j \leq m$ be the endpoint connected to $m+1$. Note that m and $m+1$ are not connected by an arc of M, since M is connected. Let M' be the matching obtained from M by replacing the two arcs (m, i) and $(j, m+1)$ by the arcs $(m+1, i)$ and (j, m).

We now construct the required matching T, by adding new arcs to N. For every two consecutive endpoints a and $a+1$ of N such that a is a right endpoint and $a+1$ is a left endpoint, we insert into N an isomorphic copy of M' whose endpoints all belong to the interval $(a, a+1)$. Let N' be the resulting matching. Note that N' is $\widehat{\sigma}$-avoiding, since $\widehat{\sigma}$ is connected and each connected component of N' is a subset of N or a subset of an isomorphic copy of M'.

Next, we create the envelope matching T, by inserting a new short arc between any pair of consecutive endpoints $b < c$ of N' such that b is a left endpoint and c is a right endpoint. This means that the long arcs of T are precisely the arcs of N'. Let τ be the permutation whose envelope matching is T, and therefore its reduced envelope matching is N'.

We claim that τ has the required properties. By Lemma 3.10, τ avoids π. It remains to show that if ρ is a π-avoiding permutation containing τ, then the reduced envelope matching of ρ contains N. Suppose that ρ is a π-avoiding permutation which contains τ, and suppose for contradiction that the matching $R = R(\rho)$ avoids N.

By Lemma 3.12, the envelope matching of ρ can be created from T by a sequence of tanglings and insertions of new long arcs. Since the long arcs of T contain the arcs of N as a subset while $R(\rho)$ avoids N, we see that one of the tanglings used to create $E(\rho)$ from T must have altered the relative order of the endpoints of N. This means that the tangling was performed on an interval I that contained a right endpoint a of N situated to the left of a left endpoint $a+1$ of N. By construction, N' contains a copy of M' in the interval $(a, a+1)$. The tangling on I will change this copy of M' into a copy of M, and insert a new short arc nested below all the arcs of M, thus creating a copy of $\widehat{\pi}$. Since any further tangling or any further insertion of long arcs cannot affect the relative order of endpoints in this copy of $\widehat{\pi}$, we see that $E(\rho)$ contains $\widehat{\pi}$, and therefore ρ contains π, a contradiction.

This completes the proof of Lemma 3.14 and of Theorem 3.1. $\square$

3.3 More natural splittings

The splittings obtained by applying Theorem 3.1 involve patterns which are large and have complicated structure. It is unlikely that such splittings could be directly used for enumeration purposes. In special cases, it is however possible to apply a different argument which yields more natural and more explicit splittings. One such result, stated in the next theorem, is the focus of this subsection.

Theorem 3.15. *Let π be an indecomposable permutation of order n. If the class $Av(\pi)$ admits a splitting*

$$\{Av(\pi_1), Av(\pi_2), \ldots, Av(\pi_k)\}$$

for a multiset $\{\pi_1, \ldots, \pi_k\}$ of indecomposable permutations, then $Av(1 \oplus \pi)$ admits the splitting

$$\{Av(1 \oplus \pi_1), Av(1 \oplus \pi_2), \ldots, Av(1 \oplus \pi_k)\}^{*K},$$

for some $K \leq 16^n$.

Before we prove this theorem, we show that the assumption that $\pi_1, \ldots, \pi_k$ are indecomposable does not restrict the theorem's applicability.

Lemma 3.16. *Let π be an indecomposable permutation. Suppose that $Av(\pi)$ has the splitting*

$$\mathcal{S} = \{Av(\pi_1), Av(\pi_2), \ldots, Av(\pi_k)\}, \tag{1}$$

for some permutations $\pi_1, \ldots, \pi_k$. Suppose furthermore that π_1 can be written as $\pi_1 = \pi_1' \oplus \pi_1''$. Then $Av(\pi)$ also admits the splitting

$$\mathcal{S}' = \{Av(\pi_1'), Av(\pi_1''), Av(\pi_2), \ldots, Av(\pi_k)\}. \tag{2}$$

