On the increase of Grönwall function value at the multiplication of its argument by a prime

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Abstract. We consider the function $G(n) = \frac{\sigma(n)}{n \log\log n}$ (where $\sigma(n) = \sum_{d|n} d$) and set an imposed condition on its argument n, the fulfillment of which is sufficient for the existence of a prime p, at which $G(np) > G(n)$. This inequality is of interest in connection with the Robin's inequality. The paper also presents the results of numerical experiment conducted with superabundant numbers.

Keywords: Riemann hypothesis, Robin's inequality, superabundant numbers

MSC 2010: 11A41, 11M26, 11Y55

1 Introduction

The well-known Robin's theorem [9] proclaims the equivalence of the Riemann hypothesis of non-trivial zeros of the Riemann zeta function and elementary statement that the inequality

$$G(n) < e^{\gamma} \tag{1}$$

is true at any $n > 5040$. Function G, featured here, is called the Grönwall function [5]. It is defined by the equality

$$G(n) = \frac{\sigma(n)}{n \log\log n},$$

in which $\sigma(n)$ denotes the sum of divisors of n and γ — the Euler-Mascheroni constant.

Many publications are devoted to the study of function G behavior. Their review can be found in [4] and [6]. The work [3] contains the following remarkable theorem.

Theorem 1. *Riemann hypothesis is true if and only if for each $n > 5040$ there exists such a positive integer m, at which $G(mn) > G(n)$.*

Our main result is a theorem, which sets an imposed condition on n sufficient for the existence of a *prime* factor m:

Theorem 2. *Let $n = \prod_{i=1}^{k} p_i^{a_i}$, where p_i denotes the i-th prime number ($p_1 = 2$, $p_2 = 3$ etc.). If for certain l the inequality*

$$p_l^{a_l+1} < \log n \tag{2}$$

is true, then the inequality

$$G(np_l) > G(n) \tag{3}$$

is also true.

Material of the paper is as follows. In the second section we give a proof of Theorem 2, the third section describes a numerical experiment which deals with the so-called superabundant numbers, and the last section contains concluding remarks.

2 Proof of the Theorem 2

We need a proposition that establishes a necessary and sufficient condition for the inequality (3), and an auxiliary lemma.

Proposition 3. *Let l be positive integer and $n = \prod_{i=1}^{k} p_i^{a_i} \geq 3$ (p_i is i-th prime). Then*

$$G(np_l) > G(n) \iff (\log n)^{\left(\sum\limits_{\alpha=1}^{a_l+1} p_l^{\alpha}\right)^{-1}} > 1 + \frac{\log p_l}{\log n}.$$

Proof. Let us consider the ratio $\frac{G(np_l)}{G(n)}$. Provided that $\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{a_i+1}-1}{p_i-1}$, we have:

$$\frac{G(np_l)}{G(n)} = \frac{\sigma(np_l)}{\sigma(n)} \cdot \frac{\log\log n}{p_l \cdot \log\log(np_l)} = \frac{p_l^{a_l+2}-1}{p_l^{a_l+2}-p_l} \cdot \frac{\log\log n}{\log\log(np_l)}.$$

Thus,

$$G(np_l) > G(n) \iff (\log n)^{\frac{p_l^{a_l+2}-1}{p_l^{a_l+2}-p_l}} > \log n + \log p_l \iff (\log n)^{\left(\sum\limits_{\alpha=1}^{a_l+1} p_l^{\alpha}\right)^{-1}} > 1 + \frac{\log p_l}{\log n}. \quad \square$$

Lemma 4. *Let t and s be positive integers, and ξ — real number greater than t^s. Then the following inequality holds:*

$$\xi^{\left(\sum\limits_{\alpha=1}^{s} t^{\alpha}\right)^{-1}} > 1 + \frac{\log t}{\xi}.$$

Proof. Since $\xi > t^s$ and $s\left(\sum_{\alpha=1}^{s} t^{\alpha}\right)^{-1} \geq t^{-s}$, we have:

$$\xi^{\left(\sum\limits_{\alpha=1}^{s} t^{\alpha}\right)^{-1}} > t^{s\left(\sum\limits_{\alpha=1}^{s} t^{\alpha}\right)^{-1}} \geq 1 + \log t^{t^{-s}} = 1 + \frac{\log t}{t^s} > 1 + \frac{\log t}{\xi}. \quad \square$$

