MORPHIC WORDS AND NESTED RECURRENCE RELATIONS

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Abstract. We explore a family of nested recurrence relations with arbitrary levels of nesting, which have an interpretation in terms of fixed points of morphisms over a countably infinite alphabet. Recurrences in this family are related to a number of well-known sequences, including Hofstadter's G sequence and the Conolly and Tanny sequences. For a recurrence $a(n)$ in this family with only finitely terms, we provide necessary and sufficient conditions for the limit $a(n)/n$ to exist.

1. Introduction

When Hofstadter described his G sequence in [7], defined to be $g(0) = 0$ and $g(n) = n - g(g(n-1))$ for $n \geq 1$, he mentions his discovery of a curious interpretation of $g(n)$ in terms of an infinite rooted tree $\mathcal{T}$. Starting with disconnected nodes labelled $1, 2, 3, \ldots$, The tree $\mathcal{T}$ is constructed step-by-step as follows: step 1 places node 1 as the root and on step $n > 1$, node n is attached to $\mathcal{T}$ as the right-most child of node $g(n)$. Hofstadter notes that $\mathcal{T}$ has a very interesting structure; for instance, the number of nodes at each depth is determined by the Fibonacci sequence. A proof of this interpretation has recently been given by Mustazee Rahman in [13].

Such a tree interpretation has been used successfully to shed light on the behaviour of various other nested recurrences as well. In [11], Kubo and Vakil provide an elegant recursive decomposition of the tree $\mathcal{T}$ representing the Hofstadter-Conway sequence

$$c(n) = c(n - c(n-1)) + c(c(n-1))\,;\, c(1) = c(2) = 1,$$

and use it to prove a number of interesting theorems about $c(n)$. Also of relevance is Golomb's self describing sequence, which is the unique increasing sequence $b(n)$ for which $b(1) = 1$ and every $n \geq 1$ appears $b(n)$ times. The interpretation for $\mathcal{T}$ in the case of $b(n)$ is inherent in the definition of $b(n)$; in $\mathcal{T}$, every child of node $n \geq 2$ has n children (node 2 is a child of itself). The nested recurrence for $b(n)$, due to Colin Mallows [5], is

$$b(n) = b(n - b(b(n-1))) + 1;\, b(1) = 1.$$

Interestingly, the discovery of this recurrence came after $b(n)$ was introduced in [4].

Hofstadter's G sequence, however, serves as a canonical example of the kind of sequences we explore in this paper. The tree $\mathcal{T}$ arising from $g(n)$, which appears as the right subtree of the tree in Figure 1.1, has a very specific structure. In $\mathcal{T}$, every square node has two children: a square node followed by a circle node. On the other hand, every circle node has only one child, a square node. This suggests that $\mathcal{T}$ can be completely described by a simple morphism, namely $\texttt{1} \to \texttt{10}$ and $\texttt{0} \to \texttt{1}$, with $\texttt{1}$'s ($\texttt{0}$'s) representing the square (circle) nodes. This is the morphism whose

Date: January 9, 2012.

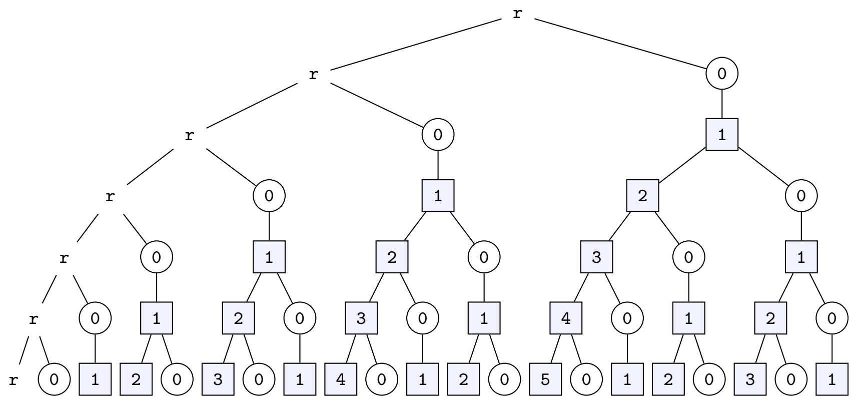

FIGURE 1.1. The tree $\mathcal{T}$ representing $a(n) = n - 1 - a(a(n-1))$, which is a translation of Hofstadter's G sequence (A005206) [16].

unique fixed point is the Fibonacci word, a fact which is hardly a coincidence. In this paper, we identify a family of nested recursions which admit a similar kind of "morphic" tree interpretation.

1.1. **A note on notation.** We adopt a number of conventions in this paper. For a proposition P, we define $[\![P]\!]$ to be 1 when P is true and 0 when P is false. For a given sequence $a(n)$, we let $a^k(n)$ denote k-fold composition and $\nabla a(n)$ denote the backward difference $a(n) - a(n-1)$. For a symbol $\mathtt{s}$, we write $\mathtt{s}^k$ to mean the word $\mathtt{ss}\ldots\mathtt{s}$ of length k. We write $|\mathtt{W}|$ to denote the number of symbols in the word $\mathtt{W}$, and $|\mathtt{W}|_{\mathtt{s}}$ to denote the number of occurrences of the symbol $\mathtt{s}$ in the word $\mathtt{W}$. To avoid ambiguity, we distinguish symbols $\mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots$ from nonnegative integers by using square brackets; $[2^3]$ is the symbol $\mathtt{8}$ while $[2]^3$ is the word $\mathtt{222}$.

2. MAIN THEOREM

Let Σ, Π be alphabets (sets of letters). The set of all finite words with letters from Σ, including the empty word ϵ, is denoted by Σ^*. A *morphism* $\sigma : \Sigma^* \to \Pi^*$ is a function such that if $\mathtt{W}_1, \mathtt{W}_2$ are words in Σ^*, then $\sigma(\mathtt{W}_1\mathtt{W}_2) = \sigma(\mathtt{W}_1)\,\sigma(\mathtt{W}_2)$ and $\sigma(\epsilon) = \epsilon$. Any such morphism σ is uniquely determined by it's action on the individual letters of Σ. Hence, to define σ, it is sufficient to specify $\sigma(\mathtt{x})$ for each letter $\mathtt{x} \in \Sigma$.

Let $R = \langle s, r_1, r_2, r_3, \ldots\rangle$ be a sequence of nonnegative integers with $s \geq 1$. Let σ_R be the following morphism on the infinite alphabet $\Sigma = \{\mathtt{r}, \mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots\}$:

$$\mathtt{r} \to \mathtt{r0}^s$$
$$[j] \to [j+1]\,\mathtt{0}^{r_{j+1}}; \quad j \geq 0.$$

Figure 1.1 above depicts this morphism when $R = \langle 1, 0, 1, 1, 1, \ldots\rangle$.

Let $\mathcal{T}$ be a rooted tree in which each node is labelled with a symbol in Σ. If $V = \{v_1, \ldots, v_k\}$ is a contiguous collection of nodes all on the same row in $\mathcal{T}$, then we say that V *spells* the word $\mathtt{w}_1 \ldots \mathtt{w}_k \in \Sigma^*$ if the label of node v_i is $\mathtt{w}_i$ for $1 \leq i \leq k$.

Consider the infinite rooted tree $\mathcal{T}_R$ in which the root gets labelled $\mathtt{r}$, and the children of each node labelled $\mathtt{x} \in \Sigma$ spell the word $\sigma_R(\mathtt{x})$. If v is a node in $\mathcal{T}_R$, let

$\ell(v)$ count the number of nodes to the left of v. Now define $a_R : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ to be the unique function that satisfies $a_R(n) = 0$ for $n < 0$ and

$$a_R(\ell(v)) = \ell(\text{parent}(v))$$

for any node v in $\mathcal{T}_R$ which is not the root. Put another way, if there are n nodes sitting to the left of some node v in $\mathcal{T}_R$, then $a_R(n)$ counts how many nodes are sitting to the left of the parent of v. This quantity doesn't depend on v, a fact we prove in the following lemma.

Remark. When the sequence R is understood in context, the subscripts in a_R, σ_R and $\mathcal{T}_R$ are omitted as is done below.

Lemma 1. *The function $a(n)$ is well-defined.*

Proof. The k^{th} row in $\mathcal{T}$ spells the word $\sigma^k(\mathtt{r})$ for any $k \geq 0$. Moreover, σ is *nonerasing*, that is, $\sigma(\mathtt{x}) \neq \epsilon$ for all $\mathtt{x} \in \Sigma$. We also have $\mathtt{r} \in \sigma^k(\mathtt{r})$ for any k, and $|\sigma(\mathtt{r})| \geq 2$. Thus $n \mapsto |\sigma^n(\mathtt{r})|$ is a strictly increasing function, which means the size of the rows in $\mathcal{T}$ is unbounded. In particular, for any $n \geq 0$ there exists some node v such that $\ell(v) = n$.