Proof. To show that $\mathcal{S}'$ is a splitting of $\mathrm{Av}(\pi)$, we will prove the following stronger statement:

$$\mathrm{Av}(\pi) \subseteq (\mathrm{Av}(\pi_1') \odot \mathrm{Av}(\pi_2) \odot \cdots \odot \mathrm{Av}(\pi_k)) \cup (\mathrm{Av}(\pi_1'') \odot \mathrm{Av}(\pi_2) \odot \cdots \odot \mathrm{Av}(\pi_k)).$$

Let ρ be a π-avoiding permutation, and let τ be the permutation $\rho \oplus \rho$. Since π is indecomposable, τ avoids π, and therefore it can be colored by k colors $c_1, \ldots, c_k$ so that the elements of color c_i avoid π_i. The permutation τ is a direct sum of two copies of ρ; we refer to them as the bottom-left copy and the top-right copy. We consider the restrictions of the coloring of τ to the two copies. If the bottom-left copy contains π_1' in color c_1 and the top-right copy contains π_1'' in color c_1, then τ contains π_1 in color c_1, which is impossible.

Suppose that the bottom-left copy has no π_1' in color c_1. Then the coloring of the bottom-left copy of ρ demonstrates that $\rho \in \mathrm{Av}(\pi_1') \odot \mathrm{Av}(\pi_2) \odot \cdots \odot \mathrm{Av}(\pi_k)$, as claimed. Similarly, if the top-right copy of ρ avoids π_1'' in color c_1, we see that $\rho \in \mathrm{Av}(\pi_1'') \odot \mathrm{Av}(\pi_2) \odot \cdots \odot \mathrm{Av}(\pi_k)$. This proves the lemma. $\square$

Corollary 3.17. *If π is indecomposable and the class $Av(\pi)$ is splittable, then $Av(\pi)$ has a splitting of the form $\{Av(\pi_1), \ldots, Av(\pi_k)\}$, where each π_i is an indecomposable π-avoiding permutation.*

The rest of this subsection is devoted to the proof of Theorem 3.15. As in the proof of Theorem 3.1, we use the formalism of ordered matchings. Note that by Observation 3.5, a class $\mathrm{Av}(\pi)$ has a splitting $\{\mathrm{Av}(\pi_1), \ldots, \mathrm{Av}(\pi_k)\}$ if and only if the class of permutation matchings $\widehat{\mathrm{Av}(\pi)}$ has a splitting $\{\widehat{\mathrm{Av}(\pi_1)}, \ldots, \widehat{\mathrm{Av}(\pi_k)}\}$. On the other hand, by Lemma 3.10, a splitting of $\mathrm{Av}(1 \oplus \pi)$ into parts of the form $\mathrm{Av}(1 \oplus \pi_i)$ can be obtained from a splitting of $\mathrm{MAv}(\widehat{\pi})$ into parts of the form $\mathrm{MAv}(\widehat{\pi}_i)$. Therefore, what we need is an argument showing how a splitting of the class $\widehat{\mathrm{Av}(\pi)}$ can be transformed into a splitting of the larger class $\mathrm{MAv}(\widehat{\pi})$. For this, we state and prove the next theorem.

Theorem 3.18. *Let π be a permutation of order n. Suppose that the class $\widehat{Av(\pi)}$ has a splitting*

$$\{\widehat{Av(\pi_1)}, \widehat{Av(\pi_2)}, \ldots, \widehat{Av(\pi_k)}\},$$

where each π_i is a nonempty indecomposable permutation. Then there is a constant $K \leq 16^n$ such that the class of matchings $MAv(\widehat{\pi})$ has a splitting

$$\{MAv(\widehat{\pi}_1), MAv(\widehat{\pi}_2), \ldots, MAv(\widehat{\pi}_k)\}^{*K}.$$

The theorem essentially says that if we can split a $\widehat{\pi}$-avoiding permutation matching into matchings that avoid some other permutation patterns, then we can have an analogous splitting also for non-permutation $\widehat{\pi}$-avoiding matchings. The splitting of non-permutation matchings may need a K-fold increase in the number of parts used.

The bound $K \leq 16^n$ can be slightly improved by a more careful proof, but the particular approach we will use in the proof seems to always produce a bound of the form c^n for some c. Getting a subexponential upper bound for K would be a major achievement, as this would imply, as a special case, a subexponential bound on the chromatic number of circle graphs avoiding a clique of size n, as we shall see in Subsection 3.4.