Proof of the Theorem 2. Provided that in Lemma 4 $\xi = \log n$, $t = p_l$, and $s = a_l + 1$, we can write the following:

$$\log n > p_l^{a_l+1} \Longrightarrow (\log n)^{\left(\sum\limits_{\alpha=1}^{a_l+1} p_l^{\alpha}\right)^{-1}} > 1 + \frac{\log p_l}{\log n}.$$

Combining this with Proposition 3 follows the statement of the Theorem 2. $\square$

3 Numerical experiment

In connection with obtaining Theorem 2 it is natural to ask how common the numbers satisfying the condition (2) are, and how the fact of their presence may be helpful for proving the Robin's inequality (1) in the general case.

When studying behavior of the Grönwall function so-called superabundant numbers (SA numbers) are of special interest. The positive integer n is called superabundant [2, 8], if the inequality $\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}$ holds for any $m < n$. In paper [1] it has been proven that the least (exceeding 5040) counterexample to inequality (1) (if exists) is a SA number. Thus, if we can prove the inequality (1) for SA numbers greater than 5040, it will be proven in the general case.

In our numerical experiment, we implement an orderly search of SA numbers, introducing the function $\Omega\colon \mathbb{N} \to \mathbb{N}$, defined by the equality $\Omega(n) = \sum_{i=1}^{k} a_i$. For each value ω of this function (from 9 to 90), we find the maximum value of the Grönwall function

$$G_{\max}(\omega) = \max \{G(n) \mid \Omega(n) = \omega\}$$

and mark the number $n_\omega^* \in \Omega^{-1}(\omega)$, for which this maximum is reached: $G(n_\omega^*) = G_{\max}(\omega)$.

Values $\omega < 9$ are of no interest, since the greatest value $\Omega(n)$ for SA numbers n, not exceeding 5040, is equal $\Omega(5040) = \Omega(2^4 \cdot 3^2 \cdot 5 \cdot 7) = 8$. At $\omega > 90$, the search for number is too cumbersome for us.

The computation results are shown in Table 1. In the second column the factorization of n_ω^* is written. Dots mean missed primes, each of which occurs in the first power. The third column shows the number of n_ω^* in the sequence of SA-numbers [7]. Each value of n_ω^* is followed by its natural logarithm and the least prime number $p(\omega)$ not included in the factorization of n_ω^*. Ticks in the last column of the table indicate that the following inequality is true

$$p(\omega) < \log n_\omega^*, \tag{4}$$

resulting from inequality (2) at $n = n_\omega^*$ and $p_l = p(\omega)$; exponent a_l is equal to zero. Our calculations show that for the considered values of ω (at $n = n_\omega^*$) the inequality (2) holds only if $a_l = 0$.

Table 1

ω	n^*_ω	SA	$\log n^*_\omega$	$p(\omega)$	Inequality (4)
9	$2^5 \cdot 3^2 \cdot 5 \cdot 7$	20	9.2	11	
10	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	25	11.6	13	
11	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	32	14.2	17	
12	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	35	15.3	17	
13	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	39	16.9	17	
14	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 17$	46	19.7	19	✓
15	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 19$	55	22.7	23	
16	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 23$	62	25.8	29	
17	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 23$	63	26.5	29	
18	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 29$	74	29.9	31	
19	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 31$	85	33.3	37	
20	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	91	35.2	37	
21	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	94	36.3	37	
22	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 37$	106	40.0	41	
23	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 41$	116	43.7	43	✓
24	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 43$	127	47.4	47	✓
25	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 47$	137	51.2	53	
26	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 47$	138	52.0	53	
27	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 53$	149	55.9	59	
28	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 59$	162	60.0	61	
29	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 61$	176	64.1	67	
30	$2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11 \cdots 61$	181	65.7	67	
31	$2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11 \cdots 67$	196	69.9	71	
32	$2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11 \cdots 71$	212	74.2	73	✓
33	$2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11 \cdots 73$	224	78.5	79	
34	$2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdots 73$	231	80.9	79	✓
35	$2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdots 79$	246	85.3	83	✓
36	$2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdots 83$	259	89.7	89	✓
37	$2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdots 89$	272	94.2	97	
38	$2^8 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdots 89$	273	94.9	97	
39	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdots 89$	276	96.0	97	
40	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdots 97$	288	100.5	101	
41	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdots 101$	299	105.2	103	✓
42	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdots 103$	311	109.8	107	✓
43	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 103$	317	112.4	107	✓
44	$2^7 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 109$	341	121.0	113	✓
45	$2^7 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 113$	354	125.8	127	
46	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 113$	356	126.4	127	
47	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 127$	368	131.3	131	✓
48	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 131$	380	136.2	137	
49	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 137$	394	141.1	139	✓
50	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 139$	408	146.0	149	