We have that σ is nonerasing and $\sigma(\mathtt{r}) = \mathtt{rZ}$ for some nonempty word $\mathtt{Z}$, and so σ is said to be *prolongable on* $\mathtt{r}$ [1, p. 10]. Hence $\sigma^\infty(\mathtt{r})$ is a well-defined right-infinite word. Moreover, the nodes in each row in $\mathcal{T}$ collectively spell out a prefix of $\sigma^\infty(\mathtt{r})$. Now suppose v is a non-root node. Consider the word $\mathtt{P}$ spelled by all nodes to the left of and including the parent of v. This $\mathtt{P}$ is the smallest prefix $\mathtt{R}$ of $\sigma^\infty(\mathtt{r})$ of such that $|\sigma(\mathtt{R})| \geq \ell(v) + 1$. Thus, $\ell(\text{parent}(v)) = |\mathtt{P}| - 1$. This quantity does not depend on v, merely $\ell(v)$. □

Throughout this paper, the word $\sigma^\infty(\mathtt{r})$ appearing in Lemma 1 will be very important in our analysis of the sequence $a(n)$. Let $\mathtt{W}$ be the right-infinite word which satisfies $\mathtt{rW} = \sigma^\infty(\mathtt{r})$. Equivalently, write $\mathtt{W} = \mathtt{r}^{-1}\sigma^\infty(\mathtt{r})$. We now give two alternative but related interpretations of $a(n)$ in terms of $\mathtt{W}$. The first says that $a(n)$ counts the non-zero symbols in the length-n prefix of $\mathtt{W}$. The second, which follows immediately from the first, says that the first-difference sequence $\{\nabla a(n)\}_{n\geq 1}$ is the binary sequence obtained by replacing all nonzero symbols in $\mathtt{W}$ with 1.

Lemma 2. *Let $\mathtt{W}_n$ be the prefix of $\mathtt{r}^{-1}\sigma^\infty(\mathtt{r})$ of length n. For $n \geq 0$,*

$$|\mathtt{W}_n|_{\mathtt{0}} = n - a(n).$$

Proof. Pick a non-root node v in $\mathcal{T}$ such that $\ell(v) = n$. By definition of σ, every node to the left of and including $v^1 := \text{parent}(v)$ has exactly one child which is not labelled $\mathtt{0}$. Moreover, each such child is always the left-most node among its siblings. There is therefore a 1-1 correspondence between the nodes to the left of and including v^1, and the nodes to the left of and including v *that are not labelled* $\mathtt{0}$. The number of symbols not labelled $\mathtt{0}$ in $\mathtt{rW}_n$ is therefore $\ell(v^1) + 1$. Hence,

$$|\mathtt{W}_n|_{\mathtt{0}} = |\mathtt{rW}_n|_{\mathtt{0}} = n + 1 - \left(\ell\left(v^1\right) + 1\right) = n - a(\ell(v)) = n - a(n).$$

□

A *coding* is a morphism $\beta : \Sigma^* \to \Pi^*$ such that $|\beta(\mathtt{s})| = 1$ for all $\mathtt{s} \in \Sigma$. In the following lemma, define $\beta : \Sigma^* \to \{\mathtt{0}, \mathtt{1}\}^*$ to be the coding $\mathtt{s} \mapsto [[\![\mathtt{s} \neq \mathtt{0}]\!]]$.

Lemma 3. *Let $\mathtt{B} = \mathtt{b}_1\mathtt{b}_2\mathtt{b}_3 \ldots = \beta\left(\mathtt{r}^{-1}\sigma^\infty(\mathtt{r})\right)$ For all $n \geq 1$,*

$$\mathtt{b}_n = [\nabla a(n)].$$

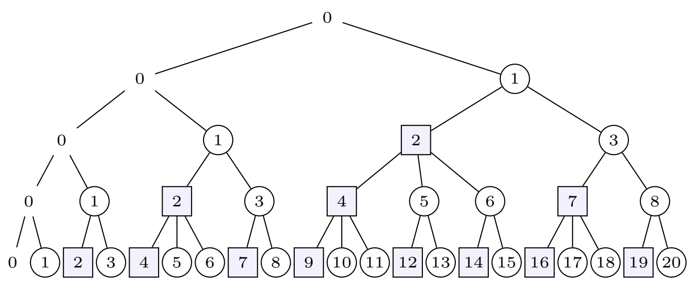

FIGURE 3.1. The tree $\mathcal{T}$ arising from $R = \langle 1, 1, 2, 2, 2, \ldots \rangle$, with the value $\ell(v)$ shown on each node v. The corresponding recurrence is $a(n) = n - 1 - a(n-1) - a(a(n-2))$. This recurrence satisfies $a(n) = \lfloor (\sqrt{2} - 1)(n+1) \rfloor$ (A097508) [16].

Proof. Let $\mathtt{W}_n = \mathtt{w}_1 \ldots \mathtt{w}_n$ be as in Lemma 2. Then,

$$\begin{aligned} \nabla a(n) = a(n) - a(n-1) &= 1 - \left(|\mathtt{W}_n|_{\mathtt{0}} - |\mathtt{W}_{n-1}|_{\mathtt{0}} \right) \\ &= 1 - [\![\mathtt{w}_n = \mathtt{0}]\!] \\ &= [\![\mathtt{w}_n \neq \mathtt{0}]\!]. \end{aligned}$$

Hence

$$[\nabla a(n)] = [[\![\mathtt{w}_n \neq \mathtt{0}]\!]] = \beta(\mathtt{w}_n) = \mathtt{b}_n.$$

□

The following theorem is the main result of this paper. This theorem, as well as its proof, makes use of the following two doubly-indexed quantities:

$$\begin{aligned} r_{i,j} &:= [\![r_i \geq j]\!] \\ c_{i,j} &:= r_{i,j} - r_{i-1,j}. \end{aligned}$$

Here, and throughout the rest of the paper, we define $r_0 = -1$.

Theorem 4. *For $n < s$, $a(n) = 0$ and for $n \geq s$,*

$$a(n) = n - s - \sum_{i,j \geq 1} c_{i,j} a^i (n - j). \quad (2.1)$$

We'll call a recurrence *morphic* if it admits an interpretation in terms of a tree $\mathcal{T}$ as described above.

3. Some Examples

In this section we present what we consider to be interesting examples of morphic recurrences.

3.1. **Beatty Sequences.** A Beatty sequence is a sequence which has the form $\{\lfloor \alpha n \rfloor : n \geq 1\}$, where α is some irrational constant. Of particular interest is the case when α has the continued fraction expansion $[0; k, k, k, \ldots]$, or equivalently, $\alpha = \frac{1}{2}\left(\sqrt{k^2+4} - k\right)$. In this case, the corresponding Beatty sequence is the solution to a morphic recurrence.

Corollary 5. *Let $k \geq 1$, and let $\alpha = [0; k, k, k, \ldots]$. Assume $a(n) = 0$ for $n < k$, and for $n \geq k$, let*

$$a(n) = n - k + 1 - \left(\sum_{i=1}^{k-1} a(n-i)\right) - a(a(n-k)).$$

Then for nonnegative n, $a(n) = \lfloor \alpha(n+1) \rfloor$.

Proof. When $k = 1$, this recurrence is simply Hofstadter's G sequence, and the conclusion has been proven independently by several authors [2, 3, 6]. For $k > 1$, we consider the sequence $R = \langle s, r_1, r_2, \ldots \rangle$ which gives rise to this recurrence. A little calculation shows this recurrence arises precisely when $s = r_1 = k - 1$ and $r_i = k$ for $i \geq 2$. The corresponding morphism σ is

$$\begin{aligned} \mathtt{r} &\to \mathtt{r0}^{k-1} \\ \mathtt{0} &\to \mathtt{10}^{k-1} \\ \mathtt{1} &\to \mathtt{10}^{k}, \end{aligned}$$

where, for brevity, the symbols $\mathtt{1}, \mathtt{2}, \mathtt{3}, \ldots$ are all identified as $\mathtt{1}$. This identification is not a problem, since the underlying structure of the tree $\mathcal{T}$ remains the same.