Recall that for a matching M, an arc $\alpha \in M$ is long if at least one endpoint of M is nested below α; otherwise it is short. Let $\ell(M)$ be the number of long arcs of M. Define the *weight* of M, denoted by $\mathrm{w}(M)$, as $\mathrm{w}(M) = |M| + \ell(M)$. To prove Theorem 3.18, we will in fact prove the following more general result.

Lemma 3.19. *Let π be a permutation of order n. Suppose that the class $\widehat{Av(\pi)}$ has a splitting*

$$\{\widehat{Av(\pi_1)}, \widehat{Av(\pi_2)}, \ldots, \widehat{Av(\pi_k)}\},$$

where each π_i is a nonempty indecomposable permutation. Let M be a matching. There is a constant $K \leq 4^{w(M)}$ such that $MAv(\{\widehat{\pi}, M\})$ has a splitting

$$\{MAv(\widehat{\pi}_1), MAv(\widehat{\pi}_2), \dots, MAv(\widehat{\pi}_k)\}^{*K}.$$

To deduce Theorem 3.18 from Lemma 3.19, we simply choose $M = \widehat{\pi}$ and observe that $\mathrm{w}(M) \leq 2|M|$.

Proof of Lemma 3.19. We proceed by induction on $\mathrm{w}(M)$. Let $\mathcal{S}$ denote the multiset $\{\mathrm{MAv}(\widehat{\pi}_1), \mathrm{MAv}(\widehat{\pi}_2), \dots, \mathrm{MAv}(\widehat{\pi}_k)\}$, and let $K(M)$ denote the smallest value of K for which $\mathcal{S}^{*K}$ is a splitting of $\mathrm{MAv}(\{\widehat{\pi}, M\})$. We want to prove that this value is finite for each M, and in particular, that $K(M) \leq 4^{\mathrm{w}(M)}$. We will assume, without loss of generality, that the endpoints of M form the set $[2m]$, where m is the size of M.

If $\mathrm{w}(M) = 1$, then M consists of a single arc, so the class $\mathrm{MAv}(\{\widehat{\pi}, M\})$ only contains the empty matching, and we have $K(M) = 1$. Suppose now that $\mathrm{w}(M) > 1$. We will distinguish several cases, based on the structure of M.

Suppose first that M has the form $M_1 \uplus M_2$ for some nonempty matchings M_1 and M_2. Note that both M_1 and M_2 have smaller weight than M.

Choose a matching $N \in \mathrm{MAv}(\{\widehat{\pi}, M\})$ with q arcs. Suppose that N has the point set $[2q]$. Let $N[<i]$ be the submatching of N consisting of all those arcs whose both endpoints are smaller than i, and symmetrically, $N[>i]$ consists of those arcs whose both endpoints are larger than i. $N[\leq i]$ and $N[\geq i]$ are defined analogously.

Fix the smallest value of i such that $N[\leq i]$ contains a copy of M_1. If no such value exists, it means that N belongs to $\mathrm{MAv}(\{\widehat{\pi}, M_1\})$, and admits the splitting $\mathcal{S}^{*K(M_1)}$ by induction. Suppose then that i has been fixed. Then $N[>i]$ avoids M_2, otherwise N would contain M. We partition N into three disjoint matchings $N[<i]$, $N[>i]$, and $N_0 = N \setminus (N[<i] \cup N[>i])$. The first of these matchings avoids M_1, the second avoids M_2, and the third one is a permutation matching. It follows that these matchings admit the splittings $\mathcal{S}^{*K(M_1)}$, $\mathcal{S}^{*K(M_2)}$, and $\mathcal{S}$, respectively, and hence N admits the splitting $\mathcal{S}^{*(K(M_1)+K(M_2)+1)}$. We conclude that $K(M) \leq 2(4^{\mathrm{w}(M)-1}) + 1 \leq 4^{\mathrm{w}(M)}$.

Suppose from now on that M is $\uplus$-indecomposable. This part of our argument is analogous to the proof of Lemma 3.13. Fix a matching $N \in \mathrm{MAv}(\{\widehat{\pi}, M\})$. Recall that the permutations $\pi_1, \dots, \pi_k$ are indecomposable by assumption, and therefore the matchings $\widehat{\pi}_i$ are all connected. Therefore, if for some K each connected component of N admits a splitting $\mathcal{S}^{*K}$, then N admits this splitting as well. We will therefore assume from now on that the matching N is connected. Recall that G_N is the intersection graph of N.