Continue of Table 1

ω	n^*_ω	SA	$\log n^*_\omega$	$p(\omega)$	Inequality (4)
51	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 139$	409	146.7	149	
52	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 151$	438	156.0	157	
53	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 151$	440	156.7	157	
54	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 157$	458	163.0	163	✓
55	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 157$	459	163.7	163	✓
56	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 163$	476	168.9	167	✓
57	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 167$	493	173.9	173	✓
58	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdots 173$	518	181.2	179	✓
59	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdots 173$	519	181.9	179	✓
60	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdots 181$	554	191.6	191	✓
61	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdots 181$	555	192.3	191	✓
62	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdots 191$	575	197.8	193	✓
63	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdots 193$	596	202.8	197	✓
64	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdots 197$	613	208.1	199	✓
65	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdots 199$	628	213.4	211	✓
66	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdots 211$	643	218.8	223	
67	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 211$	653	221.7	223	
68	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 223$	670	227.1	227	✓
69	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 227$	685	232.5	229	✓
70	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 229$	701	238.0	233	✓
71	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 233$	717	243.4	239	✓
72	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 239$	733	248.9	241	✓
73	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 241$	748	254.4	251	✓
74	$2^9 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 241$	752	256.0	251	✓
75	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 241$	755	257.1	251	✓
76	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 251$	774	262.6	257	✓
77	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 257$	791	268.2	263	✓
78	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 263$	808	273.7	269	✓
79	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 269$	825	279.3	271	✓
80	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 271$	842	284.9	277	✓
81	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 277$	859	290.6	281	✓
82	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 281$	874	296.2	283	✓
83	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 283$	889	301.8	293	✓
84	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 293$	903	307.5	307	✓
85	$2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdots 293$	904	308.2	307	✓
86	$2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdots 293$	912	311.4	307	✓
87	$2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdots 307$	927	317.1	311	✓
88	$2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdots 311$	942	322.8	313	✓
89	$2^9 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdots 317$	971	333.6	331	✓
90	$2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdots 317$	972	334.3	331	✓

4 Concluding remarks

From Table 1 we can see that at the increase in parameter ω, the proportion of its values $\omega' \leq \omega$, for which the inequality (4) is satisfied, rises (which, however, does not rule out a fundamental change in the situation at greater values of ω). If we can prove that

$$\exists \tilde{\omega} \quad \text{s. t.} \quad \forall \omega \in [9, \tilde{\omega}] \quad G_{\max} < e^{\gamma} \quad \text{and} \quad \forall \omega \geq \tilde{\omega} \quad p(\omega) < \log n^*_{\omega},$$

we get a proof of the Riemann hypothesis.

In fact, the fulfillment of the inequality (4) for certain ω value means that the inequality $G_{\max}(\omega+1) > G_{\max}(\omega)$ also holds:

$$G_{\max}(\omega+1) = G(n^*_{\omega+1}) \geq G(n^*_{\omega} p(\omega)) > G_{\max}(\omega).$$

Let us consider any SA number n such that $\Omega(n) \geq \tilde{\omega}$. If we construct an infinite sequence

$$u_1 = n, \; u_2 = n^*_{\Omega(n)+1}, \; u_3 = n^*_{\Omega(n)+2}, \; \dots \;, \; u_i = n^*_{\Omega(n)+i-1}, \; \dots$$

we will note that it has the following property:

$$\forall i, j \qquad i < j \Longrightarrow G(u_i) < G(u_j).$$

Given the Grönwall equality [5]

$$\limsup_{n \to \infty} G(n) = e^{\gamma},$$

we can state that Robin's inequality (1) is true for all SA numbers that exceed 5040, and thus holds in the general case, which is equivalent to the Riemann hypothesis.

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