Let $\mathtt{C}$ be the infinite word $\mathtt{c}_1\mathtt{c}_2\mathtt{c}_3\ldots$, where $\mathtt{c}_n = [\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor]$. Applying a theorem of A. A. Markov, Stolarsky showed in [17] that $\mathtt{C} = \gamma^{\infty}(\mathtt{0})$, where γ is the morphism $\mathtt{0} \to \mathtt{0}^{k-1}\mathtt{1}$, $\mathtt{1} \to \mathtt{0}^{k-1}\mathtt{10}$. The two morphisms σ and γ appear to be similar in their action on $\mathtt{0}$ and $\mathtt{1}$. Indeed, for any right-infinite word $\mathtt{W}$ on $\{\mathtt{0}, \mathtt{1}\}$,

$$\gamma(\mathtt{W}) = \mathtt{0}^{k-1}\sigma(\mathtt{W}). \tag{3.1}$$

This is because if $\mathtt{W} = \mathtt{w}_1\mathtt{w}_2\mathtt{w}_3\ldots$, and ϕ is the morphism $\mathtt{1} \to \mathtt{10}$, $\mathtt{0} \to \mathtt{1}$,

$$\begin{aligned} \gamma(\mathtt{W}) &= \gamma(\mathtt{w}_1)\gamma(\mathtt{w}_2)\gamma(\mathtt{w}_3)\ldots \\ &= \mathtt{0}^{k-1}\phi(\mathtt{w}_1)\,\mathtt{0}^{k-1}\phi(\mathtt{w}_2)\,\mathtt{0}^{k-1}\phi(\mathtt{w}_3)\ldots \\ &= \mathtt{0}^{k-1}\sigma(\mathtt{w}_1)\,\sigma(\mathtt{w}_2)\,\sigma(\mathtt{w}_3)\ldots \\ &= \mathtt{0}^{k-1}\sigma(\mathtt{W}). \end{aligned}$$

Define the infinite word $\mathtt{B} := \mathtt{b}_1\mathtt{b}_2\mathtt{b}_3\ldots$ so that $\mathtt{b}_n = [\nabla a(n)]$. By Lemma 3, $\mathtt{rB} = \sigma^{\infty}(\mathtt{r})$, and so

$$\mathtt{rB} = \sigma(\mathtt{rB}) = \sigma(\mathtt{r})\,\sigma(\mathtt{B}) = \mathtt{r0}^{k-1}\sigma(\mathtt{B}).$$

This implies

$$\mathtt{B} = \mathtt{0}^{k-1}\sigma(\mathtt{B}) = \gamma(\mathtt{B}).$$

As γ has only one fixed point,

$$\mathtt{B} = \gamma^{\infty}(\mathtt{0}) = \mathtt{C}.$$

∎

3.2. ***k*-ary Meta-Fibonacci Sequences.** Define the recurrence

$$b(n) = \sum_{i=1}^{k} b(n - i - b(n-i)) + s,$$

which has initial conditions $b(n) = \max(0, n)$ for $n < s$ and is parametrized by two constants $k, s \geq 1$.

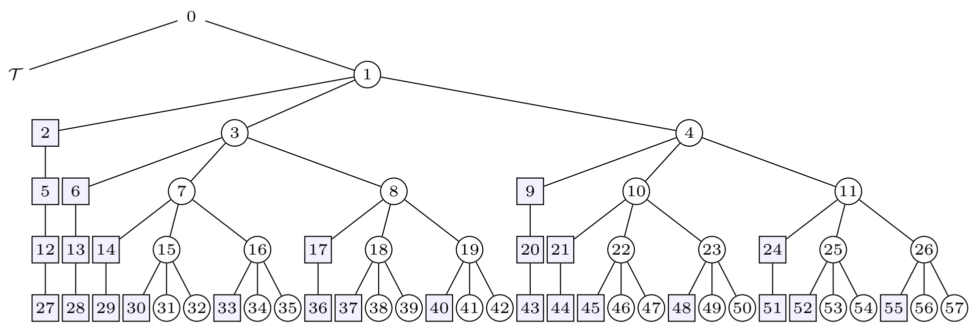

FIGURE 3.2. The tree $\mathcal{T}$ representing the sequence (3.2), with the left subtree, itself a copy of $\mathcal{T}$, stubbed out. As before, $\ell(v)$ is shown for each node v.

Recurrences similar to this one have been studied extensively in recent years [9, 14]. In fact, $b(n)$ is a special case of those which are described in [9], and from there a combinatorial interpretation is given for $b(n)$ in terms of an infinite rooted tree not unlike the one presented in this paper.

It turns out that $b(n)$ can also be described in terms of a morphic recurrence.

Corollary 6. *If $a(n)$ is the sequence $n-b(n)$, then for $n \geq s$, $a(n)$ satisfies*

$$a(n) = n - s - \sum_{i=1}^{k} a(n-i) + \sum_{i=1}^{k} a(a(n-i)),$$

which is the (2.1) recurrence arising from letting $R = \langle s, k, 0, 0, 0, \ldots \rangle$.

Proof. The recurrence for $a(n)$ follows directly from the definitions of $a(n)$ and $b(n)$. We illustrate this calculation when $s=1$ and $k=2$, so that

$$a(n) = n - 1 - a(n-1) - a(n-2) + a(a(n-1)) + a(a(n-2)) \tag{3.2}$$

with $a(n) = 0$ for $n \leq 1$. If we consider the sequence $b(n) = n - a(n)$, then we have for $n \geq 1$,

$$\begin{aligned} b(n) &= n - (n - 1 - a(n-1) + a(a(n-1)) - a(n-2) + a(a(n-2))) \\ &= 1 + a(n-1) - a(a(n-1)) + a(n-2) - a(a(n-2)) \\ &= 1 + b(a(n-1)) + b(a(n-2)) \\ &= 1 + b(n - 1 - b(n-1)) + b(n - 2 - b(n-2)). \end{aligned}$$

□

It can be directly shown that when $s=1$ and $k=2$, $b(n)$ is equal to $c(n+1)-1$, where $c(n)$ is the well-known Conolly sequence (A046699) [16] introduced in [19].

4. Proof of the Theorem

We define a *left-most* node in $\mathcal{T}$ to be a node which has no siblings on its left. We also define an *n-node* in $\mathcal{T}$ to be a node v which satisfies $\ell(v) = n$. Finally, we denote the parent of a node v by v^1, the grandparent by v^2, and so on. We assume $v = v^0$.

To prove the theorem, we introduce the sequence

$$d(n) = \max\left\{k : a^k(n) - a^k(n-1) = 1\right\}$$

for $n \geq 1$. It is a consequence of Lemma 3 that $a(n)$ is *slow-growing*; that is, a is a monotone increasing sequence with successive differences equal to either zero or one. For each n, the number zero lies in this set. Moreover, since $a(0) = 0$, the set is bounded above by the depth of any arbitrary n-node. Therefore, $d(n)$ is well-defined for all n. If we consider an n-node and its adjacent $(n-1)$-node in $\mathcal{T}$, we may interpret $d(n)$ as the length of the path from either of these nodes to their lowest common ancestor.

We begin with the following easy lemma:

Lemma 7. *Let v be an n-node. The following statements are equivalent:*

(1) v is labelled $[k]$ for some $k \geq 1$.
(2) v is a left-most node, and $n \geq 1$.
(3) $\nabla a(n) = 1$.
(4) $d(n) \geq 1$.

Nodes which satisfy these conditions will be called *square* nodes.

Proof. (1) $\implies$ (2): If v is labelled $[k]$ for $k \geq 1$, then v^1 must exist and must be labelled $[k-1]$. The child nodes of v^1 spell the word $\sigma([k-1]) = [k]\,\mathtt{0}^{r_k}$, and so v, which has the unique $[k]$ label, must be the left-most child of v^1. The fact that $n > 0$ follows from the fact that all 0-nodes have the label $\mathtt{r}$, not $[k]$.

(2) $\implies$ (3): Node v^1 must exist since otherwise v would be a 0-node. For the same reason, v^1 must not be a 0-node. But this means that v^1 is positioned to the right of another node w^1, and since σ is nonerasing, every node in $\mathcal{T}$ has a child, and so w^1 is the parent of a $(n-1)$-node w sitting next to v. Therefore, we have

$$\nabla a(n) = a(n) - a(n-1) = \ell\left(v^1\right) - \ell\left(w^1\right) = 1.$$

(3) $\implies$ (4): This implication is immediate from the definitions.