Let $\alpha_0 \in N$ be the arc that contains the leftmost endpoint of N. We partition the arcs of N into sets $L_0, L_1, L_2, \dots,$ where an arc β belongs to L_i if and only if the shortest path between α_0 and β in G_N has length i. We refer to the elements of L_i as *arcs of level i*.

Since $\widehat{\pi}_i$ is connected for each i, we see that if each level L_i admits a splitting $\mathcal{S}^{*K}$, then the union $L_0 \cup L_2 \cup L_4 \cup \cdots$ of all the even levels admits this splitting as well, and the same is true for the union of the odd layers. It then follows that the matching N admits the splitting $\mathcal{S}^{*2K}$.

It is thus sufficient to find, for each fixed i, a splitting of L_i of the form $\mathcal{S}^{*K}$. Since L_0 only contains the arc α_0, we focus on $i \geq 1$. As in the proof of Lemma 3.13, for an arc $\beta \in L_i$, we let $\nu(\beta)$ be an arbitrary arc of level $i-1$ crossed by β. We again partition L_i into two sets L_i^+ and L_i^-, with

$$L_i^+ = \{\beta \in L_i \colon \nu(\beta) \text{ crosses } \beta \text{ from the left}\}, \text{ and}$$
$$L_i^- = \{\beta \in L_i \colon \nu(\beta) \text{ crosses } \beta \text{ from the right}\}.$$

We now define two matchings M^+ and M^-, both of strictly smaller weight than M, and show that the blocks of L_i^+ and L_i^- avoid M^+ and M^-, respectively.

Let γ be the arc of M that contains the leftmost endpoint of M, and let x be its other endpoint; in other words, we have $\gamma = \{1, x\}$. Note that γ is long, otherwise M would be $\uplus$-decomposable. Let M^+ be the matching obtained from M by replacing the arc γ with the arc $\{x - 0.5, x\}$. Since we replaced a long arc $\gamma \in M$ by a short arc $\{x - 0.5, x\} \in M^+$, we see that $\mathrm{w}(M^+) = \mathrm{w}(M) - 1$.

We define the matching M^- symmetrically. Let γ' be the arc of M containing the rightmost endpoint of M, and let y be the left endpoint of γ'. We let M^- denote the matching obtained by replacing γ' by the arc $\{y, y + 0.5\}$.

Repeating the arguments of Claim 1 and Claim 2 in the proof of Lemma 3.13, we again see that every block of L_i^+ avoids M^+ and every block of L_i^- avoids M^-. By induction, every block of L_i^+ admits the splitting $\mathcal{S}^{*K(M^+)}$, and consequently, the whole matching L_i^+ admits such splitting as well. Similarly, L_i^- admits the splitting $\mathcal{S}^{*K(M^-)}$.

Consequently, L_i admits the splitting $\mathcal{S}^{*(K(M^+)+K(M^-))}$. This implies that both $L_0 \cup L_2 \cup L_4 \cup \cdots$ and $L_1 \cup L_3 \cup L_5 \cup \cdots$ admit the splitting $\mathcal{S}^{*(K(M^+)+K(M^-))}$. We conclude that N has a splitting $\mathcal{S}^{*K}$, with $K \leq 2(K(M^+) + K(M^-)) \leq 4^{\mathrm{w}(M)}$, as claimed.

This completes the proof of Lemma 3.19 and therefore also of Theorem 3.18 and Theorem 3.15. $\square$

3.4 Splittings and χ-boundedness of circle graphs

Let δ_n denote the decreasing permutation $n(n-1)\cdots 1$. It is well known that any δ_n-avoiding permutation can be merged from at most $n-1$ increasing sequences; in other words, $\mathrm{Av}(\delta_n)$ has a splitting $\{\mathrm{Av}(21)\}^{*(n-1)}$. By Theorem 3.15, we conclude that $\mathrm{Av}(1 \oplus \delta_n)$ has a splitting $\{\mathrm{Av}(132)\}^{*K(n-1)}$ for $K \leq 16^n$.