(4) $\implies$ (1): The fact that $d(n) \geq 1$ means that, in particular, $a(n) - a(n-1) = 1$. Since a is nonnegative, $a(n)$ must be positive, which implies that v^1 exists and $\ell\left(v^1\right) \geq 1$. In other words, v^1 is not the left-most node in its row in $\mathcal{T}$. This implies that the label of v^1 is $[j]$ for some $j \geq 0$, since the only nodes with the $\mathtt{r}$ label are the 0-nodes. This also implies that v is not a 0-node, since the children of the nodes to the left of v^1 must come before v. In particular, there exists an $(n-1)$-node w sitting next to v on the same level as v. The fact that $\nabla a(n) = 1$ implies that v is the left-most node of v^1, since if it wasn't then w would also be a child of v^1 and we'd have $a(n) = \ell\left(v^1\right) = a(n-1)$. Therefore, node v must have label $[j+1]$. □

Lemma 8. *Suppose v is a square n-node. Then v^1 has $r_{d(n)} + 1$ children.*

Proof. We show that v^1 is labelled $[d(n) - 1]$. In fact, we show the stronger statement that the label of v^i is $[d(n) - i]$ for $1 \leq i \leq d(n)$. The result will follow from the fact that the children of v^1 spell the word $\sigma([d(n) - 1]) = [d(n)]\,\mathtt{0}^{r_{d(n)}}$, which has $r_{d(n)} + 1$ symbols.

Let w be the $(n-1)$-node to the left of v. Our initial goal is to prove that node $v^{d(n)}$ is labelled $\mathtt{0}$. By definition of $d(n)$, we have $v^{d(n)} \neq w^{d(n)}$ but $v^{d(n)+1} = w^{d(n)+1}$. Therefore, $v^{d(n)}$ and $w^{d(n)}$ are siblings. Since $v^{d(n)}$ sits to the right of

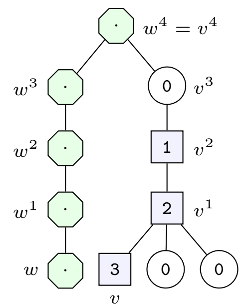

FIGURE 4.1. A depiction of Lemma 8 when $d(n) = 3$ and $r_3 = 2$. The green-shaded octagonal nodes may or may not be square.

$w^{d(n)}$, $v^{d(n)}$ must have the label $\mathtt{0}$. This is because for any $\mathtt{x} \in \Sigma$, the word $\sigma(\mathtt{x})$ contains only zeros after the first symbol.

Our next goal is to show, for $1 \leq i \leq d(n) - 1$, that the label of node $v^{d(n)-i}$ is $[i]$. Observe that nodes $v^{d(n)-i}$ and $w^{d(n)-i}$ must have different parents. Indeed, if this were not the case, then we'd have $v^{d(n)} = w^{d(n)}$, a contradiction. As there are no nodes sitting between $v^{d(n)-i}$ and $w^{d(n)-i}$, $v^{d(n)-i}$ must be the left-most node of its parent, $v^{d(n)-(i-1)}$. If we assume inductively that $v^{d(n)-(i-1)}$ is labelled $[i-1]$, then $v^{d(n)-i}$ is labelled the first symbol of $\sigma([i-1])$, which is $[i]$. □

Lemma 9. *Suppose $n > s$ and $i \geq 0$. Let v be an n-node and w the $(n-i)$-node on the same level as v. Then $r_{d(n-i),i} = 1$ if and only if w is the left-most child of v^1.*

Proof. By definition, $r_{d(n-i),i} = 1$ means that $d(n-i) \geq 1$ and $r_{d(n-i)} \geq i$. We therefore have that w is a square node, so by Lemma 8, w^1 has at least $i+1$ children. Hence, w^1 has enough children so that v is included among them.

Conversely, assume w is the left-most child of v^1. Since $n > s$, v^1 is not a 0-node. Since v and w are siblings, we further have that $n - i \geq 1$. Thus, w is a square node. By Lemma 8, w^1 has $r_{d(n-i)} + 1$ children, and by assumption, w^1 has at least $i+1$ children; this gives $r_{d(n-i)} \geq i$. By Lemma 7, $d(n-i) \geq 1$. These two inequalities imply $r_{d(n-i),i} = 1$. □

Proof of theorem. We proceed by induction on n. When $n = s$, we expect that $a(n) = 0$, and indeed this is the case:

$$a(r_0) = 0 - \sum_{i,j \geq 1} c_{i,j} a^i (r_0 - j) = 0.$$

Suppose, then, that $n > s$ and $a(n-1)$ satisfies the recurrence. Then,

$$a(n) = \nabla a(n) + a(n-1) = \nabla a(n) + n - 1 - s - \sum_{i,j \geq 1} c_{i,j} a^i (n-1-j).$$

Define, for $i, j \geq 0$,

$$e_{i,j} = a^i (n-j) - a^i (n-1-j).$$

Observe that the sequence $(e_{1,j}, e_{2,j}, e_{3,j}, \ldots)$ consists of a finite number of consecutive ones, followed by an infinite number of consecutive zeros. Moreover, the number of ones in this sequence is $d(n-j)$. Thus,

$$\begin{aligned} a(n) &= \nabla a(n) + n - 1 - s - \sum_{i,j\geq 1} c_{i,j}\left(a^i(n-j) - e_{i,j}\right) \\ &= \nabla a(n) + n - 1 - s + \sum_{i,j\geq 1} c_{i,j} e_{i,j} - \sum_{i,j\geq 1} c_{i,j} a^i(n-j). \end{aligned}$$

But

$$\sum_{i,j\geq 1} c_{i,j} e_{i,j} = \sum_{j\geq 1} \sum_{i=1}^{d(n-j)} c_{i,j} = \sum_{j\geq 1} \sum_{i=1}^{d(n-j)} \left(r_{i,j} - r_{i-1,j}\right) = \sum_{j\geq 1} r_{d(n-j),j},$$

and since

$$r_{d(n),0} = [\![r_{d(n)} \geq 0]\!] = [\![d(n) \geq 1]\!] = \nabla a(n),$$

we can write $a(n)$ as

$$a(n) = n - 1 - s + \sum_{i\geq 0} r_{d(n-i),i} - \sum_{i,j\geq 1} c_{i,j} a^i(n-j).$$

Lemma 9 provides both the existence and uniqueness of a nonnegative integer i such that $r_{d(n-i),i} = 1$. Consequently,

$$\sum_{i\geq 0} r_{d(n-i),i} = 1.$$

Substituting this into the above expression for $a(n)$, we get

$$a(n) = n - s - \sum_{i,j\geq 1} c_{i,j} a^i(n-j).$$

□

5. Generating Functions

Let $R = \langle s, r_1, r_2, \ldots \rangle$ with $s \geq 1$, and write $\sigma = \sigma_R$. Let $\mathtt{L}_n = \sigma^n(\mathtt{0}^s)$ and $\mathtt{T}_n = \mathtt{r}^{-1}\sigma^n(\mathtt{r})$. One can easily show by induction that $\mathtt{T}_n$ can be also written as follows:

$$\mathtt{T}_n = \mathtt{L}_0 \mathtt{L}_1 \mathtt{L}_2 \ldots \mathtt{L}_{n-1} = \mathtt{0}^s \sigma(\mathtt{0}^s) \sigma^2(\mathtt{0}^s) \ldots \sigma^{n-1}(\mathtt{0}^s).$$

Given $[x] \in \Sigma \setminus \{\mathtt{r}\}$, we obtain the following recurrence relation from the definition of σ:

$$|\mathtt{L}_{n+1}|_{[x]} = \begin{cases} s & \text{if } n = 0 \\ |\mathtt{L}_n|_{[x-1]} & \text{if } n \geq 1 \text{ and } x \geq 1 \\ \sum_{j\geq 0} r_{j+1} |\mathtt{L}_n|_{[j]} & \text{if } n \geq 1 \text{ and } x = 0. \end{cases}$$

From this recurrence relation we deduce that for $n \geq 1$,

$$|\mathtt{L}_{n+1}|_{\mathtt{0}} = \sum_{i=0}^{n} r_{i+1} |\mathtt{L}_{n-i}|_{\mathtt{0}}.$$

Define the generating functions

$$R(z) := \sum_{n\geq 0} r_n z^n$$

$$N(z) := \sum_{n\geq 0} |\mathtt{L}_n|_{\mathtt{0}} z^n$$

Note that

$$\begin{aligned}
N(z)\,R(z) &= \left(\sum_{n\geq 0}\left|\mathtt{L}_n\right|_{\mathtt{0}} z^n\right)\left(\sum_{n\geq 0} r_n z^n\right)\\
&= \sum_{n\geq 0}\left(\sum_{i=0}^{n} r_i \left|\mathtt{L}_{n-i}\right|_{\mathtt{0}}\right) z^n\\
&= \sum_{n\geq 0}\left(-\left|\mathtt{L}_n\right|_{\mathtt{0}} + \sum_{i=1}^{n} r_i \left|\mathtt{L}_{n-i}\right|_{\mathtt{0}}\right) z^n\\
&= -N(z) + \sum_{n\geq 0}\left(\sum_{i=0}^{n-1} r_{i+1} \left|\mathtt{L}_{n-1-i}\right|_{\mathtt{0}}\right) z^n\\
&= -N(z) + \sum_{n\geq 1}\left|\mathtt{L}_n\right|_{\mathtt{0}} z^n\\
&= -N(z) + N(z) - s\\
&= -s.
\end{aligned}$$