Let $f(n)$ denote the smallest integer such that $\mathrm{Av}(1 \oplus \delta_n)$ admits the splitting $\{\mathrm{Av}(132)\}^{*f(n)}$. Recall that a *circle graph* is a graph that can be obtained as the intersection graph of a matching. A *proper coloring* of a graph G is the coloring of the vertices of G in such a way that no two adjacent vertices have the same color. The *chromatic number* of G is the smallest number of colors needed for a proper coloring of G. A graph is K_n*-free* if it has no complete subgraph on n vertices.

Proposition 3.20. *Let k and n be integers. Then the class $Av(1 \oplus \delta_n)$ admits the splitting $\{Av(132)\}^{*k}$ if and only if every K_n-free circle graph can be properly colored by k colors. In particular, $f(n)$ is equal to the largest chromatic number of K_n-free circle graphs.*

Proof. Let M be a matching, and let G be its intersection graph. Note that G is K_n-free if and only if M has no n pairwise crossing arcs, i.e., M is $\widehat{\delta_n}$-avoiding. Note also that G can be properly k-colored if and only if M admits the splitting $\{\mathrm{MAv}(\widehat{\delta_2})\}^{*k}$. Thus, if every K_n-free circle graph can be properly k-colored, then $\mathrm{MAv}(\widehat{\delta_n})$ admits the splitting $\{\mathrm{MAv}(\widehat{\delta_2})\}^{*k}$, and by Lemma 3.10, $\mathrm{Av}(1 \oplus \delta_n)$ admits the splitting $\{\mathrm{Av}(132)\}^{*k}$.

Conversely, assume that $\mathrm{Av}(1 \oplus \delta_n)$ admits the splitting $\{\mathrm{Av}(132)\}^{*k}$. We will show that every K_n-free circle graph can be properly k-colored. It is enough to show this for the smallest possible value of k, i.e., for $k = f(n)$. In particular, we may assume that there is a permutation $\rho \in \mathrm{Av}(1 \oplus \delta_n)$ that does not admit the splitting $\{\mathrm{Av}(132)\}^{*(k-1)}$.

Let G be a K_n-free circle graph, and suppose that G is the intersection graph of a matching M. Let σ be a permutation whose reduced envelope matching is M. By Lemma 3.10, σ is $(1 \oplus \delta_n)$-avoiding. Let π be the permutation obtained by simultaneously inflating each LR-minimum of σ by a copy of ρ. Note that π is also $(1 \oplus \delta_n)$-avoiding. Therefore π has a coloring c by k colors with no monochromatic copy of 132. Every vertex v of G corresponds to an arc of M, which will be denoted by $\alpha[v]$. In turn, the arc $\alpha[v]$ corresponds to a covered element $\sigma[v]$ of σ, and this corresponds to an element $\pi[v]$ of π. Let us define a coloring c' of the vertices of G by putting $c'(v) = c(\pi[v])$.

We claim that c' is a proper coloring. To see this, pick two adjacent vertices u and v of G. Then $\alpha[u]$ and $\alpha[v]$ are two crossing arcs in M, and $\sigma[u]$ and $\sigma[v]$ are two covered elements of σ with the property that σ has an LR-minimum $\sigma(i)$ that covers both $\sigma[u]$ and $\sigma[v]$.

Note that in every occurrence of ρ in π, the coloring c must use all k colors, because ρ does not admit the splitting $\{\mathrm{Av}(132)\}^{*(k-1)}$. In particular, the copy of ρ formed by inflating $\sigma(i)$ must contain all k colors. Consequently, $\pi[u]$ and $\pi[v]$ must have distinct colors in order to avoid a monochromatic copy of 132. Thus, $c'(u) \neq c'(v)$, showing that c' is a proper coloring. □

The problem of estimating the value of $f(n)$, i.e., the largest chromatic number of K_n-free circle graphs, has been studied by graph theorists since the 1980s. Gyárfás [21, 22] was the first to prove that this chromatic number is bounded, and showed that $f(n) \leq n^2 2^n (2^n - 2)$. This has been later improved by Kostochka and Kratochvíl [26], who proved that $f(n) \leq 50 \cdot 2^n - O(n)$. They in fact showed that this bound is also applicable to the so-called polygon-circle graphs, which are intersection graphs of sets of polygons inscribed into a common circle, and are easily seen to be a generalization of circle graphs. Currently, the best known upper bound for $f(n)$ is due to Černý [16], who proved that $f(n) \leq 21 \cdot 2^n - O(n)$. All these bounds are still far away from the best known general lower bound $f(n) \geq \Omega(n \log n)$, proven by Kostochka [25].