Thus,

$$N(z) = -\frac{s}{R(z)}.$$

Let $L(z) := \sum_{n\geq 0}\left|\mathtt{L}_n\right| z^n$. This is the generating function for the number of nodes of the $(n+1)^{\text{th}}$ row of $\mathcal{T}_R'$, where $\mathcal{T}_R'$ is the tree that results from pruning the left-most subtree of $\mathcal{T}_R$. The recurrence relation gives $\left|\mathtt{L}_n\right| = \sum_{i=0}^{n}\left|\mathtt{L}_n\right|_{[i]} = \sum_{i=0}^{n}\left|\mathtt{L}_i\right|_{\mathtt{0}}$, thus

$$L(z) = -\frac{s}{(1-z)\,R(z)}.$$

Similarly, let $T(z) := \sum_{n\geq 0}\left|\mathtt{T}_n\right| z^n$, which is the generating function for the total number of nodes of row n in $\mathcal{T}_R$ excluding the left-most node labelled $\mathtt{r}$. Since $\left|\mathtt{T}_n\right| = \sum_{i=0}^{n-1}\left|\mathtt{L}_n\right|$, we have

$$T(z) = -\frac{sz}{(1-z)^2\,R(z)}.$$

5.1. **Recurrences with finitely many terms.** The examples of morphic recurrences analyzed thus far have only finitely many terms. Saying that a recurrence $a_R(n)$ of the form (2.1) has finitely many terms is equivalent to saying that $R = \langle s, r_1, r_{2,} \ldots\rangle$ is eventually constant; this follows directly from the statement of Theorem 4. If k is the largest integer such that $r_k \neq r_{k+1}$, then the generating function $R(z)$ is rational. Indeed,

$$R(z) = r_0 + r_1 z + \ldots + r_k z^k + \frac{r_{k+1} z^{k+1}}{1-z}.$$

It follows that $T(z)$ is also rational, and in particular

$$T(z) = -\frac{sz}{(1-z)\,q(z)},$$

where

$$q(z) = (1-z)\left(r_0 + r_1 z + \ldots + r_k z^k\right) + r_{k+1} z^{k+1}. \tag{5.1}$$

5.2. **An example of a recurrence with infinitely many terms.** Theorem 4 implies that nested recurrences can have infinitely many terms and still be well-defined. Suppose, for instance, that $R = \langle 1, 1, 2, 3, 4, \ldots \rangle$. The recurrence given by this sequence is

$$a(n) = n - 1 - a(n-1) - a(a(n-2)) - a(a(a(n-3))) - \ldots$$

with $a(n) = 0$ for $n \leq 0$.

For this particular example we have $R(z) = -1 + z/(1-z)^2$, and hence

$$T(z) = \frac{z}{1 - 3z + z^2}.$$

The coefficients of this generating function are $0, 1, 3, 8, 21, 55, \ldots$, the even Fibonacci numbers starting with $F_0 = 0$. Hence, the length of the word $\mathtt{T}_n$ is F_{2n}. We can use this observation to prove the following statement:

Proposition 10. *Let F_n denote the n^{th} Fibonacci number. For $n \geq 1$,*

$$a(F_{2n}) = F_{2n-2}.$$

Proof. Let v be the right-most node in $\mathcal{T}$ on row n. There are $|\mathtt{T}_n|$ nodes to the left of v. The parent of v is also a right-most node in $\mathcal{T}$, and so there are $|\mathtt{T}_{n-1}|$ nodes to the left of it. Then by definition,

$$a(F_{2n}) = a(|\mathtt{T}_n|) = |\mathtt{T}_{n-1}| = F_{2n-2}.$$

□

Somewhat mysteriously, it also appears that $a(n)$ has the same shifting property on the odd Fibonacci numbers; it seems $a(F_{2n+1}) = F_{2n-1}$ for $n \geq 1$. We do not have a proof of this claim.

6. Asymptotics

In this section, we analyze the asymptotics of recurrences of the form (2.1) which have finitely many terms. In particular, we give sufficient and necessary conditions for such a recurrence to be asymptotically linear and determine its limiting slope. This question has been looked at by Kiss and Zay [10] in the particular case when $R = \langle 1, 0, \ldots, 0, 1, 1, \ldots \rangle$ with k zeros. They show that $\lim_{n\to\infty} a_R(n)/n$ equals the unique positive root of the polynomial $x^k + x - 1$.

When $a_R(n)$ has finitely many terms, that is, when it is the case that there exists some k such that $r_i = r_{k+1}$ for all $i \geq k+1$, then, as done previously in this paper, we can indentify the symbols $[k], [k+1], [k+2], \ldots$ as just one symbol, namely $[k]$. This allows us to define σ_R on a *finite* alphabet $\Sigma_k := \{\mathtt{r}, \mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots, [k]\}$ as follows:

$$\begin{aligned} \mathtt{r} &\to \mathtt{r}\mathtt{0}^s \\ [j] &\to [j+1]\,\mathtt{0}^{r_{j+1}}; \quad 0 \leq j < k \\ [k] &\to [k]\,\mathtt{0}^{r_{k+1}}. \end{aligned}$$

As before, the underlying structure of $\mathcal{T}_R$, upon which the definition of $a_R(n)$ is based, remains unchanged in this alternative definition of σ_R which we now use.

Theorem 11. *Let $R = \langle s, r_1, r_2, \ldots \rangle$, $s \geq 1$ be a nonnegative integer sequence which is eventually constant; that is, there exists some $k \geq 1$ such that $r_i = r_{k+1}$ for all $i \geq k+1$. Then the limit*

$$\lim_{n \to \infty} \frac{a_R(n)}{n}$$

exists if and only if at least one of the following two conditions holds:

(1) $r_1 + r_2 + r_3 + \ldots = 0$ *or* 1.
(2) $\gcd \{i \geq 1 : r_i \geq 1\} = 1$.

If it exists, it is equal to the smallest positive root of the polynomial

$$(1 - x)\left(r_0 + r_1 x + \ldots + r_k x^k\right) + r_{k+1} x^{k+1}.$$

6.1. **Results on nonnegative matrices.** Before giving the proof, we review some preliminaries on nonnegative matrices. See [1, Ch. 8] or [12, Ch. 8] for further details. A nonnegative square matrix M is said to be *reducible* if there exists square matrices A and B, possibly of different sizes, and a permutation matrix P such that

$$PMP^T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where C is an arbitrary matrix and 0 is a zero matrix. M is said to be *irreducible* if it is not reducible. If M is integral and interpreted as the adjancency matrix of a digraph D, then a sufficient and necessary condition for the irreducibility of M is that D is strongly connected [12, p. 671].

Denote the characteristic polynomial of a matrix M by $p_M(x)$. If M is reducible with A, B as above, then

$$p_M(x) = p_A(x)\, p_B(x).$$

Hence, the eigenvalues of M are just the combined eigenvalues of A and B.

Given a nonnegative square matrix M, there exists an eigenvalue λ called the Perron-Frobenius eigenvalue which is equal to the largest modulus of all eigenvalues of M. That λ is itself an eigenvalue is a consequence of the Perron-Frobenius theorem. If M is irreducible with Perron-Frobenius eigenvalue λ, then there exists a positive integer h such that the collection

$$\left\{\lambda, \lambda\omega, \lambda\omega^2, \ldots, \lambda\omega^{h-1}\right\},\ \omega := e^{2\pi i/h},$$

is the collection of all eigenvalues of M with modulus λ. Moreover, every eigenvalue in this collection is simple [1, Theorem 8.3.10]. The number h is called the *index of imprimitivity of* M. If $p_M(x)$ is written as

$$p_M(x) = c_n + c_{n-1}x + c_{n-2}x^2 + \ldots + c_1 x^{n-1} + x^n,$$

then $h = \gcd \{j : c_j \neq 0\}$ [1, Theorem 8.3.9].

An even stronger notion than irreducibility is that of primitivity. A nonnegative square matrix M is said to be *primitive* if there exists some integer $n \geq 1$ such that M^n has only positive entries. If M is integral and interpreted as the adjancency matrix of a digraph D, then a sufficient and necessary condition for the primitivity of M is that there exists an integer n such that between any two vertices u, v of D there exists a walk starting at u and ending at v that has length n.

A key property of primitive matrices is that the Perron-Frobenius eigenvalue λ of M strictly dominates in modulus all other eigenvalues of M. All primitive matrices are irreducible, and the index of imprimitivity of a primitive matrix is 1.

The converse is also true; an irreducible matrix with an index of imprimitivity of 1 is necessarily primitive [1, Theorem 8.3.10].

The proof of the theorem in this section relies on a result due to K. Saari [15], part of which is given in the following proposition. For any morphism $\gamma : \Gamma^* \to \Gamma^*$ defined on a finite alphabet $\Gamma = \{\mathtt{s}_1, \dots, \mathtt{s}_k\}$, we define the *incidence matrix of* γ, denoted M_γ, to be the matrix

$$(M_\gamma)_{i,j} := |\gamma(\mathtt{s}_i)|_{\mathtt{s}_j}.$$

Proposition 12. *Let γ be a nonerasing morphism on a finite alphabet Γ such that:*

1. *There exists an $\mathtt{s} \in \Gamma$ such that $\gamma(\mathtt{s}) = \mathtt{sZ}$ for some nonempty word* $\mathtt{Z}$.
2. *In the Jordan canonical form of M_γ, there is a Jordan block associated with the Perron-Frobenius eigenvalue λ of M_σ which is strictly larger in dimension than any other Jordan block associated with an eigenvalue of modulus λ.*

Then letting $\mathtt{W}_n$ denote the length-n prefix of $\gamma^\infty(\mathtt{s})$, the limit

$$\lim_{n\to\infty} \frac{|\mathtt{W}_n|_{\mathtt{t}}}{n}$$

exists for all $\mathtt{t} \in \Gamma$.

6.2. **Proof of theorem.** To simplify the proof of the theorem a little, we only consider the case when $s = 1$. The arguments are the same in the general case. As before we drop the subscripts from $a_R(n)$ and σ_R.

Proof. We begin by proving that the limit, when it exists, is the smallest positive root of the given polynomial. We have by Theorem 4 that

$$a(n) = n - 1 - \sum_{i,j\geq 1} c_{i,j} a^i(n-j)$$

for $n \geq 1$. Dividing both sides by n and taking limits, we get

$$\begin{aligned}
\alpha := \lim_{n\to\infty} \frac{a(n)}{n} &= \lim_{n\to\infty} \frac{1}{n}\left(n - 1 - \sum_{i,j\geq 1} (r_{i,j} - r_{i-1,j})\, a^i(n-j)\right) \\
&= 1 - \sum_{i,j\geq 1} (r_{i,j} - r_{i-1,j})\, \alpha^i \\
&= 1 - \sum_{i\geq 1} r_i \alpha^i + \sum_{i\geq 1} r_i \alpha^{i+1} \\
&= 1 - r_1\alpha - \sum_{i=1}^{k} (r_{i+1} - r_i)\, \alpha^{i+1}.
\end{aligned}$$

Hence, α is a root of the polynomial

$$\begin{aligned}
q(x) &:= 1 - \sum_{i=0}^{k} (r_{i+1} - r_i)\, x^{i+1} \\
&= r_{k+1}x^{k+1} + (1-x)\sum_{i=0}^{k} r_i.
\end{aligned}$$

To see that α is the smallest such root, note that $q(x)$ is the polynomial (5.1) which appears in the denominator of $T(z)$. Considering $T(z)$ as a complex rational function, then, observe that $T(z)$ has a nonzero radius of convergence c. If u is the smallest positive root of $q(x)$, then $c \leq u$ since u is a pole of $T(z)$. The ratio test implies

$$\lim_{n\to\infty} \frac{|\mathtt{T}_{n-1}|}{|\mathtt{T}_n|} = c.$$

By Lemma 2, however, $|\mathtt{T}_{n-1}| = |\mathtt{T}_n| - |\mathtt{T}_n|_{\mathtt{0}} = a(|\mathtt{T}_n|)$. Hence, the LHS is just α.

It is worth noting that $q(x)$ has a root that is strictly less than 1 if $r_1+r_2+\ldots \geq 2$. Thus, the only time we can have $\alpha = 1$ is when $r_1 + r_2 + \ldots = 0$ or 1. We use this fact later.

We next prove that each of the given conditions imply the existence of the limit. Suppose first that condition (1) holds. If $r_1 + r_2 + \ldots = 0$, then $a(n) = n - 1$ and clearly the theorem holds. Otherwise, suppose $r_1+r_2+\ldots = 1$; that is, there exists some $h \geq 1$ such that $r_h = 1$ and $r_i = 0$ for all positive $i \neq h$. This is something of a degenerate case, and the morphism σ has a very simple structure. In particular, if i is an integer written as $i = qh + r$ for integers q, r with $0 \leq r < h$, then

$$\sigma^i(\mathtt{0}) = [h]^q [r].$$

It follows that $\left|\sigma^i(\mathtt{0})\right|$ is equal to $\lfloor i/h \rfloor + 1$.

Now let m be the smallest integer such that

$$n \leq \left|\mathtt{0}\sigma(\mathtt{0})\sigma^2(\mathtt{0})\ldots\sigma^{mh-1}(\mathtt{0})\right|.$$

The above inequality holds if $m = \left\lceil \sqrt{2n/h} \right\rceil$, since

$$\left|\mathtt{0}\sigma(\mathtt{0})\ldots\sigma^{mh-1}(\mathtt{0})\right| = \sum_{i=0}^{mh-1} (\lfloor i/h \rfloor + 1) = \sum_{i=0}^{m-1} (i+1)h = \frac{hm(m+1)}{2} \geq \frac{hm^2}{2} \geq n.$$

It follows that

$$m \leq \left\lceil \sqrt{2n/h} \right\rceil.$$

Recall Lemma 2, which states that

$$a(n) = n - |\mathtt{W}_n|_{\mathtt{0}}$$

where $\mathtt{W}_n$ is the length-n prefix of

$$\mathtt{0}\sigma(\mathtt{0})\sigma^2(\mathtt{0})\sigma^3(\mathtt{0})\ldots = \mathtt{r}^{-1}\sigma^\infty(\mathtt{r}).$$

For any $i \geq 0$, $\sigma^i(\mathtt{0})$ has at most one $\mathtt{0}$, and so

$$\begin{aligned} a(n) &\geq n - \sum_{i=0}^{mh-1} \left|\sigma^i(\mathtt{0})\right|_{\mathtt{0}} \\ &\geq n - mh \\ &\geq n - \left\lceil \sqrt{2n/h} \right\rceil h. \end{aligned}$$

It is clear that $a(n)/n \leq 1$ for all n. Thus,

$$\liminf_{n\to\infty} \frac{a(n)}{n} \geq \liminf_{n\to\infty} \frac{n - \lceil \sqrt{2n/h} \rceil h}{n} = 1 \geq \limsup_{n\to\infty} \frac{a(n)}{n},$$

and hence the limit exists.

We next show that condition (2) implies the existence of the limit. For the morphism σ defined on Σ_k, the incidence matrix of σ is

$$M_\sigma = \begin{pmatrix} 1 & & & & & \\ 1 & r_1 & r_2 & \cdots & r_k & r_{k+1} \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \end{pmatrix}.$$

Let λ denote the Perron-Frobenius eigenvalue of M_σ. Our goal is to show that λ is simple and is the *only* eigenvalue of M_σ with modulus λ. Then we can apply Proposition 12 and Lemma 2 to conclude

$$\lim_{n\to\infty} \frac{a(n)}{n} = \lim_{n\to\infty} \frac{n - |\mathtt{W}_n|_{\mathtt{0}}}{n} = 1 - \lim_{n\to\infty} \frac{|\mathtt{W}_n|_{\mathtt{0}}}{n}$$

is well-defined. To do this, we consider two cases based on the value of r_{k+1}. Define M_1 and M_2 to be the two submatrices of M_σ as shown:

$$M_\sigma = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 1 & & & \\ \vdots & & M_1 & \\ 0 & & & \end{array}\right) = \left(\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \hline 1 & & & & r_{k+1} \\ \vdots & & M_2 & & \vdots \\ 0 & & & & 0 \\ \hline 0 & 0 & \cdots & 1 & 1 \end{array}\right).$$

If $r_{k+1} \geq 1$, then we show that M_1 is primitive. If $r_{k+1} = 0$, then we show that M_2 is primitive. In either case, we show that the eigenvalues of the primitive submatrix in question are also eigenvalues of M_σ. The remaining eigenvalues of M_σ are then shown to be smaller in modulus than λ, thus proving our goal. Without loss of generality, we assume $r_k \neq r_{k+1}$.