For specific values of n, better estimates are known. For instance, $f(3) = 5$, as shown by Kostochka [24], who proved that $f(3) \leq 5$, and Ageev [1], who constructed an example of K_3-free 5-chromatic circle graph on 220 vertices. Recently, Nenashev [29] has also shown that $f(4) \leq 30$.

In general, the problem of estimating $f(n)$ appears rather challenging, and one might therefore expect that the more general problem of estimating the value of K will be hard as well. On the other hand, the previously unknown connection between circle graphs and permutations might lead to new useful insights.

4 Open problems and further directions

We have seen that the class $\mathrm{Av}(\sigma)$ is splittable when σ is decomposable (up to small exceptions), and unsplittable when σ is a simple permutation. Obviously, the most natural open problem is to extend this study to the remaining permutation patterns.

Problem 1. For which pattern σ is the class $\mathrm{Av}(\sigma)$ splittable?

The number of simple permutations of order n is asymptotically $\frac{n!}{e^2}(1 - \frac{4}{n} + O(n^{-2}))$, as shown by Albert, Atkinson and Klazar [3, Theorem 5] (see also [35, sequence A111111]). The number of decomposable permutations of order n is easily seen to be $n!(\frac{2}{n} + O(n^{-2}))$ (see [35, sequence A003319]). Thus both these classes form a significant proportion of all permutations, and one might wonder what splittability behavior we should expect for $\mathrm{Av}(\sigma)$ when σ is a 'typical' pattern of large size. We may phrase this formally as follows.

Problem 2. What is the asymptotic probability that $\mathrm{Av}(\sigma)$ is splittable, assuming that σ is chosen uniformly at random among the permutations of order n?

For certain permutation classes, e.g. for $\mathrm{Av}(1342)$, our results imply splittability, but only provide splittings whose parts are defined by avoidance of rather large patterns. We may hope that this is an artifact of our proof technique, and that these classes admit more 'natural' splittings. It is possible to show that every permutation of order at most 3 is unavoidable for $\mathrm{Av}(1342)$, as is every layered permutation or a complement of a layered permutation. On the other hand, it is not clear whether e.g. the permutations 1423 or 2413 are unavoidable in $\mathrm{Av}(1342)$.

Problem 3. Which permutations are unavoidable in the class $\mathrm{Av}(1342)$? Is, e.g., 1423 or 2413 unavoidable in $\mathrm{Av}(1342)$? Does the class $\mathrm{Av}(1342)$ even admit a splitting with all parts of the form $\mathrm{Av}(1423)$ or $\mathrm{Av}(2413)$?

We showed in Lemma 1.5 that an unsplittable class of permutations is necessarily 1-amalgamable. This offers a simple approach to prove splittability of a class C by exhibiting an example of permutations $\pi, \sigma \in C$ that fail to amalgamate in C. Although we never used such approach in the present paper, we believe that amalgamability is a concept worth exploring.

For some splittable classes, it is easy to see that they fail to be 1-amalgamable: e.g., a class of the form $\mathrm{Av}(\rho \oplus 1 \oplus \tau)$ has no amalgamation identifying the leftmost element of $\rho \oplus 1$ with the leftmost element of $1 \oplus \tau$, showing that the class $\mathrm{Av}(\rho \oplus 1 \oplus \tau)$ is not 1-amalgamable. However, for more general splittable classes, there does not seem to be any such obvious argument.

Problem 4. Is there an example of a splittable permutation class that is 1-amalgamable?

We may also ask about higher-order amalgamations or Ramsey properties. As we pointed out in the introduction, by results of Cameron [14] and Böttcher and Foniok [9], any 3-amalgamable permutation class is amalgamable and Ramsey. It is not hard to verify that some classes, e.g. the class $\mathrm{Av}(132)$, or the class $\mathrm{Av}(2413, 3142)$ of separable permutations, are 2-amalgamable but not 3-amalgamable. Beyond that, we do not know much about 2-amalgamable classes.

Problem 5. Which permutation classes are 2-amalgamable? Are there infinitely many of them?

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