Case 1: $r_{k+1} \geq 1$. Observe that M_1 is the indicence matrix of the morphism σ_1 defined on $\Sigma_k \setminus \{\mathtt{r}\}$ by:

$$\begin{aligned} [j] &\to [j+1]\, \mathtt{0}^{r_{j+1}}; \quad 0 \leq j < k \\ [k] &\to [k]\, \mathtt{0}^{r_{k+1}}. \end{aligned}$$

We show that M_1 is irreducible. The fact that it is primitive then follows from the fact that it has nonzero trace; see [1, Theorem 8.3.9]. To show that M_1 is irreducible, it is sufficient to show that for any $[x], [y] \in \Sigma_k \setminus \{\mathtt{r}\}$, there exists an integer n such that $\sigma_1^n([x])$ contains at least one $[y]$. For such $[x], [y]$, the word $\sigma_1^{k+1-x}([x])$ contains at least one $\mathtt{0}$ since it is assumed that $r_{k+1} \geq 1$. However, the word $\sigma_1^y(\mathtt{0})$ also contains at least one $[y]$. It follows that the word $\sigma_1^{y+k+1-x}([x])$ must contain at least one $[y]$, and so M_1 is irreducible.

The matrix M_σ is a reducible matrix with two irreducible matrices along the diagonal: M_1 and a 1×1 matrix each consisting of a single 1. Thus, the characteristic polynomial of M_σ is

$$p_{M_\sigma}(x) = (x-1)\, p_{M_1}(x).$$

Hence, the eigenvalues of M_σ are the eigenvalues of M_1 and 1.

Case 2: $r_{k+1} = 0$. In this case, M_2 is the incidence matrix of the morphism σ_2 defined on $\Sigma_{k-1} \setminus \{\mathtt{r}\}$ by:

$$\begin{aligned} [j] &\to [j+1]\, \mathtt{0}^{r_{j+1}}; \quad 0 \le j < k-1 \\ [k-1] &\to \mathtt{0}^{r_k}. \end{aligned}$$

The argument that M_2 is irreducible is essentially the same as in the previous case. For $[x], [y] \in \Sigma_{k-1} \setminus \{\mathtt{r}\}$, $\sigma_2^{k-x}([x])$ contains at least one $\mathtt{0}$ since $r_k \neq r_{k+1} = 0$ and $\sigma_2^{y}(\mathtt{0})$ contains at least one $[y]$. Thus $\sigma_2^{y+k-x}([x])$ must contain at least one $[y]$. Now one can check that the characteristic polynomial of M_2 is equal to

$$p_{M_2}(x) = x^k - r_1 x^{k-1} - r_2 x^{k-2} - \ldots - r_{k-1} x - r_k.$$

Thus, the index of imprimitivity of M_2 is equal to $\gcd\{j \ge 1 : r_j \neq 0\}$. By condition (2), however, this is equal to 1. It follows that M_2 is primitive.

When $r_{k+1} = 0$, M_σ has three irreducible matrices along the diagonal: M_2 and two 1×1 matrices each consisting of a single 1. Thus, the characteristic polynomial of M_σ is

$$p_{M_\sigma}(x) = (x-1)^2\, p_{M_2}(x).$$

As in the previous case, the eigenvalues of M_σ are the eigenvalues of M_2 and 1.

In both cases, to show that λ is a simple eigenvalue of M_σ that dominates all others in modulus it remains to show that $\lambda > 1$. The characteristic polynomial of M_σ is equal to

$$p_{M_\sigma}(x) = (x-1)^2 \left(x^k - r_1 x^{k-1} - r_2 x^{k-2} - \ldots - r_{k-1} x - r_k\right) - r_{k+1}(x-1).$$

Without loss of generality, we may assume that $r_1 + r_2 + r_3 + \ldots \ge 2$ since the cases where this sum is 0 or 1 are treated separately in condition (1). If $r_{k+1} \ge 1$, then $p'_{M_\sigma}(1) = -r_{k+1} < 0$. If $r_{k+1} = 0$, then $p'_{M_\sigma}(1) = 0$ but

$$p''_{M_\sigma}(1) = 2(1 - r_1 - r_2 - \ldots - r_k) = 2(1 - r_1 - r_2 - \ldots) < 0.$$

In either case, there exists $\epsilon > 0$ so that $p_{M_\sigma}(1+\epsilon) < 0$. Since $p_{M_\sigma}(x)$ is monic it must be positive for x large enough. By the intermediate value theorem, therefore, there exists $x > 1$ which is an eigenvalue of M_σ. It follows that $\lambda \ge x$.

Finally, we prove the converse of the theorem, that is, we show that the the existence of the limit implies either condition (1) or (2). Suppose $\alpha := \lim a(n)/n$ exists, and consider the integer sequence

$$R' = \langle s, r_h, r_{2h}, \ldots \rangle$$

where $h = \gcd\{j \ge 1 : r_j \ge 1\}$. Note that if $r_1 + r_2 + \ldots = 0$, then h is not defined but the theorem holds trivially. Let $\gamma := \sigma_{R'}$ (as before, we write $\sigma = \sigma_R$). We use the following observations:

(1) Any symbol in $\sigma^n(\mathtt{0})$ other than $[k]$ is congruent to $n \mod h$. In particular, $|\sigma^n(\mathtt{0})|_{\mathtt{0}} = 0$ if $n \not\equiv 0 \mod h$.

(2) If $x \not\equiv -1 \mod h$, then $r_{x+1} = 0$ and so $|\sigma([x])| = |[x+1]| = 1$. It follows that for all $n \ge 0$ and $0 \le j < h$,
$$\left|\sigma^{nh+j}(\mathtt{0})\right| = \left|\sigma^{nh}(\mathtt{0})\right|.$$

(3) We also have that $\sigma^{nh}(\mathtt{0}) = \gamma^n(\mathtt{0})$ after relabelling each symbol $[x]$ in $\sigma^{hn}(\mathtt{0})$ to $[x/h]$. In particular, for all $n \ge 0$,
$$\left|\sigma^{nh}(\mathtt{0})\right| = |\gamma^n(\mathtt{0})| \quad \text{and} \quad \left|\sigma^{nh}(\mathtt{0})\right|_{\mathtt{0}} = |\gamma^n(\mathtt{0})|_{\mathtt{0}}.$$

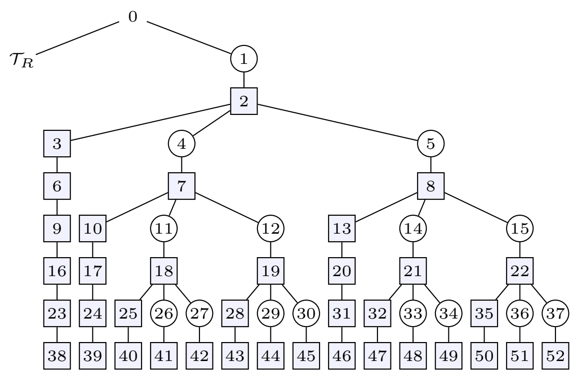

FIGURE 6.1. The tree $\mathcal{T}_R$ representing $R = \langle 1, 0, 2, 0, 0, \ldots \rangle$. In this example, $h = 2$. The tree $\mathcal{T}_{R'}$ is depicted in Figure 3.2.

(4) For $n \geq 1$,

$$\sigma^n(\mathtt{r}) = \mathtt{r0}\sigma(\mathtt{0})\sigma^2(\mathtt{0})\ldots\sigma^{n-1}(\mathtt{0})$$
$$\gamma^n(\mathtt{r}) = \mathtt{r0}\gamma(\mathtt{0})\gamma^2(\mathtt{0})\ldots\gamma^{n-1}(\mathtt{0}).$$

(5) For $n \geq 0$,

$$|\gamma^n(\mathtt{r})| = \left|\gamma^{n+1}(\mathtt{r})\right| - \left|\gamma^{n+1}(\mathtt{r})\right|_{\mathtt{0}}.$$

□

Proof. Because α exists, we must have that

$$1 - \alpha = \lim_{n\to\infty} \frac{\left|\sigma^{nh}(\mathtt{r})\right|_{\mathtt{0}}}{\left|\sigma^{nh}(\mathtt{r})\right|} = \lim_{n\to\infty} \frac{\left|\sigma^{nh+1}(\mathtt{r})\right|_{\mathtt{0}}}{\left|\sigma^{nh+1}(\mathtt{r})\right|}.$$

Let $\beta = \lim_{n\to\infty} a_{R'}(n)/n$, which exists since $\gcd\{hj \geq 1 : r_{hj} \geq 1\} = 1$. We compute both of these limits in terms of β and h, and conclude that at least one of β, h must equal 1. If $\beta = 1$, then $\alpha = 1$ and so condition (1) holds by previous remarks. Assume therefore that $\beta < 1$.

By observations (4), (2), (3), and (4),

$$\begin{aligned}\left|\sigma^{nh}(\mathtt{r})\right| = 1 + \sum_{i=0}^{nh-1}\left|\sigma^{i}(\mathtt{0})\right| = 1 + h\sum_{i=0}^{n-1}\left|\sigma^{ih}(\mathtt{0})\right| &= 1 + h\sum_{i=0}^{n-1}\left|\gamma^{i}(\mathtt{0})\right| \\ &= 1 + h\left(\left|\gamma^n(\mathtt{r})\right| - 1\right)\end{aligned}$$

and by observations (4), (1), (3), and (4),

$$\left|\sigma^{nh}(\mathtt{r})\right|_{\mathtt{0}} = \sum_{i=0}^{nh-1}\left|\sigma^{i}(\mathtt{0})\right|_{\mathtt{0}} = \sum_{i=0}^{n-1}\left|\sigma^{ih}(\mathtt{0})\right|_{\mathtt{0}} = \sum_{i=0}^{n-1}\left|\gamma^{i}(\mathtt{0})\right|_{\mathtt{0}} = \left|\gamma^n(\mathtt{r})\right|_{\mathtt{0}}.$$

Thus

$$\lim_{n\to\infty} \frac{\left|\sigma^{nh}(\mathtt{r})\right|_{\mathtt{0}}}{\left|\sigma^{nh}(\mathtt{r})\right|} = \lim_{n\to\infty} \frac{\left|\gamma^n(\mathtt{r})\right|_{\mathtt{0}}}{1 + h\left(\left|\gamma^n(\mathtt{r})\right| - 1\right)} = \frac{1-\beta}{h}.$$

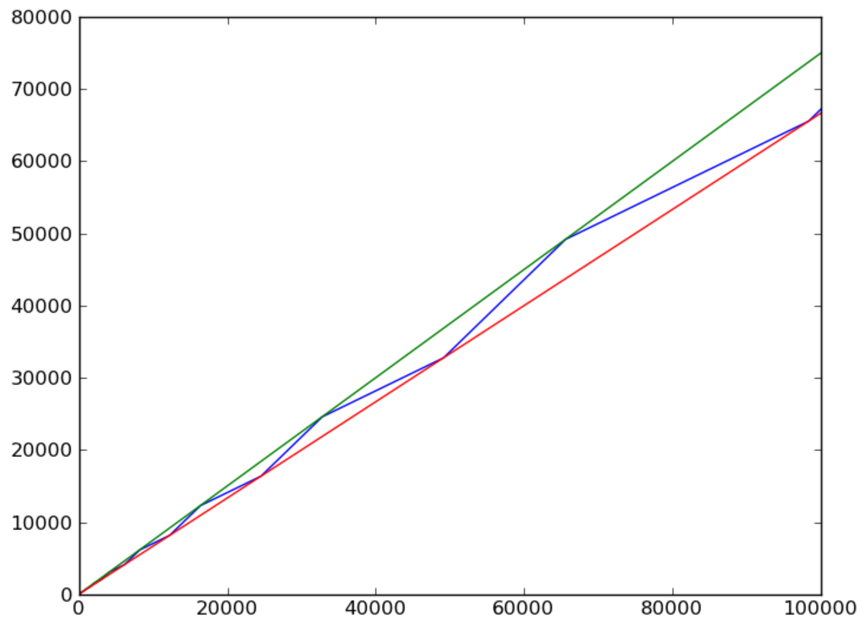

FIGURE 6.2. A plot of $a_R(n)$ when $R = \langle 1, 0, 2, 0, 0, \ldots \rangle$. The sequence is bounded by the two lines shown, but it has no limit.

On the other hand,

$$\begin{aligned}
\left|\sigma^{nh+1}(\mathtt{r})\right| &= \left|\sigma^{nh}(\mathtt{r})\right| + \left|\sigma^{nh}(\mathtt{0})\right| && (4)\\
&= 1 + h\left(\left|\gamma^{n}(\mathtt{r})\right| - 1\right) + \left|\gamma^{n}(\mathtt{0})\right| && (3)\\
&= 1 + (h-1)\left|\gamma^{n}(\mathtt{r})\right| - h + \left|\gamma^{n}(\mathtt{r})\right| + \left|\gamma^{n}(\mathtt{0})\right| \\
&= 1 + (h-1)\left|\gamma^{n}(\mathtt{r})\right| - h + \left|\gamma^{n+1}(\mathtt{r})\right| && (4)\\
&= (h-1)\left(\left|\gamma^{n}(\mathtt{r})\right| - 1\right) + \left|\gamma^{n+1}(\mathtt{r})\right| \\
&= (h-1)\left(\left|\gamma^{n+1}(\mathtt{r})\right| - \left|\gamma^{n+1}(\mathtt{r})\right|_{\mathtt{0}} - 1\right) + \left|\gamma^{n+1}(\mathtt{r})\right| && (5)\\
&= h\left|\gamma^{n+1}(\mathtt{r})\right| - (h-1)\left(\left|\gamma^{n+1}(\mathtt{r})\right|_{\mathtt{0}} - 1\right),
\end{aligned}$$

and by observations (4), (3), and (4),

$$\left|\sigma^{nh+1}(\mathtt{r})\right|_{\mathtt{0}} = \left|\sigma^{nh}(\mathtt{r})\right|_{\mathtt{0}} + \left|\sigma^{nh}(\mathtt{0})\right|_{\mathtt{0}} = \left|\gamma^{n}(\mathtt{r})\right|_{\mathtt{0}} + \left|\gamma^{n}(\mathtt{0})\right|_{\mathtt{0}} = \left|\gamma^{n+1}(\mathtt{r})\right|_{\mathtt{0}}.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|\sigma^{nh+1}\left(\mathtt{r}\right)\right|_{\mathtt{0}}}{\left|\sigma^{nh+1}\left(\mathtt{r}\right)\right|} &= \lim_{n\to\infty}\left(\frac{h\left|\gamma^{n+1}\left(\mathtt{r}\right)\right|-(h-1)\left(\left|\gamma^{n+1}\left(\mathtt{r}\right)\right|_{\mathtt{0}}-1\right)}{\left|\gamma^{n+1}\left(\mathtt{r}\right)\right|_{\mathtt{0}}}\right)^{-1}\\ &= \lim_{n\to\infty}\left(\frac{h\left|\gamma^{n+1}\left(\mathtt{r}\right)\right|}{\left|\gamma^{n+1}\left(\mathtt{r}\right)\right|_{\mathtt{0}}}-\frac{(h-1)\left(\left|\gamma^{n+1}\left(\mathtt{r}\right)\right|_{\mathtt{0}}-1\right)}{\left|\gamma^{n+1}\left(\mathtt{r}\right)\right|_{\mathtt{0}}}\right)^{-1}\\ &= \left(\frac{h}{1-\beta}-(h-1)\right)^{-1}\\ &= \frac{1-\beta}{h-(h-1)\left(1-\beta\right)}.\end{aligned}$$

The two limits are equal, however, and since $\beta \neq 1$ we must have that the two denominators are equal, that is,

$$h = h-(h-1)\left(1-\beta\right).$$

But this implies $(h-1) = \beta\left(h-1\right)$, and so $h = 1$ which is precisely condition (2). □

7. Concluding Remarks

An observation one can make in the study of nested recurrence relations is that they tend to follow come in one of two flavours: either they are highly chaotic and unpredictable, or they appear to have some regular structure. Recurrences of the latter form quite often admit some sort of combinatorial interpretation, and the ones studied in this paper are no exception. Using our morphism and tree interpretations, we are able to better understand the behaviour of our mysterious recurrences. The picture remains incomplete, however, since our asymptotic analysis only applies when $a_R\left(n\right)$ has only finitely many terms. The general case remains unsolved, but we suspect that our asymptotic results no longer hold in the general case and the situation becomes more subtle.

An interesting open problem is to determine whether or not these recurrences have closed forms, since to date very few nested recurrences have been found with closed-form solutions (for some recent developments on this front see [8]). Moreover, not all morphic nested recurrences are of the form (2.1). Indeed, Hofstadter noted in [7] that his "married" functions (see [18]) have a morphism interpretation as well. It would be interesting to see if Theorem 4 could be generalized to include mutually defined nested recurrences. What would such morphisms look like?

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